Research Article

Local Well-Posedness and Blow-Up for the Solutions to the Axisymmetric Inviscid Hall-MHD Equations

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In this paper, we consider the regularity problem of the solutions to the axisymmetric, inviscid, and incompressible Hall-magnetohydrodynamics (Hall-MHD) equations. First, we obtain the local-in-time existence of sufficiently regular solutions to the axisymmetric inviscid Hall-MHD equations without resistivity. Second, we consider the inviscid axisymmetric Hall equations without fluids and prove that there exists a finite time blow-up of a classical solution due to the Hall term. Finally, we obtain some blow-up criteria for the axisymmetric resistive and inviscid Hall-MHD equations.

1. Introduction

Magnetohydrodynamics is the study of the dynamics of the electrically conducting fluids. The dynamics of the fluids can be described by the Navier-Stokes equations and the dynamics of the magnetic field can be described by the Maxwell equations for a perfect conductor. The Hall-magnetohydrodynamics (Hall-MHD) equations differ from the standard incompressible MHD equations by the Hall term $\nabla \times ((\nabla \times B) \times B)$, which plays an important role in the study of the magnetic reconnection in the case of the large magnetic shear (see [1, 2]). In [3], Hall-MHD equations have been formally derived from using the generalized Ohm's law instead of the usual simplified Ohm's law. The Cauchy problem for three-dimensional incompressible Hall-MHD equations reads as follows:

$$
\begin{aligned}
&\partial_t u + (u \cdot \nabla) u + \nabla p = (\nabla \times B) \times B + \nu \Delta u, \\
&\nabla \cdot u = 0, \quad \text{in } \mathbb{R}^3 \times (0, \infty), \\
&\partial_t B - \nabla \times (u \times B) + \nabla \times ((\nabla \times B) \times B) = \kappa \Delta B, \\
&(u(0, x), B(0, x)) = (u_0(x), B_0(x)), \quad \text{in } \mathbb{R}^3,
\end{aligned}
\tag{1}
$$

where u, B, and p represent three-dimensional velocity vector field, the magnetic field, and scalar pressure, respectively. The initial data u_0 and B_0 satisfy

$$
\nabla \cdot u_0 = \nabla \cdot B_0 = 0. \tag{2}
$$

Note that if $\nabla \cdot B_0 = 0$, then the divergence free condition is propagated by $(1)_3$. We only consider $\mathbb{R}^3$ for a spatial domain with vanishing at infinity condition for simplicity.

The Hall magnetohydrodynamics were studied systematically by Lighthill [2]. The Hall-MHD is important, describing many physical phenomena, e.g., space plasmas, star formation, neutron stars, and geo-dynamo (see [1, 4–8] and references therein).

The Hall-MHD equations have been mathematically investigated in several works. In [9], Acheritogaray, Degond, Frouvelle, and Liu derived the Hall-MHD equations from either two fluids' model or kinetic models in a mathematically more rigorous way. In [10], the global existence of weak solutions to (1) and the local well-posedness of classical solution are established when $\nu, \kappa > 0$. Also, a blow-up criterion for smooth solution to (1) and the global existence of smooth solution for small initial data are obtained (see [10, Theorem 2.2 and 2.3]). Some of the results have been refined by many authors (see [11–13] and references therein). Recently, temporal decay for the weak solution and smooth

solution with small data to Hall-MHD are also established in [14]. Spatial and temporal decays of solutions to (1) have been investigated in [15].

Using vector identity, we can rewrite (1) as follows:

$$\partial_t u + (u \cdot \nabla) u + \nabla \left(p + \frac{|B|^2}{2} \right) = (B \cdot \nabla) B + \nu \Delta u,$$

$$\nabla \cdot u = 0,$$

$$\text{in } \mathbb{R}^3 \times (0, \infty), \quad (3)$$

$$\partial_t B + (u \cdot \nabla) B + \nabla \times ((\nabla \times B) \times B) = (B \cdot \nabla) u + \kappa \Delta B,$$

$$(u(0, x), B(0, x)) = (u_0(x), B_0(x)),$$

$$\text{in } \mathbb{R}^3.$$

Note that a weak solution (u, B) to (1) satisfies the following energy inequality (see [10]):

$$\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2$$

$$+ 2 \int_0^t \nu \|\nabla u(\cdot, \tau)\|_{L^2}^2 + \kappa \|\nabla B(\cdot, \tau)\|_{L^2}^2 \, d\tau \quad (4)$$

$$\leq \|u_0\|_{L^2}^2 + \|B_0\|_{L^2}^2,$$

for almost every $t \in [0, \infty)$.

Next we consider the mathematical setting for the axisymmetric Hall-MHD equations. Introducing the cylindrical coordinates

$$r = \sqrt{x_1^2 + x_2^2},$$

$$\theta = \arctan \frac{x_2}{x_1}, \quad (5)$$

$$z = x_3,$$

and standard basis vectors for the cylindrical coordinates

$$e_r = (\cos \theta, \sin \theta, 0),$$

$$e_\theta = (-\sin \theta, \cos \theta, 0), \quad (6)$$

$$e_z = (0, 0, 1),$$

we set

$$u = u^r(r, z, t) e_r + u^z(r, z, t) e_z,$$

$$B = B^\theta(r, z, t) e_\theta. \quad (7)$$

It is well-known that the local-in-time classical solutions to axisymmetric Navier-Stokes equations without swirl persist to any time (see [16, 17]). But the global well-posedness for the axisymmetric Navier-Stokes equations with swirl component is widely open and has been one of the most fundamental open problems in the Navier-Stokes equations.

The axisymmetric MHD equations can be written as follows:

$$\partial_t u^r + (u^r \partial_r + u^z \partial_z) u^r - \nu \left(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) u^r$$

$$+ \partial_r P = -\frac{\left(B^\theta \right)^2}{r},$$

$$\partial_t u^z + (u^r \partial_r + u^z \partial_z) u^z - \nu \left(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 \right) u^z$$

$$+ \partial_z P = 0, \quad (8)$$

$$\partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0,$$

$$\partial_t B^\theta + (u^r \partial_r + u^z \partial_z) B^\theta - \frac{u^r B^\theta}{r}$$

$$- \kappa \left(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) B^\theta = 0.$$

Lei [18] proved the global well-posedness of classical solutions to system (8) when $\kappa \geq 0$.

Then axisymmetric Hall-MHD equations are reduced to the following:

$$\partial_t u^r + (u^r \partial_r + u^z \partial_z) u^r - \nu \left(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) u^r$$

$$+ \partial_r P = -\frac{\left(B^\theta \right)^2}{r},$$

$$\partial_t u^z + (u^r \partial_r + u^z \partial_z) u^z - \nu \left(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 \right) u^z$$

$$+ \partial_z P = 0, \quad (9)$$

$$\partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0,$$

$$\partial_t B^\theta + (u^r \partial_r + u^z \partial_z) B^\theta - \frac{u^r B^\theta}{r}$$

$$- \kappa \left(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) B^\theta = \partial_z \left(\frac{\left(B^\theta \right)^2}{r} \right).$$

For axisymmetric Hall-MHD equations, the global well-posedness of the axisymmetric solutions to the viscous case $(\nu, \kappa > 0)$ was first established by Fan, Huang, and Nakamura [19]. Recently, Chae and Weng [20] showed that the incompressible Hall-MHD system without resistivity is not globally in time well-posed in any Sobolev space $H^m(\mathbb{R}^3)$ with $m > 7/2$. But local-in-time existence of smooth solution to (1) is totally open when $\kappa \equiv 0$. Compared with the work in [18], it seems very surprising that Hall term plays a dominant role for the occurrence of the singularity and even for the local well-posedness of the partially viscous Hall-MHD problems. In this paper, we intend to investigate the blow-up problem for the solutions to the partially viscous axisymmetric Hall-MHD equations and local-in-time existence of solutions to

such solution with the axisymmetry. Setting $\omega^\theta = \partial_z u^r - \partial_r u^z$, $\Omega = \omega^\theta/r$, and $\Pi = B^\theta/r$, (9) are equivalent to the following equations:

$$\frac{\partial\Omega}{\partial t} + (u^r\partial_r + u^z\partial_z)\,\Omega + 2\Pi\partial_z\Pi$$
$$= \nu\left(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\right)\Omega,$$
$$\frac{\partial\Pi}{\partial t} + (u^r\partial_r + u^z\partial_z)\,\Pi - 2\Pi\partial_z\Pi$$
$$= \kappa\left(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\right)\Pi. \tag{10}$$

$$\frac{\operatorname{curl} u}{r} = \Omega,$$
$$\frac{B}{r} = \Pi, \tag{13}$$
$$\Omega(t) \in C\left([0, T_0); H^m\right) \cap \operatorname{Lip}\left([0, T_0); H^{m-1}\right) \text{ and } u(t), B(t), \Pi(t) \in C\left([0, T_0); H^m\right) \cap \operatorname{Lip}\left([0, T_0); H^{m-1}\right).$$

Remark 2. Since the local-in-time regularity of solution to (1) is necessary to preserve the axisymmetry of the Hall-MHD equations locally in time, Theorem 1 cannot resolve the open question raised from [20]. We remark that the relation between (11)–(12) and (1) cannot be justified without local well-posedness of solution to (1) ($\kappa = \nu \equiv 0$).

Next, we consider the local well-posedness/blow-up problem for the axisymmetric Hall equations with zero fluid velocity and $\kappa = 0$. We rewrite the Hall equation for $\Pi = B^\theta/r$:

$$\partial_t\Pi = -2\Pi\partial_z\Pi,$$
$$\Pi(x, 0) = \Pi_0(x). \tag{14}$$

The above equation has similar features to the inviscid Burgers equation.

Theorem 3. *Assume $\Pi_0 \in H^m(\mathbb{R}^3)$ for any integer $m > 5/2$. Then there exist $T_0 > 0$ and a classical solution to (14) such that*

$$\Pi(t) \in C\left([0, T_0); H^m\left(\mathbb{R}^3\right)\right)$$
$$\cap \operatorname{Lip}\left([0, T_0); H^{m-1}\left(\mathbb{R}^3\right)\right). \tag{15}$$

Furthermore, for any $\Pi_0 \neq 0$, there exists $T^* > 0$ such that the above local solution $\Pi(t)$ has singularity at a finite time $t = T^*$.

Remark 4. In [20], the authors showed that if the initial data Π_0 satisfies $\partial_z\Pi_0(0,0) \geq 10^4 C_*^2$ for some constant C_* and $\Pi_0(0,0) > 0$, then the singularity of Π and Ω to axisymmetric inviscid Hall-MHD equations happens in a finite time. Theorem 3 implies that the singularity of Π which is a solution to (14) happens in a finite time without any restriction of the initial data.

First, we consider the local well-posedness of the axisymmetric Hall MHD equations with $\nu \equiv 0$ and $\kappa \equiv 0$, and (10) can be rewritten as the equations

$$\partial_t\Pi + (u^r\partial_r + u^z\partial_z)\,\Pi = 2\Pi\partial_z\Pi \tag{11}$$
$$\partial_t\Omega + (u^r\partial_r + u^z\partial_z)\,\Omega = -2\Pi\partial_z\Pi. \tag{12}$$

Theorem 1. *Let $(u_0, B_0, B_0/r, \operatorname{curl} u_0/r) \in H^m(\mathbb{R}^3) \times H^m(\mathbb{R}^3) \times H^m(\mathbb{R}^3) \times H^{m-1}(\mathbb{R}^3)$ with integer $m > 9/2$ be axisymmetric initial data. There exist $T_0 > 0$ and classical and axisymmetric solution (Π, Ω) to (11)–(12) such that*

Finally, we consider the incompressible Hall-MHD equations with zero fluid viscosity, for simplicity, assuming that $\nu \equiv 0$ and $\kappa \equiv 1$.

For the solutions to (10), global a priori bounds can be obtained; that is,

$$\|\Pi\|_{L^\infty(0,T;L^\infty)} + \|\Omega\|_{L^\infty(0,T;L^2)} < \infty \quad \text{for all } T > 0. \tag{16}$$

We assume that our initial data (u_0, B_0) is axisymmetric and satisfies

$$(u_0, B_0, \Omega_0, \Pi_0) \in H^m\left(\mathbb{R}^3\right)^4 \quad \text{with } m > \frac{5}{2},$$
$$\nabla \cdot u_0 = \nabla \cdot B_0 = 0. \tag{17}$$

The local-in-time existence of a smooth solution to (1) was already obtained by Chae, Wan, and Wu [21]. We obtain the following blow-up criterion for the local-in-time solutions to the Hall-MHD equations with $\nu \equiv 0$ and $\kappa \equiv 1$.

Theorem 5. *Let (u, B, p) be a local-in-time classical solution to the axisymmetric Hall-MHD equations (9) with $\nu = 0, \kappa = 1$. Then, for the first blow-up time $T^* < \infty$ of the classical solution to (9), it holds that*

$$\limsup_{t \nearrow T^*} \left(\|u(t)\|_{H^m}^2 + \|B(t)\|_{H^m}^2 + \|\Omega(t)\|_{H^m}^2\right.$$
$$\left. + \|\Pi(t)\|_{H^m}^2\right) = \infty, \tag{18}$$

if and only if one of the following conditions holds:
(i)

$$\int_0^{T^*} \left\|\frac{u_+^r}{r}\right\|_{L^\infty(\mathbb{R}^3\setminus C_R)} dt + \int_0^{T^*} \|ru_+^r\|_{L^\infty(C_R)} dt = \infty. \tag{19}$$

(ii)

$$\int_0^{T^*} \left\| \omega^\theta \right\|_{L^2(\mathbb{R}^3 \backslash C_R)}^4 dt = \infty. \tag{20}$$

In the above, C_R denotes the inside of infinite cylinder such that $C_R = \{(x, y, z) \mid x^2 + y^2 < R^2\}$ for any $R > 0$ and $f_+(x)$ is defined by $\max\{f(x), 0\}$.

Remark 6. For the usual MHD equations, Lei [18] proved the global well-posedness for the axisymmetric MHD equations even for the case that $\nu \equiv 1$ and $\kappa \equiv 0$. For Hall-MHD equations, even local well-posedness is widely open for this zero resisivity case due to the Hall term (see [20]). Theorem 5 indicates that if there exists a finite time singularity to the axisymmetric equations with $\nu \equiv 0$ and $\kappa \equiv 1$, then some norms of velocity and vorticity should approach infinity even for the outside of any infinite cylinder.

For simplicity, we denote C for the harmless constant which changes from line to line, and $\| \cdot \|_m$ for H^m-norm.

2. Proof of Theorem 1: Local-in-Time Existence

In this section, we consider the local-in-time existence of regular solution to (11)–(12). Even if this problem does not seem complicated, we have a few technical difficulties raised from the axisymmetry; e.g., mollifying equations do not preserve the axial symmetry. We briefly explain some steps to prove Theorem 1: First, we consider system (21) without giving any symmetry. We can obtain the regularized system (25) by using standard mollifier. Then we can obtain various estimates and local-in-time existence of a solution for (21). Finally, we consider the initial data which is axial symmetry and axisymmetry is also preserved by (21) and this argument gives a proof of local-in-time existence of solution to (11)–(12).

We consider the equations

$$\begin{aligned} \partial_t \omega + (\mathbf{u} \cdot \nabla) \omega &= u^r \Omega - 2B \partial_z \Pi, \\ \partial_t B + (\mathbf{u} \cdot \nabla) B &= u^r \Pi + 2B \partial_z \Pi, \\ \partial_t \Pi + (\mathbf{u} \cdot \nabla) \Pi &= 2\Pi \partial_z \Pi, \\ \partial_t \Omega + (\mathbf{u} \cdot \nabla) \Omega &= -2\Pi \partial_z \Pi, \end{aligned} \tag{21}$$

where ω, B, Π, and Ω are assumed to be independent scalar valued functions without assuming symmetry for a while, and the divergence free velocity field $\mathbf{u} = u^r(r, \theta, z)e_r + u^z(r, \theta, z)e_z$ is assumed to be obtained from the equation

$$-\Delta \mathbf{u} = \nabla \times (\omega e_\theta). \tag{22}$$

Thus, we have

$$\mathbf{u}(x) = \Phi * (\nabla \times (\omega e_\theta)), \quad \text{where } \Phi(x) = \frac{1}{4\pi |x|}. \tag{23}$$

If $\omega \in H^1(\mathbb{R}^3) \cap C(\mathbb{R}^3)$, then the divergence theorem and trace theorem induce the following estimates:

$$\begin{aligned} |\mathbf{u}(x)| &\leq \|\Phi\|_{L^2(B(0;1))} \|\nabla \omega\|_{L^2(\mathbb{R}^3)} \\ &\quad + \|\nabla \Phi\|_{L^2(\mathbb{R}^3 \backslash B(0;1))} \|\omega\|_{L^2(\mathbb{R}^3)} \\ &\quad + \int_{\partial B(0;1)} \Phi(y) \omega(x - y) \, dS \\ &\leq \|\Phi\|_{L^2(B(0;1))} \|\nabla \omega\|_{L^2(\mathbb{R}^3)} \\ &\quad + \|\nabla \Phi\|_{L^2(\mathbb{R}^3 \backslash B(0;1))} \|\omega\|_{L^2(\mathbb{R}^3)} \\ &\quad + C \left(\int_{\partial B(0;1)} |\omega(x - y)|^2 \, dS \right)^{1/2} \\ &\leq \|\Phi\|_{L^2(B(0;1))} \|\nabla \omega\|_{L^2(\mathbb{R}^3)} \\ &\quad + \|\nabla \Phi\|_{L^2(\mathbb{R}^3 \backslash B(0;1))} \|\omega\|_{L^2(\mathbb{R}^3)} + C \|\omega\|_{H^1(\mathbb{R}^3)} \\ &\leq C \|\omega\|_{H^1(\mathbb{R}^3)}. \end{aligned} \tag{24}$$

We define a regularized system of (21) as follows:

$$\begin{aligned} \partial_t \omega_\epsilon &+ \mathscr{J}_\epsilon (\mathscr{J}_\epsilon \mathbf{u} \cdot \nabla) \mathscr{J}_\epsilon \omega_\epsilon \\ &= \mathscr{J}_\epsilon (\mathscr{J}_\epsilon u_\epsilon^r \mathscr{J}_\epsilon \Omega_\epsilon) - 2 \mathscr{J}_\epsilon (\mathscr{J}_\epsilon B_\epsilon \partial_z \mathscr{J}_\epsilon \Pi_\epsilon), \\ \partial_t B_\epsilon &+ \mathscr{J}_\epsilon (\mathscr{J}_\epsilon \mathbf{u} \cdot \nabla) \mathscr{J}_\epsilon B_\epsilon \\ &= \mathscr{J}_\epsilon (\mathscr{J}_\epsilon u_\epsilon^r \mathscr{J}_\epsilon \Pi_\epsilon) + 2 \mathscr{J}_\epsilon (\mathscr{J}_\epsilon B_\epsilon \partial_z \mathscr{J}_\epsilon \Pi_\epsilon), \\ \partial_t \Pi_\epsilon &+ \mathscr{J}_\epsilon (\mathscr{J}_\epsilon \mathbf{u} \cdot \nabla) \mathscr{J}_\epsilon \Pi_\epsilon = 2 \mathscr{J}_\epsilon (\mathscr{J}_\epsilon \Pi_\epsilon \partial_z \mathscr{J}_\epsilon \Pi_\epsilon), \\ \partial_t \Omega_\epsilon &+ \mathscr{J}_\epsilon (\mathscr{J}_\epsilon \mathbf{u} \cdot \nabla) \mathscr{J}_\epsilon \Omega_\epsilon = -2 \mathscr{J}_\epsilon (\mathscr{J}_\epsilon \Pi_\epsilon \partial_z \mathscr{J}_\epsilon \Pi_\epsilon), \end{aligned} \tag{25}$$

where $\mathscr{J}_\epsilon$ is a standard mollifier as in [22]. Next, we obtain apriori estimates to derive a time T_0 which does not depend on $\epsilon > 0$. Then we prove that (25) have a local-in-time solution $S_\epsilon = (\omega_\epsilon, B_\epsilon, \Pi_\epsilon, \Omega_\epsilon) \in C^1(0, T_0; H^m) \times C^1(0, T_0; H^m) \times C^1(0, T_0; H^{m+1}) \times C^1(0, T_0; H^m)$ space for each $\epsilon > 0$.

Proposition 7. *Let*

$$\|S_\epsilon\|_{X^m}(t) := \|\omega_\epsilon\|_m + \|B_\epsilon\|_m + \|\Pi_\epsilon\|_{m+1} + \|\Omega_\epsilon\|_m \tag{26}$$

where $X^m = H^m(\mathbb{R}^3) \times H^m(\mathbb{R}^3) \times H^{m+1}(\mathbb{R}^3) \times H^m(\mathbb{R}^3)$ with an integer $m > 5/2$. Then, for some positive constant C_0 and $T < 1/C_0 S_0$ with $S_0 = \|S_\epsilon(0)\|_{X^m}$, we have

$$\sup_{0 < t < T} \|S_\epsilon\|_{X^m}(t) \leq \frac{S_0}{1 - C_0 S_0 T}, \tag{27}$$

$$\sup_{0 < t < T} \|\partial_t S_\epsilon\|_{X^{m-1}}(t) \leq \left(\frac{S_0}{1 - C_0 S_0 T} \right)^2.$$

Proof. For $m > 5/2$ and $1 \leq s \leq m$, we have the following inequality by the calculus inequality and Hölder's inequality:

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla^s \omega_\epsilon\right\|_{L^2}^2$$

$$\leq \int_{\mathbb{R}^3}\left|\mathcal{J}_\epsilon\nabla^s\omega_\epsilon\right|\left|\nabla^{s-1}\left(\mathcal{J}_\epsilon\nabla\mathbf{u}_\epsilon\cdot\mathcal{J}_\epsilon\nabla\omega_\epsilon\right)\right|dx$$

$$+ \int_{\mathbb{R}^3}\left|\mathcal{J}_\epsilon\nabla^s\omega_\epsilon\right|\left|\nabla^s\left(\mathcal{J}_\epsilon u_\epsilon^r\mathcal{J}_\epsilon\Omega_\epsilon\right)\right|dx$$

$$+ \int_{\mathbb{R}^3}\left|\mathcal{J}_\epsilon\nabla^s\omega_\epsilon\right|\left|\nabla^s\left(\mathcal{J}_\epsilon B_\epsilon\partial_z\mathcal{J}_\epsilon\Pi_\epsilon\right)\right|dx \leq \left\|\omega_\epsilon\right\|_m \tag{28}$$

$$\cdot\left(\left\|\nabla\mathbf{u}_\epsilon\right\|_{L^\infty}\left\|\nabla^s\omega_\epsilon\right\|_{L^2} + \left\|\nabla^s\mathbf{u}_\epsilon\right\|_{L^2}\left\|\nabla\omega_\epsilon\right\|_{L^\infty}\right)$$

$$+ \left\|\omega_\epsilon\right\|_m\left(\left\|u_\epsilon^r\right\|_{L^\infty}\left\|\nabla^s\Omega_\epsilon\right\|_{L^2} + \left\|\Omega_\epsilon\right\|_{L^\infty}\left\|\nabla^s u_\epsilon^r\right\|_{L^2}\right)$$

$$+ 2\left\|\omega_\epsilon\right\|_m\left(\left\|B_\epsilon\right\|_{L^\infty}\left\|\nabla^{s+1}\Pi_\epsilon\right\|_{L^2}\right.$$

$$+ \left\|\nabla\Pi_\epsilon\right\|_{L^\infty}\left\|\nabla^s B_\epsilon\right\|_{L^2}\right) \leq C\left\|\omega_\epsilon\right\|_m\left(\left\|\omega_\epsilon\right\|_{m-1}\left\|\omega_\epsilon\right\|_m\right.$$

$$+ \left\|\Omega_\epsilon\right\|_m\left\|\omega_\epsilon\right\|_m + \left\|\Pi_\epsilon\right\|_{m+1}\left\|B_\epsilon\right\|_m\Big).$$

Similarly, for $m > 5/2$ and $1 \leq s \leq m$, we can obtain

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla^s B_\epsilon\right\|_{L^2}^2 \leq \left\|B_\epsilon\right\|_m\left(\left\|\omega_\epsilon\right\|_{m-1}\left\|B_\epsilon\right\|_m\right.$$

$$+ \left\|\omega_\epsilon\right\|_m\left\|\Pi_\epsilon\right\|_m + \left\|\Pi_\epsilon\right\|_{m+1}\left\|B_\epsilon\right\|_m\Big), \tag{29}$$

and $1 \leq s \leq m + 1$,

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla^s\Pi_\epsilon\right\|_{L^2}^2 \tag{30}$$

$$\leq \left\|\Pi_\epsilon\right\|_{m+1}\left(\left\|\omega_\epsilon\right\|_m\left\|\Pi_\epsilon\right\|_{m+1} + \left\|\Pi_\epsilon\right\|_{m+1}^2\right).$$

For an integer $m > 5/2$ and all integers $1 \leq s \leq m$, we have

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla^s\left(\Pi_\epsilon + \Omega_\epsilon\right)\right\|_{L^2}^2 \leq \left\|\Pi_\epsilon + \Omega_\epsilon\right\|_m^2\left\|\omega_\epsilon\right\|_{m-1}. \tag{31}$$

Hence, we conclude that, for $m > 5/2$,

$$\frac{d}{dt}\left\|S_\epsilon\right\|_{X^m}(t)$$

$$\leq \frac{1}{2}C_0\left(\left\|\omega_\epsilon\right\|_m^2 + \left\|B_\epsilon\right\|_m^2 + \left\|\Pi_\epsilon\right\|_{m+1}^2 + \left\|\Pi_\epsilon + \Omega_\epsilon\right\|_m^2\right) \tag{32}$$

$$\leq C_0\left\|S_\epsilon\right\|_{X^m}^2(t).$$

Inequality (32) implies that (27) holds true. □

Proposition 8. *Assume C_0 and S_0 are the same as in Proposition 7. Let $T_0 = 3/4C_0 S_0$. Then for given initial data $\omega_0, B_0, \Omega_0 \in H^m(\mathbb{R}^3)$, and $\Pi_0 \in H^{m+1}(\mathbb{R}^3)$ with an integer $m > 5/2$, there exists a unique solution $(\omega_\epsilon, B_\epsilon, \Pi_\epsilon, \Omega_\epsilon)$ to regularized system (25) such that $\omega_\epsilon, B_\epsilon, \Omega_\epsilon \in C^1(0, T_0; H^m)$, $\Pi_\epsilon \in C^1(0, T_0; H^{m+1})$.*

Proof. We set

$$F_\epsilon^1\left(\omega_\epsilon, B_\epsilon, \Pi_\epsilon, \Omega_\epsilon\right) = -\mathcal{J}_\epsilon\left(\mathcal{J}_\epsilon\mathbf{u}_\epsilon\cdot\nabla\right)\mathcal{J}_\epsilon\omega_\epsilon$$

$$+ \mathcal{J}_\epsilon\left(\mathcal{J}_\epsilon u_\epsilon^r\mathcal{J}_\epsilon\Omega_\epsilon\right)$$

$$- 2\mathcal{J}_\epsilon\left(\mathcal{J}_\epsilon B_\epsilon\partial_z\mathcal{J}_\epsilon\Pi_\epsilon\right),$$

$$F_\epsilon^2\left(\omega_\epsilon, B_\epsilon, \Pi_\epsilon, \Omega_\epsilon\right) = -\mathcal{J}_\epsilon\left(\mathcal{J}_\epsilon\mathbf{u}_\epsilon\cdot\nabla\right)\mathcal{J}_\epsilon B_\epsilon$$

$$+ \mathcal{J}_\epsilon\left(\mathcal{J}_\epsilon u_\epsilon^r\mathcal{J}_\epsilon\Pi_\epsilon\right)$$

$$+ 2\mathcal{J}_\epsilon\left(\mathcal{J}_\epsilon B_\epsilon\partial_z\mathcal{J}_\epsilon\Pi_\epsilon\right), \tag{33}$$

$$F_\epsilon^3\left(\omega_\epsilon, \Pi_\epsilon\right) = -\mathcal{J}_\epsilon\left(\mathcal{J}_\epsilon\mathbf{u}_\epsilon\cdot\nabla\right)\mathcal{J}_\epsilon\Pi_\epsilon$$

$$+ 2\mathcal{J}_\epsilon\left(\mathcal{J}_\epsilon\Pi_\epsilon\partial_z\mathcal{J}_\epsilon\Pi_\epsilon\right),$$

$$F_\epsilon^4\left(\omega_\epsilon, \Pi_\epsilon, \Omega_\epsilon\right) = -\mathcal{J}_\epsilon\left(\mathcal{J}_\epsilon\mathbf{u}_\epsilon\cdot\nabla\right)\mathcal{J}_\epsilon\Omega_\epsilon$$

$$- 2\mathcal{J}_\epsilon\left(\mathcal{J}_\epsilon\Pi_\epsilon\partial_z\mathcal{J}_\epsilon\Pi_\epsilon\right).$$

First, we show that F_ϵ^1 is Lipschitz continuous on H^m space. We estimate for $\epsilon < 1, m \geq 3$

$$\left\|F_\epsilon^1\left(\omega_\epsilon, B_\epsilon, \Pi_\epsilon, \Omega_\epsilon\right) - F_\epsilon^1\left(\widetilde{\omega}_\epsilon, B_\epsilon, \Pi_\epsilon, \Omega_\epsilon\right)\right\|_m$$

$$\leq \left\|\left(\mathcal{J}_\epsilon\left(\mathbf{u}_\epsilon - \widetilde{\mathbf{u}}_\epsilon\right)\cdot\nabla\right)\mathcal{J}_\epsilon\omega_\epsilon\right\|_m$$

$$+ \left\|\left(\mathcal{J}_\epsilon\widetilde{\mathbf{u}}_\epsilon\cdot\nabla\right)\left(\mathcal{J}_\epsilon\omega_\epsilon - \mathcal{J}_\epsilon\widetilde{\omega}_\epsilon\right)\right\|_m$$

$$+ \left\|\mathcal{J}_\epsilon\left(u_\epsilon^r - \widetilde{u}_\epsilon^r\right)\mathcal{J}_\epsilon\Omega_\epsilon\right\|_m$$

$$\leq \frac{C}{\epsilon}\left(\left\|\mathbf{u}_\epsilon - \widetilde{\mathbf{u}}_\epsilon\right\|_{L^\infty} + \left\|\omega_\epsilon - \widetilde{\omega}_\epsilon\right\|_{m-1}\right) \tag{34}$$

$$\cdot\left(\left\|\omega_\epsilon\right\|_m + \epsilon\left\|\Omega_\epsilon\right\|_m\right) + \frac{C}{\epsilon}\left(\left\|\widetilde{\mathbf{u}}_\epsilon\right\|_{L^\infty} + \left\|\widetilde{\omega}_\epsilon\right\|_{m-1}\right)$$

$$\cdot\left\|\omega_\epsilon - \widetilde{\omega}_\epsilon\right\|_m \leq \frac{C}{\epsilon}\left(\left\|\omega_\epsilon\right\|_m + \left\|\Omega_\epsilon\right\|_m + \left\|\widetilde{\omega}_\epsilon\right\|_{m-1}\right)$$

$$\cdot\left\|\omega_\epsilon - \widetilde{\omega}_\epsilon\right\|_m.$$

By the similar estimates as in (34), we obtain

$$\left\|F_\epsilon^1\left(\omega_\epsilon, B_\epsilon, \Pi_\epsilon, \Omega_\epsilon\right) - F_\epsilon^1\left(\omega_\epsilon, \widetilde{B}_\epsilon, \Pi_\epsilon, \Omega_\epsilon\right)\right\|_m$$

$$\leq 2\left\|\left(\mathcal{J}_\epsilon B_\epsilon - \mathcal{J}_\epsilon\widetilde{B}_\epsilon\right)\partial_z\mathcal{J}_\epsilon\Pi_\epsilon\right\|_m \tag{35}$$

$$\leq C\left\|\Pi_\epsilon\right\|_{m+1}\left\|B_\epsilon - \widetilde{B}_\epsilon\right\|_m,$$

$$\left\|F_\epsilon^1\left(\omega_\epsilon, B_\epsilon, \Pi_\epsilon, \Omega_\epsilon\right) - F_\epsilon^1\left(\omega_\epsilon, B_\epsilon, \widetilde{\Pi}_\epsilon, \Omega_\epsilon\right)\right\|_m$$

$$\leq 2\left\|\mathcal{J}_\epsilon B_\epsilon\left(\partial_z\mathcal{J}_\epsilon\Pi_\epsilon - \partial_z\mathcal{J}_\epsilon\widetilde{\Pi}_\epsilon\right)\right\|_m \tag{36}$$

$$\leq C\left\|B_\epsilon\right\|_m\left\|\Pi_\epsilon - \widetilde{\Pi}_\epsilon\right\|_{m+1},$$

$$\left\|F_\epsilon^1\left(\omega_\epsilon, B_\epsilon, \Pi_\epsilon, \Omega_\epsilon\right) - F_\epsilon^1\left(\omega_\epsilon, B_\epsilon, \Pi_\epsilon, \widetilde{\Omega}_\epsilon\right)\right\|_m$$

$$\leq \left\|\mathcal{J}_\epsilon u_\epsilon^r\left(\mathcal{J}_\epsilon\Omega_\epsilon - \mathcal{J}_\epsilon\widetilde{\Omega}_\epsilon\right)\right\|_m \tag{37}$$

$$\leq C\left\|\omega_\epsilon\right\|_{m-1}\left\|\Omega_\epsilon - \widetilde{\Omega}_\epsilon\right\|_m.$$

By the virtue of properties of mollifier, Lipschitz continuity of the remaining functions F_ϵ^j, $j = 2, 3, 4$, can be obtained with constant C/ϵ. Thus, we can deduce the following for $\| \cdot \|_{X^m}, \|\widetilde{\cdot}\|_{X^m} \leq R$,

$$\left\| F_\epsilon \left(\cdot \right) - F_\epsilon \left(\widetilde{\cdot} \right) \right\|_{X^m} \leq \frac{C}{\epsilon} R \left\| \cdot - \widetilde{\cdot} \right\|_{X^m} \tag{38}$$

with $m \geq 3$ and $F_\epsilon = (F_\epsilon^1, F_\epsilon^2, F_\epsilon^3, F_\epsilon^4)$. Now we use the Picard theorem with domain X^m. By picking any initial data $S_\epsilon(0) \in X^m$ and choosing $T_0 = 3/4C_0S_0$, we have, for $R = S_0/(1 - C_0T_0S_0) = 4S_0$,

$$\frac{d}{dt} \|S_\epsilon\|_{X^m} (t) = F_\epsilon \left(S_\epsilon (t) \right),$$

$$\left\| F_\epsilon \left(S_\epsilon \right) - F_\epsilon \left(\widetilde{S}_\epsilon \right) \right\|_{X^m} \leq \frac{4CS_0}{\epsilon} \left\| S_\epsilon - \widetilde{S}_\epsilon \right\|_{X^m} \tag{39}$$

where $\|S_\epsilon\|_{X^m}, \|\widetilde{S}_\epsilon\|_{X^m} \leq 4S_0$. Therefore, the Picard theorem implies that, for each $0 < \epsilon < 1$, there exists a unique solution $S_\epsilon(t) \in C^1(0, T_\epsilon; X^m)$ for a fixed time $T_\epsilon > 0$. For simplicity, let T_ϵ be the maximal existence time of such solution. Suppose that, for some $0 < \epsilon < 1$, we have $T_\epsilon < T_0$. Then by Proposition 7, for arbitrarily small $\delta > 0$, we have

$$\|S_\epsilon\|_{X^m} \left(T_\epsilon - \delta \right) \leq \frac{S_0}{1 - C_0 S_0 T_\epsilon} < 4S_0. \tag{40}$$

If we apply the standard continuation argument, then we can have local-in-time solution S_ϵ at least until T_0. This contradicts the assumption that $T_\epsilon < T_0$. Hence we prove that, for any $0 < \epsilon < 1$, there is a unique solution $S_\epsilon(t)$ with a uniform time T_0, such that $S_\epsilon(t) \in C^1(0, T_0; X^m)$. This completes the proof. □

Proposition 9. *For an integer $m > 7/2$, the solutions obtained in Proposition 8 form the Cauchy sequences in the following spaces:*

$$\{S_\epsilon\} \in C\left(0, T_0; X^1\right),$$
$$\{\partial_t S_\epsilon\} \in C\left(0, T_0; X^0\right). \tag{41}$$

Proof. Taking ∂_i operator ($i = 1, 2, 3$) on both sides of $(25)_1$ and multiplying $\partial_i \omega_\epsilon$, we deduce that

$$\frac{1}{2} \frac{\partial}{\partial t} \left| \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right|^2$$
$$= \partial_i \left(F_\epsilon^1 \left(\omega_\epsilon, B_\epsilon, \Pi_\epsilon, \Omega_\epsilon \right) - F_{\epsilon'}^1 \left(\omega_{\epsilon'}, B_{\epsilon'}, \Pi_{\epsilon'}, \Omega_{\epsilon'} \right) \right) \tag{42}$$
$$\cdot \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) = A_1 + A_2 + A_3 + A_4 + A_5 + A_6,$$

where

A_1

$$= \left(\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \left(\mathcal{J}_\epsilon \partial_i \mathbf{u}_\epsilon \cdot \nabla \right) \mathcal{J}_\epsilon \omega_\epsilon$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \left(\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \partial_i \mathbf{u}_\epsilon \cdot \nabla \right) \mathcal{J}_\epsilon \omega_\epsilon$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \left(\mathcal{J}_{\epsilon'} \partial_i \left(\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon'} \right) \cdot \nabla \right) \mathcal{J}_\epsilon \omega_\epsilon \tag{43}$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \left(\mathcal{J}_{\epsilon'} \partial_i \mathbf{u}_{\epsilon'} \cdot \nabla \right) \left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \omega_\epsilon$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \left(\mathcal{J}_{\epsilon'} \partial_i \mathbf{u}_{\epsilon'} \cdot \nabla \right) \mathcal{J}_{\epsilon'} \left(\omega_\epsilon - \omega_{\epsilon'} \right),$$

A_2

$$= \left(\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \left(\mathcal{J}_\epsilon \mathbf{u}_\epsilon \cdot \nabla \right) \mathcal{J}_\epsilon \partial_i \omega_\epsilon$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \left(\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \mathbf{u}_\epsilon \cdot \nabla \right) \mathcal{J}_\epsilon \partial_i \omega_\epsilon$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \left(\mathcal{J}_{\epsilon'} \left(\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon'} \right) \cdot \nabla \right) \mathcal{J}_\epsilon \partial_i \omega_\epsilon \tag{44}$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \left(\mathcal{J}_{\epsilon'} \mathbf{u}_{\epsilon'} \cdot \nabla \right) \left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \partial_i \omega_\epsilon$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \left(\mathcal{J}_{\epsilon'} \mathbf{u}_{\epsilon'} \cdot \nabla \right) \mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right),$$

A_3

$$= \left(\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \mathcal{J}_\epsilon \partial_i u_\epsilon^r \mathcal{J}_\epsilon \Omega_\epsilon$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \partial_i u_\epsilon^r \mathcal{J}_\epsilon \Omega_\epsilon$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \mathcal{J}_{\epsilon'} \partial_i \left(u_\epsilon^r - u_{\epsilon'}^r \right) \mathcal{J}_\epsilon \Omega_\epsilon \tag{45}$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \mathcal{J}_{\epsilon'} \partial_i u_{\epsilon'}^r \left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \Omega_\epsilon$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \mathcal{J}_{\epsilon'} \partial_i u_{\epsilon'}^r \mathcal{J}_{\epsilon'} \left(\Omega_\epsilon - \Omega_{\epsilon'} \right),$$

A_4

$$= \left(\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \mathcal{J}_\epsilon u_\epsilon^r \mathcal{J}_\epsilon \partial_i \Omega_\epsilon$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) u_\epsilon^r \mathcal{J}_\epsilon \partial_i \Omega_\epsilon$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \mathcal{J}_{\epsilon'} \left(u_\epsilon^r - u_{\epsilon'}^r \right) \mathcal{J}_\epsilon \partial_i \Omega_\epsilon \tag{46}$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \mathcal{J}_{\epsilon'} u_{\epsilon'}^r \left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \partial_i \Omega_\epsilon$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \mathcal{J}_{\epsilon'} u_{\epsilon'}^r \mathcal{J}_{\epsilon'} \partial_i \left(\Omega_\epsilon - \Omega_{\epsilon'} \right),$$

A_5

$$= \left(\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \mathcal{J}_\epsilon \partial_i B_\epsilon \mathcal{J}_\epsilon \partial_z \Pi_\epsilon$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \partial_i B_\epsilon \mathcal{J}_\epsilon \partial_z \Pi_\epsilon$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \mathcal{J}_{\epsilon'} \partial_i \left(B_\epsilon - B_{\epsilon'} \right) \mathcal{J}_\epsilon \partial_z \Pi_\epsilon \tag{47}$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \mathcal{J}_{\epsilon'} \partial_i B_{\epsilon'} \left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \partial_z \Pi_\epsilon$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \mathcal{J}_{\epsilon'} \partial_i B_{\epsilon'} \mathcal{J}_{\epsilon'} \partial_z \left(\Pi_\epsilon - \Pi_{\epsilon'} \right),$$

$$A_6$$

$$= \left(\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \mathcal{J}_\epsilon B_\epsilon \mathcal{J}_\epsilon \partial_i \partial_z \Pi_\epsilon$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) B_\epsilon \mathcal{J}_\epsilon \partial_i \partial_z \Pi_\epsilon$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \mathcal{J}_{\epsilon'} \left(B_\epsilon - B_{\epsilon'} \right) \mathcal{J}_\epsilon \partial_i \partial_z \Pi_\epsilon \qquad (48)$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \mathcal{J}_{\epsilon'} B_{\epsilon'} \left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \partial_i \partial_z \Pi_\epsilon$$
$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right) \mathcal{J}_{\epsilon'} B_{\epsilon'} \mathcal{J}_{\epsilon'} \partial_i \partial_z \left(\Pi_\epsilon - \Pi_{\epsilon'} \right).$$

$A_1, \ldots, A_6$ can be estimated as follows:

$$\int_{\mathbb{R}^3} \left| A_1 \right| dx$$

$$\leq \left(\epsilon + \epsilon' \right) \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left\| \left(\mathcal{J}_\epsilon \partial_i \mathbf{u}_\epsilon \cdot \nabla \right) \mathcal{J}_\epsilon \omega_\epsilon \right\|_1$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left(\epsilon + \epsilon' \right) \left\| \nabla \mathbf{u}_\epsilon \right\|_1 \left\| \nabla \omega_\epsilon \right\|_{L^\infty}$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2}^2 \left\| \nabla \omega_\epsilon \right\|_{L^\infty} \qquad (49)$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left\| \nabla \mathbf{u}_{\epsilon'} \right\|_{L^\infty} \left(\epsilon + \epsilon' \right) \left\| \nabla \omega_\epsilon \right\|_1$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2}^2 \left\| \nabla \mathbf{u}_{\epsilon'} \right\|_{L^\infty},$$

$$\int_{\mathbb{R}^3} \left| A_2 \right| dx$$

$$\leq \left(\epsilon + \epsilon' \right) \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left\| \left(\mathcal{J}_\epsilon \mathbf{u}_\epsilon \cdot \nabla \right) \mathcal{J}_\epsilon \partial_i \omega_\epsilon \right\|_1$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left(\epsilon + \epsilon' \right) \left\| \nabla \mathbf{u}_\epsilon \right\|_{L^2} \left\| \nabla^2 \omega_\epsilon \right\|_{L^\infty} \qquad (50)$$

$$+ 2 \left\| \omega_\epsilon - \omega_{\epsilon'} \right\|_1^2 \left\| \nabla^2 \omega_\epsilon \right\|_{L^2}$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left\| \mathbf{u}_{\epsilon'} \right\|_{L^\infty} \left(\epsilon + \epsilon' \right) \left\| \nabla^2 \omega_\epsilon \right\|_1,$$

$$\int_{\mathbb{R}^3} \left| A_3 \right| dx$$

$$\leq \left(\epsilon + \epsilon' \right) \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left\| \mathcal{J}_\epsilon \partial_i u_\epsilon^r \mathcal{J}_\epsilon \Omega_\epsilon \right\|_1$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left(\epsilon + \epsilon' \right) \left\| \nabla \mathbf{u}_\epsilon \right\|_1 \left\| \Omega_\epsilon \right\|_{L^\infty}$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2}^2 \left\| \Omega_\epsilon \right\|_{L^\infty} \qquad (51)$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left\| \nabla \mathbf{u}_{\epsilon'} \right\|_{L^\infty} \left(\epsilon + \epsilon' \right) \left\| \Omega_\epsilon \right\|_1$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left\| \nabla \mathbf{u}_{\epsilon'} \right\|_{L^\infty} \left\| \Omega_\epsilon - \Omega_{\epsilon'} \right\|_{L^2},$$

$$\int_{\mathbb{R}^3} \left| A_4 \right| dx$$

$$\leq \left(\epsilon + \epsilon' \right) \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left\| \mathcal{J}_\epsilon u_\epsilon^r \mathcal{J}_\epsilon \partial_i \Omega_\epsilon \right\|_1$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left(\epsilon + \epsilon' \right) \left\| \nabla \mathbf{u}_\epsilon \right\|_{L^2} \left\| \nabla \Omega_\epsilon \right\|_{L^\infty}$$

$$+ \left\| \omega_\epsilon - \omega_{\epsilon'} \right\|_1^2 \left\| \nabla \Omega_\epsilon \right\|_{L^2}$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left\| \mathbf{u}_{\epsilon'} \right\|_{L^\infty} \left(\epsilon + \epsilon' \right) \left\| \nabla \Omega_\epsilon \right\|_1$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left\| \mathbf{u}_{\epsilon'} \right\|_{L^\infty} \left\| \nabla \left(\Omega_\epsilon - \Omega_{\epsilon'} \right) \right\|_{L^2}, \qquad (52)$$

$$\int_{\mathbb{R}^3} \left| A_5 \right| dx$$

$$\leq \left(\epsilon + \epsilon' \right) \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left\| \mathcal{J}_\epsilon \partial_i B_\epsilon \mathcal{J}_\epsilon \partial_z \Pi_\epsilon \right\|_1$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left(\epsilon + \epsilon' \right) \left\| \nabla B_\epsilon \right\|_1 \left\| \nabla \Pi_\epsilon \right\|_{L^\infty}$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left\| \nabla \left(B_\epsilon - B_{\epsilon'} \right) \right\|_{L^2} \left\| \nabla \Pi_\epsilon \right\|_{L^\infty} \qquad (53)$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left\| \nabla B_{\epsilon'} \right\|_{L^\infty} \left(\epsilon + \epsilon' \right) \left\| \nabla \Pi_\epsilon \right\|_1$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left\| \nabla B_{\epsilon'} \right\|_{L^\infty} \left\| \nabla \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \right\|_{L^2},$$

$$\int_{\mathbb{R}^3} \left| A_6 \right| dx$$

$$\leq \left(\epsilon + \epsilon' \right) \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left\| \mathcal{J}_\epsilon B_\epsilon \mathcal{J}_\epsilon \partial_i \partial_z \Pi_\epsilon \right\|_1$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left(\epsilon + \epsilon' \right) \left\| B_\epsilon \right\|_1 \left\| \nabla^2 \Pi_\epsilon \right\|_{L^\infty}$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left\| B_\epsilon - B_{\epsilon'} \right\|_{L^2} \left\| \nabla^2 \Pi_\epsilon \right\|_{L^\infty} \qquad (54)$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left\| B_{\epsilon'} \right\|_{L^\infty} \left(\epsilon + \epsilon' \right) \left\| \nabla^2 \Pi_\epsilon \right\|_1$$

$$+ \left\| \nabla \left(\omega_\epsilon - \omega_{\epsilon'} \right) \right\|_{L^2} \left\| B_{\epsilon'} \right\|_{L^\infty} \left\| \nabla^2 \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \right\|_{L^2}.$$

Similarly, we can obtain the estimates for Π.

$$\frac{1}{2} \frac{d}{dt} \left| \partial_i \partial_j \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \right|^2$$

$$= \partial_i \partial_j \left(F_\epsilon^3 \left(\omega_\epsilon, \Pi_\epsilon \right) - F_{\epsilon'}^3 \left(\omega_{\epsilon'}, \Pi_{\epsilon'} \right) \right) \partial_i \partial_j \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \qquad (55)$$

$$= B_1 + B_2 + B_3 + B_4 + B_5 + B_6$$

where

$$B_1 = \left(\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \partial_i \partial_j \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \right) \left(\mathcal{J}_\epsilon \partial_i \partial_j \mathbf{u}_\epsilon \cdot \nabla \right)$$

$$\cdot \mathcal{J}_\epsilon \Pi_\epsilon + \left(\mathcal{J}_{\epsilon'} \partial_i \partial_j \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \right)$$

$$\cdot \left(\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \partial_i \partial_j \mathbf{u}_\epsilon \cdot \nabla \right) \mathcal{J}_\epsilon \Pi_\epsilon$$

$$+ \left(\mathcal{J}_{\epsilon'} \partial_i \partial_j \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \right) \left(\mathcal{J}_{\epsilon'} \partial_i \partial_j \left(\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon'} \right) \cdot \nabla \right) \qquad (56)$$

$$\cdot \mathcal{J}_\epsilon \Pi_\epsilon + \left(\mathcal{J}_{\epsilon'} \partial_i \partial_j \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \right) \left(\mathcal{J}_{\epsilon'} \partial_i \partial_j \mathbf{u}_{\epsilon'} \cdot \nabla \right)$$

$$\cdot \left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \Pi_\epsilon + \left(\mathcal{J}_{\epsilon'} \partial_i \partial_j \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \right)$$

$$\cdot \left(\mathcal{J}_{\epsilon'} \partial_i \partial_j \mathbf{u}_{\epsilon'} \cdot \nabla \right) \mathcal{J}_{\epsilon'} \left(\Pi_\epsilon - \Pi_{\epsilon'} \right),$$

$$B_2 = \left(\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \partial_i \partial_j \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \right) \left(\mathcal{J}_\epsilon \partial_i \mathbf{u}_\epsilon \cdot \nabla \right)$$

$$\cdot \mathcal{J}_\epsilon \partial_j \Pi_\epsilon + \left(\mathcal{J}_\epsilon \partial_i \partial_j \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \right)$$

$$\cdot \left(\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \partial_i \mathbf{u}_\epsilon \cdot \nabla \right) \mathcal{J}_\epsilon \partial_j \Pi_\epsilon$$

$$+ \left(\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\left(\mathcal{J}_{\epsilon'}\partial_i\left(\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon'}\right)\cdot\nabla\right)$$

$$\cdot\, \mathcal{J}_\epsilon\partial_j\Pi_\epsilon + \left(\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\left(\mathcal{J}_{\epsilon'}\partial_i\mathbf{u}_{\epsilon'}\cdot\nabla\right)$$

$$\cdot\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'}\right)\partial_j\Pi_\epsilon + \left(\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)$$

$$\cdot\left(\mathcal{J}_{\epsilon'}\partial_i\mathbf{u}_{\epsilon'}\cdot\nabla\right)\mathcal{J}_{\epsilon'}\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right), \tag{57}$$

$$\begin{aligned}
B_3 &= \left(\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'}\right)\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\left(\mathcal{J}_\epsilon\mathbf{u}_\epsilon\cdot\nabla\right)\\
&\quad\cdot \mathcal{J}_\epsilon\partial_i\partial_j\Pi_\epsilon + \left(\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\\
&\quad\cdot\left(\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'}\right)\mathbf{u}_\epsilon\cdot\nabla\right)\mathcal{J}_\epsilon\partial_i\partial_j\Pi_\epsilon\\
&\quad+ \left(\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\left(\mathcal{J}_{\epsilon'}\left(\mathbf{u}_\epsilon - \mathbf{u}_{\epsilon'}\right)\cdot\nabla\right)\\
&\quad\cdot \mathcal{J}_\epsilon\partial_i\partial_j\Pi_\epsilon + \left(\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\left(\mathcal{J}_{\epsilon'}\mathbf{u}_{\epsilon'}\cdot\nabla\right)\\
&\quad\cdot\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'}\right)\partial_i\partial_j\Pi_\epsilon + \left(\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\\
&\quad\cdot\left(\mathcal{J}_{\epsilon'}\mathbf{u}_{\epsilon'}\cdot\nabla\right)\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right),
\end{aligned} \tag{58}$$

$$\begin{aligned}
B_4 &= \left(\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'}\right)\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\mathcal{J}_\epsilon\partial_i\partial_j\Pi_\epsilon\mathcal{J}_\epsilon\partial_z\Pi_\epsilon\\
&\quad+ \left(\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'}\right)\partial_i\partial_j\Pi_\epsilon\mathcal{J}_\epsilon\partial_z\Pi_\epsilon\\
&\quad+ \left(\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\mathcal{J}_\epsilon\partial_z\Pi_\epsilon\\
&\quad+ \left(\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\mathcal{J}_{\epsilon'}\partial_i\partial_j\Pi_{\epsilon'}\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'}\right)\partial_z\Pi_\epsilon\\
&\quad+ \left(\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\\
&\quad\cdot \mathcal{J}_{\epsilon'}\partial_i\partial_j\Pi_{\epsilon'}\mathcal{J}_{\epsilon'}\partial_z\left(\Pi_\epsilon - \Pi_{\epsilon'}\right),
\end{aligned} \tag{59}$$

$$\begin{aligned}
B_5 &= \left(\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'}\right)\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\mathcal{J}_\epsilon\partial_i\Pi_\epsilon\mathcal{J}_\epsilon\partial_j\partial_z\Pi_\epsilon\\
&\quad+ \left(\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'}\right)\partial_i\Pi_\epsilon\mathcal{J}_\epsilon\partial_j\partial_z\Pi_\epsilon\\
&\quad+ \left(\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\mathcal{J}_{\epsilon'}\partial_i\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\mathcal{J}_\epsilon\partial_j\partial_z\Pi_\epsilon\\
&\quad+ \left(\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\mathcal{J}_{\epsilon'}\partial_i\Pi_{\epsilon'}\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'}\right)\partial_j\partial_z\Pi_\epsilon\\
&\quad+ \left(\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\\
&\quad\cdot \mathcal{J}_{\epsilon'}\partial_i\Pi_{\epsilon'}\mathcal{J}_{\epsilon'}\partial_j\partial_z\left(\Pi_\epsilon - \Pi_{\epsilon'}\right),
\end{aligned} \tag{60}$$

$$\begin{aligned}
B_6 &= \left(\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'}\right)\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\mathcal{J}_\epsilon\Pi_\epsilon\mathcal{J}_\epsilon\partial_i\partial_j\partial_z\Pi_\epsilon\\
&\quad+ \left(\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'}\right)\Pi_\epsilon\mathcal{J}_\epsilon\partial_i\partial_j\partial_z\Pi_\epsilon\\
&\quad+ \left(\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\mathcal{J}_{\epsilon'}\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\mathcal{J}_\epsilon\partial_i\partial_j\partial_z\Pi_\epsilon\\
&\quad+ \left(\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\mathcal{J}_{\epsilon'}\Pi_{\epsilon'}\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'}\right)\partial_i\partial_j\partial_z\Pi_\epsilon\\
&\quad+ \left(\mathcal{J}_{\epsilon'}\partial_i\partial_j\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right)\\
&\quad\cdot \mathcal{J}_{\epsilon'}\partial_i\Pi_{\epsilon'}\mathcal{J}_{\epsilon'}\partial_i\partial_j\partial_z\left(\Pi_\epsilon - \Pi_{\epsilon'}\right).
\end{aligned} \tag{61}$$

These $B_1, \ldots, B_6$ can be estimated similarly.

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left|B_1\right|dx\\
&\leq \left(\epsilon + \epsilon'\right)\left\|\nabla^2\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_{L^2}\left\|\left(\mathcal{J}_\epsilon\partial_i\partial_j\mathbf{u}_\epsilon\cdot\nabla\right)\mathcal{J}_\epsilon\Pi_\epsilon\right\|_1\\
&\quad+ \left\|\nabla^2\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_{L^2}\left(\epsilon + \epsilon'\right)\left\|\nabla^2\mathbf{u}_\epsilon\right\|_1\left\|\nabla\Pi_\epsilon\right\|_{L^\infty}\\
&\quad+ \left\|\nabla^2\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_{L^2}\left\|\nabla\left(\omega_\epsilon - \omega_{\epsilon'}\right)\right\|_{L^2}\left\|\nabla\Pi_\epsilon\right\|_{L^\infty}\\
&\quad+ \left\|\nabla^2\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_{L^2}\left\|\nabla^2\mathbf{u}_{\epsilon'}\right\|_{L^\infty}\left(\epsilon + \epsilon'\right)\left\|\nabla\Pi_\epsilon\right\|_1\\
&\quad+ \left\|\Pi_\epsilon - \Pi_{\epsilon'}\right\|_2^2\left\|\nabla^2\mathbf{u}_{\epsilon'}\right\|_{L^\infty},
\end{aligned} \tag{62}$$

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left|B_2\right|dx\\
&\leq \left(\epsilon + \epsilon'\right)\left\|\nabla^2\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_{L^2}\left\|\left(\mathcal{J}_\epsilon\partial_i\mathbf{u}_\epsilon\cdot\nabla\right)\mathcal{J}_\epsilon\partial_j\Pi_\epsilon\right\|_1\\
&\quad+ \left\|\nabla^2\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_{L^2}\left(\epsilon + \epsilon'\right)\left\|\nabla\mathbf{u}_\epsilon\right\|_1\left\|\nabla^2\Pi_\epsilon\right\|_{L^\infty}\\
&\quad+ \left\|\nabla^2\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_{L^2}\left\|\omega_\epsilon - \omega_{\epsilon'}\right\|_{L^2}\left\|\nabla^2\Pi_\epsilon\right\|_{L^\infty}\\
&\quad+ \left\|\nabla^2\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_{L^2}\left\|\nabla\mathbf{u}_{\epsilon'}\right\|_{L^\infty}\left(\epsilon + \epsilon'\right)\left\|\nabla^2\Pi_\epsilon\right\|_1\\
&\quad+ \left\|\nabla^2\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_2^2\left\|\nabla\mathbf{u}_{\epsilon'}\right\|_{L^\infty},
\end{aligned} \tag{63}$$

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left|B_3\right|dx\\
&\leq \left(\epsilon + \epsilon'\right)\left\|\nabla^2\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_{L^2}\left\|\left(\mathcal{J}_\epsilon\mathbf{u}_\epsilon\cdot\nabla\right)\mathcal{J}_\epsilon\partial_i\partial_j\Pi_\epsilon\right\|_1\\
&\quad+ \left\|\nabla^2\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_{L^2}\left(\epsilon + \epsilon'\right)\left\|\nabla\mathbf{u}_\epsilon\right\|_{L^2}\left\|\nabla^3\Pi_\epsilon\right\|_{L^\infty}\\
&\quad+ \left\|\nabla^2\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_{L^2}\left\|\omega_\epsilon - \omega_{\epsilon'}\right\|_1\left\|\nabla^3\Pi_\epsilon\right\|_{L^2}\\
&\quad+ \left\|\nabla^2\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_{L^2}\left\|\mathbf{u}_{\epsilon'}\right\|_{L^\infty}\left(\epsilon + \epsilon'\right)\left\|\nabla^3\Pi_\epsilon\right\|_1,
\end{aligned} \tag{64}$$

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left|B_4\right|dx\\
&\leq \left(\epsilon + \epsilon'\right)\left\|\nabla^2\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_{L^2}\left\|\mathcal{J}_\epsilon\partial_i\partial_j\Pi_\epsilon\mathcal{J}_\epsilon\partial_z\Pi_\epsilon\right\|_1\\
&\quad+ \left\|\nabla^2\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_{L^2}\left(\epsilon + \epsilon'\right)\left\|\nabla^2\Pi_\epsilon\right\|_1\left\|\nabla\Pi_\epsilon\right\|_{L^\infty}\\
&\quad+ \left\|\nabla^2\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_2^2\left\|\nabla\Pi_\epsilon\right\|_{L^\infty}\\
&\quad+ \left\|\nabla^2\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_{L^2}\left\|\nabla^2\Pi_{\epsilon'}\right\|_{L^\infty}\left(\epsilon + \epsilon'\right)\left\|\nabla\Pi_\epsilon\right\|_1\\
&\quad+ \left\|\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_2^2\left\|\nabla^2\Pi_{\epsilon'}\right\|_{L^\infty},
\end{aligned} \tag{65}$$

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left|B_5\right|dx\\
&\leq \left(\epsilon + \epsilon'\right)\left\|\nabla^2\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_{L^2}\left\|\mathcal{J}_\epsilon\partial_i\Pi_\epsilon\mathcal{J}_\epsilon\partial_j\partial_z\Pi_\epsilon\right\|_1\\
&\quad+ \left\|\nabla^2\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_{L^2}\left(\epsilon + \epsilon'\right)\left\|\nabla\Pi_\epsilon\right\|_1\left\|\nabla^2\Pi_\epsilon\right\|_{L^\infty}\\
&\quad+ \left\|\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_2^2\left\|\nabla^2\Pi_\epsilon\right\|_{L^\infty}
\end{aligned}$$

$$+ \left\| \nabla^2 \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \right\|_{L^2} \left\| \nabla \Pi_{\epsilon'} \right\|_{L^\infty} \left(\epsilon + \epsilon' \right) \left\| \nabla^2 \Pi_\epsilon \right\|_1$$

$$+ \left\| \nabla^2 \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \right\|_{L^2}^2 \left\| \nabla \Pi_{\epsilon'} \right\|_{L^\infty},$$

$$\tag{66}$$

$$\int_{\mathbb{R}^3} \left| B_6 \right| dx$$

$$\leq \left(\epsilon + \epsilon' \right) \left\| \nabla^2 \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \right\|_{L^2} \left\| \mathscr{J}_\epsilon \Pi_\epsilon \mathscr{J}_\epsilon \partial_i \partial_j \partial_z \Pi_\epsilon \right\|_1$$

$$+ \left\| \nabla^2 \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \right\|_{L^2} \left(\epsilon + \epsilon' \right) \left\| \Pi_\epsilon \right\|_1 \left\| \nabla^3 \Pi_\epsilon \right\|_{L^\infty}$$

$$+ \left\| \nabla^2 \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \right\|_2^2 \left\| \nabla^3 \Pi_\epsilon \right\|_{L^\infty} \tag{67}$$

$$+ \left\| \nabla^2 \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \right\|_{L^2} \left\| \Pi_{\epsilon'} \right\|_{L^\infty} \left(\epsilon + \epsilon' \right) \left\| \nabla^3 \Pi_\epsilon \right\|_1.$$

The other terms B and $\Pi + \Omega$ can be estimated similarly, so we omit the details. Then we have

$$\frac{1}{2} \frac{d}{dt} \|S\|_{X^1} (t) := \frac{1}{2} \frac{d}{dt} \left(\left\| \omega_\epsilon - \omega_{\epsilon'} \right\|_1 + \left\| B_\epsilon - B_{\epsilon'} \right\|_1 \right.$$

$$+ \left\| \Pi_\epsilon - \Pi_{\epsilon'} \right\|_2 + \left\| \left(\Pi_\epsilon + \Omega_\epsilon \right) - \left(\Pi_{\epsilon'} + \Omega_{\epsilon'} \right) \right\|_1 \right)$$

$$\leq \left(\left(\epsilon + \epsilon' \right) + \|S\|_{X^1} (t) \right) \left(\left\| \nabla^3 \Pi_\epsilon \right\|_{L^\infty} + \left\| \nabla^2 \omega_\epsilon \right\|_{L^\infty} \right. \tag{68}$$

$$+ \left\| \nabla^2 B_\epsilon \right\|_{L^\infty} + \left\| \nabla^2 \left(\Pi_\epsilon + \Omega_\epsilon \right) \right\|_{L^\infty} \right) \leq C \left(\left(\epsilon + \epsilon' \right) \right.$$

$$+ \|S\|_{X^1} (t) \right),$$

for $m > 7/2$. Gronwall's inequality gives us

$$\sup_{0 \leq t \leq T_0} \|S\|_{X^1} (t) \leq \left(\epsilon + \epsilon' \right) e^{CT_0}, \tag{69}$$

which implies that $\{S_\epsilon\} \in C([0, T_0], X^1)$ and this information completes the proof. $\square$

Proof of Theorem 1. With the bounds in Proposition 7, if we use the Sobolev inequality, then we can obtain the higher order convergence, i.e., $S \in C(0, T_0; X^s)$ for all $s < m$ by the following inequality

$$\sup_{0 \leq t \leq T_0} \left\| S_{\epsilon'} - S_\epsilon \right\|_{X^s} \leq C \left\| S_{\epsilon'} - S_\epsilon \right\|_{X^1}^{1-s/m} \left\| S_{\epsilon'} - S_\epsilon \right\|_m^{s/m}. \tag{70}$$

Now, to show $S \in C(0, T_0; X^m) \cap \mathrm{Lip}(0, T_0; X^{m-1})$ where S satisfies our equations in classical sense almost every time, we begin the process of obtaining the right continuity at $t = 0$ first. Because X^m is a reflexive Banach space, by Proposition 7, there exist a subsequence and limit functions $S(t) \in L^\infty(0, T_0; H^m)$ which satisfies $\partial_t S(t) \in L^\infty(0, T_0; H^{m-1})$ for any $0 \leq t \leq T_0$, $S_{\epsilon_n}(t) \rightharpoonup S(t)$ in X^m and $\partial_t S_{\epsilon_n}(t) \rightharpoonup \partial_t S(t)$ in X^{m-1}. This implies that $\|S(t)\|_{X^m} \leq S_0/(1 - C_0 S_0 t)$, $\|\partial_t S(t)\|_{X^{m-1}} \leq (S_0/(1 - C_0 S_0 t))^2$. Thus we have

$$\limsup_{t \to 0+} \|S\|_{X^m} (t) \leq S_0. \tag{71}$$

If we use the above result, $S(t) \in C(0, T_0; X^s)$ for any $s < m$, then $S \in C_w(0, T_0; X^m)$ is obtained by the following estimate.

For arbitrary $\delta > 0$ and $\varphi \in X^{-m}$, there exists $\widetilde{\varphi} \in X^{-s}$ such that $\|\varphi - \widetilde{\varphi}\|_{X^{-m}} \leq \delta/8 S_0$,

$$\left| [\varphi, S_\epsilon - S]_{X^m} \right| \leq \|\varphi - \widetilde{\varphi}\|_{X^{-m}} \|S_\epsilon - S\|_{X^m}$$

$$+ \left| [\widetilde{\varphi}, S_\epsilon - S]_{X^s} \right| \tag{72}$$

$$\leq \delta + \left| [\widetilde{\varphi}, S_\epsilon - S]_{X^s} \right|,$$

where $[\cdot, \cdot]_{X^k}$ is a dual pairing on $X^{-k} \times X^k$. If we choose $\varphi = \nabla^{2m} \omega_0 \in H^{-m}$, then by the weak continuity,

$$\left| [\varphi, \omega_0]_{H^m} \right| \leq \left| [\varphi, \omega(t) - \omega_0]_{H^m} \right| + \left| [\varphi, \omega(t)]_{H^m} \right|$$

$$\leq \delta + \left| [\varphi, \omega(t)]_{H^m} \right|, \tag{73}$$

$$\left\| \nabla^m \omega_0 \right\|_{L^2}^2 \leq \delta + \left\| \nabla^m \omega(t) \right\|_{L^2} \left\| \nabla^m \omega_0 \right\|_{L^2}$$

$$\leq \delta + \frac{\left\| \nabla^m \omega(t) \right\|_{L^2}^2}{2} + \frac{\left\| \nabla^m \omega_0 \right\|_{L^2}^2}{2}. \tag{74}$$

Similarly, we have

$$\liminf_{t \to 0+} \|S\|_{X^m} (t) \geq S_0. \tag{75}$$

By inequalities (71) and (75), we have the right continuity of $S(t)$ at $t = 0$. If we apply the standard time translation invariant property and the time reversal techique, we also have $S(t) \in C(0, T_0; H^m)$ without any difficulty. Lipschitz continuity also can be obtained by

$$\left\| S(t_1) - S(t_2) \right\|_{X^{m-1}} \leq \delta + \left\| S_\epsilon (t_1) - S_\epsilon (t_2) \right\|_{X^{m-1}}$$

$$\leq \delta + \left\| \int_{t_2}^{t_1} \partial_t S_\epsilon (\tau) \, d\tau \right\|_{X^{m-1}}$$

$$\leq \delta \tag{76}$$

$$+ \left| t_1 - t_2 \right| \sup_{0 \leq \tau \leq T_0} \left\| \partial_t S_\epsilon (\tau) \right\|_{X^{m-1}}$$

$$\leq \delta + \left(4 S_0 \right)^2 \left| t_1 - t_2 \right|,$$

which means $S \in C(0, T_0; X^m) \cap \mathrm{Lip}(0, T_0; X^{m-1})$. Hence it is a classical solution to (21) almost every time.

Next we assume that the initial data $(\mathbf{u}_0, B_0, \Pi_0, \Omega_0)$ is axisymmetric. Then the axisymmetry of a classical solution to (21) is preserved and (ω, B, Π, Ω) is axisymmetric solution to (21). Now we go back to (9) with $\nu = \kappa = 0$, and set

$$u_0 \in H^m \left(\mathbb{R}^3 \right),$$

$$\frac{\mathrm{curl}\, u_0}{r} \in H^{m-1} \left(\mathbb{R}^3 \right),$$

$$B_0 \in H^m \left(\mathbb{R}^3 \right), \tag{77}$$

$$\frac{B_0}{r} \in H^m \left(\mathbb{R}^3 \right),$$

$$m > \frac{9}{2}.$$

Then for

$$\omega_0 = \text{curl } u_0,$$
$$\Omega_0 = \frac{\omega_0}{r},$$
$$\Pi_0 = \frac{B_0}{r},$$

(78)

we know that there exists a unique solution $(\omega, B, \Omega) \in C(0, T_0; H^{m-1}) \cap \text{Lip}(0, T_0; H^{m-2})$, $\Pi \in C(0, T_0; H^m) \cap \text{Lip}(0, T_0; H^{m-1})$. But if we replace Ω with ω/r and Π with B/r, then $\omega, B, \omega/r, B/r$ also satisfy (21) with the initial data. So by the uniqueness, $\Omega = \omega/r$ in $C(0, T_0; H^{m-1}) \cap \text{Lip}(0, T_0; H^{m-2})$ and $\Pi = B/r$ in $C(0, T_0; H^m) \cap \text{Lip}(0, T_0; H^{m-1})$. Next, we can show that $\mathbf{u} = \Phi * \text{curl} \omega \in C(0, T_0; H^m) \cap \text{Lip}(0, T_0; H^{m-1})$. By the Poincaré lemma, $\text{curl} \mathbf{u} = \omega$ satisfies the ω equation of (9). Then we can deduce that $\mathbf{u}$ and B satisfy (9) almost every time by finding the axisymmetric scalar pressure p. Then the energy inequality (4) implies that $u \in L^\infty(0, T_0; L^2)$ and, almost every time,

$$\|u(t_1) - u(t_2)\|_{L^2(\mathbb{R}^3)} \leq |t_1 - t_2|$$
$$\cdot \int_{t_2}^{t_1} \|\partial_t u(t_1) - \partial_t u(t_2)\|_{L^2(\mathbb{R}^3)}(t) dt \leq |t_1 - t_2|$$
$$\cdot \sup_{0 \leq t \leq T_0} \|\partial_t u\|_{L^2} \leq |t_1 - t_2| \sup_{0 \leq t \leq T_0} (\|u\|_{L^\infty} \|\nabla u\|_{L^2}$$
$$+ \|(\nabla \times B) \times B\|_{L^2} + \|\nabla p\|_{L^2}) \leq |t_1 - t_2|$$
$$\cdot \sup_{0 \leq t \leq T_0} (\|\omega\|_1^2 + \|B\|_2^2) < C |t_1 - t_2|,$$

(79)

which implies $\mathbf{u} \in C(0, T_0; H^m) \cap \text{Lip}(0, T_0; H^{m-1})$. The uniqueness of $\mathbf{u}$ can be obtained from the standard techniques and we omit the details. Finally we can show that $B \in C(0, T_0; H^m) \cap \text{Lip}(0, T_0; H^{m-1})$. Almost every time, we can rewrite the B equation with

$$\partial_t B + (u^r \partial_r + u^z \partial_z) B = u^r \Pi + 2\Pi \partial_z B, \quad \left(\frac{B}{r} = \Pi\right). \quad (80)$$

Then we can obtain the conclusion through the standard H^m estimate with Gronwall's inequality. □

3. Proof of Theorem 3: Blow-Up of Axisymmetric Hall Equations

The proof of Theorem 3 is split into two propositions: local-in-time existence of a regular solution to (14) and the finite time blow-up of the local-in-time solution.

Proposition 10. *The equation*

$$\partial_t \Pi = -2\Pi \partial_z \Pi$$
$$\Pi(x, 0) = \Pi_0(x) \in H^m, \quad m > \frac{5}{2}$$

(81)

has a unique local-in-time solution.

Proof. First, we find the global solution to the following regularized equation of (14) without assuming the axisymmetry,

$$\partial_t \Pi_\epsilon = -2\mathscr{J}_\epsilon (\mathscr{J}_\epsilon \Pi_\epsilon \cdot \partial_z \mathscr{J}_\epsilon \Pi_\epsilon)$$
$$\Pi(x, 0) = \Pi_0(x), \quad \Pi_0 \in H^m, \ m > \frac{5}{2}$$

(82)

Before proceeding further, we note that the divergence theorem can be applicable due to the mollifier. Let

$$F(\Pi_\epsilon) = -2\mathscr{J}_\epsilon (\mathscr{J}_\epsilon \Pi_\epsilon \cdot \partial_z \mathscr{J}_\epsilon \Pi_\epsilon).$$

(83)

Hence, the image of the function F defined on H^m is included in H^m for $m > 3/2$.

To use the Picard theorem on H^m space ($m > 5/2$), we first obtain that F is Lipschitz continuous on H^m, i.e.,

$$\left\|F\left(\Pi_\epsilon^1\right) - F\left(\Pi_\epsilon^2\right)\right\|_m$$
$$\leq 2 \left\|\mathscr{J}_\epsilon \left\{\mathscr{J}_\epsilon \Pi_\epsilon^1 \left(\partial_z \mathscr{J}_\epsilon \Pi_\epsilon^1 - \partial_z \mathscr{J}_\epsilon \Pi_\epsilon^2\right)\right\}\right\|_m$$
$$+ 2 \left\|\mathscr{J}_\epsilon \left\{\left(\mathscr{J}_\epsilon \Pi_\epsilon^1 - \mathscr{J}_\epsilon \Pi_\epsilon^2\right) \partial_z \mathscr{J}_\epsilon \Pi_\epsilon^2\right\}\right\|_m$$
$$\leq 2 \left\|\mathscr{J}_\epsilon \Pi_\epsilon^1\right\|_m \left\|\partial_z \mathscr{J}_\epsilon \left(\Pi_\epsilon^1 - \Pi_\epsilon^2\right)\right\|_m$$
$$+ 2 \left\|\mathscr{J}_\epsilon \left(\Pi_\epsilon^1 - \Pi_\epsilon^2\right)\right\|_m \left\|\partial_z \mathscr{J}_\epsilon \Pi_\epsilon^2\right\|_m$$
$$\leq \frac{C}{\epsilon^{m+1}} \left(\left\|\Pi_\epsilon^1\right\|_{L^2} + \left\|\Pi_\epsilon^2\right\|_{L^2}\right) \left\|\Pi_\epsilon^1 - \Pi_\epsilon^2\right\|_m.$$

(84)

F is a Lipschitz continous function on a bounded open set O in H^m. Now we can apply Picard theorem. For each $\epsilon > 0$, there exist a unique solution Π_ϵ and a finite time T_ϵ, such that $\Pi_\epsilon \in C^1(0, T_\epsilon : H^m)$. Following the standard process of constructing local-in-time solution, we obtain an implicit form of the solution

$$\Pi_\epsilon(t) = \Pi_0 + \int_0^t F(\Pi_\epsilon(s)) ds, \quad 0 \leq t < T_\epsilon. \quad (85)$$

Since $F(0) = 0$, we have

$$\|\Pi_\epsilon\|_m(t) \leq \|\Pi_0\|_m + \int_0^t \|F(\Pi_\epsilon(s))\|_m ds$$
$$\leq \|\Pi_0\|_m + \int_0^t C_{m,\epsilon} \|\Pi_\epsilon\|_{L^2}(s) \|\Pi_\epsilon\|_m(s) ds,$$
$$0 \leq t < T_\epsilon.$$

(86)

Since the above regularized equation satisfies an energy estimate, we deduce that

$$\frac{d}{dt} \int_{\mathbb{R}^3} \Pi_\epsilon^2 dx = -\frac{2}{3} \int_{\mathbb{R}^3} \partial_z (\mathscr{J}_\epsilon \Pi_\epsilon)^3 dx = 0, \quad (87)$$

and hence

$$\|\Pi_\epsilon\|_{L^2}(t) = \|\Pi_0\|_{L^2}. \quad (88)$$

For the higher order norm, Gronwall's inequality implies

$$\|\Pi_\epsilon\|_m(t) \leq \|\Pi_0\|_m e^{C_{m,\epsilon} \|\Pi_0\|_{L^2(\mathbb{R}^3)} t}, \quad 0 \leq t < T_\epsilon. \quad (89)$$

The above inequality justifies that each solution Π_ϵ is a global solution to regularized equation, and

$$\Pi_\epsilon \in C^1\left(0, \infty : H^m\right), \quad \text{for all } \Pi_0 \in H^m \tag{90}$$

Second, we show that, for some finite time T, the sequence $\{\Pi_\epsilon\}_{\epsilon > 0}$ is a Cauchy sequence in $C(0, T; L^2)$. We note that, for $T < 1/C_m \|\Pi_0\|_m$,

$$\sup_{0 \leq t \leq T} \|\Pi_\epsilon\|_m (t) \leq \frac{1}{1 - C_m T \|\Pi_0\|_m},$$

$$\sup_{0 \leq t \leq T} \|\partial_t \Pi_\epsilon\|_{m-1} (t) \leq C \left(\frac{1}{1 - C_m T \|\Pi_0\|_m} \right)^2. \tag{91}$$

By the standard energy estimates, we have, for $0 \leq s \leq m$,

$$\frac{1}{2} \frac{d}{dt} \|\nabla^s \Pi_\epsilon\|_{L^2}^2$$

$$= -\int_{\mathbb{R}^3} \nabla^s \left\{ \partial_z \left(\mathcal{J}_\epsilon \Pi_\epsilon \right)^2 \right\} \nabla^s \left(\mathcal{J}_\epsilon \Pi_\epsilon \right) dx \tag{92}$$

$$= -\int_{\mathbb{R}^3} \partial_z \left\{ \nabla^j \left(\mathcal{J}_\epsilon \Pi_\epsilon \right) \nabla^{s-j} \left(\mathcal{J}_\epsilon \Pi_\epsilon \right) \right\} \nabla^s \left(\mathcal{J}_\epsilon \Pi_\epsilon \right) dx.$$

If $j \neq 0$ or $j \neq s$, then we obtain easily that

$$\frac{1}{2} \frac{d}{dt} \|\nabla^s \Pi_\epsilon\|_{L^2}^2$$

$$\leq \int_{\mathbb{R}^3} \left| \nabla^{j+1} \left(\mathcal{J}_\epsilon \Pi_\epsilon \right) \right| \left| \nabla^{s-j} \left(\mathcal{J}_\epsilon \Pi_\epsilon \right) \right| \left| \nabla^s \left(\mathcal{J}_\epsilon \Pi_\epsilon \right) \right| dx \tag{93}$$

$$\leq C \|\nabla \Pi_\epsilon\|_{L^\infty} \|\Pi_\epsilon\|_m^2 \leq C \|\Pi_\epsilon\|_m^3.$$

If $j = 0$ or $j = s$, then we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla^s \Pi_\epsilon\|_{L^2}^2$$

$$= \int_{\mathbb{R}^3} \left(\mathcal{J}_\epsilon \Pi_\epsilon \right) \cdot \partial_z \nabla^s \left(\mathcal{J}_\epsilon \Pi_\epsilon \right) \cdot \nabla^s \left(\mathcal{J}_\epsilon \Pi_\epsilon \right) dx$$

$$= \int_{\mathbb{R}^3} \left(\mathcal{J}_\epsilon \Pi_\epsilon \right) \partial_z \left(\nabla^s \left(\mathcal{J}_\epsilon \Pi_\epsilon \right) \right)^2 dx \tag{94}$$

$$= -\int_{\mathbb{R}^3} \partial_z \left(\mathcal{J}_\epsilon \Pi_\epsilon \right) \left| \nabla^s \left(\mathcal{J}_\epsilon \Pi_\epsilon \right) \right|^2 dx$$

$$\leq \|\partial_z \Pi_\epsilon\|_{L^\infty(\mathbb{R}^3)} \|\Pi_\epsilon\|_m^2 \leq C \|\Pi_\epsilon\|_m^3.$$

Combining the above inequalities (93) and (94), we have

$$\frac{d}{dt} \|\Pi_\epsilon\|_m \leq C_m \|\Pi_\epsilon\|_m^2. \tag{95}$$

The above inequality gives us

$$\|\Pi_\epsilon\|_m (t) \leq \frac{\|\Pi_0\|_m}{1 - t C_m \|\Pi_0\|_m}$$

$$\text{for } 0 \leq t \leq T < \frac{1}{C_m \|\Pi_0\|_m},$$

$$\sup_{0 \leq t \leq T} \|\Pi_\epsilon\|_m (t) \leq \frac{1}{1 - C_m T \|\Pi_0\|_m}$$

$$\text{for } T < \frac{1}{C_m \|\Pi_0\|_m}. \tag{96}$$

By applying H^{m-1} *norm* at the regularized equation, we deduce that

$$\|\partial_t \Pi_\epsilon\|_{m-1} = \|F\left(\Pi_\epsilon \right)\|_{m-1} \leq C'_m \|\Pi_\epsilon\|_{m-1} \|\partial_z \Pi_\epsilon\|_{m-1}$$

$$\leq C'_m \|\Pi_\epsilon\|_m^2 (t)$$

$$\leq C'_m \left(\frac{1}{1 - C_m T \|\Pi_0\|_m} \right)^2 \tag{97}$$

$$\text{for } T < \frac{1}{C_m \|\Pi_0\|_m}.$$

Now we are ready to show that $\{\Pi_\epsilon\} \subset C(0, T; L^2)$ is a Cauchy sequence (as a sequence for $\epsilon_n \longrightarrow 0$), where T is chosen as above.

$$\frac{1}{2} \frac{d}{dt} \|\Pi_\epsilon - \Pi_{\epsilon'}\|_{L^2(\mathbb{R}^3)}^2$$

$$= -\int_{\mathbb{R}^3} \left\{ \mathcal{J}_\epsilon \partial_z \left(\mathcal{J}_\epsilon \Pi_\epsilon \right)^2 - \mathcal{J}_{\epsilon'} \partial_z \left(\mathcal{J}_{\epsilon'} \Pi_{\epsilon'} \right)^2 \right\}$$

$$\cdot \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) dx = -\int \mathcal{J}_\epsilon \left(\Pi_\epsilon - \Pi_{\epsilon'} \right)$$

$$\cdot \partial_z \left(\mathcal{J}_\epsilon \Pi_\epsilon \right)^2 + \int \mathcal{J}_{\epsilon'} \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \partial_z \left(\mathcal{J}_{\epsilon'} \Pi_{\epsilon'} \right)^2$$

$$= -2 \int \left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'} \right) \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \mathcal{J}_\epsilon \Pi_\epsilon \partial_z \mathcal{J}_\epsilon \Pi_\epsilon \tag{98}$$

$$- 2 \int \mathcal{J}_{\epsilon'} \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \left(\mathcal{J}_\epsilon \Pi_\epsilon - \mathcal{J}_{\epsilon'} \Pi_{\epsilon'} \right) \partial_z \left(\mathcal{J}_\epsilon \Pi_\epsilon \right)$$

$$- 2 \int \left\{ \mathcal{J}_{\epsilon'} \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \right\}^2 \partial_z \left(\mathcal{J}_\epsilon \Pi_\epsilon \right)$$

$$- 2 \int \mathcal{J}_{\epsilon'} \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \mathcal{J}_{\epsilon'} \Pi_{\epsilon'} \partial_z \left(\mathcal{J}_\epsilon \Pi_\epsilon - \mathcal{J}_{\epsilon'} \Pi_{\epsilon'} \right)$$

$$- \int \mathcal{J}_{\epsilon'} \Pi_{\epsilon'} \partial_z \left\{ \mathcal{J}_{\epsilon'} \left(\Pi_\epsilon - \Pi_{\epsilon'} \right) \right\}^2$$

By the properties of regularizer $\mathcal{J}_\epsilon$, for $m > 5/2$, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\|\Pi_\epsilon - \Pi_{\epsilon'}\right\|_{L^2(\mathbb{R}^3)}^2 &\leq C\left\|\Pi_\epsilon - \Pi_{\epsilon'}\right\|_{L^2}\left(\epsilon + \epsilon'\right)\\
&\cdot\left(\left\|\partial_z\Pi_\epsilon\right\|_{L^\infty}\left\|\Pi_\epsilon\right\|_1 + \left\|\partial_z\Pi_\epsilon\right\|_1\left\|\Pi_\epsilon\right\|_{L^\infty}\right) + C\left(\epsilon\right.\\
&\left.+ \epsilon'\right)\left\|\Pi_\epsilon\right\|_1\left\|\Pi_\epsilon - \Pi_{\epsilon'}\right\|_{L^2}\left\|\partial_z\Pi_\epsilon\right\|_{L^\infty} + C\left\|\Pi_\epsilon\right.\\
&\left.- \Pi_{\epsilon'}\right\|_{L^2}^2\left\|\partial_z\Pi_\epsilon\right\|_{L^\infty} + C\left(\epsilon + \epsilon'\right)\left\|\Pi_\epsilon\right\|_2\left\|\Pi_\epsilon - \Pi_{\epsilon'}\right\|_{L^2}\\
&\cdot\left\|\Pi_{\epsilon'}\right\|_{L^\infty} + C\left\|\Pi_\epsilon - \Pi_{\epsilon'}\right\|_{L^2}^2\left\|\partial_z\Pi_{\epsilon'}\right\|_{L^\infty} \leq C\left(\epsilon\right.\\
&\left.+ \epsilon'\right)\left\|\Pi_\epsilon\right\|_m\left\|\Pi_{\epsilon'}\right\|_m\left\|\Pi_\epsilon - \Pi_{\epsilon'}\right\|_{L^2} + C\left(\left\|\Pi_\epsilon\right\|_m\right.\\
&\left.+ \left\|\Pi_{\epsilon'}\right\|_m\right)\left\|\Pi_\epsilon - \Pi_{\epsilon'}\right\|_{L^2}^2 \leq C\left\|\Pi_\epsilon - \Pi_{\epsilon'}\right\|_{L^2}\\
&\cdot\left\{\left(\epsilon + \epsilon'\right)\left\|\Pi_\epsilon\right\|_m\left\|\Pi_{\epsilon'}\right\|_m\right.\\
&\left.+ \left\|\Pi_\epsilon - \Pi_{\epsilon'}\right\|_{L^2}\left(\left\|\Pi_\epsilon\right\|_m + \left\|\Pi_{\epsilon'}\right\|_m\right)\right\}
\end{aligned}
\tag{99}
$$

In summary, we have

$$
\begin{aligned}
\frac{d}{dt}&\left\|\Pi_\epsilon - \Pi_{\epsilon'}\right\|_{L^2(\mathbb{R}^3)}\\
&\leq C\left(\epsilon + \epsilon'\right)\left\|\Pi_\epsilon\right\|_m\left\|\Pi_{\epsilon'}\right\|_m\\
&\quad + C\left\|\Pi_\epsilon - \Pi_{\epsilon'}\right\|_{L^2}\left(\left\|\Pi_\epsilon\right\|_m + \left\|\Pi_{\epsilon'}\right\|_m\right)
\end{aligned}
\tag{100}
$$

By Gronwall's inequality, we can conclude that $\{\Pi_\epsilon\}$ is a Cauchy sequence in the $C(0, T; L^2)$ space. And by H^m boundedness, if we apply the interpolation inequality, then we can see that $\{\Pi_\epsilon\}$ is a cauchy sequence in $C(0, T; H^s)$, $\forall s < m$. So we have the limit function $\Pi \in C(0, T; H^s)$. And $\{\partial_t\Pi_\epsilon\}$ is also a cauchy sequence in the $C(0, T; L^2)$ space by the following estimates:

$$
\begin{aligned}
&\left\|\partial_t\Pi_\epsilon - \partial_t\Pi_{\epsilon'}\right\|_{L^2}\\
&\leq 2\left\|\mathcal{J}_\epsilon\left(\mathcal{J}_\epsilon\Pi_\epsilon\partial_z\mathcal{J}_\epsilon\Pi_\epsilon\right) - \mathcal{J}_{\epsilon'}\left(\mathcal{J}_{\epsilon'}\Pi_{\epsilon'}\partial_z\mathcal{J}_{\epsilon'}\Pi_{\epsilon'}\right)\right\|_{L^2}\\
&\leq 2\left\|\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'}\right)\left(\mathcal{J}_\epsilon\Pi_\epsilon\partial_z\mathcal{J}_\epsilon\Pi_\epsilon\right)\right\|_{L^2}\\
&\quad + 2\left\|\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'}\right)\Pi_\epsilon\partial_z\mathcal{J}_\epsilon\Pi_\epsilon\right\|_{L^2}\\
&\quad + 2\left\|\mathcal{J}_{\epsilon'}\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\partial_z\mathcal{J}_\epsilon\Pi_\epsilon\right\|_{L^2}\\
&\quad + 2\left\|\mathcal{J}_{\epsilon'}\Pi_{\epsilon'}\left(\mathcal{J}_\epsilon - \mathcal{J}_{\epsilon'}\right)\partial_z\Pi_\epsilon\right\|_{L^2}\\
&\quad + 2\left\|\mathcal{J}_{\epsilon'}\Pi_{\epsilon'}\mathcal{J}_{\epsilon'}\partial_z\left(\Pi_\epsilon - \Pi_{\epsilon'}\right)\right\|_{L^2}\\
&\leq 6\left(\epsilon + \epsilon'\right)\left(\left\|\Pi_\epsilon\right\|_s^2 + \left\|\Pi_{\epsilon'}\right\|_s^2\right)\\
&\quad + 4\left(\left\|\Pi_\epsilon\right\|_s + \left\|\Pi_{\epsilon'}\right\|_s\right)\left\|\Pi_\epsilon - \Pi_{\epsilon'}\right\|_1
\end{aligned}
\tag{101}
$$

For $5/2 < s < m$, we have the limit function $\partial_t\Pi \in C(0, T; H^{s-1})$.

Finally, we can show that $\Pi \in C(0, T; H^m) \cap \text{Lip}(0, T; H^{m-1})$. By the Banach Alaoglu theorem, we have $\Pi \in L^\infty(0, T_0; H^m)$ and $\partial_t\Pi \in L^\infty(0, T_0; H^{m-1})$, because we know

that, for any $s < m$, $\Pi \in C(0, T; H^s)$. It implies $\Pi \in C_w(0, T; H^m)$ by the following estimate:

$$
\begin{aligned}
[\varphi, \Pi_\epsilon - \Pi] &\leq \left\|\varphi - \tilde{\varphi}\right\|_{H^{-m}}\left\|\Pi_\epsilon - \Pi\right\|_{H^m}\\
&\quad + [\tilde{\varphi}, \Pi_\epsilon - \Pi] \leq \delta + [\tilde{\varphi}, \Pi_\epsilon - \Pi]\\
&\leq 2\delta
\end{aligned}
\tag{102}
$$

for any given $\varphi \in H^{-m}$, for some $\tilde{\varphi} \in H^{-s}$, $s < m$. Now we show that

$$
\liminf_{t \to 0+}\left\|\Pi(t)\right\|_m = \limsup_{t \to 0+}\left\|\Pi(t)\right\|_m = \left\|\Pi_0\right\|_m.
\tag{103}
$$

By the weak continuity, for any $\delta > 0$, there exists $r > 0$ such that if $0 < t < r$, then $-\delta < [\Pi(t) - \Pi_0, \varphi] < \delta$, for all $\varphi \in H^{-m}$. Choose $\varphi = \nabla^{2s}\Pi_0$ with $s \leq m$. Then it gives us

$$
\begin{aligned}
\left\|\nabla^s\Pi_0\right\|_{L^2}^2 - \delta &\leq \left\|\nabla^s\Pi(t)\right\|_{L^2}\left\|\nabla^s\Pi_0\right\|_{L^2}\\
&\leq \frac{\left\|\nabla^s\Pi(t)\right\|_{L^2}^2}{2} + \frac{\left\|\nabla^s\Pi_0\right\|_{L^2}^2}{2}.
\end{aligned}
\tag{104}
$$

Also we have

$$
\left\|\Pi_0\right\|_m^2 \leq \left\|\Pi(t)\right\|_m^2 + 2^{m+1}\delta,
\tag{105}
$$

which implies that

$$
\liminf_{t \to 0+}\left\|\Pi(t)\right\|_m \geq \left\|\Pi_0\right\|_m
\tag{106}
$$

By the H^m boundedness with weak convergence, it is deduced that

$$
\left\|\Pi(t)\right\|_m \leq \frac{\left\|\Pi_0\right\|_m}{1 - tC_m\left\|\Pi_0\right\|_m}, \quad 0 \leq t \leq T
\tag{107}
$$

which implies

$$
\limsup_{t \to 0+}\left\|\Pi(t)\right\|_m \leq \left\|\Pi_0\right\|_m.
\tag{108}
$$

Thus we have the time continuity of Π at 0. For any $t_0 \leq T$ and initial value $\Pi(t_0)$, we can obtain a right continuity at t_0 by the time translation invariant property. By the fact that $\Pi(-x, t_0 - t)$ is also a solution to the euqation for $0 \leq t \leq t_0$, we have a left continuity at t_0. Of course by the above process, the left continuity at T also can be obtained. We have proved that $\Pi \in C(0, T; H^m)$. □

Proposition 11. *Let Π be an axisymmetric global classical solution to*

$$
\partial_t\Pi + 2\Pi\partial_z\Pi = 0.
\tag{109}
$$

Then $\Pi \equiv 0$.

Proof. Define $\phi_{r_0}(t) : [0, \infty) \longrightarrow \mathbb{R}$ which satisfies the equation $\phi'_{r_0}(t) = \Pi(r_0, 2\phi_{r_0}(t), t)$. By our assumption, $\Pi \in C([0, \infty); H^m(\mathbb{R}^3)) \cap \text{Lip}([0, \infty); H^{m-1}(\mathbb{R}^3))$ for $m > 5/2$,

$\phi_{r_0}''(t) = 0$ almost every time. So we can find the explicit form of it by

$$\phi_{r_0}(t) = t\Pi_0\left(r_0, 2\phi_{r_0}(0)\right) + \phi_{r_0}(0). \tag{110}$$

Now we choose initial values $\phi_{r_0}(0)$ and $\tilde{\phi}_{r_0}(0)$ such that $\phi_{r_0}(0) \le \tilde{\phi}_{r_0}(0)$. Then it satisfies

$$\Pi_0\left(r_0, 2\phi_{r_0}(0)\right) \le \Pi_0\left(r_0, 2\tilde{\phi}_{r_0}(0)\right). \tag{111}$$

Because if we suppose that $\Pi_0(r_0, 2\phi_{r_0}(0)) > \Pi_0(r_0, 2\tilde{\phi}_{r_0}(0))$, then by the explicit form of ϕ, for some $t_0 > 0$, we have $\phi_{r_0}(t_0) = \tilde{\phi}_{r_0}(t_0)$ which makes a contradiction. Hence, $\Pi_0(r_0, z)$ is a nondecreasing funcion with respect to z. Since this process is independent of the choice of r_0, we can find that $\Pi_0 \equiv 0$ by the continuity and L^2 boundness. $\quad\square$

4. Proof of Theorem 5: Blow-Up Criterion

In this section, we provide the proof of Theorem 5 which is the blow-up criterion for the axisymmetric Hall-MHD equations with $\nu = 0$ and $\kappa = 1$:

$$\frac{\partial \Omega}{\partial t} + \left(u^r \partial_r + u^z \partial_z\right)\Omega + 2\Pi\partial_z\Pi = 0, \tag{112}$$

$$\begin{aligned}\frac{\partial \Pi}{\partial t} &+ \left(u^r \partial_r + u^z \partial_z\right)\Pi - 2\Pi\partial_z\Pi \\ &= \left(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\right)\Pi,\end{aligned} \tag{113}$$

where $\Omega = \omega^\theta / r$ and $\Pi = B^\theta / r$.

Known blow-up criterion for the partial viscous Hall-MHD equations (1) without symmetry ($\nu = 0$ and $\kappa = 1$) is as follows (see [13]).

Proposition 12. *Assume that $(u_0, B_0) \in H^m(\mathbb{R}^3)$, $m \ge 3$ with $\nabla \cdot u_0 = \nabla \cdot B_0 = 0$. Let (u, B) be a smooth solution to (1) ($\nu = 0$ and $\kappa = 1$) for $0 \le t < T$. If (u, B) satisfies*

$$\int_0^T \left(\|\nabla \times u(t)\|_{L^\infty} + \|\nabla \times B(t)\|_{L^\infty}^2\right) dt < \infty, \tag{114}$$

then the solution (u, B) can be extended beyond $t = T$.

With the axial symmetry, we can derive the following apriori estimates.

Proposition 13. *If (Ω, Π) is a solution to (112)–(113) satisfying $(\Omega, \Pi) \in C([0, T]; H^m)$, with $m \ge 3$ then it holds that*

$$\|\Pi\|_{L^\infty(0,T;L^p(\mathbb{R}^3))} \le C(\Pi_0),$$

$$\|\nabla\Pi\|_{L^2(0,T;L^2(\mathbb{R}^3))}^2 \le \|\Pi_0\|_{L^2}^2, \tag{115}$$

$$\|\Omega\|_{L^\infty(0,T;L^2)} \le C\left(\|\Omega_0\|_{L^2}, \|\Pi_0\|_{L^2}\right) < \infty.$$

Proof. We first consider $p = 2n$ with $n \in \mathbb{N}$. Taking scalar product of (113) with Π^{p-1}, we deduce that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\Pi(t)\|_{L^p}^p \\ &\quad - 2\pi \int_{-\infty}^\infty \int_0^\infty \left(\left(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\right)\Pi\right) \cdot \Pi^{p-1} r\,dr\,dz \\ &= \frac{2}{p+1}\int_{\mathbb{R}^3} \partial_z(\Pi)^{p+1}\,dx \\ &\quad - \int_{\mathbb{R}^3}\left(u^r\partial_r + u^z\partial_z\right)\Pi \cdot \Pi^{p-1}\,dx.\end{aligned} \tag{116}$$

From the divergence free condition and the decay conditions like

$$\lim_{r\to\infty}\Pi(r, z, t) = \lim_{|z|\to\infty}\Pi(r, z, t) = 0, \tag{117}$$

(116) can be reduced to

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\Pi(t)\|_{L^p}^p + \frac{4(p-1)}{p^2}\int_{\mathbb{R}^3}\left|\partial_r\Pi^{p/2}\right|^2 \\ &\quad + \left|\partial_z\Pi^{p/2}\right|^2\,dx = 0.\end{aligned} \tag{118}$$

It implies that

$$\|\Pi(t)\|_{L^\infty(0,T;L^p)}^p \le \|\Pi_0\|_{L^p}^p. \tag{119}$$

For any $R > 0$, we have

$$\left(\int_{B_R}\left|\Pi_0(x)\right|^{2n}\,dx\right)^{1/2n} \le C\|\Pi_0\|_{L^\infty} R^{3/2n}. \tag{120}$$

As $n \to \infty$, we have $\|\Pi\|_{L^\infty(0,T;L^\infty)} \le C\|\Pi_0\|_{L^\infty}$. If $p \in (2n, 2(n+1))$, then we have

$$\|\Pi\|_{L^\infty(0,T;L^p)}^p \le \|\Pi_0\|_{L^{2n}}^{n(2n+2-p)}\|\Pi_0\|_{L^{2(n+1)}}^{(n+1)(p-2n)}. \tag{121}$$

Taking L^2 scalar product of (112) with Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Omega\|_{L^2}^2 &= -2\int_{\mathbb{R}^3}\Pi\partial_z\Pi\Omega\,dx \\ &\le 2\|\Pi\|_{L^\infty}\|\partial_z\Pi\|_{L^2}\|\Omega\|_{L^2}\end{aligned} \tag{122}$$

Then we have

$$\begin{aligned}\|\Omega\|_{L^\infty(0,T;L^2)} &\lesssim \|\Omega_0\|_{L^2} \\ &\quad + \int_0^T \|\Pi(t)\|_{L^\infty}^2 + \|\partial_z\Pi(t)\|_{L^2}^2\,dt\end{aligned} \tag{123}$$

This completes the proof. $\quad\square$

From the energy estimates of the velocity and magnetic fields, we have

$$\begin{aligned}&\|u^r\|_{L^\infty(0,T;L^2)}^2 + \|u^z\|_{L^\infty(0,T;L^2)}^2 + \|B^\theta\|_{L^\infty(0,T;L^2)}^2 \\ &\quad + \int_0^T \|\partial_r B^\theta\|_{L^2}^2 + \|\partial_z B^\theta\|_{L^2}^2 + \left\|\frac{B^\theta}{r}\right\|_{L^2}^2\,dt \\ &\le \|u_0^r\|_{L^2}^2 + \|u_0^z\|_{L^2}^2 + \|B_0^\theta\|_{L^2}^2.\end{aligned} \tag{124}$$

Proof of Theorem 5. First, we assume that assumption (19) holds. If we consider the equation of the vorticity ω^θ, then we have

$$\partial_t \omega^\theta + (u^r \partial_r + u^z \partial_z)\omega^\theta = \frac{u^r}{r}\omega^\theta - 2\left(\frac{B^\theta}{r}\right)\partial_z B^\theta. \quad (125)$$

Taking L^2 scalar product of (125) with ω^θ, we have

$$\frac{d}{dt}\left\|\omega^\theta\right\|_{L^2}^2 \leq \int_{\mathbb{R}^3} \frac{u^r_+}{r}\left(\omega^\theta\right)^2 dx$$
$$+ 2\int_{\mathbb{R}^3}\left|\partial_z B^\theta\right|\left|\Pi\right|\left|\omega^\theta\right| dx$$
$$\leq \left\|\frac{u^r_+}{r}\right\|_{L^\infty(\mathbb{R}^3\backslash C_R)}\left\|\omega^\theta\right\|_{L^2(\mathbb{R}^3\backslash C_R)}^2 \quad (126)$$
$$+ \left\|ru^r_+\right\|_{L^\infty(C_R)}\left\|\Omega\right\|_{L^2(C_R)}^2$$
$$+ 2\left\|\partial_z B^\theta\right\|_{L^2}\left\|\Pi\right\|_{L^\infty}\left\|\omega^\theta\right\|_{L^2}.$$

Using Gronwall's inequality, we have

$$\left\|\omega^\theta\right\|_{L^\infty(0,T;L^2)} \leq \left(\left\|\omega_0^\theta\right\|_{L^2} + \int_0^T \left\|ru^r_+\right\|_{L^\infty(C_R)} dt + C\right)$$
$$\cdot \exp\left(\int_0^T\left\|\frac{u^r_+}{r}\right\|_{L^\infty(\mathbb{R}^3\backslash C_R)} dt + C\right). \quad (127)$$

Hence we have $\omega^\theta \in L^\infty(0,T;L^2)$ if we assume (19).

If we consider the equations for $\widetilde{\nabla}\Pi$ ($\widetilde{\nabla} = (\partial_r, \partial_z)$), then we obtain

$$\partial_t\partial_z\Pi + (u^r\partial_r + u^z\partial_z)\partial_z\Pi$$
$$+ (\partial_z u^r\partial_r + \partial_z u^z\partial_z)\Pi = \left(\Delta + \frac{2}{r}\partial_r\right)\partial_z\Pi + \partial_z^2\Pi^2$$
$$\partial_t\partial_r\Pi + (u^r\partial_r + u^z\partial_z)\partial_r\Pi + (\partial_r u^r\partial_r + \partial_r u^z\partial_z)\Pi \quad (128)$$
$$+ \frac{2}{r^2}\partial_r\Pi = \left(\Delta + \frac{2}{r}\partial_r\right)\partial_r\Pi + \partial_r\partial_z\Pi^2$$

Via an interpolation inequality and Young's inequality, we have

$$\left\|\omega^\theta\right\|_{L^2(\mathbb{R}^3)}\left\|\widetilde{\nabla}\Pi\right\|_{L^6(\mathbb{R}^3)}\left\|\widetilde{\nabla}\Pi\right\|_{L^3(\mathbb{R}^3)}$$
$$\leq \left\|\omega^\theta\right\|_{L^2(\mathbb{R}^3)}\left\|\widetilde{\nabla}^2\Pi\right\|_{L^2(\mathbb{R}^3)}^{3/2}\left\|\widetilde{\nabla}\Pi\right\|_{L^2(\mathbb{R}^3)}^{1/2} \quad (129)$$
$$\leq C\left\|\omega^\theta\right\|_{L^2(\mathbb{R}^3)}^4\left\|\widetilde{\nabla}\Pi\right\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2}\left\|\widetilde{\nabla}^2\Pi\right\|_{L^2(\mathbb{R}^3)}^2.$$

Taking scalar product of (128) with $\widetilde{\nabla}\Pi$, we deduce that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left|\widetilde{\nabla}\Pi\right|^2 dx + \int_{\mathbb{R}^3}\left|\widetilde{\nabla}^2\Pi\right|^2 dx$$
$$+ \int_{\mathbb{R}^3}\frac{2}{r^2}\left|\partial_r\Pi\right|^2 dx$$
$$\leq C\left(\left\|\omega^\theta\right\|_{L^2}\left\|\widetilde{\nabla}\Pi\right\|_{L^6}\left\|\widetilde{\nabla}\Pi\right\|_{L^3}\right.$$
$$+ \int_{\mathbb{R}^3}\frac{1}{r}\partial_r\left(\left|\widetilde{\nabla}\Pi\right|^2\right)dx + \int_{\mathbb{R}^3}\left|\partial_z\Pi\right|^3 dx \quad (130)$$
$$+ C\left\|\widetilde{\nabla}^2\Pi\right\|_{L^2}\left\|\widetilde{\nabla}\Pi\right\|_{L^2}\left\|\Pi\right\|_{L^\infty}$$
$$+ \int_{\mathbb{R}^3}\left|\partial_r\Pi\right|^2\left|\partial_z\Pi\right|dx\right) \leq C\left(\left\|\omega^\theta\right\|_{L^2}^4 + \left\|\widetilde{\nabla}\Pi\right\|_{L^2}^2\right.$$
$$+ \left\|\Pi\right\|_{L^\infty}^2\right)\left\|\widetilde{\nabla}\Pi\right\|_{L^2}^2 + \frac{1}{2}\left\|\widetilde{\nabla}\Pi\right\|_{L^2}^2.$$

In the above, we used the fact that $(1/r)\partial_r(|\partial_r\Pi|^2 + |\partial_z\Pi|^2) \in L^1(\mathbb{R}^3)$ when $t < T^*$. Gronwall's inequality again gives us

$$\widetilde{\nabla}\Pi \in L^\infty\left(0,T;L^2\right),$$
$$\widetilde{\nabla}^2\Pi \in L^2\left(0,T;L^2\right). \quad (131)$$

Multiplying both sides of (112) with $|\Omega|^4\Omega$ and integrating over $\mathbb{R}^3$, we have

$$\frac{1}{6}\frac{d}{dt}\left\|\Omega\right\|_{L^6(\mathbb{R}^3)}^6 \leq 2\int_{\mathbb{R}^3}\left|\partial_z\Pi\right|\left|\Pi\right|\left|\Omega\right|^5 dx$$
$$\leq \left\|\widetilde{\nabla}\Pi\right\|_{L^6}\left\|\Pi\right\|_{L^\infty}\left\|\Omega\right\|_6^5. \quad (132)$$

Then it is immediate that $\Omega \in L^\infty(0,T;L^6)$.

Following the ideas in [18], we introduce the angular stream function ψ^θ such that

$$-\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2} + \partial_z^2\right)\psi^\theta = \omega^\theta. \quad (133)$$

For all $1 < p < \infty$, we have

$$\int\left|\nabla^2\frac{\psi^\theta}{r}\right|^p dx \leq C\int\left|\frac{\omega^\theta}{r}\right|^p dx. \quad (134)$$

By the interpolation inequality $\|f\|_{L^\infty} \leq C\|\nabla f\|_{L^2}^{1/2}\|\nabla f\|_{L^6}^{1/2}$, we have

$$\int_0^T\left\|\frac{u^r}{r}\right\|_{L^\infty} dt$$
$$\leq \int_0^T\left\|\nabla\partial_z\left(\frac{\psi^\theta}{r}\right)\right\|_{L^2}^{1/2}\left\|\nabla\partial_z\left(\frac{\psi^\theta}{r}\right)\right\|_{L^6}^{1/2} dt \quad (135)$$
$$\leq \sup_t\left\|\Omega\right\|_{L^2}^{1/2}\int_0^T\left\|\Omega\right\|_{L^6}^{1/2} dt.$$

Therefore, we have $u^r/r \in L^1(0, T; L^\infty)$. Also we can have $u^r/r \in L^p(0, T; L^\infty)$ for all $p < \infty$.

If we multiply $(B^\theta)^{2n-1}$ on the both sides of $(9)_4$ and integrate over $\mathbb{R}^3$, then we have

$$\frac{1}{2n}\frac{d}{dt}\left\|B^\theta\right\|_{L^{2n}}^{2n} + \frac{(2n-1)}{n^2}\left\|\widetilde{\nabla}\left|B^\theta\right|^n\right\|_{L^2}^2 + \left\|\frac{\left(B^\theta\right)^n}{r}\right\|_{L^2}^2$$

$$= \int_{\mathbb{R}^3} \frac{u^r}{r}\left(B^\theta\right)^{2n} dx$$

$$+ 2\pi \int_0^\infty \int_{-\infty}^\infty \partial_z\left(B^\theta\right)^2 \left(B^\theta\right)^{2n-1} dz dr \qquad (136)$$

$$\leq \left\|\frac{u^r}{r}\right\|_{L^\infty}\left\|B^\theta\right\|_{L^{2n}}^{2n}.$$

Hence we deduce that

$$\left\|B^\theta\right\|_{L^\infty(0,T;L^{2n})} \leq \left\|B_0^\theta\right\|_{L^{2n}}\exp\left(\int_0^T \left\|\frac{u^r}{r}\right\|_{L^\infty} dt\right). \qquad (137)$$

Similarly, we have $\|B^\theta\|_{L^\infty(0,T;L^\infty)} \leq \|B_0^\theta\|_{L^{2n}}\exp(\int_0^T \|u^r/r\|_{L^\infty}dt)$.

Setting $\mathbf{B} = B^\theta e_\theta$, we have

$$\partial_t\mathbf{B} + (u\cdot\nabla)\mathbf{B} - \Delta\mathbf{B} = \frac{u^r}{r}\mathbf{B} + 2\partial_z\mathbf{B}\Pi. \qquad (138)$$

By the maximal inequality, we have

$$\|\Delta\mathbf{B}\|_{L^{4,4}_{x,t}(\mathbb{R}^3\times(0.T))}^4 \leq C\left(\|\mathbf{B_0}\|_{W^{2,4}}^4\right.$$

$$+ \|u\|_{L^{6,\infty}_{x,t}(\mathbb{R}^3\times(0.T))}^4 \|\nabla\mathbf{B}\|_{L^{12,4}_{x,t}(\mathbb{R}^3\times(0.T))}^4$$

$$+ \left\|\frac{u^r}{r}\right\|_{L^{\infty,4}_{x,t}(\mathbb{R}^3\times(0.T))}^4 \|\mathbf{B}\|_{L^{4,\infty}_{x,t}(\mathbb{R}^3\times(0.T))}^4 \qquad (139)$$

$$\left.+ \|\Pi\|_{L^{\infty,\infty}_{x,t}(\mathbb{R}^3\times(0.T))}^4 \|\nabla\mathbf{B}\|_{L^{4,4}_{x,t}(\mathbb{R}^3\times(0.T))}^4\right).$$

By the Gagliardo-Nirenberg inequality

$$\|\nabla\mathbf{B}\|_{L^{12}} \leq C\|\mathbf{B}\|_{L^\infty}^{2/5}\|\Delta\mathbf{B}\|_{L^4}^{3/5},$$

$$\|\nabla\mathbf{B}\|_{L^4} \leq C\|\mathbf{B}\|_{L^\infty}^{4/5}\|\Delta\mathbf{B}\|_{L^4}^{1/5}, \qquad (140)$$

(139) can be reduced to

$$\|\Delta\mathbf{B}\|_{L^{4,4}_{x,t}(\mathbb{R}^3\times(0.T))}^4 \leq C\left(\|\mathbf{B_0}\|_{W^{2,4}}^4 + \|\mathbf{B}\|_{L^{4,\infty}_{x,t}(\mathbb{R}^3\times(0.T))}^4\right.$$

$$+ \|\mathbf{B}\|_{L^{\infty,4}_{x,t}(\mathbb{R}^3\times(0.T))}^{8/5}\|\Delta\mathbf{B}\|_{L^{4,4}_{x,t}(\mathbb{R}^3\times(0.T))}^{12/5}$$

$$\left.+ \|\mathbf{B}\|_{L^{\infty,4}_{x,t}(\mathbb{R}^3\times(0.T))}^{16/5}\|\Delta\mathbf{B}\|_{L^{4,4}_{x,t}(\mathbb{R}^3\times(0.T))}^{4/5}\right) \leq C\left(1\right.$$

$$\left.+ \|\mathbf{B}\|_{L^{4,\infty}_{x,t}(\mathbb{R}^3\times(0.T))}^4 + \|\mathbf{B}\|_{L^{\infty,4}_{x,t}(\mathbb{R}^3\times(0.T))}^4\right)$$

$$+ \frac{1}{2}\|\Delta\mathbf{B}\|_{L^{4,4}_{x,t}(\mathbb{R}^3\times(0.T))}^4. \qquad (141)$$

Since the last term in the above can be absorbed in the left hand side, we have

$$\Delta\mathbf{B} \in L^4\left(0, T; L^4\right),$$

$$\nabla\mathbf{B} \in L^4\left(0, T; L^\infty\right). \qquad (142)$$

Then, from (125), we have, for all $1 < p < \infty$,

$$\frac{1}{p}\frac{d}{dt}\left\|\omega^\theta\right\|_{L^p}^p \leq \left\|\frac{u^r}{r}\right\|_{L^\infty}\left\|\omega^\theta\right\|_{L^p}^p$$

$$+ 2\|\Pi\|_{L^\infty}\left\|\nabla B^\theta\right\|_{L^p}\left\|\omega^\theta\right\|_{L^p}^{p-1}. \qquad (143)$$

By Gronwall's inequality, we have

$$\left\|\omega^\theta\right\|_{L^\infty(0,T;L^p)} \leq \left(\left\|\omega_0^\theta\right\|_{L^p} + 2\int_0^T \|\Pi\|_{L^\infty}\left\|\nabla B^\theta\right\|_{L^p} dt\right)$$

$$\cdot \exp\left(\int_0^T \left\|\frac{u^r}{r}\right\|_{L^\infty} dt\right). \qquad (144)$$

If we let $p \longrightarrow \infty$, we obtain $\omega^\theta \in L^\infty(0, T; L^\infty)$. Hence, for any $T < \infty$, we obtain that $\omega^\theta, \nabla\mathbf{B} \in L^2(0, T; L^\infty)$ and conclude that there does not exist a finite time blow-up if we assume (19).

Next, we assume that condition (20) holds. If we apply (129) to (128), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left|\widetilde{\nabla}\Pi\right|^2 dx + \int_{\mathbb{R}^3}\left|\widetilde{\nabla}^2\Pi\right|^2 dx$$

$$+ \int_{\mathbb{R}^3}\frac{2}{r^2}\left|\partial_r\Pi\right|^2 dx \leq C\left(\left\|\omega^\theta\right\|_{L^2(\mathbb{R}^3)}^4\left\|\widetilde{\nabla}\Pi\right\|_{L^2(\mathbb{R}^3)}^2\right.$$

$$+ \frac{1}{2}\left\|\widetilde{\nabla}^2\Pi\right\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3}\frac{1}{r}\partial_r\left(\left|\partial_r\Pi\right|^2 + \left|\partial_z\Pi\right|^2\right)dx \qquad (145)$$

$$+ \int_{\mathbb{R}^3}\left|\partial_z\Pi\right|^3 dx + \int_{\mathbb{R}^3}\left|\partial_z^2\Pi\right|\left|\partial_z\Pi\right||\Pi| dx$$

$$\left.+ \int_{\mathbb{R}^3}\left|\partial_r\Pi\right|^2\left|\partial_z\Pi\right| dx + \int_{\mathbb{R}^3}\left|\widetilde{\nabla}^2\Pi\right|\left|\partial_r\Pi\right||\Pi| dx\right)$$

Using an inequality

$$\left\|\omega^\theta\right\|_{L^2(\mathbb{R}^3)}^4 \leq 2\left\|\omega^\theta\right\|_{L^2(\mathbb{R}^3\backslash C_R)}^4 + 2R^4\|\Omega\|_{L^2(C_R)}^4, \qquad (146)$$

we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}&\int_{\mathbb{R}^3}\left|\widetilde{\nabla}\Pi\right|^2 dx + \int_{\mathbb{R}^3}\left|\widetilde{\nabla}^2\Pi\right|^2 dx \\
&+ \int_{\mathbb{R}^3}\frac{2}{r^2}\left|\partial_r\Pi\right|^2 dx \\
&\leq C\left(\left\|\omega^\theta\right\|_{L^2(\mathbb{R}^3\setminus C_R)}^4 + R^4\left\|\Omega\right\|_{L^2(C_R)}^4\right)\left\|\widetilde{\nabla}\Pi\right\|_{L^2(\mathbb{R}^3)}^2 \\
&+ \frac{1}{2}\left\|\widetilde{\nabla}^2\Pi\right\|_{L^2(\mathbb{R}^3)}^2 \\
&+ \int_{\mathbb{R}^3}\frac{1}{r}\partial_r\left(\left|\partial_r\Pi\right|^2 + \left|\partial_z\Pi\right|^2\right)dx \\
&+ \int_{\mathbb{R}^3}\left|\partial_z\Pi\right|^3 dx + \int_{\mathbb{R}^3}\left|\partial_z^2\Pi\right|\left|\partial_z\Pi\right|\left|\Pi\right|dx \\
&+ \int_{\mathbb{R}^3}\left|\partial_r\Pi\right|^2\left|\partial_z\Pi\right|dx \\
&+ \int_{\mathbb{R}^3}\left|\widetilde{\nabla}^2\Pi\right|\left|\partial_r\Pi\right|\left|\Pi\right|dx.
\end{aligned}
\tag{147}
$$

By Gronwall's inequality, we conclude that

$$
\begin{aligned}
\widetilde{\nabla}\Pi &\in L^\infty\left(0,T;L^2\right), \\
\widetilde{\nabla}^2\Pi &\in L^2\left(0,T;L^2\right).
\end{aligned}
\tag{148}
$$

The estimate of Ω, u^r/r, $\mathbf{B}$, and $\widetilde{\nabla}\Pi$ can be obtained similarly to the proof of the condition of (19). This completes the proof. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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