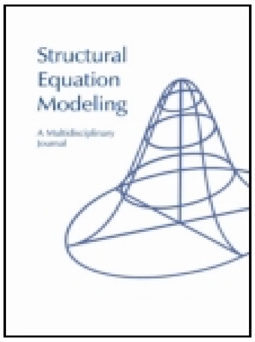

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Parameter Influence in Structural Equation Modeling

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In applications of structural equation modeling (SEM), investigators obtain and interpret parameter estimates that are computed so as to produce optimal model fit. The obtained parameter estimates are optimal in the sense that model fit would deteriorate to some degree if any of those estimates were changed. If a small change of a parameter estimate has large influence on model fit, such a parameter can be called highly influential, whereas if a substantial perturbation of a parameter estimate has negligible influence on model fit, that parameter can be called uninfluential. This is the idea of parameter influence. This article covers 2 approaches to quantifying parameter influence. One existing approach determines the direction vector of parameter perturbation causing maximum deterioration in model fit. In this article, we propose a new approach for quantifying the influence of individual parameters on model fit. In this new approach, the influence of individual parameters is quantified as the degree of perturbation required to produce a prespecified value of change in model fit. Using empirical examples, we illustrate how these 2 methods can be effectively employed, complementing each other and as a complement to conventional approaches to interpretation of parameter estimates obtained in empirical data analyses.

Keywords: model fit, parameter influence, parameter interpretation, sensitivity analysis, structural equation modeling

The process of changing various aspects of a model or data to assess subsequent impact on results or inferences is called sensitivity analysis. Some examples of various perturbation schemes include case deletion, alteration of the model, and perturbation of parameter estimates. In certain modeling contexts, sensitivity analysis can provide a useful way of evaluating a statistical model and its characteristics by examining the degree of change in the results of statistical analysis induced by minor perturbations of various aspects of the data or the model. For example, when a minor perturbation of a certain aspect of the model results in major changes in outputs, there is surely cause for concern about the model (Cook, 1986).

In the context of structural equation modeling (SEM), researchers have studied various perturbation schemes including case weights perturbation, additive or multiplicative perturbation of components of the manifest variables, and additive or multiplicative perturbation of latent variables. Effects of such perturbations on outcomes of subsequent statistical analysis have been assessed in specific modeling contexts such as restricted factor analysis models (Kwan & Fung, 1998), full information item factor models (Lee & Xu, 2003a), and linear and nonlinear structural equation models with continuous or polytomous manifest variables (Cadigan, 1995; Lee & Lu, 2003; Lee & Tang, 2004; Lee & Wang, 1996; Lee & Xu, 2003b; Poon, Wang, & Lee, 1999).

However, the issue of the relation of perturbation of parameter estimates to changes in model fit, although Lee and Wang (1996) first proposed a method to investigate this issue with a few follow-up papers (Poon et al., 1999; Lee & Tang, 2004; Lee & Xu, 2003b), has drawn relatively little attention among applied researchers using SEM. Rather, recent focus has tended to be on the study of statistical outcomes resulting from the perturbation of data to identify influential cases (Pek & MacCallum, 2011; Yuan & Hayashi, 2010; Yuan & Zhang, 2012; Yuan & Zhong, 2008).

In applications of SEM, investigators routinely obtain and interpret parameter estimates that are computed so as to produce optimal model fit. The obtained parameter

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estimates are optimal only in the sense that substituting a nonoptimal solution necessarily deteriorates the fit of the model. If a small perturbation of the optimal value of a parameter estimate has a large influence on model fit, such a parameter can be called highly influential and estimates of such parameters could be interpreted more rigorously. On the other hand, if a substantial perturbation of a parameter estimate has negligible influence on model fit, that parameter can be called uninfluential. Rigorous interpretations of such parameter estimates might be withheld because values very different from the optimal estimates would yield nearly the same level of model fit. Therefore, systematically designed sensitivity analysis seems warranted to answer such questions as "To what extent would model fit deteriorate if parameter estimates were changed from the optimal solution?" Methods for answering such questions will create an opportunity for identifying parameters having higher (lower) impact on model fit that might require more (less) rigorous interpretation.

In this article we focus on the sensitivity of model fit with respect to perturbation of parameter estimates in the context of SEM. We propose and illustrate methods for identifying single parameters or combinations of parameters that might have high or low influence on model fit. To that end, two approaches to perturbation are considered. The first approach involves perturbing all parameter estimates simultaneously and the second involves perturbing one parameter estimate at a time.

The remainder of this article is organized as follows: First, after a brief review of estimation and inferences for SEM, the definition of *parameter influence* is operationalized for subsequent development of a method for determining influential parameters. Next, two methods for examining parameter influence are introduced with illustrative examples. The first is a vector method and the second is a single parameter method. The illustrative examples are designed to demonstrate how the two methods can be used in identifying (un)influential parameters and how the results should be interpreted. Pros and cons of each method, as well as practical implications, are also discussed.

ESTIMATION AND INFERENCES FOR STRUCTURAL EQUATION MODELING

We consider an $N \times p$ data matrix $\boldsymbol{Y}$ where N and p represent the sample size and the number of manifest variables, respectively. Then the ith row of $\boldsymbol{Y}$, denoted by $\boldsymbol{y}_i'$, is assumed to be a vector of observations drawn from a p-dimensional multivariate normal distribution with mean vector $\boldsymbol{\mu}_0$ and covariance matrix $\boldsymbol{\Sigma}_0$.

In SEM, it is assumed that there exists a $q \times 1$ vector $\boldsymbol{\vartheta}_0$ that resides in the parameter space Θ (i.e., $\boldsymbol{\vartheta}_0 \in \boldsymbol{\Theta} \subset \mathcal{R}^d$) such that $\boldsymbol{\mu}_0 = \boldsymbol{\mu}(\boldsymbol{\vartheta}_0)$ and $\boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)$, where $\boldsymbol{\mu}(\cdot)$ and $\boldsymbol{\Sigma}(\cdot)$, respectively, represent a vector-valued and a matrix-valued function specifying the functional relationship between the model-implied mean and covariance structure and the parameter vector $\boldsymbol{\vartheta}\ (\in \boldsymbol{\Theta})$. In other words, it is assumed that there exists a unique vector of parameters $\boldsymbol{\vartheta}_0\ (\in \boldsymbol{\Theta})$ such that the model-implied mean vector $\boldsymbol{\mu}(\boldsymbol{\vartheta}_0)$ and the model-implied covariance matrix $\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)$ can reproduce the population mean vector $\boldsymbol{\mu}_0$ and the population covariance matrix $\boldsymbol{\Sigma}_0$, respectively.

For the estimation of $\boldsymbol{\vartheta}_0$, a specified model is fit to the sample mean vector $\bar{\boldsymbol{y}}$ and sample covariance matrix $\boldsymbol{S}$, producing a vector of parameter estimates $\hat{\boldsymbol{\vartheta}}$ that makes the model-implied mean and covariance structure as similar to $\bar{\boldsymbol{y}}$ and $\boldsymbol{S}$ as possible. A number of discrepancy functions have been proposed to measure this (dis)similarity, including generalized least squares (GLS) or asymptotically distribution-free (ADF), and normal theory maximum likelihood (ML) discrepancy functions.

Under multivariate normality, we can obtain the ML estimates for $\boldsymbol{\vartheta}_0$ by minimizing the normal theory ML discrepancy function

$$F_{ML}\left[\bar{\boldsymbol{y}}, \boldsymbol{S}; \boldsymbol{\mu}(\boldsymbol{\vartheta}), \boldsymbol{\Sigma}(\boldsymbol{\vartheta})\right] = \ln|\boldsymbol{\Sigma}(\boldsymbol{\vartheta})| - \ln|\boldsymbol{S}| + \mathrm{tr}\left[\boldsymbol{S}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\right] - p + \left[\bar{\boldsymbol{y}} - \boldsymbol{\mu}(\boldsymbol{\vartheta})\right]' \boldsymbol{\Sigma}(\boldsymbol{\vartheta})^{-1}\left[\bar{\boldsymbol{y}} - \boldsymbol{\mu}(\boldsymbol{\vartheta})\right] \tag{1}$$

or equivalently by maximizing the log-likelihood function

$$\ell(\boldsymbol{\vartheta}\,|\bar{\boldsymbol{y}}, \boldsymbol{S}) \propto -\frac{N}{2}\left\{\ln|\boldsymbol{\Sigma}(\boldsymbol{\vartheta})| + \mathrm{tr}\left[\boldsymbol{S}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\right] + \left[\bar{\boldsymbol{y}} - \boldsymbol{\mu}(\boldsymbol{\vartheta})\right]' \boldsymbol{\Sigma}\left(\boldsymbol{\vartheta}^{-1}\right)\left[\bar{\boldsymbol{y}} - \boldsymbol{\mu}(\boldsymbol{\vartheta})\right]\right\} \tag{2}$$

Maximizing Equation 2 or minimizing Equation 1 with respect to $\boldsymbol{\vartheta}$ results in the ML estimates $\hat{\boldsymbol{\vartheta}} \in \boldsymbol{\Theta}$. The ML estimator of $\boldsymbol{\vartheta}$ can be formally defined as

$$\hat{\boldsymbol{\vartheta}} = \arg\max_{\boldsymbol{\vartheta}\in\boldsymbol{\Theta}} \ell(\boldsymbol{\vartheta}\,|\bar{\boldsymbol{y}}\,, \boldsymbol{S}) = \arg\min_{\boldsymbol{\vartheta}\in\boldsymbol{\Theta}} F_{ML}\left[\bar{\boldsymbol{y}}, \boldsymbol{S}; \boldsymbol{\mu}(\boldsymbol{\vartheta}), \boldsymbol{\Sigma}(\boldsymbol{\vartheta})\right] \tag{3}$$

Under multivariate normality of $\boldsymbol{y}_i$ and a correct model specification, the ML estimator is asymptotically unbiased, consistent, efficient, and normally distributed with its center equal to $\boldsymbol{\vartheta}_0$ and its dispersion equal to $\mathcal{I}(\boldsymbol{\vartheta}_0)^{-1}$, where $\mathcal{I}(\boldsymbol{\vartheta})$ denotes the observed information matrix, the negative of the second order partial derivative matrix of the log-likelihood function

$$\mathcal{I}(\boldsymbol{\vartheta}) = -\frac{\partial^2 \ell(\boldsymbol{\vartheta}\,|\bar{\boldsymbol{y}}\,, \boldsymbol{S})}{\partial\boldsymbol{\vartheta}\,\partial\boldsymbol{\vartheta}'} = \frac{N}{2}\left\{\frac{\partial^2 F_{ML}\left[\bar{\boldsymbol{y}}, \boldsymbol{S}; \boldsymbol{\mu}(\boldsymbol{\vartheta}), \boldsymbol{\Sigma}(\boldsymbol{\vartheta})\right]}{\partial\boldsymbol{\vartheta}\,\partial\boldsymbol{\vartheta}'}\right\} \tag{4}$$

Further, $T = (N-1)\,F_{ML}\left[\bar{\boldsymbol{y}}, \boldsymbol{S}; \boldsymbol{\mu}\left(\hat{\boldsymbol{\vartheta}}\right), \boldsymbol{\Sigma}\left(\hat{\boldsymbol{\vartheta}}\right)\right]$ forms the likelihood ratio test statistic for testing the hypothesized moment structure against the general alternative. Under multivariate normality and correct model specification, T is asymptotically distributed as central chi-square with degrees of freedom equal to $p + p(p+1)/2 - q$ (see Browne & Arminger, 1995).

MEASURING PARAMETER INFLUENCE

It should be borne in mind that the ML estimates $\hat{\boldsymbol{\vartheta}}$ defined in Equation 3 are optimal in the sense that substituting nonoptimal values $\tilde{\boldsymbol{\vartheta}} \left(\neq \hat{\boldsymbol{\vartheta}}\right)$ necessarily deteriorates the ML discrepancy function value; that is, $F_{ML}\left[\bar{\mathbf{y}}, \mathbf{S}; \boldsymbol{\mu}\left(\tilde{\boldsymbol{\vartheta}}\right), \boldsymbol{\Sigma}\left(\tilde{\boldsymbol{\vartheta}}\right)\right] > F_{ML}\left[\bar{\mathbf{y}}, \mathbf{S}; \boldsymbol{\mu}\left(\hat{\boldsymbol{\vartheta}}\right), \boldsymbol{\Sigma}\left(\hat{\boldsymbol{\vartheta}}\right)\right]$. If $\tilde{\boldsymbol{\vartheta}}$ represents a small perturbation of (a subset of) elements in $\hat{\boldsymbol{\vartheta}}$ and is associated with a substantial deterioration in the discrepancy function value the perturbed parameters can be called highly influential and estimates of those parameters can be interpreted more rigorously. On the other hand, if $\tilde{\boldsymbol{\vartheta}}$ produces only a negligibly small change in the discrepancy function value despite (a subset of) its elements being substantially different from $\hat{\boldsymbol{\vartheta}}$, it would be safe to withhold the literal interpretation of such parameter estimates. This is the idea of parameter influence.

In this article we define parameter influence as the degree of change in model fit as a function of perturbation imposed on optimal parameter estimates. A primary objective of this research is to propose methods for examining the parameter influence. To define and measure parameter influence, we adapt the likelihood displacement (LD) criterion, originally proposed by Cook (1986) in a general context of sensitivity analysis, to the study of parameter influence in the context of SEM. In Cook (1986), LD quantifies *influence* as the discrepancy between the log-likelihood under the original model and the log-likelihood under the model in which a minor perturbation is imposed. Let $\ell(\boldsymbol{\vartheta})$ denote the log-likelihood of the original model indexed by a $q \times 1$ parameter vector $\boldsymbol{\vartheta}$. The formal definition of LD can then be given as

$$LD(\boldsymbol{\omega}) = 2\left[\ell\left(\hat{\boldsymbol{\vartheta}}\right) - \ell\left(\tilde{\boldsymbol{\vartheta}}_{\omega}\right)\right] \tag{5}$$

where $\boldsymbol{\omega}$ represents any well-defined perturbation scheme. Vectors $\hat{\boldsymbol{\vartheta}}$ and $\tilde{\boldsymbol{\vartheta}}_{\omega}$ represent the $q \times 1$ vectors of the ML estimates under the original model and the perturbed parameter estimates for a given ω, respectively.

In this article, we focus on perturbations applied to the ML estimates $\hat{\boldsymbol{\vartheta}}$ in the context of SEM as defined in Equation 3. And thus, $\boldsymbol{\omega}$ represents a $q \times 1$ direction vector of unit length along which $\hat{\boldsymbol{\vartheta}}$ is to be perturbed and the consequent impact on model fit is to be examined. Vector $\boldsymbol{\omega}$ is referred to as a perturbation vector. The ML discrepancy function of Equation 1 is employed to define the influence or change in model fit caused by such perturbations. Therefore, *parameter influence* is operationalized as the difference between the ML discrepancy function under the original model and the ML discrepancy function under the model in which a minor perturbation is imposed on the ML estimates. Formally, the parameter influence in SEM can be defined as

$$\begin{aligned} F_{ML}(\boldsymbol{\omega}) = {} & F_{ML}\left[\bar{\mathbf{y}}, \mathbf{S}; \boldsymbol{\mu}\left(\tilde{\boldsymbol{\vartheta}}_{\omega}\right), \boldsymbol{\Sigma}\left(\tilde{\boldsymbol{\vartheta}}_{\omega}\right)\right] \\ & - F_{ML}\left[\bar{\mathbf{y}}, \mathbf{S}; \boldsymbol{\mu}\left(\hat{\boldsymbol{\vartheta}}\right), \boldsymbol{\Sigma}\left(\hat{\boldsymbol{\vartheta}}\right)\right] \end{aligned} \tag{6}$$

where $\tilde{\boldsymbol{\vartheta}}_{\omega}$ represents the vector of parameter values obtained by imposing a minor perturbation on $\hat{\boldsymbol{\vartheta}}$ along the direction of $\boldsymbol{\omega}$.

With Equation 6, we can systematically investigate parameter influence in SEM by studying the behavior of the function $F_{ML}(\boldsymbol{\omega})$. In the next sections, two approaches termed the vector method and the single parameter method are introduced and illustrated. Although not very well known, the vector method was originally proposed by Lee and Wang (1996). The single parameter method proposed here is a new method for investigating parameter influence.

VECTOR METHOD

The vector method is an approach for determining a direction vector maximizing (minimizing) the influence on model fit when the optimal parameter estimates are perturbed in the direction of the specified vector. In the context of SEM, the vector approach to the study of parameter influence was suggested first by Lee and Wang (1996), where the authors developed a general method for conducting sensitivity analysis with respect to a minor perturbation introduced either to the model or data. In their paper, Lee and Wang employed the general discrepancy function, denoted by $G(\boldsymbol{\vartheta})$, including the GLS and ML discrepancy functions as its special cases, to define the influence of parameters. Specifically, the measure of influence was defined as $G(\boldsymbol{\omega})$,

$$\begin{aligned} G(\boldsymbol{\omega}) = {} & G\left(\bar{\mathbf{y}}, \mathbf{S}; \boldsymbol{\mu}\left(\tilde{\boldsymbol{\vartheta}}_{\omega}\right), \boldsymbol{\Sigma}\left(\tilde{\boldsymbol{\vartheta}}_{\omega}\right)\right) \\ & - G\left(\bar{\mathbf{y}}, \mathbf{S}; \boldsymbol{\mu}\left(\hat{\boldsymbol{\vartheta}}\right), \boldsymbol{\Sigma}\left(\hat{\boldsymbol{\vartheta}}\right)\right) \end{aligned} \tag{7}$$

with

$$\tilde{\boldsymbol{\vartheta}}_{\omega} = \hat{\boldsymbol{\vartheta}} + U\boldsymbol{\omega} \tag{8}$$

where $\boldsymbol{\omega}$ represents a $q \times 1$ perturbation vector of unit length, $\hat{\boldsymbol{\vartheta}}$ represents a vector of the optimal solution minimizing $G(\boldsymbol{\vartheta})$, and $\mathbf{U}$ is a $q \times q$ diagonal matrix that will be defined shortly. That is, if $G(\boldsymbol{\omega})$ or the change in model fit is large for a small perturbation of $\hat{\boldsymbol{\vartheta}}$ in the direction of $\boldsymbol{\omega}$, it can be concluded that the perturbation scheme $\boldsymbol{\omega}$ is influential to the discrepancy function around the optimal solution. On the other hand, if a large perturbation of $\hat{\boldsymbol{\vartheta}}$ in the direction of

$\boldsymbol{\omega}$ yields a small $G(\boldsymbol{\omega})$, it can be concluded that the perturbation scheme $\boldsymbol{\omega}$ is uninfluential to the discrepancy function around the optimal solution.

It is noteworthy that Lee and Wang (1996) employed a $q \times q$ diagonal matrix, $\boldsymbol{U}$, whose elements are to be chosen according to the investigators' special concerns. For example, $\boldsymbol{U}$ can be chosen to be an identity matrix if the same degree of perturbation to all parameters is to be applied. Or the diagonal elements of $\boldsymbol{U}$ can be proportional to the absolute value of the corresponding parameter's estimated value to study the influence of a perturbation proportional to the parameters' estimated values. In other words, the matrix $\boldsymbol{U}$ is specified so as to define a degree and type of minor perturbation of parameter estimates from the optimal solution, $\hat{\boldsymbol{\vartheta}}$. Because not all parameter estimates in SEM are on the same scale, the use of the $\boldsymbol{U}$ matrix allows the user to choose among different perturbation schemes depending on the special interests in a given application. As pointed out by Lee and Wang (1996), different choices of $\boldsymbol{U}$ will produce different outcomes in subsequent analyses.

Let the Hessian matrix[1] of $G(\boldsymbol{\omega})$ evaluated at its optimal solution, $\hat{\boldsymbol{\vartheta}}$ be denoted by $H\left(\hat{\boldsymbol{\vartheta}}\right)$. Then, Lee and Wang (1996) showed that the eigenvector associated with the largest eigenvalue of the adjusted Hessian matrix, $\tilde{H}\left(\hat{\boldsymbol{\vartheta}}\right)$, defined as,

$$\tilde{H}\left(\hat{\boldsymbol{\vartheta}}\right) = UH\left(\hat{\boldsymbol{\vartheta}}\right)U \tag{9}$$

gives the directional vector of unit length for parameter perturbation, denoted by $\boldsymbol{\omega}_{\max}$, that yields the maximum value in $G(\boldsymbol{\omega})$ or maximum change in model fit. Although not explicitly discussed in Lee and Wang (1996), the eigenvector associated with the smallest eigenvalue of $\tilde{H}\left(\hat{\boldsymbol{\vartheta}}\right)$ gives the directional vector of unit length for parameter perturbation, denoted by $\boldsymbol{\omega}_{\min}$, that yields minimum change in model fit. By inserting $\boldsymbol{\omega}_{\max}$ or $\boldsymbol{\omega}_{\min}$ into Equation 8 in conjunction with the researcher-specified $\boldsymbol{U}$ matrix, we can obtain the perturbed parameter estimates that will have maximum influence, denoted by $\tilde{\boldsymbol{\vartheta}}_{\boldsymbol{\omega}_{max}}$, or minimum influence, denoted by $\tilde{\boldsymbol{\vartheta}}_{\boldsymbol{\omega}_{min}}$, on model fit.

Illustration of the Vector Method

To illustrate the vector method, we use the open-book closed-book (OBCB) test data in Mardia, Kent, and Bibby (1979).[2] The data set consists of five observed variables, which are test scores of five topics: mechanics, vectors, algebra, analysis, and statistics. The first two measures are closed-book tests, and the other measures are open-book test scores. The sample mean vector $\bar{\boldsymbol{y}}$ and covariance matrix $\boldsymbol{S}$ are presented in Table 1. The sample size is 88 ($N = 88$).

TABLE 1
Sample Means, Variances, and Covariances for Measured Variables in Open-Book Closed-Book Example

Observed Variables	*1*	*2*	*3*	*4*	*5*
1. Mechanics	305.768	—	—	—	—
2. Vectors	127.223	172.842	—	—	—
3. Algebra	101.579	85.157	112.886	—	—
4. Analysis	106.273	94.673	112.113	220.380	—
5. Statistics	117.405	99.012	121.871	155.536	297.755
Means	38.955	50.591	50.602	46.682	42.307

A confirmatory factor analysis (CFA) model with two correlated factors and no mean structure was fit to data. The first factor loaded only on the closed-book tests and the second factor loaded only on the open-book tests. The two factors were specified as being correlated. Factor variances were fixed to a value of 1.0 for model identification.

The model specification is as follows:

$$\begin{aligned} y_1 &= \lambda_{11}\eta_1 + \zeta_1 \\ y_2 &= \lambda_{21}\eta_1 + \zeta_3 \\ y_3 &= \lambda_{32}\eta_2 + \zeta_3 \\ y_4 &= \lambda_{42}\eta_2 + \zeta_4 \\ y_5 &= \lambda_{52}\eta_2 + \zeta_5 \end{aligned} \tag{10}$$

where $\boldsymbol{y} = (y_1, y_2, y_3, y_4, y_5)'$ represents a 5×1 random vector for the five test scores, and $\boldsymbol{\eta}_1$ and η_2 are the two common factors. The model-implied covariance matrix, $\boldsymbol{\Sigma}(\boldsymbol{\vartheta})$ is a function of a vector of $\boldsymbol{\vartheta}$ composed of 11 model parameters, $\lambda_{11}, \lambda_{21}, \lambda_{32}, \lambda_{42}, \lambda_{52}, \psi_{12}, \phi_{11}, \phi_{22}, \phi_{33}, \phi_{44}, \phi_{55}$, where λ_{ij} represents the factor loading of y_j on η_i, ψ_{12} represents the correlation between η_1 and η_2, and ϕ_{jj} represents unique variance for y_j.

The ML estimates $\hat{\boldsymbol{\vartheta}}$ were obtained by minimizing the ML discrepancy function of Equation 1. The optimal discrepancy function value $\hat{F}_{ML} = F_{ML}\left[\bar{\boldsymbol{y}}, \boldsymbol{S}; \boldsymbol{\mu}\left(\hat{\boldsymbol{\vartheta}}\right), \boldsymbol{\Sigma}\left(\hat{\boldsymbol{\vartheta}}\right)\right]$ for the fitted two-factor CFA model was 0.0238 with 4 degrees of freedom, resulting in the point estimate of root mean square error of approximation (RMSEA) value being .00. The parameter estimates $\hat{\boldsymbol{\vartheta}}$ and the associated standard error estimates $S.E.\left(\hat{\boldsymbol{\vartheta}}\right)$ are presented in Table 2.

To apply the vector method and identify the direction of parameter perturbation for the maximum (minimum) change in model fit, we calculated the Hessian matrix of Equation 1 and evaluated at $\hat{\boldsymbol{\vartheta}}$. In this example, we specify $\boldsymbol{U}$ as the

[1] The symmetric matrix composed of second-order partial derivatives of a function is called the Hessian matrix.

[2] This data set has been a canonical example in sensitivity analysis in the context of factor analysis and SEM since it was first used in Lee and Wang (1996). The data set received considerable attention later by Cadigan (1995), Fung and Kwan (1995), Lee and Tang (2004), Kwan and Fung (1998), and Poon and Poon (2002). See also Tanaka and Odaka (1989a, 1989b).

TABLE 2
Parameter Estimates, Standard Errors, and the Results of Parameter Influence Analysis for Open-Book Closed-Book Example

Parameter	*MLE*	*SE*	$\boldsymbol{\omega}_{max}$	$\boldsymbol{\omega}_{min}$	*LB*	*UB*
λ_{11}	12.253	1.832	0.048	−0.006	9.764	14.897
λ_{21}	10.383	1.371	0.105	0.008	8.519	12.363
λ_{32}	9.834	0.923	−0.893	−0.072	8.600	11.197
λ_{42}	11.490	1.395	0.287	0.059	9.616	13.550
λ_{52}	12.517	1.658	0.178	0.057	10.285	14.958
ψ_{12}	0.818	0.072	0.256	0.012	0.709	0.907
ϕ_{11}	155.632	31.499	−0.011	0.013	116.001	204.746
ϕ_{22}	65.035	17.996	−0.010	−0.026	41.056	91.784
ϕ_{33}	16.187	7.219	−0.089	0.967	6.320	27.260
ϕ_{44}	88.352	16.677	0.035	−0.186	67.023	115.419
ϕ_{55}	141.072	24.739	0.022	−0.130	109.932	181.270

Note. The model is fit to the covariance matrix only. $N = 88$. MLE = maximum likelihood estimates; SE = standard error estimates; $\boldsymbol{\omega}_{max}$ = the eigenvector associated with the largest eigenvalue of the adjusted Hessian matrix; $\boldsymbol{\omega}_{min}$ = the eigenvector associated with the smallest eigenvalue of the adjusted Hessian matrix; LB (UB) = the lower (upper) bound of perturbation for individual parameters hat yields .001 increase in root mean square error of approximation.

diagonal matrix whose elements are proportional to the absolute values of the associated parameter estimates to take into account the difference in scales of parameter estimates. Specifically, the diagonal elements in the $\boldsymbol{U}$ matrix are set to be the vector of $\frac{\sqrt{|\hat{\boldsymbol{\vartheta}}|}}{2}$. Then we calculated the eigenvectors for the adjusted Hessian matrix defined in Equation 9.

Table 2 provides the results from the vector method. The two columns $\boldsymbol{\omega}_{\max}$ and $\boldsymbol{\omega}_{\min}$, respectively, show the eigenvectors associated with the largest and smallest eigenvalues for the adjusted Hessian matrix. These eigenvectors are embedded in the 11-dimensional parameter space pointing to the directions in which the ML discrepancy function changes the most or the least under the particular choice of $\boldsymbol{U}$ matrix we made. In other words, each eigenvector indicates the direction in which the ML parameter estimates shall be perturbed simultaneously to produce the maximum or minimum change in model fit.

The overall pattern of the two eigenvectors indicates that, under the scheme of perturbing all parameters simultaneously *proportional to the absolute value* of the corresponding parameter estimates, the perturbation of factor loadings $\lambda_{32}, \lambda_{42}, \lambda_{52}$, and factor intercorrelation φ_{12} from the optimal solution contributes the most to the maximum change in model fit, whereas the smallest change in model fit is mostly due to the perturbation of unique variances ϕ_{33}, ϕ_{44}, and ϕ_{55} from the optimal parameter estimates. Recall that the ML solution produced a discrepancy function value of .0238 and RMSEA of .000. In comparison, the ML discrepancy function value and the associated RMSEA for $\tilde{\boldsymbol{\vartheta}}_{\boldsymbol{\omega}_{\max}}$ are .184 and .186, respectively, whereas the ML discrepancy function value and the associated RMSEA for $\tilde{\boldsymbol{\vartheta}}_{\boldsymbol{\omega}_{\min}}$ are .0248 and .000, respectively. Therefore, it can be concluded that a group of parameters $\lambda_{32}, \lambda_{42}, \lambda_{52}$, and φ_{12}, as a whole, are highly influential, allowing more rigorous interpretation of their estimated values. These parameters are the loadings for the three open-book tests and the correlation between the two factors. It can also be concluded that a group of parameters ϕ_{33}, ϕ_{44}, and ϕ_{55}, as a whole, are uninfluential and their estimated values should not be interpreted in the literal sense because changes in these parameters have little impact on model fit. These are the unique variances for the open-book tests.

To reiterate, the vector method allows the researcher to identify, under the specific perturbation scheme, a group of parameters that will exhibit more or less impact on model fit than others by focusing on the overall pattern of the size of the numerical values in the obtained eigenvectors. Admittedly, the subjective nature of the interpretation of the pattern in the eigenvectors must be recognized especially when the patterns of the eigenvectors do not offer a clear structure for simple interpretation. In a sense, such difficulties naturally arise due to the multidimensional nature of the solution (e.g., 11×1 perturbation vectors).

SINGLE PARAMETER METHOD

The vector method has two characteristics that should be noticed. First, as pointed out by Lee and Wang (1996), the results of the vector method depend on the choice of the $\boldsymbol{U}$ matrix. In the previous example, if the $\boldsymbol{U}$ matrix were chosen to be an identity matrix, $\boldsymbol{\omega}_{\max}$, and $\boldsymbol{\omega}_{\min}$, respectively, would have been pointing to different directions for maximum and minimum change in model fit. Second, as seen in the previous empirical example, the nature of the solution provided by the vector method is multidimensional so that the overall pattern of the eigenvector defines a combination of influential parameters that will have greater impact on model fit.

In this section we propose an alternative method for investigating parameter influence that allows for examining the influence of individual parameters separately. This method, termed the *single parameter method,* defines the influence of individual parameters as the degree of perturbation in the estimate of the selected parameter required to produce a prespecified value of change in model fit. In other words, more influential parameters are those that, given the specified level of change in model fit, require smaller perturbation from its optimal solution than a less influential parameter does.

Because the single parameter method evaluates the influence of individual parameters, scale differences among parameters are not an issue and the choice of the $\boldsymbol{U}$ matrix becomes irrelevant. Further, due to the unidimensional nature of the solution, users of the single parameter method can directly interpret the results without having to interpret the overall pattern of the solution given by the vector method. For example, if a factor correlation with its ML estimate equal to .56 can be perturbed to values of .20 or .90, yielding changes in RMSEA of less than .005, then it is clear that the factor correlation has little influence on model fit and its estimate .56 cannot (and should not) be interpreted in a literal sense. This interpretation for the factor correlation is valid regardless of the degree of influence of other parameters given by either the single parameter or the vector method.

To reiterate, the single parameter method provides a mechanism for determining the lower and upper bounds (e.g., .20 and .90) of the perturbation for individual parameters that will produce a prespecified value of change in model fit (e.g., increase in RMSEA by .005). In the following section, an algorithm is proposed and described for determining the *perturbation bounds* for individual parameters or the extent to which individual parameters ought to be perturbed to produce a specified level of deterioration in model fit.

An Algorithm to Determine the Degree of Perturbation Given a Prespecified Change in Fit

Let F^* denote a prespecified value of model fit, slightly larger than the optimal fit function value $\hat{F} = F_{ML}\left[\bar{\mathbf{y}}, \mathbf{S}; \boldsymbol{\mu}\left(\hat{\boldsymbol{\vartheta}}\right), \boldsymbol{\Sigma}\left(\hat{\boldsymbol{\vartheta}}\right)\right]$. And let $\boldsymbol{\vartheta}_1$ denote the jth parameter of interest or the focal parameter for which the degree of perturbation is to be determined. Also let $\boldsymbol{\vartheta}_2$ denote a $(q-1) \times 1$ vector of all the other parameters but the jth focal parameter. In this article, we call $\boldsymbol{\vartheta}_2$ the vector of nuisance parameters. Then, we can always rearrange the $q \times 1$ model parameter $\boldsymbol{\vartheta}$ as $\boldsymbol{\vartheta} = \left(\boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2'\right)'$. Further, let κ^+ and κ^- denote constants representing the degree of perturbation of the focal parameter $\boldsymbol{\vartheta}_1$ from its optimal solution $\hat{\boldsymbol{\vartheta}}_1$ in the positive and negative direction, respectively, to yield the specified value of F^*. Then, the goal of the algorithm is to determine the constants κ^+ and κ^- satisfying

$$F\left[\bar{\mathbf{y}}, \mathbf{S}; \boldsymbol{\mu}\left(\tilde{\boldsymbol{\vartheta}}_+\right), \sum\left(\tilde{\boldsymbol{\vartheta}}_+\right)\right] = F^* \text{ and } F\left[\bar{\mathbf{y}}, \mathbf{S}; \boldsymbol{\mu}\left(\tilde{\boldsymbol{\vartheta}}_-\right), \sum\left(\tilde{\boldsymbol{\vartheta}}_-\right)\right] = F^* \tag{11}$$

where $\tilde{\boldsymbol{\vartheta}}_+ = \left(\hat{\vartheta}_1 + \kappa^+, \tilde{\boldsymbol{\vartheta}}_2'\right)'$, $\tilde{\boldsymbol{\vartheta}}_- = \left(\hat{\vartheta}_1 + \kappa^-, \tilde{\boldsymbol{\vartheta}}_2'\right)$, and $\tilde{\boldsymbol{\vartheta}}_2 = \left(\hat{\vartheta}_1, \ldots, \hat{\vartheta}_{j-1}, \tilde{\vartheta}_{j+1}, \cdots, \tilde{\vartheta}_q\right)'$.

It is worth noticing that the preceding equation becomes a straightforward one-dimensional root finding problem with respect to κ^+ or κ^- if the $\tilde{\boldsymbol{\vartheta}}_2$ is a priori known or fixed at some values such as its optimal solution $\hat{\boldsymbol{\vartheta}}_2 = \left(\hat{\vartheta}_1, \ldots, \hat{\vartheta}_{j-1}, \hat{\vartheta}_{j+1}, \cdots, \hat{\vartheta}_q\right)'$. In fact, the problem of finding κ^+ or κ^- that yields F^* with $\tilde{\boldsymbol{\theta}}_2$ fixed at a priori known values was successfully addressed by adopting an algorithm known as the Brent method in the context of studying the power for tests of model fit in SEM (MacCallum, Lee, & Browne, 2010). In essence, the Brent algorithm finds a value approximating the root of a function within a prespecified level of precision, given an interval containing the root of the function, by systematically and iteratively searching the interval provided by the user. The Brent algorithm is described in detail in Press, Flannery, Teukolsky, and Vetterling (1992).

In the current context of studying parameter influence, it should be borne in mind that, due to correlations among parameter estimates, changes in $\boldsymbol{\vartheta}_1$ from $\hat{\boldsymbol{\vartheta}}_1$ will necessarily induce changes in $\hat{\boldsymbol{\vartheta}}_2$. Therefore, $\boldsymbol{\vartheta}_2$ must be adjusted to a new vector of optimal values as perturbation is imposed on $\hat{\boldsymbol{\vartheta}}_1$. One approach to adjusting $\boldsymbol{\vartheta}_2$ would be to refit the posited model with $\boldsymbol{\vartheta}_1$ fixed at a perturbed value, and take the resulting estimates of $\boldsymbol{\vartheta}_2$ as the new optimal solution for $\boldsymbol{\vartheta}_2$. In other words, we can make the adjustment for $\boldsymbol{\vartheta}_2$ by maximizing the profile-likelihood of $\boldsymbol{\vartheta}_2$ at a given value of $\boldsymbol{\vartheta}_1$. However, when this profile-likelihood-based approach is embedded in the Brent algorithm that iteratively finds the root κ^+ or κ^-, the posited model must be refitted at every iteration during the root finding processes for the adjustment of the nuisance parameters. In other words, iterative model refitting processes are to be nested within each cycle of the iterative root finding processes. In this iteration-within-iteration approach, the computations involved in the model refitting processes can quickly become burdensome as the model gets more complex and can easily dominate the computational cost required for the root finding process.

In response to the problem of the computational burden involved in the adjustment of nuisance parameters, we use an alternative method due to Pawitan (2001, p. 267) for the adjustment of the nuisance parameter values at each iteration in the root finding processes. The computational burden of this alternative approach is minimal because it does not require model refitting but simply recycles the results of the ML estimation. In essence, based on the standard multivariate normal distribution theory, Pawitan showed that the change in the nuisance parameters associated with the change in the focal parameter can be approximated by

$$\tilde{\boldsymbol{\vartheta}}_2^{(t)} = \hat{\boldsymbol{\vartheta}}_2 + \mathcal{I}_{22}^{-1}\mathcal{I}_{21}\left(\hat{\vartheta}_1 - \tilde{\vartheta}_1^{(t)}\right) \tag{12}$$

where $\tilde{\vartheta}_1^{(t)} = \hat{\vartheta}_1 + \kappa_{(t)}^+$ or $\tilde{\vartheta}_1^{(t)} = \hat{\vartheta}_1 + \kappa_{(t)}^-$ represents the perturbed value of $\boldsymbol{\vartheta}_1$ from its optimal solution $\hat{\boldsymbol{\vartheta}}_1$ to the degree of $\kappa_{(t)}^+$ or $\kappa_{(t)}^-$ at the tth iteration of the Brent algorithm. $\mathcal{I}_{22}$ and $\mathcal{I}_{21}$, respectively, indicate the information matrix for the vector of nuisance parameters $\boldsymbol{\vartheta}_2$ and joint information for the vector of nuisance parameters $\boldsymbol{\vartheta}_2$ and the focal parameter ϑ_1 evaluated at $\hat{\vartheta}$. That is,

$$\mathcal{I}_{22} = -\frac{\partial^2 \ell\left(\hat{\boldsymbol{\vartheta}}\,|\bar{\mathbf{y}}, \boldsymbol{S}\right)}{\partial \boldsymbol{\vartheta}_2 \partial \boldsymbol{\vartheta}_2'} \quad \text{and} \quad \mathcal{I}_{21} = -\frac{\partial^2 \ell\left(\hat{\boldsymbol{\vartheta}}\,|\bar{\mathbf{y}}, \boldsymbol{S}\right)}{\partial \boldsymbol{\vartheta}_2 \partial \boldsymbol{\vartheta}_1} \tag{13}$$

Equation 12 shows that the adjustment of $\boldsymbol{\vartheta}_2$ at the tth iteration can be very simple if we recycle the results of parameter estimation: the ML estimates of Equation 3 and information matrix of Equation 4. Equation 12 also shows that the adjustment is proportional to the degree of perturbation of the focal parameter from its optimal solution at the tth iteration: $\left(\hat{\vartheta}_1 - \tilde{\vartheta}_1^{(t)}\right)$. Equation 12 provides a linear approximation to the change of nuisance parameters at the tth iteration as a function of change in the focal parameter.

Inserting Equation 12 into Equation 11, the target equations in 11 are again reduced to a one-dimensional root finding problem with respect to the unknown constants κ^+ and κ^- while taking into account correlations among parameter estimates.

Illustration of the Single Parameter Method

In this section, we illustrate the single parameter method by computing perturbation bounds for each of the 11 parameters specified in the CFA model fitted to the OBCB data set. In this illustration, F^* or the desired level of model fit is specified in terms of the point estimate of RMSEA because RMSEA has an empirical scale for reference, whereas the fit function value itself does not (Browne & Cudeck, 1993). Then the perturbation bounds for individual parameters are obtained by solving Equation 11 with respect to κ^+ and κ^- while taking into account correlations among parameter estimates using Equation 12.

Given that RMSEA for the two-factor model was .000, we chose to set the perturbed value of RMSEA at .001. The corresponding value of F^* would be .046. We then compute lower and upper bounds for each parameter that yield $F^* =$ .046. The last two columns in Table 2 show these lower and upper bounds of perturbation for each of the 11 parameters. For example, λ_{11} has lower and upper perturbation bounds of 9.76 and 14.90, respectively. This implies that λ_{11} can take any values within the range given by the perturbation bounds, yielding RMSEA less than .001. Perturbation bounds for other parameters can be interpreted in the same manner.

Inspection of Table 2 indicates that perturbation bounds for factor loadings all look reasonable in that the values in the perturbation bounds are not anomalously wide enough to call into question substantive interpretations of the parameter estimates, whereas perturbation bounds for unique variances are rather wide, questioning whether the estimates can be interpreted in the literal sense. These results are quite consistent with those obtained from the vector method. The upper bound of factor correlation indicates that the correlation can take the value of .91 and still yield RMSEA of .001, suggesting that adequate discriminatory validity for the two method factors might well be doubted. This result might appear to conflict with that of the vector method where ψ_{12} is a member of the group of influential parameters. But when it comes to the influence of an individual parameter ψ_{12}, what can be concluded from the result of the vector method is that ψ_{12} will exercise significant influence on model fit only when perturbed simultaneously with other parameters including λ_{32}, λ_{42}, and λ_{52}. In this sense, the vector method is not well suited for studying the influence of individual parameters separately. Being able to examine the influence of individual parameters separately is a significant advantage of the single parameter method. According to the result of the single parameter method, ψ_{12} turns out to have little influence on model fit when considered in isolation.

Influence Mappings for Single Parameters

Recall that, when computing perturbation intervals for single parameters, nuisance parameters are adjusted using Equation 12 without refitting the model each time a focal parameter value changes in the processes of solving Equation 11. To evaluate the quality of the approximate adjustment, we created a plot termed *influence mapping for single parameters* (IMSP) for each of the focal parameters, depicting the change of the ML discrepancy function value in the vicinity of the optimal estimate of the focal parameter. Two types of IMSP are created. In one type of IMSP, termed IMSP-A, the nuisance parameter values are adjusted so as to maximize the profile-likelihood by refitting the model at each change in the focal parameter values. In the other type of IMSP, termed IMSP-B, the nuisance parameters are adjusted using Equation 12 without refitting the model.

To save space, we present IMSP for four selected focal parameters from the OBCB example: two factor loadings (λ_{21}, λ_{32}), a factor correlation (ψ_{12}), and a unique variance (ϕ_{33}). Four panels in Figure 1 present IMSPs for the four parameters. In Figure 1, the horizontal axis represents the amount of perturbation of the focal parameter, from its optimal solution and the vertical axis represents the associated change in the ML discrepancy function value. In each of the four panels, the solid and dotted U-curves represent IMSP-A and IMSP-B for the focal parameter, respectively. Perturbation bounds yielding .001 increase in RMSEA are also marked in each panel. As can be seen, IMSP-A and IMSP-B for the four parameters are virtually identical in a small neighborhood of the optimal solution of each of the focal parameters, although a slight discrepancy begins to occur as the degree of perturbation gets larger. The absolute magnitudes of discrepancies between IMSP-A and IMSP-B, however, remain very small in the vicinity of the optimal solution. And thus, considering the minimal level of computation, the use of Equation 12 for the adjustment of nuisance parameter looks promising.

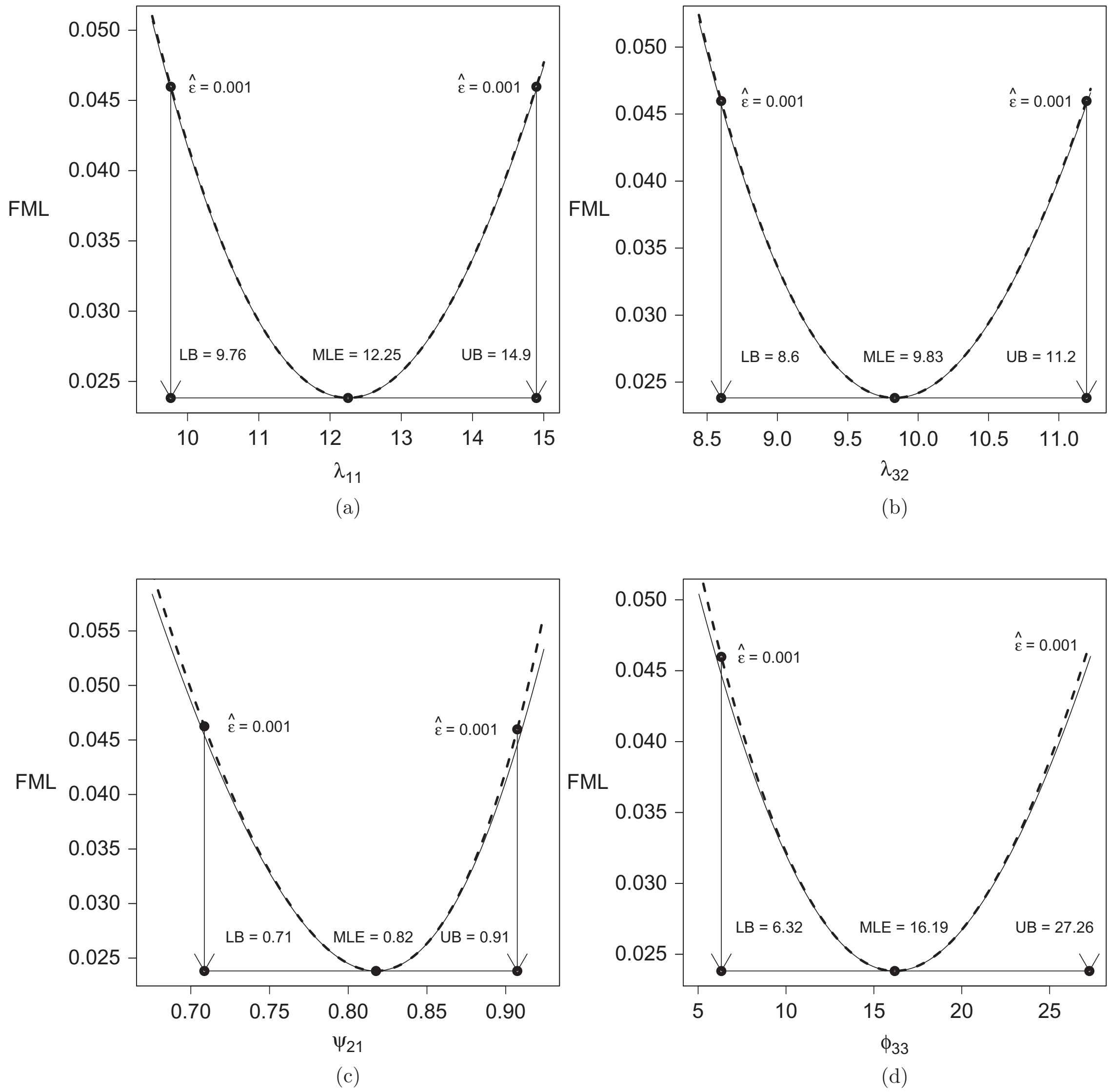

FIGURE 1 Influence mappings for single parameters. (a) IMSP for λ_{11}. (b) IMSP for λ_{32}. (c) IMSP for ψ_{12}. (d) IMSP for ϕ_{33}.

Note. In each of the four panels, the solid and dotted curves represent IMSP-A and IMSP-B, respectively. IMSP-A is obtained by refitting the model at every perturbed value of the focal parameter, whereas IMSP-B is obtained by adjusting the nuisance parameters using Equation 12 without refitting the model. The lower bound (LB) and upper bound (UB) of perturbation to yield root mean square error of approximation increase by .001 are marked on IMSP-B for each parameter. MLE = maximum likelihood estimates.

TWO ILLUSTRATIVE EXAMPLES

As examples of how the results of parameter influence studies of a given model can be effectively employed in real data analysis, we consider parameter estimates in CFA models fitted to a multitrait–multimethod (MTMM) data matrix. It is common practice under CFA approaches for investigating construct validity that sizable and statistically significant factor loadings are considered as supportive evidence for convergent validity of the purported psychological constructs and that factor correlations offer indications of discriminant validity of those psychological constructs. In the following sections, we present the results of two parameter influence studies obtained from two MTMM data matrices. In both examples, the $\boldsymbol{U}$ matrix employed in the vector method was set to be a diagonal matrix with elements that are proportional to the absolute values of the corresponding parameter estimates to take into account the scale differences among parameter estimates.

Kenny and Kashy (1992) Data Set

As the first example, we present the results of parameter influence studies obtained from analyzing the parameter

TABLE 3
Parameter Estimates, Standard Errors, and the Results of Parameter Influence Analysis for Kenny and Kashy (1992) Example

Parameter	*MLE*	*SE*	$\boldsymbol{\omega}_{max}$	$\boldsymbol{\omega}_{min}$	*LB*	*UB*
λ_{11}	1.350	0.266	−0.007	0.001	1.254	1.449
λ_{41}	1.225	0.248	0.005	−0.003	1.135	1.317
λ_{71}	0.597	0.170	−0.006	−0.003	0.534	0.659
λ_{22}	0.599	0.286	0.001	−0.011	0.496	0.703
λ_{52}	1.120	0.495	0.001	0.005	0.954	1.310
λ_{82}	0.338	0.215	−0.018	−0.012	0.260	0.419
λ_{33}	0.841	0.185	0.095	−0.004	0.773	0.909
λ_{63}	0.902	0.209	0.032	0.000	0.825	0.979
λ_{93}	0.730	0.162	−0.174	0.000	0.671	0.790
ψ_{12}	0.476	0.196	0.014	−0.016	0.403	0.547
ψ_{13}	0.020	0.171	−0.001	−0.999	−0.043	0.082
ψ_{23}	0.353	0.260	0.016	−0.033	0.257	0.446
ϕ_{88}	1.818	0.310	−0.693	0.001	1.707	1.936
ϕ_{99}	0.931	0.236	−0.447	−0.001	0.845	1.018
ϕ_{21}	0.520	0.296	0.010	0.004	0.412	0.630
ϕ_{31}	0.172	0.192	−0.004	0.023	0.102	0.243
ϕ_{32}	0.522	0.231	0.052	0.007	0.439	0.608
ϕ_{54}	0.316	0.366	0.000	0.006	0.180	0.449
ϕ_{64}	0.224	0.223	0.003	0.008	0.143	0.307
ϕ_{65}	0.196	0.278	−0.001	0.006	0.095	0.299
ϕ_{87}	0.444	0.207	0.053	0.004	0.370	0.522
ϕ_{97}	0.239	0.164	−0.021	0.007	0.180	0.301
ϕ_{98}	0.674	0.198	0.511	0.001	0.603	0.749

Note. The model is fit to the covariance matrix only. $N = 80$. In the interest of saving space, the results for seven unique variances ($\phi_{11}, \cdots, \phi_{77}$) are omitted. MLE = maximum likelihood estimates; SE = standard error estimates; $\boldsymbol{\omega}_{max}$ = the eigenvector associated with the largest eigenvalue of the adjusted Hessian matrix; $\boldsymbol{\omega}_{min}$ = the eigenvector associated with the smallest eigenvalue of the adjusted Hessian matrix; LB (UB) = the lower (upper) bound of perturbation for individual parameters hat yields .001 increase in root mean square error of approximation.

estimates given in Table 3 of Kenny and Kashy (1992). Original data were collected to measure three traits: administrative ability, ability to give feedback to subordinates, and consideration when dealing with others. Each trait was assessed by three methods—ratings by supervisors, self, and subordinates—producing a 9 × 9 MTMM correlation matrix. We reproduced the results in Table 3 of Kenny and Kashy by fitting a CFA model with three correlated trait factors and correlated uniqueness. Trait factor variances were fixed at unity for identification purposes. The model yields $\chi^2(15, N = 80) = 18.698$ and RMSEA = .056, which is virtually identical within rounding errors with the original result in Kenny and Kashy.

The second and third columns in our Table 3 present the ML parameter estimates and the associated standard error estimates, respectively. As can been seen, all of the trait factor loadings (i.e., λ_{11}, λ_{41}, λ_{71}, λ_{22}, λ_{52}, λ_{33}, λ_{63}, λ_{93}) are in general large and significant except λ_{82}, which is relatively small and only marginally significant. These results can be viewed in support of convergent validity of the three traits. Small to moderate factor correlations among trait factors (ψ_{12}, ψ_{13}, ψ_{23}) are indicative of adequate discriminant validity among the three traits. There are nonzero and statistically significant covariances among unique factors (i.e., ϕ_{21}, ϕ_{32}, ϕ_{87}, ϕ_{98}), reflecting the existence of nontrivial method effects. Overall, according to conventional standards, the results in Table 3 provide evidence that the indicators are strongly related to their purported latent construct (convergent validity), adjusting for the effects of assessment method. Adequate discriminant validity is evidenced by small to modest correlations among trait factors.

The results of parameter influence studies are also consistent with and supportive of such conclusions. The results of the vector method, shown in Table 3, indicate that, according to $\boldsymbol{\omega}_{max}$, a unique factor covariance ϕ_{98} is a member in the group of influential parameters (i.e., ϕ_{88}, ϕ_{99}, ϕ_{98},) offering supportive evidence for the existence of a nontrivial method factor based on conventional standards. In $\boldsymbol{\omega}_{min}$, the factor correlation between the first and the third traits (ψ_{13}) stood out as the least influential parameter, implying that changes in ψ_{13} will have little influence on model fit. Examination of the perturbation bounds of ψ_{13}, however, revealed that the absolute magnitude of the lower and upper perturbation bounds is not too large to threaten the discriminant validity of the first and the third traits. The upper bounds of the other two correlations among trait factors (ψ_{12}, ψ_{23}) also have moderate size of values, providing supportive evidence for adequate discriminant validity among the trait factors. The last two columns show the lower and upper perturbation bounds for all individual parameters, which are not unreasonably wide, given a specified value of model deterioration in terms of RMSEA perturbed by .001. Specifically, the lower bounds of all factor loadings have reasonably large values, even for statistically nonsignificant λ_{82}, and thus support the adequate convergent validity of the purported trait factors.

Figure 2 presents IMSP-A (solid line) and IMSP-B (dotted line) for four selected parameters (λ_{63}, λ_{93}, ψ_{12}, and ψ_{13}) with perturbation bounds of each parameter marked on IMSP-B. It is noteworthy that IMSP-A and IMSP-B are virtually identical in the neighborhood of the ML estimate of each parameter for the selected parameters (and all others not presented here), highlighting the quality of the approximation used in the single parameter method.

This example demonstrates how the results of parameter influence analysis can be combined with conventional statistical standards in establishing convergent and discriminant validity for measures of psychological constructs. In this example, construct validity (convergent and discriminant) of the purported constructs is evidenced by the results of parameter influence studies (i.e., sensible perturbation bounds) as well as by the conventional statistical standards (i.e., statistical significance).

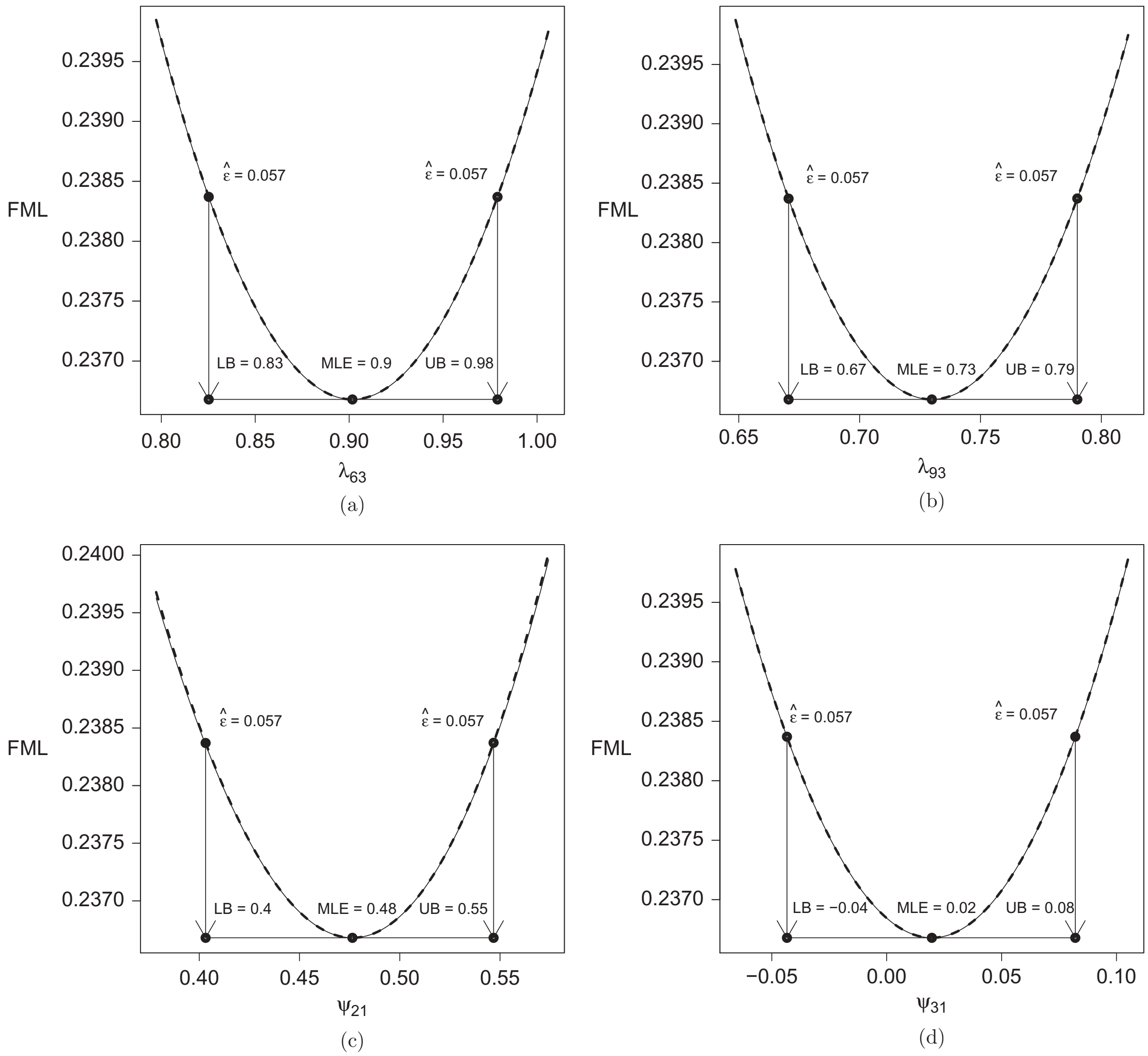

FIGURE 2 Influence mappings for single parameters. (a) IMSP for λ_{63}. (b) IMSP for λ_{93}. (c) IMSP for ψ_{12}. (d) IMSP for ψ_{13}.

Note. In each of the four panels, the solid and dotted curves represent IMSP-A and IMSP-B, respectively. IMSP-A is obtained by refitting the model at every perturbed value of the focal parameter, whereas IMSP-B is obtained by adjusting the nuisance parameters using Equation 12 without refitting the model. The lower bound (LB) and upper bound (UB) of perturbation to yield root mean square error of approximation increase by .001 are marked on IMSP-B for each parameter. MLE = maximum likelihood estimates.

Lance et al. (2002) Data Set

As the second example, we conducted parameter influence analysis on the results given in Table 3 in Lance, Noble, and Scullen (2002). Original data were collected to measure three traits: receptiveness to individual and cultural differences, consideration for the feelings and needs of others, and adaptability in varying environment. Each trait was assessed by three methods—situational interview, in-basket, and biodata—producing a 9 × 9 MTMM correlation matrix.

The results are reproduced in Table 4. Parameter estimates are slightly different from the original published results due to the fact that the model is fitted to the correlation matrix provided by Table 1 in Lance et al. (2002), rather than to the original raw data matrix. However, the exact reproduction of the original parameter estimates is not critical because the main objective of this illustration is to demonstrate the effectiveness of the parameter influence studies in interpreting the parameter estimates in a given (well-fitting) model.

Table 4 shows parameter estimates obtained by fitting a CFA model with three correlated trait factors and correlated

TABLE 4
Parameter Estimates, Standard Errors, and the Results of Parameter Influence Analysis for Lance et al. (2002) Example

Parameter	*MLE*	*SE*	$\boldsymbol{\omega}_{max}$	$\boldsymbol{\omega}_{min}$	*LB*	*UB*
λ_{11}	0.302	0.096	0.002	−0.053	0.043	0.571
λ_{41}	0.718	0.144	0.003	0.048	0.392	1.098
λ_{71}	0.219	0.094	−0.001	−0.216	−0.037	0.486
λ_{22}	0.427	0.092	−0.006	−0.026	0.171	0.689
λ_{52}	0.616	0.106	−0.009	0.015	0.338	0.914
λ_{82}	0.309	0.092	0.034	−0.193	0.054	0.568
λ_{33}	0.478	0.095	−0.006	−0.014	0.217	0.745
λ_{63}	0.581	0.102	−0.005	0.030	0.311	0.869
λ_{93}	0.137	0.095	−0.003	−0.922	−0.129	0.400
ψ_{12}	0.739	0.162	0.006	−0.071	0.327	1.000
ψ_{13}	0.736	0.164	0.003	−0.076	0.315	1.000
ψ_{23}	0.911	0.144	−0.077	−0.040	0.537	1.000
ϕ_{88}	0.902	0.102	0.628	0.018	0.665	1.243
ϕ_{99}	0.981	0.105	0.658	0.018	0.739	1.338
ϕ_{21}	0.169	0.073	0.000	0.093	−0.018	0.380
ϕ_{31}	0.239	0.073	−0.003	0.052	0.054	0.458
ϕ_{32}	0.171	0.075	−0.005	0.096	−0.026	0.387
ϕ_{87}	0.230	0.074	−0.029	0.054	0.044	0.451
ϕ_{97}	0.216	0.075	−0.011	0.061	0.027	0.437
ϕ_{98}	0.380	0.078	−0.402	0.050	0.188	0.627

Note. The model is fit to the covariance matrix only. $N = 180$. In the interest of saving space, the results for seven unique variances ($\phi_{11}, \cdots, \phi_{77}$) are omitted. MLE = maximum likelihood estimates; SE = standard error estimates; $\boldsymbol{\omega}_{max}$ = the eigenvector associated with the largest eigenvalue of the adjusted Hessian matrix; $\boldsymbol{\omega}_{min}$ = the eigenvector associated with the smallest eigenvalue of the adjusted Hessian matrix; LB (UB) = the lower (upper) bound of perturbation for individual parameters that yields .001 increase in root mean square error of approximation.

uniqueness to the 9 × 9 MTMM correlation matrix.[3] This model fits the data very well, $\chi^2(18, N = 180) = 10.349$ with RMSEA = .000. According to conventional practice, the estimated factor loadings suggest that there is convergent validity in assessing the three traits: All but one estimated factor loading (i.e., λ_{93}) are statistically significant, and some of these are large by conventional standards. The estimated factor correlation of .91 between the second and third traits (ψ_{23}), however, indicates that these two trait factors are strongly intercorrelated, implying poor discriminant validity. The correlations of .74 between the first and second traits (ψ_{12}) and between the first and third traits (ψ_{13}) can be a source of argument regarding whether or not the associated traits have adequate discriminant validity. Clearly neither statistical significance nor the size of coefficient provides a clear-cut standard to resolve the interpretation of results.

In this situation, results from parameter influence analysis can provide an effective and principled way to determine whether the two associated traits having the correlation of .74 have adequate discriminant validity. Specifically, the upper bounds of the three factor correlations indicate that the factor correlations can each take the value of 1.00 and still yield the model's RMSEA being less than .001. Figures 3a and 3b present the IMSP for ψ_{12} and ψ_{13} with lower and upper perturbation bounds yielding RMSEA of .001 marked on IMSP-B (dotted lines).[4] It can be seen that the upper perturbation bounds of 1.00 for ψ_{12} and ψ_{13} yield an RMSEA of .000. These results provide strong evidence of poor discriminant validity among the three traits.

In addition, the results of parameter influence studies on the factor loadings also call into question the convergent validity of the purported traits. That is, lower bounds of most factor loadings approach or include zero, and yet yield RMSEA of .001. Figures 3c and 3d present the IMSP for two selected factor loadings, λ_{82} and λ_{93}, with lower and upper perturbation bounds yielding an RMSEA of .001 marked on IMSP-B (dotted lines). It can be seen that the lower perturbation bound for λ_{82} is small and close to zero and the lower pertubation bound for λ_{93} is negative. The result of the vector method also indicates that, according to $\boldsymbol{\omega}_{\min}$, the factor loadings λ_{71}, λ_{82}, and λ_{93} are, as a group, uninfluential on model fit. These results provide critical and converging information when determining the quality of measurements.

In this example, construct validity of the purported psychological constructs was not supported by the results of parameter influence analysis (i.e., unfavorable perturbation bounds), despite being evidenced by conventional statistical standards (i.e., statistical significance). This particular example clearly demonstrates that researchers could benefit from considering results of parameter influence studies, in addition to conventional statistical standards, to arrive at appropriate interpretations of optimal parameter estimates even if they are embedded in a well-fitting model.

DISCUSSION

In this article, we have described and illustrated two methods for examining parameter influence in the context of SEM. First, we illustrated that the vector method, originally proposed by Lee and Wang (1996), can be useful in finding a direction of simultaneous perturbation of all parameters and identifying a group of influential parameters. We also illustrated that the vector method could be extended in such a way that a group of least influential parameters is identified by examining the eigenvector associated with the smallest eigenvalue for the adjusted Hessian matrix. Next, we proposed the single parameter method and demonstrated that this new method can be effectively employed to measure the influence of individual parameters separately. Using

[3]Readers who are interested in the characteristics of the original data set are referred to Lance et al. (2002). The model we fitted had correlated uniqueness only for two methods (situational interview and biodata), rendering the degrees of freedom equal to 18.

[4]This example shows again that IMSP-A (solid curves) and IMSP-B are virtually identical in the vicinity of the optimal parameter estimates.

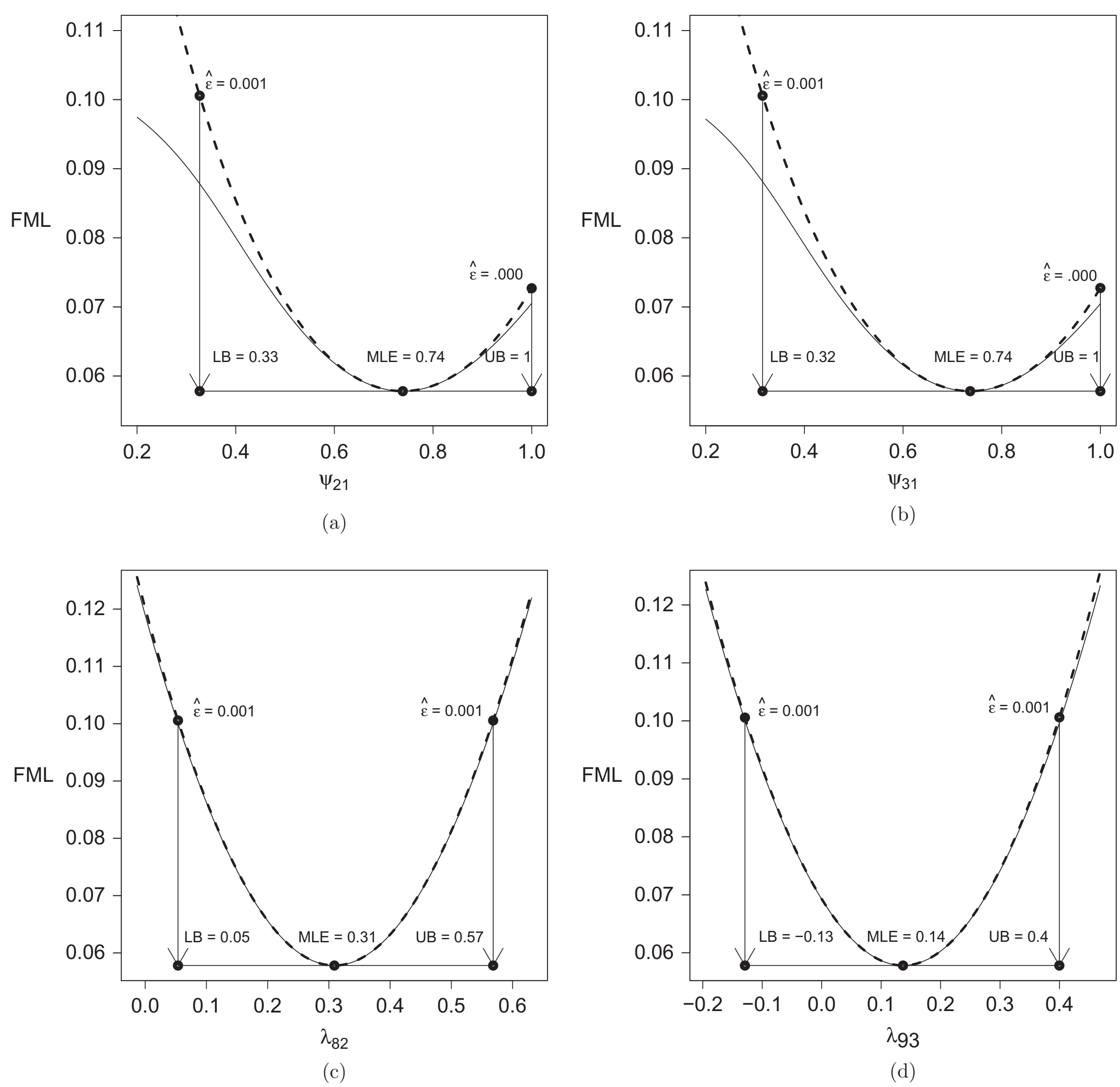

FIGURE 3 Influence mappings for single parameters. (a) IMSP for ψ_{12}. (b) IMSP for ψ_{13}. (c) IMSP for λ_{82}. (d) IMSP for λ_{93}.

Note. In each of the four panels, the solid and dotted *U* curves represent IMSP-A and IMSP-B, respectively. IMSP-A is obtained by refitting the model at every perturbed value of the focal parameter, whereas IMSP-B is obtained by adjusting the nusiance parameters using Equation 12 without refitting the model. The lower bound (LB) and upper bound (UB) of perturbation to yield root mean square error of approximation increase by .001 are marked on IMSP-B for each parameter. MLE = maximum likelihood estimates.

empirical data sets, we illustrated that parameters with wide perturbation bounds containing theoretically unreasonable values and yet exercising little negative impact on model fit should not be interpreted or used in the literal sense even if the corresponding parameter estimates are statistically significant. We also showed that the two methods complement each other in the analysis of parameter influence.

Although not explicitly discussed in detail in the body of the article, the perturbation bounds for individual parameters can be regarded as an alternative metric for quantifying uncertainties involved in parameter estimation. Uncertainties of estimated parameters have conventionally been quantified via estimating standard errors associated with the parameter estimates. The standard error estimates provide a measure of sampling error over the repeated sampling in terms of the expected distance between estimates and the corresponding true parameter values. On the other hand, perturbation bounds of individual parameters can be considered as an alternative measure of parameter uncertainty in that they quantify a different aspect of uncertainty in the estimated parameters. That is, parameter influence measures uncertainties of the optimal parameter estimates in a given sample with respect to sensitivity to the change in model fit. As shown in the illustrative examples, the results of parameter influence studies, as an alternative measure of parameter uncertainty and in conjunction with the conventional measure of sampling variability, can provide important adjunct information regarding whether or not parameter estimates can be interpreted in a rigorous or literal sense when they are embedded in a well-fitting model.

We suggest that routine use of results of parameter influence analyses could be highly informative in applications of SEM. In this regard, we, the authors, consider developing a user-friendly computational mechanism (e.g., SAS-macro or R-package) that will automate the analysis of parameter influence in SEM.

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