PAPER Special Section on Foundations of Computer Science

Path Maximum Query and Path Maximum Sum Query in a Tree*

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SUMMARY Let T be a node-weighted tree with n nodes, and let $\pi(u, v)$ denote the path between two nodes u and v in T. We address two problems: (i) Path Maximum Query: Preprocess T so that, for a query pair of nodes u and v, the maximum weight on $\pi(u, v)$ can be found quickly. (ii) Path Maximum Sum Query: Preprocess T so that, for a query pair of nodes u and v, the maximum weight sum subpath of $\pi(u, v)$ can be found quickly. For the problems we present solutions with $O(1)$ query time and $O(n \log n)$ preprocessing time.

key words: *path maximum query, path maximum sum query, tree*

1. Introduction

Let T be a rooted tree with n nodes. The lowest common ancestor of two nodes u and v in T, denoted by $LCA_T(u, v)$, is the node that is a common ancestor of u and v and is as far as possible from the root.

LCA (Lowest Common Ancestor) Preprocess T so that, for a query pair of nodes u and v, $LCA_T(u, v)$ can be found efficiently.

Solutions with $O(n)$ preprocessing time and $O(1)$ query time are presented in Harel and Tarjan [5], and Schieber and Vishkin [6].

Let $A[1..n]$ be an array with n real numbers.

RMQ (Range Maximum Query) Preprocess A so that, for a query pair of indices $i \le j$, $RM_A(i, j) = \max_{i \le k \le j} A[k]$ can be found efficiently.

Bender *et al.* [1] and Gabow *et al.* [4] show that LCA and RMQ are linearly equivalent. So, solutions with $O(n)$ preprocessing time and $O(1)$ query time are from [5], [6].

Define $S(i, j) = A[i] + \cdots + A[j]$ for $1 \le i \le j \le n$.

RMSQ (Range Maximum Sum Query) Preprocess A so that, for a query pair of indices $i \le j$, $RMS_A(i, j) = \max_{i \le k \le l \le j} S(k, l)$ can be found efficiently.

Chen and Chao [2] proves the linear equivalence between RMQ and RMSQ, and, as a consequence, gives a solution with $O(n)$ preprocessing time and $O(1)$ query time.

This paper extends RMQ and RMSQ from arrays to trees and studies PMQ (path maximum query) and PMSQ (path maximum sum query).

Let $T = (V, E)$ be a size n rooted tree with the node set V and the edge set E. Assume that $V = \{1, 2, \ldots, n\}$. Each node $v \in V$ is weighted with a real value $A[v]$. For a pair of nodes u and v, there exists a unique path in T that connects them. Let $\pi(u, v)$ denote the path.

PMQ (Path Maximum Query) Preprocess T so that, for a query pair of nodes u and v, $PM_T(u, v) = \max_{w \in \pi(u,v)} A[w]$ can be found efficiently.

Define $S(u, v) = \sum_{w \in \pi(u,v)} A[w]$.

PMSQ (Path Maximum Sum Query) Preprocess T so that, for a query pair of nodes u and v, $PMS_T(u, v) = \max_{w,x \in \pi(u,v)} S(w, x)$ can be found efficiently.

We are interested in solutions with $O(1)$ query time whose preprocessing time is as small as possible.

In Sect. 2 we describe our solution to the path maximum query, and in Sect. 3 we present our solution to the path maximum sum query. We give concluding remarks in Sect. 4.

Notation: (u, v) denotes an (undirected) edge, and $\langle u, v \rangle$ denotes a directed edge where v is the parent of u. $par(v)$ denotes the parent of v.

2. Path Maximum Query

We want to preprocess $T = (V, E)$ so that path maximum queries can be answered quickly. For that, we first consider a restricted version of PMQ. A path $\pi(u, v)$ is *lineal* if u is an ancestor of v or vice versa.

LPMQ (Lineal Path Maximum Query) Preprocess T so that, for a query pair of nodes u and v with v being an ancestor of u, $LPM_T(u, v) = \max_{w \in \pi(u,v)} A[w]$ can be found efficiently.

To solve this problem, we first introduce an artificial node $r^* = 0$ with $A[r^*] = \infty$. Let $T^* = (V^*, E^*)$ with $V^* = V \cup \{r^*\}$ and $E^* = E \cup \{\langle r, r^* \rangle\}$. r is the root of T, and r^* is the root of T^* and the parent of r. Define $B[v]$ to be the *bounding ancestor* of v, which is the node x such that $x \in \pi(r^*, par(v))$, $A[x] \ge A[v]$, and the level of x is as large as possible. The level of x is the number of edges in $\pi(r^*, x)$.

Figure 1 shows our algorithm for computing the array $B[1..n]$. Q is a queue that stores nodes, insert(x, Q) appends node x to the rear end of Q, and delete(Q) removes the node at the front end of Q and returns it. The algorithm traverses T^* in level-order. Let v be the currently visited node. At this point, we may assume that $B[w]$ has been computed for each $w \in \pi(r^*, v)$. Suppose it has Δ_v children $u_1, \ldots, u_{\Delta_v}$. We are to compute $B[u_1], \ldots, B[u_{\Delta_v}]$. Relabel so that $A[u_1] \le \ldots \le A[u_{\Delta_v}]$.

Manuscript received March 18, 2008.
Manuscript revised June 23, 2008.

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*This research was supported by the Chung-Ang University Research Scholarship in 2008.

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DOI: 10.1587/transinf.E92.D.166

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A[r*] ← ∞;
B[r] ← r*;
insert(r, Q);
while(Q ≠ ∅)
    v ←delete(Q);
    if(v is not a leaf)
        let u_1, ..., u_{Δ_v} be the children of v;
        sort A[u_1], ..., A[u_{Δ_v}] and relabel the indices so that
            A[u_1] ≤ ··· ≤ A[u_{Δ_v}];
        B[u_1] ← v;
        for(i = 1 ~ Δ_v)
            while(A[u_i] > A[B[u_i]])
                B[u_i] ← B[B[u_i]];
            insert(u_i, Q);
            if(i < Δ_v)
                B[u_{i+1}] ← B[u_i];
```

Fig. 1 Computing $B[1\,..\,n]$.

Consider the sequence $v, B[v], B[B[v]], \ldots, B[\cdots B[v]\cdots] = r^*$. Notice that $A[v] \le A[B[v]] \le \cdots \le A[r^*]$. We merge the sorted list $A[u_1], \ldots, A[u_{\Delta_v}]$ with another sorted list $A[v], A[B[v]], \ldots, A[r^*]$. From the merged list we can easily compute $B[u_1], \ldots, B[u_{\Delta_v}]$. The algorithm does not explicitly merge the lists, but this is done implicitly.

$B[1\,..\,n]$, computed by the algorithm in Fig. 1, may have duplicate integers. Let $b_1 < \cdots < b_k$ be the distinct integers appearing in B. Obviously, $b_1 = 0$. Let $B_i = \{j \mid B[j] = b_i\} = \{j_{i,1}, \ldots, j_{i,|B_i|}\}$ for $1 \le i \le k$. Rearrange the integers in each B_i so that $A[j_{i,1}] \le \cdots \le A[j_{i,|B_i|}]$. In the sorted list $j_{i,1}, \cdots, j_{i,|B_i|}$, let each element point to the element to its right by assigning $L[j_{i,1}] \leftarrow j_{i,2}$, $L[j_{i,2}] \leftarrow j_{i,3}$, …, and $L[j_{i,|B_i|-1}] \leftarrow j_{i,|B_i|}$. Finally, let the last element of the list point to b_i by assigning $L[j_{i,|B_i|}] \leftarrow b_i$.

Let T_L be the tree defined by L. Its node set is V^* and an edge $\langle j, j'\rangle$ exists if $L[j] = j'$. An edge $\langle j, j'\rangle$ is *vertical* if j and j' belong to the same B_i (i.e., if $j, j' \in B_i$ for some i), and *nonvertical*, otherwise. Note that T_L is a binary tree. If a node in T_L has two children, one of them is connected through a vertical edge and the other through a nonvertical edge.

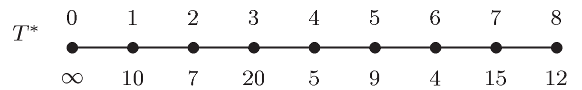
Figure 2 shows how our algorithm computes T_L when T is a tree consisting of a single path of length n; note that this actually corresponds to the case of RMQ. Fig. 3 depicts a geometric description of how $B[1\,..\,n]$ is computed for the case of a single-path tree.

Preprocess T_L for the LCA queries. Then, we claim that for a query lineal path $\pi(u, v)$, $LPM_T(u, v) = LCA_{T_L}(u, v)$. Figure 4 summarizes our algorithm for LPMQ preprocessing.

The following lemma shows that when T consists of a single path only, T_L can be used to answer PMQs correctly.

Lemma 1: Let T be a tree consisting of a single path only. Then, for a query pair of nodes u, v, $LPM_T(u, v) = LCA_{T_L}(u, v)$.

Proof: Since T is a path, we let $T = (1, 2, \ldots, n)$, where $r = 1$ and j is the child of $j-1$ for $2 \le j \le n$. Construct T_L on the array $A[1\,..\,n]$. In T_L, the root r^* has a single child, which is $j_{1,|B_1|}$. In Fig. 2, $j_{1,|B_1|} = 3$. First of all, it is easy to see that



	0	1	2	3	4	5	6	7	8
A	∞	10	7	20	5	9	4	15	12

	1	2	3	4	5	6	7	8
B	0	1	0	3	3	5	3	7

$b_1 = 0 \qquad B_1 = \{1, 3\}$
$b_2 = 1 \qquad B_2 = \{2\}$
$b_3 = 3 \qquad B_3 = \{4, 5, 7\}$
$b_4 = 5 \qquad B_4 = \{6\}$
$b_5 = 7 \qquad B_5 = \{8\}$

	1	2	3	4	5	6	7	8
L	3	1	0	5	7	5	3	7

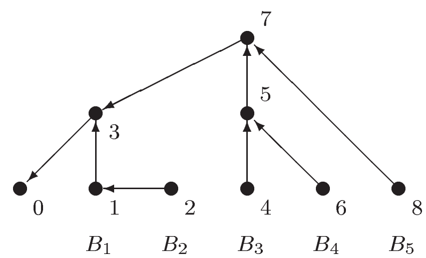


Fig. 2 Computing L and T_L when T is a single path.

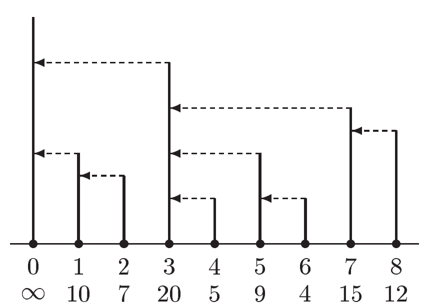


Fig. 3 Geometric description of computing $B[1\,..\,n]$ in Fig. 2. The heights of segments correspond to $A[1\,..\,n]$.

(1) Compute B.
(2) Compute $b_1, \ldots, b_k$ and $B_1, \ldots, B_k$.
(3) Sort each B_i so that $A[j_{i,1}] \le \cdots \le A[j_{i,|B_i|}]$.
(4) Compute L and T_L.
(5) Preprocess T_L for LCA queries.

Fig. 4 Preprocessing T with node values A for answering LPM queries.

$A[j_{1,|B_1|}] = \max A[1\,..\,n]$. See Fig. 3. Assume that $j_{1,|B_1|}$ has two children, p and q, and the edge $\langle p, j_{1,|B_1|}\rangle$ is vertical and the edge $\langle q, j_{1,|B_1|}\rangle$ is nonvertical. Then, $p = j_{1,|B_1|-1}$, and $q = j_{l,|B_l|}$ where l is the integer such that $b_l = j_{1,|B_1|}$. In Fig. 2, $p = 1$ and $q = 7$, $\langle 1, 3\rangle$ is vertical and $\langle 7, 3\rangle$ is nonvertical. Again, by the definition of B it is easy to see that

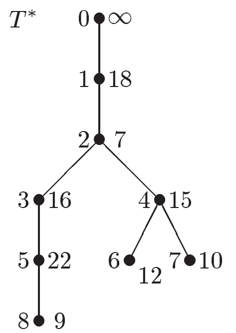


Fig. 5 Example tree.

	0	1	2	3	4	5	6	7	8
A	∞	18	7	16	15	22	12	10	9

	1	2	3	4	5	6	7	8
B	0	1	1	1	0	4	4	5

$$\begin{array}{ll} b_1 = 0 & B_1 = \{1, 5\} \\ b_2 = 1 & B_2 = \{2, 4, 3\} \\ b_3 = 4 & B_3 = \{7, 6\} \\ b_4 = 5 & B_4 = \{8\} \end{array}$$

	1	2	3	4	5	6	7	8
L	5	4	1	3	0	4	6	5

Fig. 6 Computing L for the tree in Fig. 5.

$A[p] = \max A[1 \mathinner{..} j_{1,|B_1|}-1]$ and $A[q] = \max A[j_{1,|B_1|}+1 \mathinner{..} n]$. In Fig. 3, $A[1] = \max A[1 \mathinner{..} 2]$ and $A[7] = \max A[4 \mathinner{..} 8]$.

This corresponds to the definition of a Cartesian tree [7]. The Cartesian tree on A is defined as follows: find $A[j] = \max A[1 \mathinner{..} n]$ and make $A[j]$ the root; find $A[p] = \max A[1 \mathinner{..} j-1]$ and $A[q] = \max A[j+1 \mathinner{..} n]$, and make $A[p]$ and $A[q]$ the children of $A[j]$; and recursively repeat this procedure until all elements of A are contained in the tree.

T_L is a Cartesian tree on A. Hence, by the definition of a Cartesian tree, for a query pair of nodes u, v, we have $LPM_T(u, v) = LCA_{T_L}(u, v)$. □

Figure 5 shows a tree T. Applying the algorithm in Fig. 4 to T computes B and L in Fig. 6, and T_L in Fig. 7 (a).

Consider a leaf node y of T. Let $T' = \pi(r, y)$ be the root-to-leaf path between r and y. Apply to T' our algorithm in Fig. 4 to get T'_L. Let $b'_1, \ldots, b'_{k'}$ and $B'_1, \ldots, B'_{k'}$ be the b_i's and B_i's.

Figure 7 (b) depicts T'_L for $y = 8$ and Fig. 7 (c) for $y = 7$.

Lemma 2: For any pair of nodes $u, v \in T'$, $LCA_{T'_L}(u, v) = LCA_{T_L}(u, v)$.

Proof: It is sufficient to show that, for any pair of nodes $w, x \in T'$, there is a directed edge $\langle w, x\rangle \in T'_L$ if and only if there is a directed path $\langle w, \ldots, x\rangle$ in T_L such that the nodes on the path except w and x are all in $T - T'$.

For example, in Fig. 7 the vertical edge $\langle 2, 3\rangle$ in (b) corresponds to the path $\langle 2, 4, 3\rangle$ in (a), and the nonvertical edge

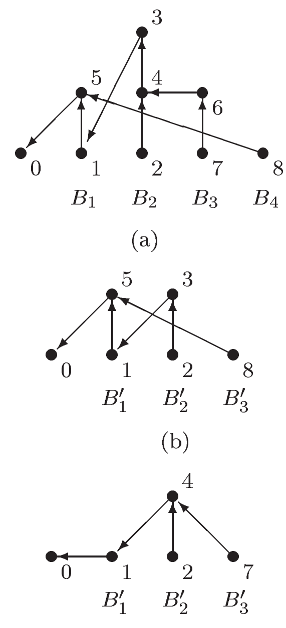


Fig. 7 (a) T_L for the tree in Fig. 5 and L in Fig. 6. (b) T'_L for $y = 8$. (c) T'_L for $y = 7$.

$\langle 4, 1\rangle$ in (c) corresponds to the path $\langle 4, 3, 1\rangle$ in (a).

$\Rightarrow$) i) $\langle w, x\rangle$ is a vertical edge:
Since $\langle w, x\rangle$ is a vertical edge in T'_L, there is an integer l' such that $w, x \in B'_{l'}$. In the sorted list $B'_{l'}$, w appears immediately before x and no other integer lies between them. By the definition of B, it is easy to show that $w, x \in B_l$ for l such that $b_l = b'_{l'}$. In B_L, w appears before x. By the definition of L there is a directed path from w to x in T_L, consisting of vertical edges only. As w and x are adjacent in $B'_{l'}$, no node in T' except w and x can appear on the path.

ii) $\langle w, x\rangle$ is a nonvertical edge:
By the definition of a nonvertical edge, $w \in B'_{l'}$ for l' such that $x = b_{l'}$, and w is the last integer in the sorted list $B'_{l'}$. Let l be the integer such that $x = b_l$. Then, $w \in B_l$. In T_L, there is a directed path $\langle w, \ldots, j_{l,|B_l|}, x\rangle$. The nodes on the path except w and x are not in T' as w is the last one in $B'_{l'}$.

$\Leftarrow$) i) $w, x \in B_l$ for some l:
In this case, $w, x \in B'_{l'}$ for l' such that $b'_{l'} = b_l$. Moreover, w and x appear adjacently because on the path $\langle w, \ldots, x\rangle$ all nodes except w and x are in $T - T'$. So, the vertical edge $\langle w, x\rangle$ is in T'_L.

ii) $w \in B_l$ and $x \in B_m$ for some $l \neq m$:
Since $w \in B_l$, we have $B[w] = b_l$. By the definition of B, b_l is an ancestor of w in T. So, $b_l \in T'$. Consider the directed path from w to x in T_L, $\pi = \langle w, \ldots, j_{l,|B_l|}, b_l, \ldots, x\rangle$. If $x \neq b_l$, then $b_l \in T'$ appears on π. This contradicts to the assumption that no node in T' except w and x appears on π. So, $x = b_l$. In T'_L, $w \in B'_{l'}$ for l' such that $b'_{l'} = x$, and w is the last element in $B'_{l'}$. Hence, $\langle w, x\rangle$ is a nonvertical edge in T'_L. □

From Lemmas 1 and 2 we have the following theorem.

Theorem 1: For any pair of nodes u, v with v being an ancestor of u, $LPM_T(u, v) = LCA_{T_L}(u, v)$.

For a query pair of nodes u, v, $PM_T(u, v)$ can be found as follows:

- Find the node w such that $w = LCA_T(u, v)$.
- Compute $a = LPM_T(u, w)$.
- Compute $b = LPM_T(v, w)$.
- Return $\max\{a, b\}$.

Since each step takes $O(1)$ time, the query time is $O(1)$.

Let us analyze the time complexity of our preprocessing algorithm in Fig. 4. In Step (1), the sorting requires $O(\Delta_v \log \Delta_v)$ time for each node v and the other requires $O(\Delta_v)$ time. Let $\Delta = \max_{v \in V} \Delta_v$. Since $\sum_{v \in V} \Delta_v \log \Delta_v \leq n \log \Delta$, Step (1) can be done in $O(n \log \Delta)$ time. Step (3) calls a sorting for each B_i, and thus needs $O(|B_i| \log |B_i|)$ time for each sorting, and $O(n \log \Delta')$ time in total, where $\Delta' = \max_{1 \leq i \leq k} |B_i|$. The other steps require $O(n)$ time. Thus, the preprocessing time is $O(n \log(\max\{\Delta, \Delta'\}))$, and since $\Delta \leq n$ and $\Delta' \leq n$, it is $O(n \log n)$.

Theorem 2: There is a solution for the path maximum query with $O(1)$ query time and $O(n \log n)$ preprocessing time.

LPMIQ (Lineal Path Minimum Query) Preprocess T so that, for a query pair of nodes u and v with v being an ancestor of u, $LPMI_T(u, v) = \min_{w \in \pi(u,v)} A[w]$ can be found efficiently.

Finding the lineal path minimum can also be solved in a similar way as finding the lineal path maximum. We introduce LPMIQ here to use in the next section.

3. Path Maximum Sum Query

In this section we preprocess T so that path maximum sum queries can be answered efficiently. Again, we are first interested in a restricted version.

LPMSQ (Lineal Path Maximum Sum Query) Preprocess T so that, for a query pair of nodes u and v with v being an ancestor of u, $LPMS_T(u, v) = \max_{w,x \in \pi(u,v)} S(w, x)$ can be found efficiently.

To solve this restricted version efficiently, we first let $C[v] = S(r, v)$ for each $v \in V$. $C[v]$ is the cumulative sum of the path from the root to v. Then, for any two nodes u, v with v being an ancestor of u, $S(u, v) = C[u] - C[par(v)]$.

Let u and v be two nodes with v being an ancestor of u. Let $x_0 = par(v)$. To find $LPMS_T(u, v)$, we first locate $x_1 \in \pi(u, v)$ such that $C[x_1] = \max_{w \in \pi(u,v)} C[w]$, and then locate $y_1 \in \pi(par(x_1), x_0)$ such that $C[y_1] = \min_{w \in \pi(par(x_1),x_0)} C[w]$. For $i = 2, \ldots$, locate $x_i \in \pi(u, x'_{i-1})$ such that $C[x_i] = \max_{w \in \pi(u,x'_{i-1})} C[w]$ where $x'_{i-1} \in \pi(u, v)$ and $par(x'_{i-1}) = x_{i-1}$, and then locate $y_i \in \pi(par(x_i), x_{i-1})$ such that $C[y_i] = \min_{w \in \pi(par(x_i),x_{i-1})} C[w]$. Refer to Fig. 8. Let $x_0, x_1, \ldots, x_l$ and $y_1, \ldots, y_l$ be the sequences of nodes thus obtained. Obviously $x_l = u$.

Lemma 3: $LPMS_T(u, v) = \max_{1 \leq i \leq l}\{C[x_i] - C[y_i]\}$.

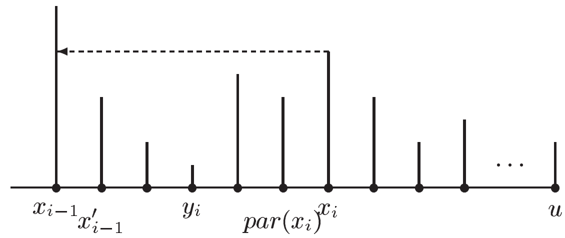


Fig. 8 x_i corresponds to the tallest one to the right of x_{i-1}, and y_i corresponds to the shortest one between x_{i-1} and $par(x_i)$.

Proof: Let x and y be the nodes such that $x, y \in \pi(u, v)$ and $S(x, y) = LPMS_T(u, v)$. Assume that y is an ancestor of x. Then, $S(x, y) = C[x] - C[par(y)]$. We shall show that $x = x_i$ and $par(y) = y_i$ for some i. Note that $\pi(par(x_l), x_0)$ is partitioned into l paths, $\pi(par(x_l), x_{l-1}), \ldots, \pi(par(x_1), x_0)$. Assume that $par(y) \in \pi(par(x_i), x_{i-1})$ for some i. Since $C[par(y)]$ should be as small as possible, we have $par(y) = y_i$ as $C[y_i]$ is the minimum of $C[par(x_i)], \ldots, C[x_{i-1}]$. Since $C[x]$ should be as large as possible, we have $x = x_i$ as $C[x_i]$ is the maximum of $C[u], \ldots, C[x'_{i-1}]$. □

In Sect. 2, $B[v]$ is defined with respect to A. Here, we redefine $B[v]$ with respect to C; $B[v]$ is the node x such that $x \in \pi(r^*, par(v))$, $C[x] \geq C[v]$, and the level of x is as large as possible.

In the sequence $x_1, \ldots, x_l$, $B[x_i] = x_{i-1}$ for $2 \leq i \leq l$ as $C[x_{i-1}] > C[x_i]$ and $C[x_i]$ is the maximum among those to the right of x_{i-1}. The definition of y_i for $2 \leq i \leq l$ can be rephrased as $C[y_i] = \min_{w \in \pi(par(x_i),B[x_i])} C[w]$.

Define $M[v]$ to be the *matching ancestor* of v, which is the node x such that $x \in \pi(par(v), B[v])$, $C[x] = \min_{w \in \pi(par(v),B[v])} C[w]$, and the level of x is as large as possible. Note that $M[x_i] = y_i$ for $2 \leq i \leq l$. Lemma 3 can be rewritten as $LPMS_T(u, v) = \max\{C[x_1] - C[y_1], \max_{2 \leq i \leq l}\{C[x_i] - C[M[x_i]]\}\}$.

For brevity, let $D[v] = C[v] - C[M[v]]$ for $v \in V - \{r\}$, and $D[r] = A[r]$. Then, $D[x_i] \geq D[w]$ for any node $w \in \pi(x_i, x'_{i-1})$. Hence, $\max_{w \in \pi(u,x'_1)} D[w] = \max_{2 \leq i \leq l} D[x_i]$.

Lemma 4: $LPMS_T(u, v) = \max\{C[x_1] - C[y_1], \max_{w \in \pi(u,x'_1)} D[w]\}$.

The algorithm in Fig. 9 computes the arrays $C[1..n]$, $B[1..n]$, $M[1..n]$ and $D[1..n]$. It works in a similar way as the one in Fig. 1 does. Let v be the current node. We assume that $B[w]$ and $M[w]$ have been computed for $w \in \pi(r^*, v)$. Suppose it has Δ_v children, $u_1, \ldots, u_{\Delta_v}$. Compute $C[u_i] = C[v] + A[u_i]$ for $1 \leq i \leq \Delta_v$, and sort them in ascending order. Consider the sequence $v, B[v], B[B[v]], \ldots, B[\cdots B[v] \cdots] = r^*$. Let $v_1 = v, v_2 = B[v], \ldots, v_m = r^*$. To compute $B[u_i]$, we assume that $B[u_{i-1}]$ has already been computed. Let $B[u_{i-1}] = v_p$. We scan the list $v_p, v_{p+1}, \ldots, v_m$ until we find v_q such that $C[v_q] \geq C[u_i]$.

To compute $M[u_i]$, we also may assume that $M[u_{i-1}]$ has already been computed. Let $B[u_{i-1}] = v_p$ and $B[u_i] = v_q$ for $p < q$. $M[u_i]$, by definition, is the node in $\pi(v, v_q)$ such that $C[M[u_i]] = \min_{w \in \pi(v,v_q)} C[w]$. This is equivalent to $C[M[u_i]] = \min_{1 \leq j \leq q-1} C[M[v_j]]$.

```
C[r*] ← ∞;
C[r] ← A[r];
B[r] ← r*;
D[r] ← A[r];
insert(r, Q);
while(Q ≠ ∅)
    v ←delete(Q);
    if(v is not a leaf)
        let u_1, ..., u_{Δ_v} be the children of v;
        for(i = 1 ∼ Δ_v)
            C[u_i] ← C[v] + A[u_i];
        sort C[u_1], ..., C[u_{Δ_v}] and relabel the indices so that
            C[u_1] ≤ ··· ≤ C[u_{Δ_v}];
        B[u_1] ← M[u_1] ← v;
        for(i = 1 ∼ Δ_v)
            while(C[u_i] > C[B[u_i]])
                if(C[M[u_i]] > C[M[B[u_i]]])
                    M[u_i] ← M[B[u_i]];
                B[u_i] ← B[B[u_i]];
            D[u_i] ← C[u_i] − C[M[u_i]];
            insert(u_i, Q);
            if(i < Δ_v)
                B[u_{i+1}] ← B[u_i];
                M[u_{i+1}] ← M[u_i];
```

Fig. 9 Computing C, B, M and D.

(1) Compute C and D using the algorithm in Fig. 9.
(2) Preprocess T for LCA queries.
(3) Preprocess T with node values D for answering $LPM_{T,D}(u,v)$ using the algorithm in Fig. 4.
(4) Preprocess T with node values C for answering $LPM_{T,C}(u,v)$ using the algorithm in Fig. 4.
(5) Preprocess T with node values C for answering $LPMI_{T,C}(u,v)$ using the algorithm in Fig. 4.

Fig. 10 Preprocessing for PMSQs and LPMSQs.

Since $C[M[u_{i-1}]] = \min_{1\le j\le p-1} C[M[v_j]]$, we have $C[M[u_i]] = \min\{C[M[u_{i-1}]], \min_{p\le j\le q-1} C[M[v_j]]\}$. Starting with $M[u_i] = M[u_{i-1}]$, the algorithm scans $M[v_p], \ldots, M[v_{q-1}]$ to find the minimum of their C values.

Figure 10 shows our algorithm for preprocessing T for PMS queries and LPMS queries. $LPM_{T,D}(u,v)$, $LPM_{T,C}(u,v)$, and $LPMI_{T,C}(u,v)$ denote $\max_{w\in\pi(u,v)} D[w]$, $\max_{w\in\pi(u,v)} C[w]$, and $\min_{w\in\pi(u,v)} A[w]$, respectively.

Given a query pair of nodes u and v with v being an ancestor of u, we can find $LPMS_T(u,v)$ using the following query-answering algorithm, which is based on Lemma 4.

- Find $x_1 \in \pi(u,v)$ such that $C[x_1] = LPM_{T,C}(u,v)$.
- Find $y_1 \in \pi(par(x_1), par(v))$ such that $C[y_1] = LPMI_{T,C}(par(x), par(v))$.
- Find $z \in \pi(u, x_1')$ such that $D[z] = LPM_{T,D}(u, x_1')$.
- Return $\max\{C[x_1] - C[y_1], D[z]\}$.

Since each step takes $O(1)$ time, the query time for $LPMS_T(u,v)$ is $O(1)$.

We are now ready to answer path maximum sum queries. Given a query pair of nodes u and v, $PMS_T(u,v)$ can be computed as follows:

- Find w such that $w = LCA_T(u,v)$.
- If $u = w$, return $LPMS_T(v,w)$.
- If $v = w$, return $LPMS_T(u,w)$.
- Compute $a = LPMS_T(u,w)$.
- Compute $b = LPMS_T(v,w)$.
- Find $x \in \pi(u,w)$ such that $C[x] = LPM_{T,C}(u,w)$.
- Find $y \in \pi(v,w)$ such that $C[y] = LPM_{T,C}(v,w)$.
- Compute $c = C[x] - C[par(w)] + C[y] - C[w]$.
- Return $\max\{a, b, c\}$.

Given u and v, we first locate $w = LCA_T(u,v)$. If $u = w$ or $v = w$, then $\pi(u,v)$ is a lineal path and its maximum sum can be found by calling either $LPMS_T(v,w)$ or $LPMS_T(u,w)$. Otherwise, we compute a, b and c, and return their maximum. a (resp., b) is the sum of a maximum sum path, both of whose end nodes are on $\pi(u,w)$ (resp., $\pi(v,w)$). c is the sum of a maximum sum path, one of whose end nodes, x, is on $\pi(u,w)$ and the other, y, is on $\pi(v,w)$.

It is easy to see that the query time is $O(1)$ as each step takes only $O(1)$ time. The time complexity of the preprocessing in Fig. 10 is as follows: Step (1) takes $O(n\log\Delta)$ time. Step (2) takes linear time. Steps (3), (4) and (5) use the same algorithm in Fig. 4 to the same tree T with different node values (C or D) or different objective functions (maximum or minimum). Step (3) takes $O(n\log(\max\{\Delta, \Delta_3'\}))$, where $\Delta_3' = \Delta'$ and Δ' is defined in the previous section. Similarly, Steps (4) and (5) take $O(n\log(\max\{\Delta, \Delta_4'\}))$ and $O(n\log(\max\{\Delta, \Delta_5'\}))$, respectively. The preprocessing requires $O(n\log(\max\{\Delta, \Delta_3', \Delta_4', \Delta_5'\})) = O(n\log n)$ time as $\Delta, \Delta_3', \Delta_4', \Delta_5' \le n$.

Theorem 3: There is a solution for the path maximum sum query with $O(1)$ query time and $O(n\log n)$ preprocessing time.

4. Concluding Remarks

We have presented solutions with $O(1)$ query time for both the path maximum query and the path maximum sum query. One immediate future work is to reduce the preprocessing time.

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