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A Free-Boundary Problem for Euler Flows with Constant Vorticity

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Abstract—A steady, two-dimensional Euler flow problem to determine the boundary of the domain given the constant tangential wall velocity and constant vorticity in the whole domain is studied. A complete answer from the argument principle is presented in complex variable formulation and another variational approach to this problem is briefly discussed.

Keywords—Inviscid limit, Vorticity, Argument principle, Variational formulation.

1. INTRODUCTION

Prandtl-Batchelor (PB) theory on steady flows characterizes the inviscid limit (i.e., viscosity $\nu \to 0$) of two-dimensional viscous flows with nested closed streamlines (see [1,2]). It asserts that under certain general circumstances, the vorticity ω (conventionally the z-component of the vorticity vector $\vec{\omega} = \nabla \times \vec{u}$) in the whole region of nested closed streamlines approaches a constant as the viscosity tends to zero [3]. Although in reality, steady flows of small viscosity are usually unstable, this result is important as one of few asymptotic properties for steady inviscid limits of Navier-Stokes flows. We mention that a version of PB theory applies to linear advection-diffusion problems involving a scalar or vector field, see, e.g., [4]. Another related area of application is to the contour dynamics of boundaries of constant vorticity [5].

We here propose and solve a kind of inverse problem in Euler flow raised by PB theory: can one determine the boundary of the domain (i.e., determine the whole flow in the domain) given only the tangential Euler wall velocity $q_e(s)$ (s is the arclength) and the assumption of the constant vorticity?

Mathematically this is a free boundary problem which is not easy to solve in general. As a first step, we study in this paper the simplest realization, i.e., the case of constant $q_e(s) = q$, which corresponds to a constant pressure distribution. We shall demonstrate that in this case, D is a disk (and the motion is a rigid body rotation with a vanishing pressure gradient throughout the region). We present two different approaches to this problem. The first one utilizes complex variables with the argument principle (Section 2) and gives a rigorous answer to our problem. We also briefly sketch another approach to this problem which adopts a corresponding variational

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formulation (Section 3). In the proof below (Section 2), we note that for a constant ω_0, the boundary ∂D represented by $\psi(x,y) = 0$ is analytic in x and y. Therefore, any additional regularity condition on ∂D is unnecessary.

2. A FORMULATION IN COMPLEX VARIABLE

The present problem is the following: determine the steady, two-dimensional Euler flow(s) with given constant vorticity (ω_0) and constant wall velocity ($q > 0$). Since the corresponding rigid body rotation $\psi(x,y) = -(\omega_0/4)(x^2+y^2)$ on a circular disk of radius $R = 2q/\omega_0$ is a solution of the problem, it is natural to ask if this is the unique solution. We answer in the affirmative.

Let us first construct the corresponding complex variable formulation of the problem.

$$\nabla^2\psi = -\omega_0, \qquad \text{on } D, \tag{1}$$

$$\psi = 0, \qquad \text{on } \partial D, \tag{2}$$

$$\frac{\partial\psi}{\partial n} = q, \qquad \text{on } \partial D. \tag{3}$$

We introduce the complex variable $z = x + iy$, and obtain the solution of (1) in the form

$$\psi(z) = -\frac{\omega_0}{4} z\bar{z} + f(z) + \bar{f}(z), \tag{4}$$

where $f(z)$ is an analytic function to be determined in D. In fact, $f(z)$ can be analytically continued on some domain D_{cont} which contains D and ∂D because $f(z)$ is bounded on ∂D. We represent the boundary of the domain D in the form $z(s) = x(s) + iy(s)$ by a parameter s (the arc length). Then, on the boundary ∂D, from (3)

$$\frac{\partial\psi}{\partial z} = \frac{1}{2}\left(\frac{\partial\psi}{\partial x} - i\frac{\partial\psi}{\partial y}\right) = -\frac{i}{2} q\bar{\dot{z}}, \tag{5}$$

where a dot denotes the differentiation on s. Combining this with (4) yields the differential equation

$$-\frac{\omega_0}{4}\bar{z} + f'(z) = -\frac{i}{2} q\bar{\dot{z}} \tag{6}$$

for $f(z)$ on ∂D. At this point, we notice that the boundary $z = z(s)$ is analytic in x and y. From the constant speed condition (3) on the boundary, we have $\dot{z}(s) = e^{i\theta(s)}$ and $\ddot{z}(s) = i\dot{\theta}(s)e^{i\theta(s)} = i\kappa(s)\dot{z}$, where $\kappa(s) = \dot{\theta}(s)$ is the curvature of the boundary. If we differentiate (6) with respect to s and multiply both sides of the result by $\bar{\dot{z}}$, we find

$$q^2 f''(z) = \left(\frac{\omega_0}{4} - \frac{q\kappa(s)}{2}\right)\bar{\dot{z}}^2. \tag{7}$$

Our key point of the remaining proof is to show that $f''(z)$ vanishes identically on ∂D. We consider first the case where $f''(z)$ never assumes zero on ∂D. Computing the change of argument along ∂D in the anticlockwise direction, we obtain

$$\Delta_{\partial D} \arg f'' \geq 0, \tag{8}$$

since $f''(z)$ is analytic on D. On the other hand, from (7), we arrive at

$$\Delta_{\partial D} \arg f'' = \Delta_{\partial D} \arg \bar{\dot{z}}^2 = -4\pi, \tag{9}$$

which is a contradiction. Therefore, $f''(z)$ has some zeros on ∂D. If the number of these zeros is finite, at points $P_1, \dots, P_n$ of ∂D, we construct small circular arcs of radius $\epsilon > 0$, $C_1, \dots, C_n$

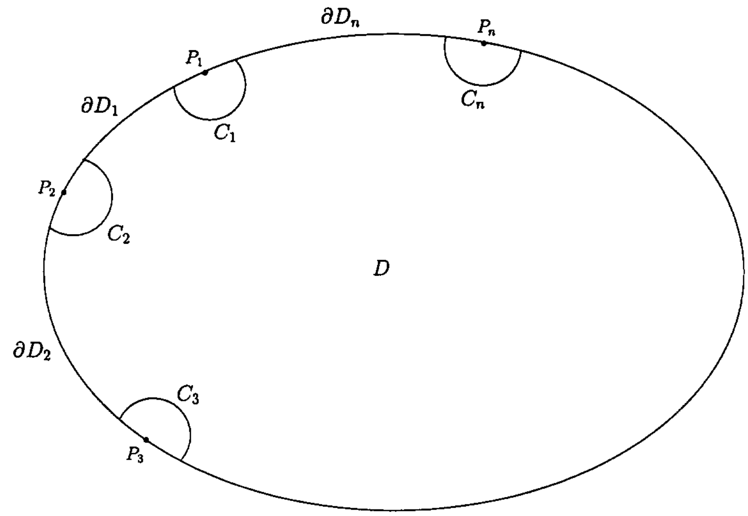

Figure 1. Cutting of ∂D.

at each of these points inside D. Also let $\partial D_1, \dots, \partial D_n$ be the remaining portions of ∂D where $f''(z)$ is not zero (Figure 1). We then compute the total variation of the argument along ∂D in two parts:

$$\Delta_{\partial D} \arg f'' = \underbrace{\Delta_{\partial D_1} \arg f'' + \cdots + \Delta_{\partial D_n} \arg f''}_{\text{I}} \tag{10}$$

$$+ \underbrace{\Delta_{C_1} \arg f'' + \cdots + \Delta_{C_n} \arg f''}_{\text{II}}. \tag{11}$$

To compute the first part I in the above, we use the expression (7) of $f(z)$ which never vanishes on these contours $\partial D_1, \dots, \partial D_n$. As $\epsilon \to 0$, the argument leading to (9) yields -4π argument changes. On the other hand, since $f''(z)$ is analytic at each point P_k, any clockwise circulation along $C_1, \dots, C_n$ produces a negative increment in argument. The second part II in (11), which is the sum of these changes is also negative. Consequently, we obtain

$$\Delta_{\partial D} \arg f'' < 0$$

and thus a contradiction.

Therefore, $f''(z) = 0$ infinitely many times on the boundary and we conclude $f''(z) \equiv 0$ since $f(z)$ is analytic on D_{cont}. Then, again from (7), we obtain

$$\kappa(s) \equiv \frac{\omega_0}{2q} \tag{13}$$

on the whole boundary and conclude that the boundary is a circle. This proves the uniqueness of the solution of solid body rotation of a disc.

3. SOME REMARKS ON VARIATIONAL FORMULATION OF THE PROBLEM

We next set up the corresponding variational formulation of the problem (1)–(3). We shall perturb the domain D fixing the area A. The variation of the Green function $G(x, y)$ of Laplace's equation under the domain perturbation is given by Hadamard's variational formula, see [6]. Let D^* be the region whose boundary ∂D^* is obtained by shifting ∂D an infinitesimal distance $\delta\nu = \epsilon\rho(s)$ along its outer normal ν, where ϵ is a small number and ρ is a real analytic function of the arclength s on ∂D. Then the 1^{st}-order term δG in the expansion of the corresponding Green's function G^* of D^* in powers of ϵ is

$$\delta G(\zeta, w) = \int_{\partial D} \frac{\partial G}{\partial \nu} \frac{\partial G^*}{\partial \nu} \, \delta\nu \, ds + O\left(\epsilon^2\right). \tag{14}$$

This is Hadamard's variational formula. Next we introduce the torsional rigidity (P) of the domain D defined by

$$P = \frac{4}{\omega_0} \iint_D \psi \, dx \, dy, \tag{15}$$

where $\psi(x, y) = 0$ represents ∂D (see [7] where ψ is the *stress function* in this case). In the present problem, we rewrite

$$P = -\frac{4}{{\omega_0}^2} \iint_D \psi \nabla^2 \psi \, dx \, dy = \frac{4}{{\omega_0}^2} \iint_D |\nabla \psi|^2 \, dx \, dy, \tag{16}$$

which involves the Dirichlet integral. From the definition of P in (15), we calculate δP, the first variation of P,

$$\delta P = 4 \iint_D \iint_D \delta G(\zeta, w) \, d\zeta \, dw + O\left(\epsilon^2\right), \tag{17}$$

since the solution of (1),(2) is $\psi(x, y) = -\omega_0 \iint_D G(x - x', y - y') \, dx' \, dy'$, where $G(x, y)$ is the Green function of D. Here ζ, w are complex coordinates of the domain D. By Hadamard's formula (14), we finally obtain

$$\delta P = \frac{4}{{\omega_0}^2} \int_{\partial D} |\nabla \psi|^2 \, \delta\nu \, ds + O\left(\epsilon^2\right). \tag{18}$$

On the other hand, the variation of area A of D is simply

$$\delta A = \int_{\partial D} \delta\nu \, ds. \tag{19}$$

Hence, an extremum of P under the perturbation of the domain fixing the area A occurs if

$$0 = \delta(P - \lambda A) = \int_{\partial D} \left(|\nabla \psi|^2 - \lambda \right) \delta\nu \, ds. \tag{20}$$

Since the variation $\delta\nu = \epsilon\rho(s)$ is arbitrary, the above is possible only if

$$|\nabla \psi|^2 = \lambda, \qquad \text{on } \partial D, \tag{21}$$

which recovers the constant speed condition (3) in the original problem with $\lambda = q$.

Among all functions with the given boundary condition (2), we seek a ψ that minimizes the Dirichlet integral (16). Then Dirichlet principle tells us that this ψ provides the desired solution of the original problem (1)–(3).

REFERENCES

1. L. Prandtl, Uber flussigkeitsbewegung bei sehr kleiner reibung, Presented at the *International Mathematical Congress, Heidelberg*, pp. 484–491, (1904); *Gesammelte Abhandlungen II*, 575–584 (1961).
2. G.K. Batchelor, On steady laminar flow with closed streamlines at large Reynolds number, *J. Fluid Mech.* **1**, 177–190 (1956).
3. S.-C. Kim, On Prandtl-Batchelor theory of steady flow at large Reynolds number, Ph.D. Thesis, New York University, (1996).
4. S. Childress and A.M. Soward, Scalar transport and alpha-effect for a family of cat's-eye flows, *J. Fluid Mech.* **205**, 99–133 (1989).
5. N.J. Zabusky, Visualizing mathematics: Evolution of vortical flows, *Physica* **18D**, 15–25 (1986).
6. P.R. Garabedian, *Partial Differential Equations*, Second edition, Chelsea, New York, (1986).
7. G. Pólya and Szegö, *Isoperimetric Inequalities of Mathematical Physics*, Princeton University Press, Princeton, NJ, (1951).