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GLOBAL WELL-POSEDNESS AND LONG TIME BEHAVIORS OF CHEMOTAXIS-FLUID SYSTEM MODELING CORAL FERTILIZATION

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Abstract. We consider generalized models on coral broadcast spawning phenomena involving diffusion, advection, chemotaxis, and reactions when egg and sperm densities are different. We prove the global-in-time existence of the regular solutions of the models as well as their temporal decays in two and three dimensions. We also show that the total masses of egg and sperm density have positive lower bounds as time tends to infinity in three dimensions.

1. **Introduction.** In this paper, we study the interaction between reactions and chemotaxis in the mathematical model of the broadcast spawning phenomenon. Broadcast spawning is a fertilization strategy used by many sea animals, like sea urchins and corals(see [6, 7, 17]). In contrast with the numerical simulations based on the turbulent eddy diffusivity, the field measurements indicate that fertilization rates are often extremely as high as 90%(see [8, 9] and references therein) and it seems plausible that the chemotaxis emitted by the egg gametes play an important role in these high fertilization rates.

The simplest and most classical models of chemotaxis equations describing the collective motion of cells or bacterias have been introduced by Patlak[18] and Keller-Segel[13, 14]. The logistic source type of reaction term is also considered in many studies for the mathematical modeling of chemotaxis equations in a bounded domain with Neumann boundary conditions(see [19, 20, 21] and references therein).

In [15, 16], Kiselev and Ryzhik initiated mathematical study on the phenomenon of broadcast spawning when males and females release sperm and egg gametes into

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the surrounding fluid. There is experimental evidence that eggs release a chemical that attracts sperm. The authors in [15] and [16] in particular have proposed the following chemotaxis model regarding the fertilization process (assuming that the densities of egg and sperm gametes are identical):

$$\partial_t n + (u\cdot\nabla)n - \Delta n = \chi\nabla\cdot(n\nabla(\Delta)^{-1}n) - \epsilon n^q, \quad \text{in } (x,t)\in\mathbb{R}^d\times(0,T), \tag{1.1}$$

where n is the density of egg (sperm) gametes, u is the smooth divergence free sea fluid velocity, and χ denotes the positive chemotactic sensitivity constant. Also, $-\epsilon n^q$ denotes the reaction (fertilization) phenomenon. In [15], the global-in-time existence of the solution to (1.1) is presented under suitable conditions. Additionally, in $\mathbb{R}^2$, they showed that the total mass $m_0(t) = \int_{\mathbb{R}^2} n(x,t)dx$ approaches a positive constant whose lower bound is $C(\chi, n_0, u)$ as $t\to\infty$ when q is an integer larger than 2. They also provided that $C(\chi, n_0, u)\to 0$ as $\chi\to\infty$. This implies that if the chemotactic sensitivity increases, then more eggs can be fertilized. The critical case of $d = q = 2$ was studied in [16]; the total mass can go to zero with a reaction term only, but not faster than a logarithmic rate when the initial data is in the Schwartz class. If chemotaxis is present, the total mass is diminished in a power of $1/\chi$, which gives a faster decay rate than $1/\log t$ in a certain time scale. Recently, the existence and total mass behaviors have been studied in [1] when the chemical concentration is governed by the parabolic equation. Espejo and Suzuki [10] considered parabolic-parabolic Keller Segel equations with reaction term coupled with Stokes equations in $\mathbb{R}^2$. They obtained the global-in-time existence of solution.

Kiselev and Ryzhik[15] also presented the following model of sperm and egg densities

$$\begin{cases} \partial_t s + (u\cdot\nabla)s = \kappa_1\Delta s - (se)^{\frac{q}{2}}, & s(x,0) = s_0(x), \\ \partial_t e + (u\cdot\nabla)e = \kappa_2\Delta e - (se)^{\frac{q}{2}}, & e(x,0) = e_0(x). \end{cases} \tag{1.2}$$

Here, s and e denote the densities of sperm and egg gametes. From [15], it is obtained that if $q > \max\{\frac{d+2}{d}, 2\}$, then there exists an absolute positive constant μ_1 such that $\|s(\cdot,t)\|_{L^1(\mathbb{R}^d)} + \|e(\cdot,t)\|_{L^1(\mathbb{R}^d)} \geq \mu_1 > 0$ for all t.

In this paper we consider more general mathematical models by allowing that egg density can differ from sperm density in $\mathbb{R}^d$ ($d = 2,3$) with $q = 2$ considering the chemotaxis effect in the s equation in (1.2). Our first model reads as follows :

$$\begin{cases} \partial_t e + (u\cdot\nabla)e - \Delta e = -\epsilon(se), \\ \partial_t s + (u\cdot\nabla)s - \Delta s = \chi\nabla\cdot(s\nabla\Delta^{-1}e) - \epsilon(se), \end{cases} \quad \text{in } (x,t)\in\mathbb{R}^d\times(0,\infty), \tag{1.3}$$

where $e\geq 0$, $s\geq 0$, and u denote the density of egg gametes, sperm gametes and the divergence free sea velocity of sea fluid, respectively. In the above, χ and ϵ are positive constants representing chemotactic sensitivity and fertilization rate, respectively. We also assume that u is in $C^\infty(\mathbb{R}^{d+1})$ and div $u = 0$.

We will obtain the apriori estimates in section 2. Initial data are given by ($e_0(x)$, $s_0(x)$) with $e_0(x)$, $s_0(x)\geq 0$.

From now on, we denote $L^{q,p}_{t,x} = L^q(0,T;L^p(\mathbb{R}^d))$ and $L^p_{t,x} = L^p(0,T;L^p(\mathbb{R}^d))$ with any given time T in the context. We mostly omit the spatial domain $\mathbb{R}^d$ in $L^p(\mathbb{R}^d)$ if there is no ambiguity. We also denote a norm

$$\|f\|_{M_n} = \int_{\mathbb{R}^d}(|f(x)| + |\nabla f(x)|)(1+|x|^n)dx,$$

and Banach space $K_{m,n}$ defined by the norm $\|f\|_{K_{m,n}} = \|f\|_{M_n} + \|f\|_{H^m}$. We also denote a function space $X^T_{m,n} \equiv C([0,T]; K_{m,n})$ and $X^\infty_{m,n} \equiv C([0,\infty); K_{m,n})$. Let $m_s(t)$ and $m_e(t)$ denote the total mass of sperm and egg gametes, respectively : $m_s(t) = \int_{\mathbb{R}^d} s(x,t)dx$ and $m_e(t) = \int_{\mathbb{R}^d} e(x,t)dx$.

Our first main result is the global-in-time existence of smooth solutions to (1.3). We also obtain the positive lower bound of the total mass for 3-dimensional case and the decay estimates of $\|e\|_{L^p}$ and $\|s\|_{L^p}$. Compared to the case of e, the temporal decay of s is a bit tricky, due to the presence of the chemotactic effect, i.e. $\chi\nabla \cdot (s\nabla\Delta^{-1}e)$. It turns out that the reaction term $-\epsilon(se)$ in the egg equation, in particular in two dimensions, plays a crucial role in controlling the chemotactic term. See the argument around (2.16).

Theorem 1. *Let $d = 2, 3$. We assume the initial data $(e_0(x),\, s_0(x)) \in K_{m,n} \times K_{m,n}$ $(m \geq [\frac{d}{2}] + 1$ and $n \geq 1)$ and a given velocity field $u(x,t) \in C([0,\infty); H^m)$ satisfies* div u $= 0$.

- **(i):** *When $d = 2, 3$, there exists a unique solution $(e,\, s) \in X^\infty_{m,n} \times X^\infty_{m,n}$ to the system (1.3).*
- **(ii):** *When $d = 3$, we have $m_s(t) \geq C(s_0, e_0) > 0$ and $m_e(t) \geq C(\chi, \epsilon, s_0, e_0) > 0$.*
- **(iii):** *When $d = 2,\ 3$, we have the following temporal decay estimates*

$$\|e(t)\|_{L^p(\mathbb{R}^d)} \leq \frac{C}{t^{\frac{d}{2}(1-\frac{1}{p})}}, \quad p \in (1, \infty], \tag{1.4}$$

and

$$\|s(t)\|_{L^p(\mathbb{R}^d)} \leq \frac{C}{t^{\frac{d}{2}(1-\frac{1}{p})}}, \quad p \in (1, \infty). \tag{1.5}$$

Remark 1. In the proof of Theorem 1, the lower bound of the mass of the egg density, $C(\chi, \epsilon, s_0, e_0)$ approaches 0 as $\chi \to \infty$. It implies that if the chemotactic sensitivity is dominant, then total mass of egg density may vanish, hence perfect fertilization may occur. On the other hand, in Theorem 1 (ii), the lower bound of the mass of the sperm is independent of χ in the proof of Theorem 1. From the fact that the difference of the total mass of sperm and egg is conserved, that is,

$$m_e(t) - m_s(t) = m_e(0) - m_s(0),$$

the lower bound of the mass of the egg is independent of χ if $m_e(0) \geq m_s(0)$. Therefore, in 3D, even if the chemotactic sensitivity $\chi \to \infty$, the egg density may vanish only if $m_s(0) > m_e(0)$.

Next, we consider the following egg-sperm chemotaxis model coupled with the incompressible fluid equations(Navier-Stokes or Stokes equations):

$$\begin{cases} \partial_t e + (u\cdot\nabla)e - \Delta e = -\epsilon(se), \\ \partial_t s + (u\cdot\nabla)s - \Delta s = -\chi\nabla\cdot(s\nabla c) - \epsilon(se), \\ \partial_t c + (u\cdot\nabla)c - \Delta c = e, \\ \partial_t u + \kappa(u\cdot\nabla)u - \Delta u + \nabla p = -(s+e)\nabla\phi, \\ \text{div u} = 0, \end{cases} \quad \text{in } \ (x,t) \in \mathbb{R}^d \times (0,\, \infty), \tag{1.6}$$

where $e,\ s,\ c \geq 0$, and u denote the density of egg gametes, sperm gametes, chemicals and the divergence free sea velocity of sea fluid governed by the fluid equations, respectively. ϕ denotes potential function, which is given by gravitational force, centrifugal force, etc. We consider the cases $\kappa = 1$(Navier-Stokes equations) when

$d = 2$ and $\kappa = 0$ (Stokes system) when $d = 3$ to avoid additional smallness assumption on the initial velocity of the fluid for the well-posedness. Chemotaxis equation coupled with the fluid equations have been considered in many studies, especially for describing the dynamics of Bacillus Subtilis in the water droplet. For recent mathematical developments in the model, please refer to [2, 4, 5, 10] and references therein.

For the system (1.6) our main aim is to establish global well-posedness of solutions. To be more precise, in two dimensions, we prove that unique regular solutions exist globally in time for large initial data, provided that the data are regular enough. On the other hand, for three dimensional case, global well-posedness can be obtained under smallness condition of L^1-norm of intial data of s, i.e. $\|s_0\|_{L^1}$ (more specifically, it suffices to assume that $\chi^2\|s_0\|^2_{L^1_x}\|\nabla\phi\|^2_{L^\infty_{x,t}}$ is small). It is worth mentioning that L^1-norm of s_0 is a *super-critical* qunatity in 3D under the scaling invariance (3.1) ($L^{3/2}$-norm of s_0 is indeed scaling invariant in 3D). In this sense, our result is beyond scaling invariance but we do not know if the smallness assumption can be removed or not. Now we are ready to state our second result, where temporal decays of solutions are also shown as well.

Theorem 2. *Let $d = 2, 3$. We assume the initial data $(e_0(x), s_0(x), c_0(x), u_0(x)) \in K_{m,n} \times K_{m,n} \times K_{m,n} \times H^m$ ($m \ge [\frac{d}{2}] + 1$ and $n \ge 1$) with $\operatorname{div} u_0 = 0$. We also assume that $s_0, e_0 \in L^1(\mathbb{R}^d)$ and $\|\nabla^l\phi\|_{L^\infty} < \infty$ for $1 \le l \le m$.*

- **(i):** *When $d = 2$ and $\kappa = 1$, there exist unique solutions $(e, s, c, u) \in X^\infty_{m,n} \times X^\infty_{m,n} \times X^\infty_{m,n} \times C([0, \infty); H^m)$ to the equations (1.6).*
- **(ii):** *When $d = 3$ and $\kappa = 0$, assuming $\chi^2\|s_0\|^2_{L^1_x}\|\nabla\phi\|^2_{L^\infty_{x,t}}$ to be sufficiently small, there exist unique solutions $(e, s, c, u) \in X^\infty_{m,n} \times X^\infty_{m,n} \times X^\infty_{m,n} \times C([0, \infty); H^m)$ to the equations (1.6). Moreover, we have $m_e(t), m_s(t) \ge C(\chi, \epsilon, s_0, \nabla e_0) > 0$. This lower bound also satisfies $C(\chi, \epsilon, s_0, \nabla e_0) \to 0$ as $\chi \to \infty$.*
- **(iii):** *We have the following decay estimates of the solutions in the above*

$$\|e(t)\|_{L^p(\mathbb{R}^d)} \le \frac{C}{t^{\left(1-\frac{1}{p}\right)\frac{d}{2}}}, \qquad \text{when } 1 < p \le \infty, \qquad \text{if } d = 2, 3, \tag{1.7}$$

$$\|c(t)\|_{L^q} \le Ct^{-\frac{3}{2}\left(\frac{1}{3}-\frac{1}{q}\right)}, \qquad 3 < q < \infty. \qquad \text{if } d = 3, \tag{1.8}$$

Furthermore, when $d = 2$ and ω is the vorticity of u, if we assume that $\|s_0\|_{L^1(\mathbb{R}^2)} + \|e_0\|_{L^1(\mathbb{R}^2)} + \|\nabla c_0\|_{L^2(\mathbb{R}^2)} + \|\omega_0\|_{L^1(\mathbb{R}^2)} \le \epsilon_1$, then we have

$$\|s(t)\|_{L^p(\mathbb{R}^2)} \le \frac{C\epsilon_1}{t^{\left(1-\frac{1}{p}\right)}}, \quad \|\nabla c(t)\|_{L^\infty} \le \frac{C\epsilon_1}{t^{\frac{1}{2}}}, \quad \|\omega(t)\|_{L^\gamma(\mathbb{R}^2)} \le \frac{C\epsilon_1}{t^{1-\frac{1}{\gamma}}}, \tag{1.9}$$

where $1 < p \le \infty$ and $1 < \gamma < \infty$.

Remark 2. Formally integrating both sides of (1.3) (or (1.6)) over $\mathbb{R}^d$ and subtracting the first equation from the second equation, we deduce that

$$\|s\|_{L^1(\mathbb{R}^d)}(t) - \|e\|_{L^1(\mathbb{R}^d)}(t) = \|s_0\|_{L^1(\mathbb{R}^d)} - \|e_0\|_{L^1(\mathbb{R}^d)}, \text{ for all } t > 0. \tag{1.10}$$

Hence the difference of the total mass of sperm and egg cells is conserved.
On the other hand, in the $2D$ case, Kiselev and Ryzhik [16, Theorem 1.1] showed that if $\rho_0 \in \mathcal{S}$ (Schwartz class) and ρ satisfy

$$\partial_t\rho + (u \cdot \nabla)\rho - \Delta\rho = -\epsilon\rho^2, \tag{1.11}$$

then, for any $\sigma > 0$ and $t \geq 1$, there exists a constant $C(\sigma, \rho_0) > 0$ such that

$$\|\rho(\cdot,t)\|_{L^1(\mathbb{R}^2)} \leq \frac{C(\sigma,\rho_0)}{(1+\epsilon\log t)^{1-\sigma}}.$$

Note that (1.11) corresponds to (1.1) when the chemotaxis is absent and $q = 2$.

If $s(x,t) > e(x,t)$ holds true for all $(x,t) \in \mathbb{R}^2 \times (0,\infty)$, then $(1.3)_1$ and $(1.6)_1$ are reduced to

$$\partial_t e + (u\cdot\nabla)e - \Delta e = -\epsilon(se) \leq -\epsilon e^2.$$

In this case, applying Kiselev and Ryzhik's result for the solution to the above (assuming u is sufficiently regular), we obtain

$$\|e(\cdot,t)\|_{L^1(\mathbb{R}^2)} \leq \frac{C}{(1+\epsilon\log t)^{1-\sigma}} \to 0 \text{ as } t\to\infty.$$

Taking into account (1.10), we infer that, in $2D$, an egg cell can be perfectly fertilized if the initial sperm cell density is much larger than that of the egg cell.

Remark 3. After completing this work, we are informed that Espejo and Winkler [11] obtained classical solvability and stabilization in a chemotaxis-Navier-Stokes system modeling coral fertilization in a smooth bounded two-dimensional domain. Our result has an essential difference from their work in the asymptotic behaviour in the whole domain.

The rest of this paper is as follows : In Section 2, we provide the proofs for the global-in-time existence of the smooth solution to (1.3) and also provide the proofs of the positive lower bounds of the total mass and decay estimates. In Section 3, we consider the global well-posedness of the system (1.6) and provide the proof of Theorem 2 and especially consider the decay properties of the solutions to (1.6) with the small initial data.

2. Global well-posedness and asymptotic behavior of total mass.

In this section, we provide some apriori estimates of solutions to (1.3). Also we provide the proof of global well-posedness of (1.3) (Theorem 1 (i)) and lower bound of the total mass(Theorem 1 (ii)). Using the standard method(contraction mapping principle), the local-in-time existence of regular solution can be shown, which reads as follows:

Proposition 1. *Let $d = 2,3$ and n be a positive integer and initial data (e_0, s_0) as in Theorem 1 belong to $K_{m,n} \times K_{m,n}$ $(m > \left[\frac{d}{2}\right] + 1)$. Suppose that $u \in C^\infty \cap L^\infty(\mathbb{R}^d \times [0,\infty))$ is divergence free and any of its spatial derivatives is uniformly bounded for all $(x,t) \in \mathbb{R}^d \times (0,\infty)$. Then there exists a maximal time of existence T_*, such that for $t < T_*$, a pair of unique regular solution (e,s) of (1.3) exists and satisfies*

$$(e,s) \in X^t_{m,n} \times X^t_{m,n}.$$

The proof of the proposition is quite standard, hence we omit it. It can be found in [15, Theorem 5.4].

In this section and throughout the paper we use the maximal $L^p - L^q$ estimates or maximal regularity estimates for the heat equations: let $1 < p, q < \infty$. If v is the solution of the heat equation

$$\partial_t v - \Delta v = f(x,t), \quad v(\cdot,0) = v_0$$

for the given function $f(x,t) \in L^q_t L^p_x(0,\infty;\mathbb{R}^d)$ and $v_0 \in W^{2,p}(\mathbb{R}^d)$, there exist a constant $C > 0$ (see [12]) such that

$$\int_0^T \|\partial_t v\|^q_{L^p_x} dt + \int_0^T \|\Delta v\|^q_{L^p_x} dt \le C \left(\|v_0\|^q_{W^{2,p}} + \int_0^T \|f\|^q_{L^p} dt \right). \tag{2.1}$$

We often denote $(0,T) \times \mathbb{R}^d$ by Q_T and $\|v\|_{L^q_t L^p_x(0,T;\mathbb{R}^d)}$ by $L^{q,p}_{t,x}(Q_T)$. When $p = q$, we simply write $L^p(Q_T)$. Also we denote $\sum_{|\alpha|\le m} \|D^\alpha v\|_{L^q_t L^p_x(0,T;\mathbb{R}^d)}$ by $L^q_t W^{m,p}_x(Q_T)$ (or $L^q_t H^m_x$ if $p = 2$).

In what follows, we derive some a priori estimates of (e, s) to prove Theorem 1.

• (L^1 estimates) First, we have the following decreasing properties for the total mass

$$\frac{d}{dt} \int_{\mathbb{R}^d} e(x,t) dx + \epsilon \int_{\mathbb{R}^d} se \; dx = 0,$$

and

$$\frac{d}{dt} \int_{\mathbb{R}^d} s(x,t) dx + \epsilon \int_{\mathbb{R}^d} se \; dx = 0.$$

Integrating with respect to time, we have

$$\sup_{0\le t\le T} \int_{\mathbb{R}^d} e(x,t) dx + \epsilon \int_0^T \int_{\mathbb{R}^d} (se) dx dt \le \|e_0\|_{L^1},$$

and

$$\sup_{0\le t\le T} \int_{\mathbb{R}^d} s(x,t) dx + \epsilon \int_0^T \int_{\mathbb{R}^d} (se) dx dt \le \|s_0\|_{L^1}.$$

• (L^p-estimates) By multiplying e^{p-1} both sides of e equation, and integrating over $\mathbb{R}^d$, we obtain that

$$\sup_{0\le t\le T} \int_{\mathbb{R}^d} e^p(x,t) dx + \frac{4(p-1)}{p} \int_0^T \|\nabla e^{p/2}\|^2_{L^2} dt + \epsilon p \int_0^T \int_{\mathbb{R}^d} se^p dx dt \le \|e_0\|^p_{L^p}.$$

Moreover, as $p \to \infty$, we have $\|e(t)\|_{L^\infty} \le \|e_0\|_{L^\infty}$.

For the sperm density, we have the following

$$\frac{1}{p}\frac{d}{dt}\|s(t)\|^p_{L^p} + \frac{4(p-1)}{p^2}\|\nabla s^{\frac{p}{2}}\|^2_{L^2} + \epsilon \int_{\mathbb{R}^d} es^p dx = \frac{p-1}{p}\chi \int_{\mathbb{R}^d} es^p dx. \tag{2.2}$$

We note that if $\epsilon \ge \chi$, then the righthand side can be absorbed to the left. Hence it is direct that

$$s \in L^\infty(0,\infty; L^p) \text{ and } \nabla s^{p/2} \in L^2(0,\infty; L^2) \text{ for } p \in (1,\infty).$$

It also holds that $s \in L^\infty(0,\infty; L^\infty)$.

If $0 < \epsilon < \chi$, then we have

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|s(t)\|^p_{L^p} + \frac{4(p-1)}{p^2}\|\nabla s^{\frac{p}{2}}\|^2_{L^2} + \epsilon \int_{\mathbb{R}^d} es^p dx \\ &= \frac{p-1}{p}\chi \int_{\mathbb{R}^d} es^p dx \le \frac{p-1}{p}\chi \|e\|_{L^\infty}\|s\|^p_{L^p}. \end{aligned}$$

Hence we deduce that

$$s \in L^\infty(0,T; L^p) \text{ and } \nabla s^{p/2} \in L^2(0,T; L^2) \text{ for any } p \in (1,\infty) \text{ and } T > 0.$$

• (H^1 estimates) Next, we consider H^1 estimates of s :

By use of the maximal regularity of heat equation, we easily deduce that

$$\|\partial_t e\|_{L^2(Q_T)} + \|\Delta e\|_{L^2(Q_T)} \le C\|e_0\|_{H^2} + C(\|\nabla e\|_{L^2(Q_T)} + \|se\|_{L^2(Q_T)}) < \infty.$$

Therefore, together with (L^p-estimates) we obtain

$$\partial_t e \in L^2_{t,x}, \text{ and } e \in L^2(0,T;H^2).$$

Taking L^2 inner product of $-\Delta s$ with s equation, we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla s\|_{L^2}^2 + \|\Delta s\|_{L^2}^2 + \epsilon \int_{\mathbb{R}^d} |\nabla s|^2 e dx \\
&\le \|\nabla u\|_{L^\infty}\|\nabla s\|_{L^2}^2 + \epsilon\|\nabla s\|_{L^2}\|\nabla e\|_{L^2}\|s\|_{L^\infty} \\
&\quad +\chi\|s\|_{L^\infty}\|\nabla e\|_{L^2}\|\nabla s\|_{L^2} + C\chi\|\nabla s\|_{L^3}^2\|e\|_{L^3} \\
&\le C(\|\nabla u\|_{L^\infty} + \|s\|_{L^\infty}^2 + 1)\|\nabla s\|_{L^2}^2 + \delta\|\Delta s\|_{L^2}^2 + C\|\nabla e\|_{L^2}^2.
\end{aligned}$$

In the above, we used the followings by the integration by parts and the Calderon-Zygmund type inequality

$$\begin{aligned}
-\chi\int_{\mathbb{R}^d} \nabla\cdot(s\nabla\Delta^{-1}e)\Delta s dx &= \frac{\chi}{2}\int_{\mathbb{R}^d} e|\nabla s|^2 + \chi\int_{\mathbb{R}^d} s\nabla s\cdot\nabla e dx \\
&\quad +\chi\int_{\mathbb{R}^d} (\nabla s)^T(\nabla^2\Delta^{-1}e)(\nabla s)dx
\end{aligned}$$

and

$$\chi\left|\int_{\mathbb{R}^d} (\nabla s)^T(\nabla^2\Delta^{-1}e)(\nabla s)dx\right| \le \chi\|\nabla^2\Delta^{-1}e\|_{L^3}\|\nabla s\|_{L^3}^2 \le C\chi\|e\|_{L^3}\|\nabla s\|_{L^3}^2.$$

We note that δ can be chosen as a sufficiently small positive constant which can be absorbed in the lefthand side.

Using the Gronwall type inequality, we have for any $T > 0$.

$$\nabla s \in L^{2,\infty}_{x,t}(Q_T) \cap H^1_x L^2_t(Q_T).$$

- (H^2 estimates) For the higher norm estimates, we proceed as follows.

We estimate similarly with the above

$$\begin{aligned}
\|\partial_t\nabla e\|_{L^2(Q_T)} + \|\nabla\Delta e\|_{L^2(Q_T)} &\le C\|e_0\|_{H^2} + C\|\nabla(u\cdot\nabla e)\|_{L^2(Q_T)} + C\|\nabla(se)\|_{L^2(Q_T)} \\
&\le C\|e_0\|_{H^2} + C\|\nabla u\|_{L^\infty(Q_T)}\|\nabla e\|_{L^2(Q_T)} + \|u\|_{L^\infty(Q_T)}\|\nabla^2 e\|_{L^2(Q_T)} \\
&\quad +C(\|\nabla s\|_{L^2(Q_T)}\|e\|_{L^\infty(Q_T)} + \|\nabla e\|_{L^2(Q_T)}\|s\|_{L^\infty(Q_T)}) < \infty.
\end{aligned}$$

For the estimates of solution s, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta s\|_{L^2}^2 + \|\nabla\Delta s\|_{L^2}^2 \\
&\le \|\nabla(u\cdot\nabla s)\|_{L^2}\|\nabla\Delta s\|_{L^2} + \epsilon\|\nabla(se)\|_{L^2}\|\nabla\Delta s\|_{L^2} \\
&\quad +C\|e\|_{L^\infty}\|\Delta s\|_{L^2}^2 + C\|e\|_{L^6}\|\nabla s\|_{L^3}\|\nabla\Delta s\|_{L^2} + C\|s\|_{L^\infty}\|\nabla e\|_{L^2}\|\nabla\Delta s\|_{L^2}.
\end{aligned}$$

Using Young's inequality, the righthand side high order term $\|\nabla\Delta s\|_{L^2}^2$ can be absorbed in the lefthand side. By integrating with respect to time, we find

$$\|\Delta s\|^2_{L^{\infty,2}_{t,x}(Q_T)} + \|\nabla\Delta s\|^2_{L^{2,2}_{t,x}(Q_T)} \le \|\Delta s_0\|_{L^2}^2 + C\|\Delta s\|^2_{L^{2,2}_{t,x}(Q_T)} + C\|\nabla e\|^2_{L^2(Q_T)} < \infty.$$

- (H^3 estimate) Finally, we can obtain the following H^3 estimates for s.

By the use of maximal regularity of the heat equation, we have

$$\begin{aligned}
\|\partial_t e\|_{L^2_t H^2_x} + \|\Delta e\|_{L^2_t H^2_x} &\le C\left(\|e_0\|_{H^3} + \|(u\cdot\nabla)e\|_{L^2_t H^2_x} + \|se\|_{L^2_t H^2_x}\right) \\
&\le C\left(\|e_0\|_{H^3} + \|u\|_{L^\infty_t H^2_x}\|\nabla e\|_{L^2_t H^2_x} + \|s\|_{L^\infty_t H^2_x}\|e\|_{L^2_t H^2_x}\right) < \infty,
\end{aligned}$$

and

$$\|\partial_t e\|_{L^2_t H^2_x} + \|\Delta e\|_{L^2_t H^2_x}$$

$$\leq C\left(\|e_0\|_{H^3}+\|u\|_{L_t^\infty(H_x^3)}\|\nabla e\|_{L_t^2(H_x^3)}+\|s\|_{L_t^2(H_x^3)}\|e\|_{L_t^\infty(H_x^3)}\right)<\infty.$$

Similarly to the previous H^2 estimates, we obtain

$$\|\nabla\Delta s\|^2_{L^{2,\infty}_{x,t}(Q_T)}+\|\Delta^2 s\|^2_{L^{2,2}_{x,t}(Q_T)}<\infty.$$

We are ready to prove Theorem 1.

Proof of Theorem 1 (i) From the previous apriori estimates, only remaining estimates are about the estimates in M_n. As in [15, Theorem 5.4.], the only nontrivial part is that the contraction constant depends on H^m norm of (s_0,e_0) and not on M_n norm of (s_0,e_0). In a different way, we provide the following direct estimates for any integer $k\geq 1$ inductively :

$$\begin{aligned}\frac{d}{dt}\||x|^k e\|^2_{L^2}+\||x|^k\nabla e\|^2_{L^2}+\epsilon\int se^2|x|^{2k}dx &\leq C\left(\||x|^{k-1}e\|^2_{L^2}+\||x|^{k-\frac12}e\|^2_{L^2}+1\right),\\ \frac{d}{dt}\||x|^k s\|^2_{L^2}+\||x|^k\nabla s\|^2_{L^2}+\epsilon\int es^2|x|^{2k}dx&\\ \leq C\left(\||x|^{k-1}s\|^2_{L^2}+\||x|^{k-1}e\|^2_{L^2}\||x|^{k-\frac12}s\|^2_{L^2}+1\right).&\end{aligned}\tag{2.3}$$

The above estimates are rather standard, and hence we provide the sketch of the estimates $-\int_{\mathbb{R}^d}|x|^{2k}s\nabla\cdot(s\nabla\Delta^{-1}e)dx$. We have

$$-\int_{\mathbb{R}^d}|x|^{2k}s\nabla\cdot(s\nabla\Delta^{-1}e)dx=\frac12\int_{\mathbb{R}^d}\nabla|x|^{2k}\cdot\nabla\Delta^{-1}es^2dx-\frac12\int_{\mathbb{R}^d}|x|^{2k}s^2edx.$$

The first term can be estimated as follows :

$$\begin{aligned}\frac12\int_{\mathbb{R}^d}\nabla|x|^{2k}\cdot\nabla\Delta^{-1}es^2dx&\leq C\||x|^{k-\frac12}s\|^2_{L^2}\|\nabla\Delta^{-1}e\|_{L^\infty}\\&\leq C\||x|^{k-\frac12}s\|^2_{L^2}\|\nabla\Delta^{-1}e\|_{W^{1,r}},\end{aligned}$$

where $r>d$. The term $\|\nabla\Delta^{-1}e\|_{W^{1,r}}$ can be bounded by $\|e\|_{L^p\cap L^r}$ for some p by the Hardy-Littlewood-Sobolev inequality.

By using Young's inequality and Gronwall's inequality, we can have for any $T>0$,

$$\||x|^k(s,e)\|_{L^{\infty,2}_{t,x}(Q_T)}+\||x|^k\nabla(s,e)\|_{L^2_{t,x}(Q_T)}<\infty.$$

Similarly, we can have $\||x|^k\nabla(s,e)\|_{L^{\infty,2}_{t,x}(Q_T)}<\infty$.

This together with the previous L^1-estimates proves for any $n>0$ and $T>0$ $\|(s,e)\|_{K_{m,n}}<\infty$. This completes the proof of Theorem 1 (i).

Proof of Theorem 1 (ii) For this regular solution obtained in Theorem 1 (i), we can investigate the asymptotic behaviors of the total mass $m_s(t)$ and $m_e(t)$, especially in $\mathbb{R}^3$.

First, we show $\|e(t)\|_{L^\infty}\to 0$ as $t\to\infty$.

To be concrete, we will show that

$$\|e(t)\|_{L^\infty}\leq\frac{C}{t^{\frac d2}}.$$

We have

$$\frac1p\frac{d}{dt}\int_{\mathbb{R}^d}|e(t)|^pdx+\frac{4(p-1)}{p^2}\int_{\mathbb{R}^d}|\nabla e^{\frac p2}|^2dx\leq 0.$$

Reminding that $\|f\|_{L^2(\mathbb{R}^d)} \le C\|f\|_{L^1(\mathbb{R}^d)}^{\frac{2}{d+2}}\|\nabla f\|_{L^2(\mathbb{R}^d)}^{\frac{d}{d+2}}$, we note that

$$C\|e\|_{L^p(\mathbb{R}^d)}^{\frac{p(d+2)}{d}}\|e\|_{L^{\frac{p}{2}}(\mathbb{R}^d)}^{-\frac{2p}{d}} \le \|\nabla e^{\frac{p}{2}}\|_{L^2(\mathbb{R}^d)}^2.$$

We have

$$\frac{d}{dt}\int_{\mathbb{R}^d}|e(t)|^p dx + C\|e\|_{L^p(\mathbb{R}^d)}^{\frac{p(d+2)}{d}}\|e\|_{L^{\frac{p}{2}}(\mathbb{R}^d)}^{-\frac{2p}{d}} \le 0. \tag{2.4}$$

For convenience, we denote $y_p(t) := \|e(t)\|_{L^p(\mathbb{R}^d)}$. We show that for sufficiently large $t > 1$, $p = 2^k$ with $k = 1, 2, \cdots$ and a uniform constant $C > 0$

$$y_{2^k}(t) \le \frac{C}{t^{\left(1-\frac{1}{2^k}\right)\frac{d}{2}}}.$$

Indeed, for $k = 1$, we have

$$\frac{d}{dt}y_2^2(t) + Cy_2^{\frac{2(d+2)}{d}} \le 0.$$

Solving the above differential inequality, we have

$$y_2(t) \le Ct^{-\frac{d}{4}}.$$

Suppose that the above is true up to $k = m-1$ with $m > 1$. Then we obtain

$$\frac{d}{dt}y_{2^m}^{2^m}(t) + \frac{C}{C_{m-1}^{\frac{2^{m+1}}{d}}}t^{2^m-2}y_{2^m}^{\frac{d+2}{d}2^m}$$

$$\le \frac{d}{dt}y_{2^m}^{2^m}(t) + Cy_{2^{m-1}}^{-\frac{2}{d}2^m}y_{2^m}^{\frac{d+2}{d}2^m} \le 0.$$

Solving the above inequality, we have

$$y_{2^m}(t) \le C_{m-1}\left(\frac{d}{2C}\right)^{\frac{d}{2^{m+1}}}t^{\left(1-\frac{1}{2^m}\right)\frac{d}{2}}.$$

Then $C_m := C_{m-1}\left(\frac{d}{2C}\right)^{\frac{d}{2^{m+1}}}$ is uniformly bounded for m and we obtain

$$\|e(t)\|_{L^\infty} \le \frac{C}{t^{\frac{d}{2}}},$$

by letting $m \to \infty$.

Then we have (1.7) by the interpolation

$$y_p(t) \le \frac{C}{t^{\left(1-\frac{1}{p}\right)\frac{d}{2}}}.$$

Letting $p \to \infty$, we have

$$\|e(t)\|_{L^\infty} \le \frac{C}{t^{\frac{d}{2}}}.$$

• (Total mass behavior of $m_s(t)$) It is ready to prove the lower bound of mass of the sperm cell density. Consider the case that $d = 3$. We have the differential inequality (t_0 is chosen to be larger than 1).

$$\frac{d}{dt}\int_{\mathbb{R}^3}s(t)dx + \frac{C}{t^{\frac{3}{2}}}\int_{\mathbb{R}^3}s(t)dx \ge 0, \text{ for } t \ge t_0.$$

Then integrating with respect to time from t_0 until t and setting $y = \int_{\mathbb{R}^3}s(t)dx$, we have

$$\frac{dy}{y} \ge -\frac{Cdt}{t^{\frac{3}{2}}},$$

and thus,

$$y(t) \geq y(t_0) \exp\left(2C\left(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t_0}}\right)\right). \tag{2.5}$$

Since $t \geq t_0$, we have

$$m_s(t) \geq \frac{1}{2} m_s(t_0).$$

• (L^2 decay estimate of $s(t)$) To prove the lower bound of the mass for the egg cell density, we should obtain L^2 decay estimates for the sperm cell density.
Similarly, we obtain

$$\frac{1}{2}\frac{d}{dt}\|s\|^2_{L^2(\mathbb{R}^3)} + \|\nabla s(t)\|^2_{L^2(\mathbb{R}^3)} + \epsilon \int_{\mathbb{R}^3} es^2 dx = \chi \int_{\mathbb{R}^3} es^2 \, dx.$$

The right hand side of the above equality can be estimated by Hölder's and Sobolev's inequality as follows :

$$\chi \int_{\mathbb{R}^3} es^2(t) \, dx \leq \chi \|e(t)\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \|s(t)\|^2_{L^6(\mathbb{R}^3)} \leq C\chi \|e(t)\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \|\nabla s(t)\|^2_{L^2(\mathbb{R}^3)}.$$

Since $\|e(t)\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \leq \frac{C}{t^{\frac{1}{2}}}$, we choose t_0 so large that $\frac{C\chi}{t_0^{\frac{1}{2}}} < \frac{1}{2}$. Hence we have

$$\frac{d}{dt}\|s(t)\|^2_{L^2(\mathbb{R}^3)} + \|\nabla s(t)\|^2_{L^2(\mathbb{R}^3)} \leq 0.$$

We infer that $\|s(t)\|_{L^2} \leq \frac{C}{t^{\frac{3}{4}}}$. By the interpolation inequality, we have

$$\|s(t)\|_{L^p} \leq \frac{C}{t^{\frac{3}{2}(1-\frac{1}{p})}} \text{ for } 1 < p \leq 2.$$

• (L^p decay estimate for s in $3D$) We recall that the solution e to $(1.3)_2$ satisfies the equation

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3} |e|^p + \frac{4(p-1)}{p^2}\int_{\mathbb{R}^3} |\nabla e^{\frac{p}{2}}|^2 + \epsilon \int_{\mathbb{R}^3} e^p s = 0. \tag{2.6}$$

Multiplying a large constant M on both sides of (2.6) (M will be specified later and depend on p), we have

$$\frac{M}{p}\frac{d}{dt}\int_{\mathbb{R}^3} |e|^p + \frac{4M(p-1)}{p^2}\int_{\mathbb{R}^3} |\nabla e^{\frac{p}{2}}|^2 + M\epsilon \int_{\mathbb{R}^3} e^p s = 0. \tag{2.7}$$

Note first that the following interpolation inequality holds (see [15])

$$\|s\|^{p+1}_{L^{p+1}} \leq C\|s\|_{L^{\frac{3}{2}}} \|\nabla s^{\frac{p}{2}}\|^2_{L^2},$$

where C is uniformly bounded for all $p \in (1, \infty)$. We compute

$$\frac{p-1}{p}\chi \int_{\mathbb{R}^3} s^p e \leq \frac{p-1}{p}\chi \left(\int_{\mathbb{R}^3} s^{\frac{p^2-1}{p}\cdot\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^3} e^p s\right)^{\frac{1}{p}}$$

$$= \frac{p-1}{p}\chi \left(\int_{\mathbb{R}^3} s^{p+1}\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^2} e^p s\right)^{\frac{1}{p}} \leq C(M\epsilon)^{-\frac{1}{p-1}} \chi^{\frac{p}{p-1}} \int_{\mathbb{R}^3} s^{p+1} + \frac{M\epsilon}{2}\int_{\mathbb{R}^3} e^p s$$

$$\leq C(M\epsilon)^{-\frac{1}{p-1}} \|s\|_{L^{\frac{3}{2}}} \|\nabla s^{\frac{p}{2}}\|^2_{L^2} + \frac{M\epsilon}{2}\int_{\mathbb{R}^3} e^p s.$$

The solution $s(t)$ satisfies that

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3}|s|^p+\frac{4(p-1)}{p^2}\int_{\mathbb{R}^3}|\nabla s^{\frac{p}{2}}|^2+\epsilon\int_{\mathbb{R}^3}s^pe \le C(M\epsilon)^{-\frac{1}{p-1}}\|s\|_{L^{\frac{3}{2}}}\|\nabla s^{\frac{p}{2}}\|_{L^2}^2+\frac{M\epsilon}{2}\int_{\mathbb{R}^3}e^ps. \tag{2.8}$$

Adding (2.7) and (2.8), we have

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3}|s|^p+\frac{M}{p}\frac{d}{dt}\int_{\mathbb{R}^3}|e|^p+\frac{4(p-1)}{p^2}\int_{\mathbb{R}^3}|\nabla s^{\frac{p}{2}}|^2+\frac{4M(p-1)}{p^2}\int_{\mathbb{R}^3}|\nabla e^{\frac{p}{2}}|^2 +\epsilon\int_{\mathbb{R}^3}s^pe+\frac{M\epsilon}{2}\int_{\mathbb{R}^3}e^ps\le C(M\epsilon)^{-\frac{1}{p-1}}\|s\|_{L^{\frac{3}{2}}}\|\nabla s^{\frac{p}{2}}\|_{L^2}^2. \tag{2.9}$$

Taking $M=\frac{1}{\epsilon}\left(\frac{Cp^2\|s_0\|_{L^{\frac{3}{2}}}}{2(p-1)}\right)^{p-1}$, we have

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3}|s|^p+\frac{M}{p}\frac{d}{dt}\int_{\mathbb{R}^3}|e|^p+\frac{2(p-1)}{p^2}\int_{\mathbb{R}^3}|\nabla s^{\frac{p}{2}}|^2+\frac{4M(p-1)}{p^2}\int_{\mathbb{R}^3}|\nabla e^{\frac{p}{2}}|^2 +\epsilon\int_{\mathbb{R}^3}s^pe+\frac{M\epsilon}{2}\int_{\mathbb{R}^3}e^ps\le 0.$$

This gives the decay estimate

$$\|s(t)\|_{L^p}\le\frac{C_p}{t^{\frac{3}{2}(1-\frac{1}{p})}}\quad\text{for } p\in(1,\infty). \tag{2.10}$$

- (L^∞ decay estimates) We prove that $\|\nabla s\|_{L^\infty_tL^p_x}<C$ for any $p\in(3,\infty)$ and $\|s(t)\|_{L^\infty}\le Ct^{-\frac{3}{2}+\epsilon}$ for any small $\epsilon>0$ by the interpolation $\|s(t)\|_{L^\infty_x}\le C\|s(t)\|_{L^q_x}^{\theta}\|\nabla s(t)\|_{L^p_x}^{1-\theta}$, where $p>3$ and θ satisfies $\frac{3}{q}\theta=\frac{p-3}{p}(1-\theta)$.

To estimate $\|\nabla s\|_{L^{\infty,p}_{t,x}}$, we perform several parabolic regularity estimates for the solution of the heat equation in the following lemmas.

Lemma 3. *Let v be a solution of the equation*

$$v_t-\Delta v=f,\qquad v(0,x)=0.$$

If $f\in L^q_{t,x}(\mathbb{R}^3\times[0,\infty))\cap L^r_{t,x}(\mathbb{R}^3\times[0,\infty))$ with $q>\frac{5p}{5+p}$ and $1<r<\frac{5p}{5+p}$ for $p\in(3,\infty)$, then we have

$$\|\nabla v\|_{L^{\infty,p}_{t,x}}<C_p\left(\|f\|_{L^q_{t,x}}+\|f\|_{L^r_{t,x}}\right). \tag{2.11}$$

Also, if $f\in L^{p,\alpha}_{t,x}(\mathbb{R}^3\times[0,\infty))\cap L^{\beta}_{t,x}(\mathbb{R}^3\times[0,\infty))$ with $\frac{1}{\alpha}<\frac{1}{p}+\frac{1}{3}$ and $1<\beta<\frac{3}{2}$, then we have

$$\|\nabla v\|_{L^p_{t,x}}<C_p\left(\|f\|_{L^{p,\alpha}_{t,x}}+\|f\|_{L^{\beta}_{t,x}}\right). \tag{2.12}$$

Proof. Note that we have $\nabla v=\int_0^t\nabla\Gamma(t-\tau)*f(\tau)d\tau$, where $\Gamma(x,t)=\frac{1}{(4\pi t)^{\frac{3}{2}}}e^{-\frac{|x|^2}{4t}}$ is a 3-dimensional heat kernel. Without loss of generality, we may assume $t>1$. Then we have

$$\|\nabla v\|_{L^p_x}\le\int_0^t\|\nabla\Gamma(t-\tau)*f(\tau)\|_{L^p}d\tau$$

$$=\int_0^{t-1}\|\nabla\Gamma(t-\tau)*f(\tau)\|_{L^p}d\tau+\int_{t-1}^t\|\nabla\Gamma(t-\tau)*f(\tau)\|_{L^p}d\tau:=I_1+I_2.$$

I_1 and I_2 can be estimated as follows:

$$\begin{aligned}
I_1 &\le C\int_0^{t-1}\|\nabla\Gamma(t-\tau)\|_{L_x^{r'}}\|f(\tau)\|_{L_x^r}d\tau \quad (\text{ where } \frac{1}{r'}=1+\frac{1}{p}-\frac{1}{r})\\
&\le C\int_0^{t-1}(t-\tau)^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{p})-\frac{1}{2}}\|f(\tau)\|_{L_x^r}d\tau\\
&\le C\left[(t-\tau)^{-\frac{5}{2r}+\frac{3}{2p}+\frac{1}{2}}\right]_{\tau=0}^{\tau=t-1}\|f\|_{L_{t,x}^r}\\
&\le C\|f\|_{L_{t,x}^r} \quad (r<\frac{5p}{5+p} \text{ is equivalent to } -\frac{5}{2r}+\frac{3}{2p}+\frac{1}{2}<0),
\end{aligned}$$

and

$$\begin{aligned}
I_2 &\le C\int_{t-1}^{t}\|\nabla\Gamma(t-\tau)\|_{L_x^{q'}}\|f(\tau)\|_{L_x^q}d\tau \quad (\text{ where } \frac{1}{q'}=1+\frac{1}{p}-\frac{1}{q})\\
&\le C\left[(t-\tau)^{-\frac{5}{2q}+\frac{3}{2p}+\frac{1}{2}}\right]_{\tau=t}^{\tau=t-1}\|f\|_{L_{t,x}^q}\\
&\le C\|f\|_{L_{t,x}^q} \quad (q>\frac{5p}{5+p} \text{ is equivalent to } -\frac{5}{2q}+\frac{3}{2p}+\frac{1}{2}>0).
\end{aligned}$$

Hence we obtain (2.11). Similarly, we have

$$\begin{aligned}
\|\nabla v\|_{L_{t,x}^p} &\le C\left\|\int_0^{t-1}\|\nabla\Gamma(t-\tau)*f(\tau)\|_{L_x^p}d\tau\right\|_{L_t^p}\\
&\quad + C\left\|\int_{t-1}^{t}\|\nabla\Gamma(t-\tau)*f(\tau)\|_{L_x^p}d\tau\right\|_{L_t^p} := J_1+J_2.
\end{aligned}$$

J_1 and J_2 can be estimated as follows:

$$\begin{aligned}
J_1 &\le C\left\|\int_0^{t-1}\|\nabla\Gamma(t-\tau)\|_{L_x^{\beta'}}\|f(\tau)\|_{L_x^\beta}d\tau\right\|_{L_t^p} \quad (\text{ where } \frac{1}{\beta'}=1+\frac{1}{p}-\frac{1}{\beta})\\
&\le C\left\|\int_0^{t-1}(t-\tau)^{-\frac{3}{2}(\frac{1}{\beta}-\frac{1}{p})-\frac{1}{2}}\|f(\tau)\|_{L_x^\beta}d\tau\right\|_{L_t^p}\\
&\le C\|f\|_{L_{t,x}^\beta} \quad (\beta<\frac{3}{2} \text{ is equivalent to } -\frac{3}{2}\left(\frac{1}{\beta}-\frac{1}{p}\right)-\frac{1}{2}+\frac{1}{\beta}<0),
\end{aligned}$$

and

$$\begin{aligned}
J_2 &\le C\left\|\int_{t-1}^{t}\|\nabla\Gamma(t-\tau)\|_{L_x^{\alpha'}}\|f(\tau)\|_{L_x^\alpha}d\tau\right\|_{L_t^p} \quad (\text{ where } \frac{1}{\alpha'}=1+\frac{1}{p}-\frac{1}{\alpha})\\
&\le C\left\|\int_{t-1}^{t}(t-\tau)^{-\frac{3}{2}(\frac{1}{\alpha}-\frac{1}{p})-\frac{1}{2}}\|f(\tau)\|_{L_x^\alpha}d\tau\right\|_{L_t^p}\\
&\le C\|f\|_{L_{t,x}^{p,\alpha}} \quad (\frac{1}{\alpha}<\frac{1}{3}+\frac{1}{p} \text{ is equivalent to } -\frac{3}{2}\left(\frac{1}{\alpha}-\frac{1}{p}\right)+\frac{1}{2}>0),
\end{aligned}$$

where we used convolution type Young's inequality by extending zero outside of the interval of integrals. This completes the proof. □

Lemma 4. *Let $(e(t), s(t))$ be a solution in Theorem 2. Then, for any $p\in(1,\infty)$, there exists a positive constant C_p such that*

$$\|\nabla^2 s\|_{L_{t,x}^p}+\|\partial_t s\|_{L_{t,x}^p}\le C_p.$$

Proof. Due to the maximal regularity, we have for $p \in (1, \infty)$

$$\|\nabla^2 s\|_{L^p_{t,x}} + \|\partial_t s\|_{L^p_{t,x}} \le C(\|s_0\|_{W^{2,p}} + \|(u\cdot\nabla)s\|_{L^p_{t,x}} + \|\nabla\cdot(s\nabla\Delta^{-1}e)\|_{L^p_{t,x}} + \|es\|_{L^p_{t,x}}).$$

We note that

$$\|se\|^p_{L^p_{t,x}} = \int_0^\infty \|se\|^p_{L^p} dt \le \int_0^\infty \|s\|^p_{L^{2p}} \|e\|^p_{L^{2p}} dt$$
$$\le C\int_0^\infty (1+t)^{\frac{3}{2}-3p} dt < C \text{ if } p \ge 1,$$

where C is independent of time.

We also have the following inequality due to the maximal regularity and (2.12),

$$\|(u\cdot\nabla)s\|_{L^p_{t,x}} \le \|u\|_{L^\infty_{t,x}} \|\nabla s\|_{L^p_{t,x}}$$
$$\le C(\|s_0\|_{W^{1,p}} + \|us\|_{L^p_{t,x}} + \|s\nabla\Delta^{-1}e\|_{L^p_{t,x}} + \|es\|_{L^{p,\alpha}_{t,x}\cap L^\beta_{t,x}}),$$

where $\frac{1}{\alpha} < \frac{1}{p} + \frac{1}{3}$ and $1 < \beta < \frac{3}{2}$. We easily see that

$$\|us\|_{L^p_{t,x}} \le \|u\|_{L^{2p}_{t,x}} \|s\|_{L^{2p}_{t,x}} \le C \quad \text{ if } p \ge 1,$$

and

$$\|es\|_{L^{p,\alpha}_{t,x}\cap L^\beta_{t,x}} \le C \quad \text{ if } \beta \ge 1.$$

We find that

$$\|s\nabla\Delta^{-1}e\|_{L^p_{t,x}} \le \Big\| \|s\|_{L^{2p}_x} \|\nabla\Delta^{-1}e\|_{L^{2p}_x} \Big\|_{L^p_t}$$
$$\le C\Big\| \|s\|_{L^{2p}_x} \|e\|_{L^{\tilde{p}}_x} \Big\|_{L^p_t} < C \qquad (\frac{1}{\tilde{p}} = \frac{1}{2p} + \frac{1}{3}).$$

Finally, we find that

$$\|\nabla\cdot(s\nabla\Delta^{-1}e)\|_{L^p_{t,x}} = \|se\|_{L^p_{t,x}} + \|\nabla s\cdot\nabla\Delta^{-1}e\|_{L^p_{t,x}}.$$

The last term in the right hand side can be estimated

$$\|\nabla s\cdot\nabla\Delta^{-1}e\|_{L^p_{t,x}} \le C\Big\| \|\nabla s\|_{L^{2p}_x} \|e\|_{L^{\tilde{p}}_x} \Big\|_{L^p_t} \le C\|\nabla s\|_{L^{2p}_{t,x}} \|e\|_{L^{2p,\tilde{p}}_{t,x}} < C,$$

where $\frac{1}{\tilde{p}} = \frac{1}{2p} + \frac{1}{3}$. This completes the proof. □

Due to (2.11) and Lemma 4, we deduce that, for any $p \in (3, \infty)$ and q, r as in Lemma 3,

$$\|\nabla s\|_{L^{\infty,p}_{t,x}} \le C\|\partial_t s - \Delta s\|_{L^q_{t,x}\cap L^r_{t,x}} \le C(\|\partial_t s\|_{L^q_{t,x}\cap L^r_{t,x}} + \|\Delta s\|_{L^q_{t,x}\cap L^r_{t,x}}) \le C_p.$$

First, we choose $p \in (3, \infty)$. By the interpolation, we have

$$\|s\|_{L^\infty_{t,x}} \le C\|s\|^\theta_{L^{\infty,mp}_{t,x}} \|\nabla s\|^{1-\theta}_{L^{\infty,p}_{t,x}},$$

where $\theta = \frac{m(p-3)}{mp-3m+3}$ and $m > 0$ will be chosen later.
Since we have uniform bound for $\|\nabla s\|_{L^{\infty,p}_{t,x}}$ and decay estimates (2.10), we deduce that

$$\|s(t)\|_{L^\infty_x} \le Ct^{-\frac{3}{2}\cdot\frac{(mp-1)(p-3)}{p(mp-3m+3)}}.$$

If we choose $m > \frac{9(p-1)}{p(p-3)}$, then $-\frac{3}{2}\cdot\frac{(mp-1)(p-3)}{p(mp-3m+3)} < -1$. This implies that

$$\|s(t)\|_{L^\infty_x} \le Ct^{-1-\sigma}$$

for some $\sigma > 0$. Also we note that $-\frac{3}{2}\cdot\frac{(mp-1)(p-3)}{p(mp-3m+3)}$ approaches to $-\frac{3}{2}$ when m approaches to ∞. It implies that $\|s(t)\|_{L^\infty_x} \le Ct^{-\frac{3}{2}+\epsilon}$ for any small $\epsilon > 0$.

• (Total mass behavior of $m_e(t)$) Finally, we deduce that

$$\frac{d}{dt}\int_{\mathbb{R}^3} e(x,t)dx = -\epsilon\int_{\mathbb{R}^3} es\,dx \ge -\epsilon\|e(t)\|_{L^1}\|s(t)\|_{L^\infty}.$$

Similarly to the estimate (2.5), we have $m_e(t) \ge \frac{1}{2}m_e(t_0)$ for some t_0 and all $t \ge t_0$. In the above, C depends on $\frac{1}{\chi}$ (see e.g., L^2 decay estimates), it implies that lower bound approaches 0 as $\chi \to \infty$.

Proof of Theorem 1 (iii) We already obtained the temporal decay of e in 2D and 3D and s in 3D, that is, (1.4), hence we only consider the temporal decay of s in 2D.

• ($2D$ case) We recall that the solution e to $(1.3)_2$ satisfies the equation

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|e|^p + \frac{4(p-1)}{p^2}\int_{\mathbb{R}^2}|\nabla e^{\frac{p}{2}}|^2 + \epsilon\int_{\mathbb{R}^2} e^p s = 0. \tag{2.13}$$

Multiplying a large constant M on both sides of (2.13) (M will be specified later), we have

$$\frac{M}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|e|^p + \frac{4M(p-1)}{p^2}\int_{\mathbb{R}^2}|\nabla e^{\frac{p}{2}}|^2 + M\epsilon\int_{\mathbb{R}^2} e^p s = 0. \tag{2.14}$$

Note first that the following interpolation inequality holds (see [15])

$$\|s\|^{p+1}_{L^{p+1}} \le C_p\|s\|_{L^1}\|\nabla s^{\frac{p}{2}}\|^2_{L^2}.$$

We compute

$$\begin{aligned}\frac{p-1}{p}\chi\int_{\mathbb{R}^2} s^p e &\le C\left(\int_{\mathbb{R}^2} s^{\frac{p^2-1}{p}\cdot\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^2} e^p s\right)^{\frac{1}{p}}\\ = C\left(\int_{\mathbb{R}^2} s^{p+1}\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^2} e^p s\right)^{\frac{1}{p}} &\le CM^{-1}\int_{\mathbb{R}^2} s^{p+1} + \frac{M\epsilon}{2}\int_{\mathbb{R}^2} e^p s\\ &\le CM^{-1}\|s\|_{L^1}\|\nabla s^{\frac{p}{2}}\|^2_{L^2} + \frac{M\epsilon}{2}\int_{\mathbb{R}^2} e^p s.\end{aligned}$$

The solution $s(t)$ satisfies that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|s|^p + \frac{4(p-1)}{p^2}\int_{\mathbb{R}^2}|\nabla s^{\frac{p}{2}}|^2 + \epsilon\int_{\mathbb{R}^2} s^p e\\ &\qquad\le CM^{-1}\|s\|_{L^1}\|\nabla s^{\frac{p}{2}}\|^2_{L^2} + \frac{M\epsilon}{2}\int_{\mathbb{R}^2} e^p s.\end{aligned} \tag{2.15}$$

Adding (2.14) and (2.15), we have

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|s|^p + \frac{M}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|e|^p + \frac{4(p-1)}{p^2}\int_{\mathbb{R}^2}|\nabla s^{\frac{p}{2}}|^2 + \frac{4M(p-1)}{p^2}\int_{\mathbb{R}^2}|\nabla e^{\frac{p}{2}}|^2\\ &\qquad + \epsilon\int_{\mathbb{R}^2} s^p e + \frac{M\epsilon}{2}\int_{\mathbb{R}^2} e^p s \le CM^{-1}\|s\|_{L^1}\|\nabla s^{\frac{p}{2}}\|^2_{L^2}.\end{aligned} \tag{2.16}$$

Taking $M = \frac{Cp^2\|s_0\|_{L^1}}{2(p-1)}$, we have

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|s|^p + \frac{M}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|e|^p + \frac{2(p-1)}{p^2}\int_{\mathbb{R}^2}|\nabla s^{\frac{p}{2}}|^2 + \frac{4M(p-1)}{p^2}\int_{\mathbb{R}^2}|\nabla e^{\frac{p}{2}}|^2$$

$$+\epsilon\int_{\mathbb{R}^2} s^p e + \frac{M\epsilon}{2}\int_{\mathbb{R}^2} e^p s \le 0.$$

This gives the decay estimate

$$\|s(t)\|_{L^p} \le \frac{C_p}{t^{1-\frac{1}{p}}} \text{ for } p \in (1,\,\infty).$$

This completes the proof of Theorem 1. □

Remark 4. In two dimensions, we have $\|e(t)\|_{L^\infty} \le \frac{C}{t}$. Then via similar computations as above, we obtain

$$m_s(t) \ge \left(\frac{t_0}{t}\right)^C m_s(t_0) \text{ for } t \ge t_0.$$

Hence, in two dimensions, we can not obtain the positive lower bound of the total mass via same method in three dimensions and leave as an open problem.

3. **Global well-posednes for the model** (1.6)**.** In this section, we prove the global well-posedness of solutions to the system (1.6).

$$\begin{cases} \partial_t e + (u\cdot\nabla)e - \Delta e = -\epsilon(se), \\ \partial_t s + (u\cdot\nabla)s - \Delta s = -\chi\nabla\cdot(s\nabla c) - \epsilon(se), \\ \partial_t c + (u\cdot\nabla)c - \Delta c = e, \\ \partial_t u + \kappa(u\cdot\nabla)u - \Delta u + \nabla p = -(s+e)\nabla\phi, \\ \text{div u} = 0, \end{cases} \quad \text{in } \ (x,t)\in\mathbb{R}^d\times(0,\,\infty),$$

We will set $\kappa = 1$ (Navier-Stokes system) when $d = 2$ and $\kappa = 0$ (Stokes system) when $d = 3$ as mentioned in Section 1.

Note that the solution $(e,\,s,\,c,\,u,\,p)$ satisfies the scaling invariant property if ϕ has the following scaling property : $\phi(x,t) = \phi^\lambda(x,t) := \phi(\lambda x,\,\lambda^2 t)$. That is,

$$\begin{aligned} &(e^\lambda(x,t),\, s^\lambda(x,t),\, c^\lambda(x,t),\, u^\lambda(x,t),\, p^\lambda(x,t)) \\ &= (\lambda^2 e(\lambda x,\lambda^2 t),\, \lambda^2 s(\lambda x,\lambda^2 t),\, c(\lambda x,\lambda^2 t),\, \lambda u(\lambda x,\lambda^2 t),\, \lambda^2 p(\lambda x,\lambda^2 t)) \end{aligned} \tag{3.1}$$

is also a solution to (1.6) if $(e,\,s,\,c,\,u,\,p)$ is a solution.

The local-in-time existence of the solutions to (1.6) is obtained by the contraction as for Proposition 1. Hence we omit its proof. Moreover similar estimates as (2.3) for the M_n norm of $(e,s,c)(\cdot,T)$ are bounded by $\|(e_0,s_0,c_0)\|_{M_n}$ and $\|(e,s,c,u)\|_{C(0,T;H^m)}$. Thus the local solution is extended if $\|(e,s,c,u))\|_{C(0,T;H^m)}$ is uniformly bounded.

Let T^* be the maximal time of existence of the local solution and T be any time until T^*. In what follows we shall establish a priori estimates for $\|(e,s,c,u)\|_{C(0,T;H^m)}$ where $m = [\frac{d}{2}]+1$. All integrations are over Q_T. We often omit Q_T in $L^q_t L^p_x(Q_T)$.

L^1 estimates of e, s, c and L^p estimates of e, c are immediate. We have

$$\int_{\mathbb{R}^d} e(T)\,dx + \epsilon\int_0^T\int_{\mathbb{R}^d}(se)dxdt = \int_{\mathbb{R}^d} e_0\,dx,$$

$$\int_{\mathbb{R}^d} s(T)\,dx + \epsilon\int_0^T\int_{\mathbb{R}^d}(se)dxdt = \int_{\mathbb{R}^d} s_0\,dx,$$

$$\int_{\mathbb{R}^d} c(T)\,dx = \int_{\mathbb{R}^d} c_0\,dx + \int_0^T\int_{\mathbb{R}^d} e(x,t)\,dxdt.$$

For $1 < p < \infty$ we have

$$\frac{1}{p}\frac{d}{dt}\|e(t)\|_{L^p}^p + \frac{4(p-1)}{p^2}\|\nabla e^{\frac{p}{2}}\|_{L^2}^2 + \epsilon\int_{\mathbb{R}^d}(se^p)(x,t)dx = 0,$$

$$\frac{1}{p}\frac{d}{dt}\|c(t)\|_{L^p}^p + \frac{4(p-1)}{p^2}\|\nabla c^{\frac{p}{2}}\|_{L^2}^2 = \int_{\mathbb{R}^d}(ec^{p-1})(x,t)dx \le \|e\|_{L^p}\|c\|_{L^p}^{p-1}.$$

Hence it holds that

$$\|e\|_{L^\infty(0,T^*;L^p)}^p + \|\nabla e^{\frac{p}{2}}\|_{L^2(0,T^*;L^2)}^2 \le C\|e_0\|_{L^p}^p,$$

$$\begin{aligned}\|c\|_{L^\infty(0,T^*;L^p)}^p + \|\nabla c^{\frac{p}{2}}\|_{L^2(0,T^*;L^2)}^2 &\le C\left(\|c_0\|_{L^p} + \int_0^T \|e\|_p dt\right)^p \\ &\le C(\|c_0\|_{L^p} + T\|e_0\|_{L^p})^p.\end{aligned}$$

To obtain other L^p and higher norm estimates we first consider the estimates of u;

$$\left\{\begin{array}{lr} \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla p = -(s+e)\nabla\phi, \quad \nabla\cdot u = 0 & \text{in } \mathbb{R}^2\times(0,T) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^2. \end{array}\right. \tag{3.2}$$

Let us denote the Stokes operator by G_t. Namely $G_t * u_0$ is the solution of the free Stokes equations ($f = 0$)

$$\partial_t u - \Delta u + \nabla p = f, \quad \nabla\cdot u = 0$$

with initial data u_0. It is well known that G_t satisfies that (see e.g. [12])

$$\|G_t * f\|_{L^p} \le Ct^{\frac{1}{p}-1}\|f\|_{L^1}, \quad \|\nabla G_t * f\|_{L^p} \le Ct^{\frac{1}{p}-\frac{3}{2}}\|f\|_{L^1} \quad 1\le p\le\infty \tag{3.3}$$

in two dimensions. For the inhomogeneous Stokes equations the following maximal regularity estimate is known [12];

$$\int_0^T \|\partial_t u\|_{L_x^p}^q dt + \int_0^T \|\Delta u\|_{L_x^p}^q dt + \int_0^T \|\nabla p\|_{L^p}^q dt \le C\left(\|u_0\|_{W^{2,p}}^q + \int_0^T \|f\|_{L^p}^q dt\right) \tag{3.4}$$

for $1 < p, q < \infty$.

Lemma 5. *Let $d = 2$ and s, e, u be the local solution of (1.6) in $K_{m,n}$. The solution u to (3.2) belongs to $L^\infty(0,T;L^2)\cap L^2(0,T;W^{1,q})\cap L^4(0,T;L^4)$ for any $q\in[1,2)$.*

Proof. We remind that total masses of s and e are decreasing. Thus, $s\nabla\phi$, $e\nabla\phi$ belong to $L^\infty([0,T_0);L^1(\mathbb{R}^2))$, since ϕ is assumed to satisfies $\left\|\nabla^l\phi\right\|_{L^\infty} < \infty$ for $1\le|l|\le m$.

Let $Q := (0,T)\times\mathbb{R}^2$. We decompose the solution u to the equations (3.2) to $v + w$ in Q, where v satisfies the Stokes system:

$$\left\{\begin{array}{lr} \partial_t v - \Delta v + \nabla p_1 = -(s+e)\nabla\phi, \qquad \text{div } v = 0 & \text{in } Q, \\ v(x,0) = u_0(x) & \text{in } \mathbb{R}^2, \end{array}\right. \tag{3.5}$$

and w satisfies a perturbed homogeneous Navier-Stokes equations with zero initial data:

$$\left\{\begin{array}{lr} \partial_t w - \Delta w + \nabla p_2 = -((v+w)\cdot\nabla)v - ((v+w)\cdot\nabla)w, \quad \text{div } w = 0, & \text{in } Q, \\ w(x,0) = 0 & \text{in } \mathbb{R}^2. \end{array}\right. \tag{3.6}$$

For convenience, we denote $f := -(s+e)\nabla\phi$. By (3.3) we have

$$\|v\|_{L^{\infty,p}_{t,x}(Q_T)} \le C\|u_0\|_{L^p} + C\left(\int_0^{T_0} t^{\frac{1}{p}-1}dt\right)\|f\|_{L^{\infty,1}_{t,x}(Q_T)} < \infty.$$

for any $p \in [1,\,\infty)$. Similarly, we have

$$\|\nabla v\|_{L^{\infty,q}_{t,x}(Q_T)} \le C\|\nabla u_0\|_{L^q} + C\left(\int_0^{T_0} t^{\frac{1}{q}-\frac{3}{2}}dt\right)\|f\|_{L^{\infty,1}_{t,x}(Q_T)} < \infty$$

for any $q \in [1,2)$. Note that $\|f\|_{L^{\infty,1}_{t,x}(Q_T)} \le C(\|s_0\|_{L^1(\mathbb{R}^2)} + \|e_0\|_{L^1(\mathbb{R}^2)})$. Summing up, we obtain

$$\|v\|_{L^{\infty,p}_{t,x}(Q_T)} + \|\nabla v\|_{L^{\infty,q}_{t,x}(Q_T)} \le C = C(T_0), \qquad p\in[1,\,\infty), \quad q\in[1,\,2), \tag{3.7}$$

which yields that

$$v \in L^\infty(0,T_0;L^2)\cap L^2(0,T_0;W^{1,q})\cap L^4(0,T_0;L^4) \quad q\in[1,2).$$

For the Navier-Stokes part w, the followings come from the facts that $\int_{\mathbb{R}^2}((v+w)\cdot\nabla)w\cdot w dx = 0$, and $\int_{\mathbb{R}^2}((v+w)\cdot\nabla)v\cdot w dx = -\int_{\mathbb{R}^2}((v+w)\cdot\nabla)w\cdot v dx$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 &\le \left|\int_{\mathbb{R}^2}((v+w)\cdot\nabla)w\cdot v dx\right|\\ &\le \|v\|_{L^4}^2\|\nabla w\|_{L^2} + \|w\|_{L^4}\|\nabla w\|_{L^2}\|v\|_{L^4}\\ &\le \|v\|_{L^4}^2\|\nabla w\|_{L^2} + C\|w\|_{L^2}^{\frac{1}{2}}\|\nabla w\|_{L^2}^{\frac{3}{2}}\|v\|_{L^4}\\ &\le \frac{1}{2}\|\nabla w\|_{L^2}^2 + C\|v\|_{L^4}^4(\|w\|_{L^2}^2+1)\end{aligned}$$

which implies

$$w \in L^{\infty,2}_{t,x}(Q_T)\cap L^2(0,T;H_0^1)$$

by the Gronwall's inequality. It remains to show that $w \in \bigcap_{1\le q<2} L^2(0,T;W^{1,q})$. Using the Stokes operator, we write w as

$$\begin{aligned}\nabla w(x,\,t) &= -\int_0^t \nabla G_{t-s} * \left((v\cdot\nabla)v + (v\cdot\nabla)w + (w\cdot\nabla)v + (w\cdot\nabla)w\right)(s)ds.\\ &= -\int_0^t \nabla G_{t-s} * ((v\cdot\nabla)v(s))ds - \int_0^t \nabla G_{t-s} * ((v\cdot\nabla)w(s))ds\\ &\quad -\int_0^t \nabla G_{t-s} * ((w\cdot\nabla)v(s))ds - \int_0^t \nabla G_{t-s} * ((w\cdot\nabla)w(s))ds := I_1+I_2+I_3+I_4.\end{aligned}$$

What it follows, we separately compute $I_i,\ i=1,2,3,4$.

$$\begin{aligned}\|I_1(t)\|_{L^q} &\le \int_0^t \|\nabla G_{t-s} * ((v\nabla)v)(s)\|_{L^q}ds \le C\int_0^t (t-s)^{\frac{1}{q}-\frac{3}{2}}\|v\nabla v\|_{L^1(\mathbb{R}^2)}(s)ds\\ &\le C\int_0^t (t-s)^{\frac{1}{q}-\frac{3}{2}}\|v\|_{L^{q'}}(s)\,\|\nabla v\|_{L^q}(s)ds \le C(T_0)\,\|v\|_{L^{\infty,q'}_{t,x}(Q_T)}\,\|\nabla v\|_{L^{\infty,q}_{t,x}(Q_T)}.\end{aligned}$$

Similarly,

$$\begin{aligned}\|I_2(t)\|_{L^q} &\le C\int_0^t (t-s)^{\frac{1}{q}-\frac{3}{2}}\|v\nabla w\|_{L^1(\mathbb{R}^2)}(s)ds\\ &\le C\,\|v\|_{L^{\infty,2}_{t,x}(Q_T)}\int_0^t (t-s)^{\frac{1}{q}-\frac{3}{2}}\|\nabla w\|_{L^2(\mathbb{R}^2)}(s)ds.\end{aligned}$$

Therefore, using the convolution inequality, we have

$$\|I_2\|_{L^{2,q}_{t,x}(Q_T)} \le C(T_0)\,\|v\|_{L^{\infty,2}_{t,x}(Q_T)}\,\|\nabla w\|_{L^{2}_{t,x}(Q_T)}\,.$$

For I_3, using $w \in L^4(Q_T)$, we observe that

$$\|I_3(t)\|_{L^q} \le C\int_0^t (t-s)^{\frac{1}{q}-\frac{3}{2}}\|w\nabla v\|_{L^1(\mathbb{R}^2)}(s)ds$$

$$\le C\int_0^t (t-s)^{\frac{1}{q}-\frac{3}{2}}\|w\|_{L^4(\mathbb{R}^2)}(s)\|\nabla v\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}(s)ds$$

$$\le C\|\nabla v\|_{L^{\infty,\frac{4}{3}}_{t,x}(Q_T)}\int_0^t (t-s)^{\frac{1}{q}-\frac{3}{2}}\|w\|_{L^4(\mathbb{R}^2)}(s)ds.$$

Using the convolution inequality again, we obtain

$$\|I_3\|_{L^{2,q}_{t,x}(Q_T)} \le C(T)\,\|\nabla v\|_{L^{\infty,\frac{4}{3}}_{t,x}(Q_T)}\,\|w\|_{L^{4}_{t,x}(Q_T)}\,.$$

Finally, we compute

$$\|I_4(t)\|_{L^q} \le C\int_0^t (t-s)^{\frac{1}{q}-\frac{3}{2}}\|w\nabla w\|_{L^1(\mathbb{R}^2)}(s)ds$$

$$\le C\int_0^t (t-s)^{\frac{1}{q}-\frac{3}{2}}\,\|w\|_{L^2}\,(s)\,\|\nabla w\|_{L^2}\,(s)ds$$

$$\le C\,\|w\|_{L^{\infty,2}_{t,x}(Q_T)}\int_0^t (t-s)^{\frac{1}{q}-\frac{3}{2}}\|\nabla w\|_{L^2(\mathbb{R}^2)}(s)ds.$$

Similarly we get

$$\|I_4\|_{L^{2,q}_{t,x}(Q_T)} \le C(T)\,\|w\|_{L^{\infty,2}_{t,x}(Q_T)}\,\|\nabla w\|_{L^{2}(Q_T)}\,.$$

Summing up estimates, we obtain that $\nabla w \in \bigcap_{1\le q<2} L^2(0,T;L^q(\mathbb{R}^2))$. This completes the proof. $\square$

Remark 5. If we consider

$$\begin{aligned} \partial_t u - \Delta u + \nabla p = -(s+e)\nabla\phi, \quad \nabla\cdot u = 0 \qquad &\text{in } \mathbb{R}^3\times(0,T)\\ u(x,0) = u_0(x) \qquad &\text{in } \mathbb{R}^3. \end{aligned} \tag{3.8}$$

then similarly to Lemma 5, we can prove that the solution u to (3.8) belongs to $L^\infty(0,T;L^p)\cap L^\infty(0,T;W^{1,q})$ for any $p\in[1,3)$ and $q\in[1,\frac{3}{2})$.

We proceed other L^p and higher order estimates to conclude the global well-posedness part of Theorem 2. We treat spatial two and three dimensional cases separately.

Proof of Theorem 2 (i) ($d=2$) If we consider the equation

$$\partial_t c - \Delta c = -\nabla\cdot(uc) + e,$$

then by the maximal regularity of the heat equation (2.1) we obtain

$$\begin{aligned} \|\nabla c\|_{L^4_{t,x}} &\le C(\|uc\|_{L^4_{t,x}} + \|\nabla\Delta^{-1}e\|_{L^4_{t,x}}) + C\|\nabla c_0\|_{L^4_x}\\ &\le C(\|c\|_{L^\infty_{t,x}}\|u\|_{L^4_{t,x}} + \|e\|_{L^4_t L^{3/2}_x}) + C\|\nabla c_0\|_{L^4_x} < \infty, \end{aligned} \tag{3.9}$$

where the last inequality is due to L^p estimates of c, e and Lemma 5. Multiplying both sides of the equation of s by s^{p-1} and integrating over $\mathbb{R}^2$, we deduce that

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|s\|_{L^p}^p+\frac{4(p-1)}{p^2}\|\nabla s^{\frac{p}{2}}\|_{L^2}^2 &\le \frac{2(p-1)}{p}\chi\left|\int_{\mathbb{R}^2} s^{\frac{p}{2}}\nabla c\cdot\nabla s^{\frac{p}{2}}\right|\\ &\le \frac{2(p-1)}{p}\chi\|s^{\frac{p}{2}}\|_{L^2}^{\frac{1}{2}}\|\nabla c\|_{L^4}\|\nabla s^{\frac{p}{2}}\|_{L^2}^{\frac{3}{2}}\\ &\le C\chi^4\|\nabla c\|_{L^4_x}^4\|s\|_{L^p}^p+\frac{2(p-1)}{p^2}\|\nabla s^{\frac{p}{2}}\|_{L^2}^2.\end{aligned}\tag{3.10}$$

Hence we have

$$\sup_{0\le t\le T}\|s(t)\|_{L^p}^p\le\|s_0\|_{L^p}^p\exp\left(C\chi^4\|\nabla c\|_{L^4_{t,x}}^4\right)<\infty \text{ for all } p\in[2,\infty).\tag{3.11}$$

Therefore, $s\in L^\infty(0,T;L^p)$ and $\nabla s^{\frac{p}{2}}\in L^2(0,T;L^2)$ for all $p\in[2,\infty)$.

On the other hand, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2+\|\nabla u\|_{L^2}^2\le C(\|s\|_{L^2}+\|e\|_{L^2})\|u\|_{L^2}.$$

It gives us that $u\in L^\infty(0,T;L^2)$ and $\nabla u\in L^2(0,T;L^2)$.

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2\le C(\|s\|_{L^2}^2+\|e\|_{L^2}^2+\|\nabla u\|_{L^2}^4)+\frac{1}{2}\|\Delta u\|_{L^2}^2.$$

Therefore, we also have $\nabla u\in L^\infty(0,T;L^2)$ and $\Delta u\in L^2(0,T;L^2)$, that is

$$u\in L^\infty(0,T;L^2),\quad \nabla u\in L^\infty(0,T;L^2),\quad \Delta u\in L^2(0,T;L^2).\tag{3.12}$$

In general the maximal regularity of the heat equation and the L^p estimates of c, e yield that

$$\begin{aligned}&\|\nabla c\|_{L^p_{t,x}}\le C(\|u\|_{L^p_{t,x}}+1)<\infty,\\ &\|\Delta c\|_{L^p_{t,x}}\le C(\|u\cdot\nabla c\|_{L^p_{t,x}}+1)\le C(\|u\|_{L^q_{t,x}}\|\nabla c\|_{L^{pq/(q-p)}_{t,x}}+1)<\infty\end{aligned}\tag{3.13}$$

for all $p\in[2,\infty)$ and $q>p$. We can replace c with e in the above. Applying the maximal regularity of the heat equation to s equation together with the previous estimates, we have

$$\|\nabla s\|_{L^p_{t,x}},\ \|\Delta s\|_{L^p_{t,x}}<\infty \text{ for all } p\in[2,\infty).\tag{3.14}$$

Then by the bootstraping argument, we complete the proof of the Case I. Indeed (3.12) and (3.14) yields L^p estimate for $\nabla c, \nabla e$. Then L^p estimate of ∇s follows from the boundedness of $\|\Delta c\|_{L^p_{t,x}}$ in (3.13) as is obtained $\|s\|_{L^p}$ in (3.11). Those L^p estimates are used to yield $\nabla u\in L^\infty(0,T;L^2), \nabla^2 u\in L^\infty(0,T;L^2), \nabla^3 u\in L^2(0,T;L^2)$, which closes the H^1 estimate of e, c, s, u. Maximal regularity estimates for $\nabla c, \nabla e, \nabla s$ prove the boundedness of $\|\nabla c, \nabla e, \nabla s\|_{L^p_t W^{2,p}_x}$ for all $p\in[2,\infty)$, which corresponds to one more derivative version of (3.13) and (3.14). The H^2 estimates can be similarly done.

(ii) ($d=3$) We assume that $\chi^2\|\nabla\phi\|_{L^\infty}^2\|s_0\|_{L^1}^2$ is sufficiently small. Note that $\chi^2\|\nabla\phi\|_{L^\infty}^2\|s_0\|_{L^1}^2$ is scaling invariant quantity.

In the three dimensional case the regularity of u obtained in Remark 5 is not enough to prove (3.9) and (3.10) as is in two dimensions. We need to prove an entropy type inequality for s (3.19) for three dimensions. Taking $\log s$ as a test function for the equation $(1.6)_2$, we have

$$\frac{d}{dt}\int_{\mathbb{R}^3} s\log s dx + 4\int_{\mathbb{R}^3} |\nabla\sqrt{s}|^2 dx = \int_{\mathbb{R}^3} \chi\nabla s\cdot\nabla c dx - \epsilon\int_{\mathbb{R}^3} se(1+\log s)dx$$

We estimate

$$\left|\int_{\mathbb{R}^3}\chi\nabla s\nabla c\right| = 2\chi\left|\int_{\mathbb{R}^3}\sqrt{s}\nabla\sqrt{s}\cdot\nabla c dx\right| \le C\chi\|\nabla\sqrt{s}\|_{L^2}\|\sqrt{s}\|_{L^{\frac{30}{11}}}\|\nabla c\|_{L^{\frac{15}{2}}}$$

$$\le 2\|\nabla\sqrt{s}\|^2_{L^2} + C\chi^2\|s\|_{L^{\frac{15}{11}}}\|\nabla c\|^2_{L^{\frac{15}{2}}}.$$

Also we note that

$$-\int_{\{x:s(x)\le 1\}} se\log s dx \le C\int_{\mathbb{R}^3} e dx.$$

Hence we deduce that

$$\frac{d}{dt}\int_{\mathbb{R}^3} s\log s dx + 2\int_{\mathbb{R}^3}|\nabla\sqrt{s}|^2 dx \le C\chi^2\|s\|_{L^{\frac{15}{11}}}\|\nabla c\|^2_{L^{\frac{15}{2}}} + C\int_{\mathbb{R}^3} e dx$$

$$\le C\left(\chi^2\|s\|^{\frac{3}{5}}_{L^1}\|s\|^{\frac{2}{5}}_{L^3}\|\nabla c\|^2_{L^{\frac{15}{2}}} + \|e_0\|_{L^1}\right) \le C\left(\chi^2\|s_0\|^{\frac{3}{5}}_{L^1}\|s\|^{\frac{2}{5}}_{L^3}\|\nabla c\|^2_{L^{\frac{15}{2}}} + \|e_0\|_{L^1}\right).$$

Integrating in time gives us that

$$\int_{\mathbb{R}^3} s(t)\log s(t) dx - \int_{\mathbb{R}^3} s_0\log s_0 dx + \frac{1}{2}\int_0^t\int_{\mathbb{R}^3}|\nabla\sqrt{s}|^2 dx ds$$

$$\le C\left[\chi^2\|s_0\|^{\frac{3}{5}}_{L^1}\left(\int_0^t\|s\|_{L^3}ds\right)^{\frac{2}{5}}\left(\int_0^t\|\nabla c\|^{\frac{10}{3}}_{L^{\frac{15}{2}}}ds\right)^{\frac{3}{5}} + t\|e_0\|_{L^1}\right].$$

Considering the equation of c

$$c_t - \Delta c = -\nabla\cdot(uc) + e,$$

and by the fact that $e\in L^\infty_{t,x}$, we have

$$\|\nabla c\|_{L^{\frac{10}{3},\frac{15}{2}}_{t,x}} \le C(\|uc\|_{L^{\frac{10}{3},\frac{15}{2}}_{t,x}} + \|e\|_{L^{\frac{10}{3},\frac{15}{2}}_{t,x}}) + \|\nabla c_0\|_{L^{\frac{15}{2}}_x}$$

$$\le C(\|u\|_{L^{\frac{10}{3},\frac{15}{2}}_{t,x}} + 1) \le C(\|\nabla^2 u\|_{L^{\frac{10}{3},\frac{5}{4}}_{t,x}} + 1) \le C(\|s\|_{L^{\frac{10}{3},\frac{5}{4}}_{t,x}}\|\nabla\phi\|_{L^\infty_{t,x}} + 1),$$

where the last inequality is from (3.4). Since we have $\|s\|_{L^{\frac{5}{4}}_x} \le C\|s\|^{\frac{7}{10}}_{L^1_x}\|s\|^{\frac{3}{10}}_{L^3_x}$, we deduce that

$$\int_{\mathbb{R}^3} s\log s dx - \int_{\mathbb{R}^3} s_0\log s_0 dx + \frac{1}{2}\int_0^t\int_{\mathbb{R}^3}|\nabla\sqrt{s}|^2 dx ds$$

$$\le C\chi^2\|s\|^{\frac{2}{5}}_{L^{1,3}_{t,x}}\left(\|s\|^{\frac{3}{5}}_{L^{1,3}_{t,x}}\|\nabla\phi\|^2_{L^\infty_{t,x}}\|s_0\|^2_{L^1_x} + 1\right) + Ct\|e_0\|_{L^1}$$

$$\le\left[C_*\chi^2\|\nabla\phi\|^2_{L^\infty_{t,x}}\|s_0\|^2_{L^1_x} + \frac{1}{8}\right]\|\nabla\sqrt{s}\|^2_{L^2_{t,x}} + Ct.$$

Therefore, from the assumption that $C_*\chi^2\|\nabla\phi\|^2_{L^\infty_{t,x}}\|s_0\|^2_{L^1_x}\le\frac{1}{8}$, then we can have

$$\int_{\mathbb{R}^3} s\log s dx + \frac{1}{4}\int_0^t\int_{\mathbb{R}^3}|\nabla\sqrt{s}|^2 dx d\tau < Ct. \tag{3.15}$$

Let $(\log s)_-$ be a negative part of $\log s$ and $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$. Decomposing the domain $\{x|s(x)\le 1\}$ into $D_1\cup D_2 =: \{x|0\le s(x)\le e^{-|x|}\}\cup\{x|e^{-|x|}\le s(x)\le 1\}$ and using $s(\log s)_- < C\sqrt{s}$ for the integral over D_1, we have

$$\int_{\mathbb{R}^3} s(\log s)_- \le C\int_{\mathbb{R}^3} e^{-\frac{|x|}{2}} + C\int_{\mathbb{R}^3}\langle x\rangle s. \tag{3.16}$$

Integration by parts gives us that

$$\frac{d}{dt}\int_{\mathbb{R}^3}\langle x\rangle s = \int_{\mathbb{R}^3} s(u\cdot\nabla)\langle x\rangle + \int_{\mathbb{R}^3} s\Delta\langle x\rangle + \int_{\mathbb{R}^3}\chi s\nabla c\cdot\nabla\langle x\rangle - \epsilon\int_{\mathbb{R}^3}\langle x\rangle se.$$

Since $|\nabla\langle x\rangle| + |\Delta\langle x\rangle|\le C$, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^3} s(u\cdot\nabla)\langle x\rangle\right| &\le C\|\sqrt{s}\|^2_{L^{\frac{12}{5}}}\|u\|_{L^6} \le C\|\sqrt{s}\|^{\frac{3}{2}}_{L^2}\|\nabla\sqrt{s}\|^{\frac{1}{2}}_{L^2}\|u\|_{L^6}\\ &\le \delta\|\nabla\sqrt{s}\|^2_{L^2} + C\|s_0\|_{L^1}\|\nabla u\|^2_{L^2} + C\end{aligned}$$

and

$$\begin{aligned}\left|\int_{\mathbb{R}^3} s\Delta\langle x\rangle\right| + \left|\int_{\mathbb{R}^3}\chi s\nabla c\cdot\nabla\langle x\rangle\right| &\le C + C\|\nabla\sqrt{s}\|^{\frac{1}{2}}_{L^2}\|\nabla c\|_{L^6}\\ &\le C + \delta\|\nabla\sqrt{s}\|^2_{L^2} + C\|\nabla c\|^2_{L^6},\end{aligned}$$

for sufficiently small $\delta > 0$.

Also from the equation $\partial_t c - \Delta c = -\nabla\cdot(uc) + e$, we have

$$\|\nabla c\|^2_{L^{2,6}_{t,x}} \le C(\|uc\|^2_{L^{2,6}_{t,x}} + 1) \le C(\|u\|^2_{L^{2,6}_{t,x}} + 1) \le C(\|\nabla u\|^2_{L^{2,2}_{t,x}} + 1).$$

Considering (3.16) and adding $2\int_{\mathbb{R}^3} s(\log s)_-$ on the both sides of (3.15), we obtain

$$\int_{\mathbb{R}^3} s(t)|\log s(t)|dx + \frac{1}{8}\int_0^t\int_{\mathbb{R}^3}|\nabla\sqrt{s}|^2 dxd\tau < C(t+1) + C_{**}\int_0^t\|\nabla u\|^2_{L^2}. \tag{3.17}$$

From the equation of u, we deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2} &\le C(\|s\|_{L^{\frac{6}{5}}} + \|e\|_{L^{\frac{6}{5}}})\|u\|_{L^6}\\ &\le C + \delta\|\nabla\sqrt{s}\|^2_{L^2} + \delta\|\nabla u\|^2_{L^2}.\end{aligned}$$

Multiplying $4C_{**}$ on the both sides of the above inequality and integrating with respect to time, we have

$$2C_{**}\int_{\mathbb{R}^3}|u|^2(t)dx + 2C_{**}\int_0^t\int_{\mathbb{R}^3}|\nabla u|^2 \le Ct + 4C_{**}\delta\int_0^t\|\nabla\sqrt{s}\|^2_{L^2}d\tau. \tag{3.18}$$

If we add (3.17) and (3.18), then we have

$$\begin{aligned}&\int_{\mathbb{R}^3} s(t)|\log s(t)|dx + 2C_{**}\int_{\mathbb{R}^3}|u|^2(t)dx\\ &+\frac{1}{16}\int_0^t\int_{\mathbb{R}^3}|\nabla\sqrt{s}|^2 + C_{**}\int_0^t\int_{\mathbb{R}^3}|\nabla u|^2 \le C(1+t).\end{aligned} \tag{3.19}$$

Hence we have

$$\nabla\sqrt{s}\in L^2(0,t;L^2(\mathbb{R}^3)) \text{ i.e., } s\in L^1(0,t;L^3(\mathbb{R}^3)).$$

From the interpolation, it gives us that

$$s\in L^{q,p}_{t,x} \quad \text{with} \quad \frac{3}{p}+\frac{2}{q} = 3,\ 1\le p\le 3. \tag{3.20}$$

By the maximal regularity estimate for Stokes equation (3.4), we obtain

$$\|\Delta u\|^5_{L^{5,\frac{15}{13}}_{t,x}} \le C(\|u_0\|^5_{W^{2,\frac{15}{13}}} + \|s\|^5_{L^{5,\frac{15}{13}}_{t,x}}) < C(\|u_0\|_{W^{2,\frac{15}{13}}} + \|s_0\|^4_{L^1} + \delta\int_0^t \|\nabla\sqrt{s}\|^2_{L^2}),$$

and hence $u \in L^5_{t,x}$ by the Sobolev embedding. Also we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla c\|^2_{L^2} + \|\Delta c\|^2_{L^2} = \int_{\mathbb{R}^3} u\nabla c\Delta c - \int_{\mathbb{R}^3} e\Delta c$$

$$\le \|u\|_{L^5}\|\nabla c\|_{L^{\frac{10}{3}}}\|\Delta c\|_{L^2} + \frac{1}{8}\|\Delta c\|^2_{L^2} + C$$

$$\le C\|u\|_{L^5}\|\nabla c\|^{\frac{2}{5}}_{L^2}\|\Delta c\|^{\frac{8}{5}}_{L^2} + \frac{1}{8}\|\Delta c\|^2_{L^2} + C$$

$$\le C\|u\|^5_{L^5}\|\nabla c\|^2_{L^2} + \frac{1}{4}\|\Delta c\|^2_{L^2} + C.$$

Hence we have $\nabla c \in L^{\infty,2}_{t,x}$ and $\Delta c \in L^2_{t,x}$.

Also from the equation $\partial_t c - \Delta c = -\nabla\cdot(uc) + e$, we have

$$\|\nabla c\|_{L^5_{x,t}} \le C(\|uc\|_{L^5_{x,t}} + 1) \le C(\|u\|_{L^5_{x,t}} + 1) < \infty.$$

Hence we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|s|^2 + \int_{\mathbb{R}^3}|\nabla s|^2 \le C\left|\int_{\mathbb{R}^3} s\nabla c\cdot\nabla s\right|$$

$$\le C\|s\|_{L^{\frac{10}{3}}}\|\nabla c\|_{L^5}\|\nabla s\|_{L^2}$$

$$\le \|\nabla s\|^{\frac{8}{5}}_{L^2}\|s\|^{\frac{2}{5}}_{L^2}\|\nabla c\|_{L^5} \le \frac{1}{2}\|\nabla s\|^2_{L^2} + C\|\nabla c\|^5_{L^5}\|s\|^2_{L^2}.$$

By using Gronwall's inequality, we have $s \in L^{\infty,2}_{t,x}$ and $\nabla s \in L^2_{t,x}$. The higher order estimates can be obtained in a similar fashion. A brief sketch of the proof is as follows : as in [3, Theorem 1], we can show that

$$\|\nabla u\|_{L^5_{t,x}} \le C(\|s\|_{L^{5,\frac{15}{8}}_{t,x}} + 1)$$

and

$$\|\nabla^2 c\|_{L^{\infty,2}_{t,x}} + \|\nabla^3 c\|_{L^2_{t,x}} \le C(\|\nabla u\|_{L^5_{t,x}} + 1).$$

Also we can show that if T^* is a finite maximal existence time, then

$$\|\nabla c\|_{L^{2,\infty}_{t,x}(Q_{T^*})} = \infty.$$

But, by the previous estimates and the standard continuation argument, we can complete the proof of existence of solution in (ii). For the positive lower bound of the total mass, we can obtain the lower bounds in (ii). The proof of the last part in (ii) is parallel to the proof of Theorem 1 (ii) and we omit the details.

4. **Decay estimates in Theorem 2.** In this section we prove the part **(iii)** of Theorem 2. From the equation of e, we have

$$\frac{d}{dt}\int_{\mathbb{R}^d}|e(t)|^p dx + C\|e\|_{L^p(\mathbb{R}^d)}^{\frac{p(d+2)}{d}}\|e\|_{L^{\frac{p}{2}}(\mathbb{R}^d)}^{-\frac{2p}{d}} \le 0.$$

Following the same proof for Theorem1 (ii) (see (2.4) below), we have (1.7)

$$\|e(t)\|_{L^p} \lesssim \frac{1}{t^{(1-\frac{1}{p})\frac{d}{2}}}, \quad 1 < p \le \infty.$$

Next we will obtain the decay estimate of $\|c\|_{L^q}$ when $d = 3$.

Noting first that $\|c(t)\|_{L^1} \le Ct$ for sufficiently large t, we have

$$\frac{1}{3}\frac{d}{dt}\int_{\mathbb{R}^3}|c|^3 + t^{-1}\left(\int_{\mathbb{R}^3}|c|^3\right)^{\frac{4}{3}} \lesssim \frac{1}{3}\frac{d}{dt}\int_{\mathbb{R}^3}|c|^3 + \frac{8}{9}\int_{\mathbb{R}^3}\left|\nabla c^{\frac{3}{2}}\right|^2 \lesssim t^{-1}\|c\|_{L^3}^2,$$

where we used that $\|c\|_{L^3} \le \|c\|_{L^1}^{\frac{1}{4}}\|c\|_{L^9}^{\frac{3}{4}}$. Setting $x(t) = \|c(t)\|_{L^3}$ and dividing both sides by $\|c(t)\|_{L^3}^2$, the above inequality can be rewritten as $x'(t) \le t^{-1}(C_1 - C_2x^2(t))$ for some constants C_1 and C_2. Since it is a separable form of 1st order ordinary differential inequality, direct computations show that $\|c(t)\|_{L^3}$ is uniformly bounded. Its verification is rather standard, and thus we skip its details.

We next show that $\|c(t)\|_{L^q}$ is uniformly bounded for $3 < q \le \infty$, $d = 3$. Let $3 < q < \infty$. Using $\|e(t)\|_{L^q} \lesssim t^{-\frac{3}{2}(1-\frac{1}{q})}$, we have

$$\frac{1}{q}\frac{d}{dt}\int_{\mathbb{R}^3}|c|^q + \frac{4(q-1)}{q^2}\int_{\mathbb{R}^3}\left|\nabla c^{\frac{q}{2}}\right|^2 = \int_{\mathbb{R}^3} ec^{q-1} \le \|e\|_{L^q}\|c\|_{L^q}^{q-1} \lesssim t^{-\frac{3}{2}(1-\frac{1}{q})}\|c\|_{L^q}^{q-1}.$$

Noting that

$$\|c\|_{L^q} \le \|c\|_{L^3}^{\frac{2}{q-1}}\|c\|_{L^{3q}}^{\frac{q-3}{q-1}} \lesssim \|c\|_{L^{3q}}^{\frac{q-3}{q-1}} \lesssim \left\|c^{\frac{q}{2}}\right\|_{L^6}^{\frac{2(q-3)}{q(q-1)}} \lesssim \left\|\nabla c^{\frac{q}{2}}\right\|_{L^2}^{\frac{2(q-3)}{q(q-1)}},$$

we see that

$$\frac{1}{q}\frac{d}{dt}\int_{\mathbb{R}^3}|c|^q + \frac{4}{q^2}\left(\int_{\mathbb{R}^3}|c|^q\right)^{\frac{q-1}{q-3}} \lesssim t^{-\frac{3}{2}(1-\frac{1}{q})}\|c\|_{L^q}^{q-1},$$

which can be rewritten as, denoting $y(t) := \|c(t)\|_{L^q}$ and dividing both sides by $y(t)^{q-1}$,

$$y'(t) + \frac{1}{q^2}(y(t))^{\frac{3(q-1)}{q-3}} \lesssim t^{-\frac{3}{2}(1-\frac{1}{q})}. \tag{4.1}$$

Since $y(t)$ is a solution to the equation

$$\frac{1}{q}\frac{d}{dt}y^q(t) = -\frac{4}{q^2}\int_{\mathbb{R}^3}\left|\nabla c^{\frac{q}{2}}\right|^2 + \int_{\mathbb{R}^3} ec^{q-1}$$

and $c(t)$ is smooth solution constructed in (ii) of Theorem 2, we can show $-\frac{4}{q^2}\int_{\mathbb{R}^3}\left|\nabla c^{\frac{q}{2}}\right|^2 + \int_{\mathbb{R}^3} ec^{q-1}$ is continuous with respect to time. For example, we estimate

$$\int|\nabla c^{\frac{q}{2}}|^2(t_2) - \int|\nabla c^{\frac{q}{2}}|^2(t_1)$$
$$\le \|c^{\frac{q}{2}-1}(t_2) - c^{\frac{q}{2}-1}(t_1)\|_{L^4}^2\|\nabla c(t_2)\|_{L^4}^2 + \|c^{\frac{q}{2}-1}\|_{L^4}^2\|\nabla c(t_2) - \nabla c(t_1)\|_{L^4}^2.$$

Then it implies that $y'(t)$ is continuous with respect to time. By integrating (4.1) with respect to time, we use comparison argument with the solution

$$z'(t) + \frac{1}{q^2}(z(t))^{\frac{3(q-1)}{q-3}} \simeq t^{-\frac{3}{2}(1-\frac{1}{q})} \tag{4.2}$$

By solving the differential equation (4.2), we can deduce (1.8)

$$\|c(t)\|_{L^q} \lesssim t^{-\frac{3}{2}(\frac{1}{3}-\frac{1}{q})}, \qquad q > 3.$$

Next, we prove (1.9) for $d = 2$. First introduce the function spaces used in [2].

$$\begin{aligned}
\|c\|_{\mathcal{N}_q} &:= \sup_t t^{\frac{1}{2}-\frac{1}{q}} \|\nabla c\|_{L^q(\mathbb{R}^2)}, \qquad 2 < q < 4, \\
\|s\|_{\mathcal{K}_p} &:= \sup_t t^{1-\frac{1}{p}} \|s\|_{L^p(\mathbb{R}^2)}, \qquad \frac{4}{3} < p < 2, \\
\|e\|_{\mathcal{K}_l} &:= \sup_t t^{1-\frac{1}{l}} \|e\|_{L^l(\mathbb{R}^2)}, \qquad 1 < l \leq \infty, \\
\|\omega\|_{\mathcal{K}_r} &:= \sup_t t^{1-\frac{1}{r}} \|\omega\|_{L^r(\mathbb{R}^2)}, \qquad 1 < r < 2.
\end{aligned} \tag{4.3}$$

From (1.7), we already obtain $\|e\|_{\mathcal{K}_l} \leq C(\epsilon_1)$.

Let $\Gamma(x,t)$ be the two dimensional heat kernel, i.e.,

$$\Gamma(x,t) = (4\pi t)^{-1} \exp\left(-|x|^2/4t\right).$$

If we set

$$S(t)u = \int_{\mathbb{R}^2} \Gamma(x-y,t)u(y)dy,$$

then we write the equations as the integral representation.

$$s(t) = S(t)s_0 - \int_0^t \nabla S(t-\tau)\cdot[\chi s(\tau)\nabla c(\tau) + u(\tau)s(\tau)]\, d\tau - \epsilon \int_0^t S(t-\tau)(s(\tau)\, e(\tau))\, d\tau,$$

$$e(t) = S(t)e_0 - \int_0^t S(t-\tau)\left[u(\tau)\cdot\nabla e(\tau) + \epsilon s(\tau)e(\tau)\right] d\tau,$$

$$c(t) = S(t)c_0 - \int_0^t S(t-\tau)(u(\tau)\cdot\nabla c(\tau) - e(\tau))\, d\tau,$$

and

$$\omega(t) = G(t)\omega_0 - \int_0^t \nabla^\perp G(t-\tau)\cdot(s(\tau)+e(\tau))\nabla\phi\, d\tau - \int_0^t \nabla G(t-\tau)u(\tau)\omega(\tau)\, d\tau.$$

Let us remind the linear heat kernel estimates in $\mathbb{R}^2$;

$$\begin{aligned}
\|\nabla^\alpha S(t)f\|_{L^q} &\leq Ct^{-(1/r-1/q)-|\alpha|/2}\|f\|_{L^r}, \qquad 1 \leq r \leq q \leq \infty, \\
\|\nabla S(t)f\|_{L^q} &\leq Ct^{-\left(\frac{1}{2}-\frac{1}{q}\right)}\|\nabla f\|_{L^2}, \qquad 2 \leq q \leq \infty,
\end{aligned} \tag{4.4}$$

and

$$\int_0^t \|\nabla S(t-\tau)f(\tau)\|_{L^q} d\tau \leq C\int_0^t \frac{1}{(t-\tau)^{\frac{3}{2}-\frac{1}{\alpha_0}}}\cdot\frac{1}{\tau^{1-\frac{1}{l}}} d\tau \|f\|_{\mathcal{K}_l} \leq C\frac{1}{t^{\frac{1}{2}-\frac{1}{q}}}\|f\|_{\mathcal{K}_l} \tag{4.5}$$

with $1 + \frac{1}{q} = \frac{1}{\alpha_0} + \frac{1}{l}$. Also we use the following elementary results on the integral for any $a > 0$, $b > 0$ and $0 < a$, $b < 1$

$$\int_0^t \frac{1}{(t-s)^{1-a}}\frac{1}{s^{1-b}} ds \leq \frac{C}{t^{1-(a+b)}}, \qquad (a > 0,\, b > 0),$$

$$\int_0^{\frac{t}{2}} \frac{1}{(t-s)^b}\frac{1}{s^{1-a}} ds \leq \frac{C}{t^{b-a}}, \quad \int_{\frac{t}{2}}^t \frac{1}{(t-s)^{1-a}}\frac{1}{s^b} ds \leq \frac{C}{t^{b-a}} \qquad (a > 0, b \geq 0).$$

Using the estimate of the heat kernel, we obtain

$$\|s(t)\|_{L^p} \lesssim t^{-1+\frac{1}{p}}\|s_0\|_{L^1} + \chi\int_0^t \|\nabla S(t-\tau)(s(\tau)\nabla c(\tau))\|_{L^p}d\tau$$

$$+\int_0^t \|\nabla S(t-\tau)(u(\tau)s(\tau))\|_{L^p}d\tau + \epsilon\int_0^t \|S(t-\tau)(s(\tau)e(\tau))\|_{L^p}d\tau$$

$$\lesssim t^{-1+\frac{1}{p}}\|s_0\|_{L^1} + \chi\int_0^t \frac{1}{(t-\tau)^{\frac{3}{2}-\frac{1}{\alpha}}}\|s(\tau)\|_{L^p}\|\nabla c(\tau)\|_{L^q}d\tau$$

$$+\int_0^t \frac{1}{(t-\tau)^{\frac{3}{2}-\frac{1}{\alpha'}}}\|u(\tau)\|_{L^{\frac{2r}{2-r}}}\|s(\tau)\|_{L^p}d\tau + \epsilon\int_0^t \frac{1}{(t-\tau)^{(1-\frac{1}{\beta})}}\|s(\tau)\|_{L^p}\|e(\tau)\|_{L^l}d\tau$$

$$:= t^{-1+\frac{1}{p}}\|s_0\|_{L^1} + I_1 + I_2 + I_3,$$

where $1+\frac{1}{p} = \frac{1}{\alpha}+\frac{1}{p}+\frac{1}{q}$, $1+\frac{1}{2}-\frac{1}{r} = \frac{1}{\alpha'}$, and $1+\frac{1}{p} = \frac{1}{\beta}+\frac{1}{p}+\frac{1}{l}$. We estimate I_1, I_2 and I_3 as follows:

$$I_1 \lesssim \chi\int_0^t \frac{1}{(t-\tau)^{\frac{3}{2}-\frac{1}{\alpha}}}\cdot\frac{1}{\tau^{\frac{3}{2}-\frac{1}{p}-\frac{1}{q}}}d\tau\|s\|_{\mathcal{K}_p}\|c\|_{\mathcal{N}_q} \lesssim \frac{\chi}{t^{1-\frac{1}{p}}}\|s\|_{\mathcal{K}_p}\|c\|_{\mathcal{N}_q},$$

$$I_2 \lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{3}{2}-\frac{1}{\alpha'}}}\cdot\frac{1}{\tau^{2-\frac{1}{r}-\frac{1}{p}}}d\tau\|\omega\|_{\mathcal{K}_r}\|s\|_{\mathcal{K}_p} \lesssim \frac{1}{t^{1-\frac{1}{p}}}\|\omega\|_{\mathcal{K}_r}\|s\|_{\mathcal{K}_p},$$

and

$$I_3 \lesssim \epsilon\int_0^t \frac{1}{(t-\tau)^{1-\frac{1}{\beta}}}\cdot\frac{1}{\tau^{2-\frac{1}{p}-\frac{1}{l}}}d\tau\|s\|_{\mathcal{K}_p}\|e\|_{\mathcal{K}_l} \lesssim \frac{\epsilon}{t^{1-\frac{1}{p}}}\|s\|_{\mathcal{K}_p}\|e\|_{\mathcal{K}_l},$$

where we use the embedding $\|u\|_{L^{\frac{2r}{2-r}}} \lesssim \|\omega\|_{L^r}$, hence $1 < r < 2$ is required. Therefore, we deduce that for any exponent p, q, r, l in (4.3)

$$\|s\|_{\mathcal{K}_p} \leq C\|s_0\|_{L^1} + C\|s\|_{\mathcal{K}_p}(\chi\|c\|_{\mathcal{N}_q} + \|\omega\|_{\mathcal{K}_r} + \epsilon\|e\|_{\mathcal{K}_l}). \tag{4.6}$$

Similarly, we obtain

$$\|e\|_{\mathcal{K}_l} \leq C\|e_0\|_{L^1} + C\|e\|_{\mathcal{K}_l}(\|\omega\|_{\mathcal{K}_r} + \epsilon\|s\|_{\mathcal{K}_p}). \tag{4.7}$$

By applying (4.4), (4.5) to the c equation we easily deduce that

$$\|c\|_{\mathcal{N}_q} \leq C\|\nabla c_0\|_{L^2} + C\|c\|_{\mathcal{N}_q}\|\omega\|_{\mathcal{K}_r} + C_*\|e\|_{\mathcal{K}_l}. \tag{4.8}$$

Next by similar computaions as in [2, Lemma 3], we obtain that

$$\|\omega\|_{\mathcal{K}_r} \leq C\|\omega_0\|_{L^1} + C\|\nabla\phi\|_{L^2}(\|s\|_{\mathcal{K}_p} + \|e\|_{\mathcal{K}_l}) + C\|\omega\|^2_{\mathcal{K}_r}. \tag{4.9}$$

Here we set $M_1 := C_*$ and $M_2 = C\|\nabla\phi\|_{L^2}$, where C_* and $C\|\nabla\phi\|_{L^2}$ are the constants in (4.8) and (4.9) respectively. Indeed,

$$\|\omega(t)\|_{L^r}$$

$$\lesssim t^{-1+\frac{1}{r}}\|\omega_0\|_{L^1} + \int_0^t \|\nabla G(t-\tau)(s+e)(\tau)\nabla\phi\|_{L^r} + \int_0^t \|\nabla G(t-\tau)u(\tau)\omega(s)\|_{L^r}$$

$$\lesssim t^{-1+\frac{1}{r}}\|\omega_0\|_{L^1} + \int_0^t \frac{1}{(t-\tau)^{\frac{3}{2}-\frac{1}{\alpha}}}\|s(\tau)\|_{L^p}\|\nabla\phi\|_{L^2} + \frac{1}{(t-\tau)^{\frac{3}{2}-\frac{1}{\beta}}}\|e(\tau)\|_{L^l}\|\nabla\phi\|_{L^2}$$

$$+\int_0^t \frac{1}{(t-\tau)^{\frac{3}{2}-\frac{1}{\alpha'}}}\|u\|_{L^{\frac{2r}{2-r}}}\|\omega\|_{L^r} = t^{-1+\frac{1}{r}}\|\omega_0\|_{L^1} + J_1 + J_2 + J_3,$$

where $\frac{1}{r} = \frac{1}{\alpha} + \frac{1}{p} - \frac{1}{2}$, $\frac{1}{r} = \frac{1}{\beta} + \frac{1}{p} - \frac{1}{2}$ and $\frac{1}{\alpha'} = \frac{3}{2} - \frac{1}{r}$. Similar estimates as above yield that

$$J_1 \lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{3}{2}-\frac{1}{\alpha}}} \frac{1}{\tau^{1-\frac{1}{p}}} ds \,\|\nabla\phi\|_{L^2} \|s\|_{\mathcal{K}_p} \lesssim \frac{1}{t^{1-\frac{1}{r}}} \|\nabla\phi\|_{L^2} \|s\|_{\mathcal{K}_p}$$

and

$$J_2 \lesssim \frac{1}{t^{1-\frac{1}{r}}} \|\nabla\phi\|_{L^2} \|e\|_{\mathcal{K}_l}.$$

On the other hand, via $\|u(t)\|_{L^s} \lesssim \|\omega(t)\|_{L^r}$ with $1/r = 1/s + 1/2$, we obtain

$$J_3 \lesssim \int_0^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{\alpha'}}} \frac{1}{s^{2(1-\frac{1}{r})}} ds \,\|\omega\|^2_{\mathcal{K}_r} \lesssim \frac{1}{t^{1-\frac{1}{r}}} \|\omega\|^2_{\mathcal{K}_r}.$$

Thus, we have (4.9). Multiplying (4.6) and (4.7) with $2M_2$ and $2(M_1+M_2)$ (M_1 and M_2 are large constants, which are larger than $\|\nabla\phi\|_{L^2}$ and C_*), respectively, and summing up above estimates, we have

$$M_2\|s\|_{\mathcal{K}_p} + (M_1+M_2)\|e\|_{\mathcal{K}_l} + \|c\|_{\mathcal{N}_q} + \|\omega\|_{\mathcal{K}_r}$$
$$\le C(\|s_0\|_{L^1} + \|e_0\|_{L^1} + \|\omega_0\|_{L^1} + \|\nabla c_0\|_{L^2}) + \left(\|s\|_{\mathcal{K}_p} + \|e\|_{\mathcal{K}_l} + \|\omega\|_{\mathcal{K}_r} + \|c\|_{\mathcal{N}_q}\right)^2.$$

Under the smallness assumption, we have

$$\|(s,e,\omega)\|_{\mathcal{K}_{p,l,r}} + \|c\|_{\mathcal{N}_q} \lesssim \|s_0\|_{L^1} + \|e_0\|_{L^1} + \|\nabla c_0\|_{L^2} + \|\omega_0\|_{L^1} \lesssim \epsilon_1. \quad (4.10)$$

Now we extend the range of p, r of $\|s\|_{\mathcal{K}_p}$, $\|\omega\|_{\mathcal{K}_r}$ and consider $\|c\|_{\mathcal{N}_\infty}$ such that

$$\|s\|_{\mathcal{K}_p} := \sup_{t\ge 0} t^{1-\frac{1}{p}} \|s(t)\|_{L^p}, \quad 2 \le p \le \infty,$$
$$\|c\|_{\mathcal{N}_\infty} := \sup_{t\ge 0} t^{\frac{1}{2}} \|\nabla c(t)\|_{L^\infty},$$
$$\|\omega\|_{\mathcal{K}_r} := \sup_{t\ge 0} t^{1-\frac{1}{r}} \|\omega(t)\|_{L^r}, \quad 1 < r \le \infty.$$

Since $\int_0^t S(t-\tau)(se)(\tau)d\tau$ is always nonnegative, we have

$$\|s\|_{L^\infty}(t) \lesssim t^{-1}\|s_0\|_{L^1} + \chi \int_0^t \|\nabla S(t-\tau)(s(\tau)\nabla c(\tau))\|_{L^\infty} d\tau$$
$$+ \int_0^t \|\nabla S(t-\tau)(u(\tau)s(\tau))\|_{L^\infty} d\tau := t^{-1}\|s_0\|_{L^1} + I_1 + I_2.$$

I_1 and I_2 can be estimated as follows:

$$I_1 \lesssim \int_0^{\frac{t}{2}} \frac{1}{(t-\tau)^{\frac{3}{2}}} \|s\nabla c\|_{L^1}(\tau) d\tau + \int_{\frac{t}{2}}^t \frac{1}{(t-\tau)^{\frac{1}{2}}} \|s\nabla c\|_{L^\infty}(\tau) d\tau$$
$$\lesssim \int_0^{\frac{t}{2}} \frac{1}{(t-\tau)^{\frac{3}{2}}} \|s\|_{L^1} \|\nabla c\|_{L^\infty} d\tau + \int_{\frac{t}{2}}^t \frac{1}{(t-\tau)^{\frac{1}{2}}} \|s\|_{L^\infty} \|\nabla c\|_{L^\infty} d\tau$$
$$\lesssim \frac{\epsilon_1}{t^1} \|c\|_{\mathcal{N}_\infty} + \frac{1}{t^1} \|s\|_{\mathcal{K}_\infty} \|c\|_{\mathcal{N}_\infty},$$

and

$$I_2 \lesssim \int_0^{\frac{t}{2}} \frac{1}{(t-\tau)^{\frac{3}{2}}} \|us\|_{L^1} d\tau + \int_{\frac{t}{2}}^t \frac{1}{(t-\tau)^{\frac{3}{2}-\frac{1}{2^-}}} \|us\|_{L^{2+}} d\tau$$
$$\lesssim \int_0^{\frac{t}{2}} \frac{1}{(t-\tau)^{\frac{3}{2}}} \|u\|_{L^{2+}} \|s\|_{L^{2-}} + \int_{\frac{t}{2}}^t \frac{1}{(t-\tau)^{\frac{3}{2}-\frac{1}{2^-}}} \|u\|_{L^{2+}} \|s\|_{L^\infty}$$

$$\lesssim \frac{1}{t^{\frac{3}{2}}}\int_0^{\frac{t}{2}} \|\omega\|_{L^{\frac{\alpha}{\alpha-1}}}\|s\|_{L^{2-}} + \int_{\frac{t}{2}}^t \frac{1}{(t-\tau)^{\frac{3}{2}-\frac{1}{2^-}}}\|\omega\|_{L^{\frac{\alpha}{\alpha-1}}}\|s\|_{L^\infty}$$

$$\lesssim \frac{1}{t}\|s\|_{\mathcal{K}_{2^-}}\|\omega\|_{\mathcal{K}_{\frac{\alpha}{\alpha-1}}} + \frac{1}{t}\|s\|_{\mathcal{K}_\infty}\|\omega\|_{\mathcal{K}_{\frac{\alpha}{\alpha-1}}} \lesssim \frac{\epsilon_1^2}{t} + \frac{\epsilon_1}{t}\|s\|_{\mathcal{K}_\infty},$$

where α satisfy $2 < \alpha$ and 2^+ and 2^- satisfy $\frac{1}{2^+} = \frac{1}{2} - \frac{1}{\alpha}$ and $\frac{1}{2^-} = \frac{1}{2} + \frac{1}{\alpha}$.

Adding these estimates, we obtain that

$$\|s\|_{\mathcal{K}_\infty} \lesssim \|s_0\|_{L^1} + \epsilon_1\|c\|_{\mathcal{N}_\infty} + \|s\|_{\mathcal{K}_\infty}(\|c\|_{\mathcal{N}_\infty} + \epsilon_1) + \epsilon_1^2.$$

Using the similar methods with above and the estimates in [2], we have

$$\|c\|_{\mathcal{N}_\infty} \lesssim \|e_0\|_{L^1} + \epsilon_1^2 + \|e\|_{\mathcal{K}_\infty} + \epsilon_1\|c\|_{\mathcal{N}_\infty}.$$

Indeed,

$$\|\nabla c\|_{L^\infty}(t) \lesssim \frac{1}{t^{\frac{1}{2}}}\|c_0\|_{L^\infty} + \int_0^t \|\nabla S(t-\tau)e\|_{L^\infty}(\tau)d\tau$$

$$+ \int_0^t \|\nabla S(t-\tau)(u\nabla c)\|_{L^\infty}(\tau)d\tau = \frac{1}{t^{\frac{1}{2}}}\|c_0\|_{L^\infty} + J_1 + J_2.$$

Firstly, we estimate J_1.

$$\begin{aligned} J_1 &\lesssim \int_0^{t/2} \frac{1}{(t-\tau)^{\frac{3}{2}}}\|e(\tau)\|_{L^1}\,ds + \int_{t/2}^t \frac{1}{(t-\tau)^{\frac{1}{2}}}\|e(\tau)\|_{L^\infty}\,d\tau \\ &\lesssim \frac{1}{t^{\frac{1}{2}}}\|e\|_{L^1} + \frac{1}{t^{\frac{1}{2}}}\|e\|_{\mathcal{K}_\infty(\mathbb{R}^2)} \lesssim \frac{\epsilon_1}{t^{\frac{1}{2}}} + \frac{1}{t^{\frac{1}{2}}}\|e\|_{\mathcal{K}_\infty(\mathbb{R}^2)}. \end{aligned} \tag{4.11}$$

Before we estimate J_2, we set $1/4^+ = 1/4 - 1/\beta$ and $1/4^- = 1/4 + 1/\beta$ with $\beta > 4$. We then estimate J_2.

$$\begin{aligned} J_2 &\lesssim \int_0^{t/2} \frac{1}{t-\tau}\|u\nabla c\|_{L^2}\,d\tau + \int_{t/2}^t \frac{1}{(t-\tau)^{\frac{3}{2}-\frac{1}{2^-}}}\|u\nabla c\|_{L^{2^+}}(\tau)d\tau \\ &\lesssim \frac{1}{t}\int_0^{t/2}\|u\|_{L^{4^+}}\|\nabla c\|_{L^{4^-}}\,d\tau + \int_{t/2}^t \frac{1}{(t-\tau)^{\frac{3}{2}-\frac{1}{2^-}}}\|u\|_{L^{2^+}}\|\nabla c\|_{L^\infty}(\tau)d\tau \\ &\lesssim \frac{1}{t}\int_0^{t/2}\|\omega\|_{L^{\frac{4\beta}{3\beta-4}}}\|\nabla c\|_{L^{4^-}}\,ds + \int_{t/2}^t \frac{1}{(t-s)^{\frac{3}{2}-\frac{1}{2^-}}}\|\omega\|_{L^{\frac{\alpha}{\alpha-1}}}\|\nabla c\|_{L^\infty}(\tau)d\tau \\ &\lesssim \frac{1}{t^{\frac{1}{2}}}\|\omega\|_{\mathcal{K}_{\frac{4\beta}{3\beta-4}}(\mathbb{R}^2)}\|c\|_{\mathcal{N}_{4^-}(\mathbb{R}^2)} + \frac{1}{t^{\frac{1}{2}}}\|\omega\|_{\mathcal{K}_{\frac{\alpha}{\alpha-1}}(\mathbb{R}^2)}\|c\|_{\mathcal{N}_\infty(\mathbb{R}^2)} \\ &\lesssim \frac{\epsilon_1^2}{t^{\frac{1}{2}}} + \frac{\epsilon_1}{t^{\frac{1}{2}}}\|c\|_{\mathcal{N}_\infty(\mathbb{R}^2)}, \end{aligned} \tag{4.12}$$

where the estimates for the low range of $\|\omega\|_{\mathcal{K}_p}$, $\|c\|_{\mathcal{N}_q}$ (4.10) is used and $2+, 2-, \alpha$ are same exponents as for I_2 before. Combining (4.11)and (4.12), we have

$$\|\nabla c\|_{L^\infty}(t) \lesssim \frac{1}{t^{\frac{1}{2}}}\|c_0\|_{L^\infty} + \frac{\epsilon_1^2}{t^{\frac{1}{2}}} + \frac{1}{t^{\frac{1}{2}}}\|e\|_{\mathcal{K}_\infty(\mathbb{R}^2)} + \frac{\epsilon_1}{t^{\frac{1}{2}}}\|c\|_{\mathcal{N}_\infty(\mathbb{R}^2)}.$$

Next, we estimate the vorticity for $2 \le r < \infty$.

$$\|\omega(t)\|_{L^r} \lesssim t^{-1+\frac{1}{r}}\|\omega_0\|_{L^1} + \int_0^t \|\nabla^\perp G(t-\tau)(s\nabla\phi)(\tau)\|_{L^r}d\tau$$

$$+ \int_0^t \|\nabla^\perp G(t-\tau)(e\nabla\phi)(\tau)\|_{L^r}d\tau + \int_0^t \|\nabla G(t-\tau)(u\omega)(\tau)\|_{L^r}d\tau$$

$$= t^{-1+\frac{1}{r}}\|\omega_0\|_{L^1} + K_1 + K_2 + K_3.$$

If we consider $r > 2$, then we obtain

$$K_1 \lesssim \int_0^{t/2} \frac{1}{(t-\tau)^{\frac{3}{2}-\frac{1}{r}}}\|s(\tau)\|_{L^1}^{\frac{1}{2}}\|s(\tau)\|_{L^\infty}^{\frac{1}{2}}\|\nabla\phi\|_{L^2} + \int_{t/2}^{t} \frac{1}{(t-\tau)^{1-\frac{1}{r}}}\|s(\tau)\|_{L^\infty}\|\nabla\phi\|_{L^2}$$

$$\lesssim \frac{\epsilon_1}{t^{1-\frac{1}{r}}} + \frac{1}{t^{1-\frac{1}{r}}}\|s\|_{\mathcal{K}_\infty}.$$

Similarly, we have

$$K_2 \lesssim \frac{\epsilon_1}{t^{1-\frac{1}{r}}} + \frac{1}{t^{1-\frac{1}{r}}}\|e\|_{\mathcal{K}_\infty}.$$

If the exponents r^*, $\tilde{r}$ are defined by $\frac{1}{r^*} = \frac{1}{2} - \frac{1}{r}$ and $\frac{1}{r^*} = \frac{1}{\tilde{r}} - \frac{1}{2}$, then we estimate

$$K_3 \lesssim \int_0^{\frac{t}{2}} \frac{1}{(t-\tau)^{\frac{3}{2}-\frac{1}{2^-}}}\|u\|_{L^{2+}}\|\omega\|_{L^r} + \int_{\frac{t}{2}}^{t} \frac{1}{(t-\tau)^{1-\frac{1}{r}}}\|u\|_{L^{r^*}}\|\omega\|_{L^r}$$

$$\lesssim \frac{1}{t^{1-\frac{1}{r}}}\|\omega\|_{\mathcal{K}_{\frac{\alpha}{\alpha-1}}}\|\omega\|_{\mathcal{K}_r} + \frac{1}{t^{1-\frac{1}{r}}}\|\omega\|_{\mathcal{K}_{\tilde{r}}}\|\omega\|_{\mathcal{K}_r} \lesssim \frac{\epsilon_1}{t^{1-\frac{1}{r}}}\|\omega\|_{\mathcal{K}_r}.$$

Thus, we have

$$\|\omega\|_{\mathcal{K}_r} \lesssim \epsilon_1 + \|s\|_{\mathcal{K}_\infty} + \|e\|_{\mathcal{K}_\infty} + \epsilon_1\|\omega\|_{\mathcal{K}_r}.$$

By collecting all the estimates in the above, we find that

$$\|s\|_{\mathcal{K}_\infty} + \|e\|_{\mathcal{K}_\infty} + \|c\|_{\mathcal{N}_\infty} + \|\omega\|_{\mathcal{K}_r} \lesssim \epsilon_1.$$

This completes the proof. □

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