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Generalization of k-Uniformly Starlike and Convex Functions Using q-Difference Operator

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Abstract: In this article we have defined two new subclasses of analytic functions $k - \mathcal{S}_q[A,B]$ and $k - \mathcal{K}_q[A,B]$ by using q-difference operator in an open unit disk. Furthermore, the necessary and sufficient conditions along with certain other useful properties of these newly defined subclasses have been calculated by using q-difference operator.

Keywords: starlike functions; analytic functions; q-difference operator; Janowski functions

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1. Introduction

Assume that $\mathcal{H}(\mathcal{U})$ represents the analytic functions class in an open unit disk $\mathcal{U}$;

$$\mathcal{U} = \{z : z \in \mathbb{C} \quad \text{with} \quad |z| < 1\}.$$

Here $\mathbb{C}$ denotes the complex numbers set.

Similarly we consider the class $\mathcal{A}$ of those analytic functions that satisfies

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (\text{for all } z \in \mathcal{U}). \tag{1}$$

The class $\mathcal{A}$ is normalized by

$$f(0) = f'(0) - 1 = 0.$$

In the literature, the univalent functions class in $\mathcal{U}$ is expressed by $\mathcal{S}$. According to [1], the starlike functions class in $\mathcal{U}$ is represented by $\mathcal{S}^*$, that include $f \in \mathcal{A}$ with given condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathcal{U}). \tag{2}$$

Furthermore, the convex functions class in $\mathcal{U}$ is represented by $\mathcal{K}$, that consists the functions $f \in \mathcal{A}$ with given condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad (z \in \mathcal{U}). \tag{3}$$

It can be deduced from conditions defined in Equations (2) and (3) (see [2]) that

$$f(z) \in \mathcal{K} \Longleftrightarrow zf'(z) \in \mathcal{S}^*.$$

The analytic functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{4}$$

are denoted by the class $\mathcal{P}$, for which

$$\Re(p(z)) > 0 \qquad (\text{for all } \ z \in \mathcal{U}).$$

Subordination between any two analytic functions f and g in $\mathcal{U}$ may be represented as

$$f(z) \prec g(z) \quad \text{or} \quad f \prec g\,.$$

In case of Schwarz function w in $\mathcal{U}$, if w is analytic and satisfies

$$|w(z)| < 1 \qquad \& \qquad w(0) = 0,$$

then

$$f(z) = g(w(z)).$$

Similarly, if g satisfies condition of univalent function in $\mathcal{U}$. The equivalence transformed into

$$f(z) \prec g(z) \quad (z \in \mathcal{U}).$$

This implies

$$f(0) = g(0)$$

and

$$f(\mathcal{U}) \subset g(\mathcal{U}).$$

The conic domain was introduced by Kanas et al., described in [3], denoted by Ω_k having the form

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}. \tag{5}$$

The extremal functions, $p_k(z)$, family of conic domain Ω_k, having $p_k(0) = 1$ with $p_k'(0) > 0$ are normalized univalent functions in the form

$$p_k(z) = \begin{cases} \frac{1+z}{1-z} & \text{for} \qquad k = 0, \\ 1 + \frac{2}{\pi^2}(\log(v(z)))^2 & \text{for} \qquad k = 1, \\ 1 + \frac{2}{1+k^2}\sinh(v_1(z)) & \text{for} \qquad 0 < k < 1, \\ 1 + A\sin(B) + A & \text{for} \qquad 1 < k, \end{cases} \tag{6}$$

where

$$A = \frac{1}{k^2 - 1},$$

$$B = \frac{1}{R(t)} \int_0^{\frac{u(z)}{\sqrt{t}}} \frac{1}{\sqrt{1+x^2}\sqrt{1-(tx)^2}} dx,$$

$$v(z) = \frac{1+\sqrt{z}}{1-\sqrt{z}},$$

$$v_1(z) = \left(\left(\frac{2}{\pi^2} \arccos k \right) \operatorname{arctanh}\sqrt{z} \right),$$

$$u(z) = \frac{z - \sqrt{t}}{1 - \sqrt{tz}}$$

and we choose $t \in (0,1)$ such that

$$k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right).$$

Here $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ given by

$$R'(t) = R(\sqrt{1-r^2})$$

is the complementary integral of $R(t)$.

Definition 1 ([4])**.** *Assume that h be an analytic function with $h(0) = 1$, then $h \in \mathcal{P}[A,B]$, iff*

$$h(z) \prec \frac{Az+1}{Bz+1} \qquad \textit{with} \qquad (-1 \le B < A \le 1).$$

The class of analytic functions represented by $\mathcal{P}[A,B]$ was initially introduced by Janowski in 1973 (see [4]). He demonstrated that if a function $p \in \mathcal{P}$ exists then $h(z) \in \mathcal{P}[A,B]$. Mathematically, it takes the form

$$h(z) = \frac{[p(z)(1+A)] - (A-1)}{[p(z)(1+B)] - (B-1)} \qquad \text{with} \qquad (-1 \le B < A \le 1).$$

Definition 2. *Assume that $q \in (0,1)$ define the q-number $[\lambda]_q$, then*

$$[\lambda]_q = \begin{cases} \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 + q^3 + \ldots + q^{(n-1)}, & (\lambda = n \in \mathbb{N}), \\ \frac{1-q^\lambda}{1-q}, & (\lambda \in \mathbb{C}). \end{cases}$$

Definition 3. *According to [5,6], the q-derivative of a function f in a subset of $\mathbb{C}$ is defined by*

$$(D_q f)(z) = \begin{cases} f'(0) & (z = 0), \\ \frac{f(z)-f(qz)}{z(1-q)}, & (z \neq 0). \end{cases} \tag{7}$$

It provided the existence $f'(0)$. Similarly from Definition 3, it is noticed that

$$\lim_{q \to 1^-} (D_q f)(z) = \lim_{q \to 1^-} \frac{f(z) - f(qz)}{z(1-q)} = f'(z),$$

which is differentiable in the subset of $\mathbb{C}$. We also deduced from Equations (1) and (7) that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \tag{8}$$

Recently, the usage of the q-derivative operator is quite significant due to its applications in many diverse areas of mathematics, physics and other sciences. According to Srivastava et al. [7], the q-difference operator (D_q) in the context of Geometric Function Theory (GFT) was first utilized by Ismail et al., described in [8]. They studied a q-extension of starlike functions in $\mathcal{U}$ (see Definition 4 below). Afterwards many mathematicians continued their research highlighting the fundamental role in GFT. Mahmood et al. in [9] presents a detail description of the q-starlike functions class in conic domain, whereas in [10], the authors provided the class of q-starlike functions associated with Janowski functions. Moreover, the problems related to upper bound of third Hankel determinant $(H_3(1))$ for the class of q-starlike functions have been investigated, available in [11]. Later on, Srivastava et al. [12] have investigated the Hankel and Toeplitz determinants of a subclass of q-starlike functions. Many other authors have studied and investigated a number of other new subclasses of

q-starlike, q-convex and q-close-to-convex functions. They obtained a number of useful results like, coefficient inequalities, sufficient conditions, partial sums results and results related to radius problems (see for example [13–16]).

Definition 4 ([8])**.** *Let a function $f \in \mathcal{S}$. Then $f \in \mathcal{S}_q^*$ if*

$$f(0) = 0 = f'(0) - 1 \tag{9}$$

and

$$\left|(D_q f)(z)\frac{z}{f(z)} - \frac{1}{1-q}\right| \leq \frac{1}{1-q}, \qquad (z \in \mathcal{U}). \tag{10}$$

As $q \to 1^-$, it is clearly noticed that

$$\left|w - \frac{1}{1-q}\right| \leq \frac{1}{1-q}.$$

Similarly the class of q-starlike functions denoted by $\mathcal{S}_q^*$ decreases to the known class $\mathcal{S}^*$. Likewise, using subordination principle among analytic functions, the conditions described in Equations (9) and (10), may be revised as follows (see also [17])

$$\frac{z}{f(z)}(D_q f)(z) \prec \widehat{p}(z) \qquad \left(\widehat{p}(z) := \frac{1+z}{1-qz}\right). \tag{11}$$

Remark 1. *For function $f \in \mathcal{A}$, the idea of Alexander's theorem [1] was used by Baricz and Swaminathan [18] to define the class C_q of q-convex function in the following way*

$$f(z) \in C_q \Longleftrightarrow z(D_q f)(z) \in \mathcal{S}_q^*.$$

In order to utilize the q-difference operator $(D_q f(z))$, this study has introduced two new subclasses of $\mathcal{A}$, i.e., $k - \mathcal{S}_q[A, B]$ and $k - \mathcal{K}_q[A, B]$.

Definition 5. *For $0 < q < 1$, $k \geq 0$ and $-1 \leq B < A \leq 1$, an analytic function $f \in k - \mathcal{S}_q[A, B]$ if it satisfies*

$$\frac{zD_q f(z)}{f(z)} \prec \frac{p_k(z) + 1 + (p_k(z) - 1)A}{p_k(z) + 1 + (p_k(z) - 1)B}, \qquad z \in \mathcal{U}, \tag{12}$$

where $p_k(z)$ is given by Equation (6).

Definition 6. *For $0 < q < 1$, $k \geq 0$ and $-1 \leq B < A \leq 1$, an analytic function $f \in k - \mathcal{K}_q[A, B]$ if it satisfies*

$$\frac{D_q(zD_q f(z))}{D_q f(z)} \prec \frac{p_k(z) + 1 + (p_k(z) - 1)A}{p_k(z) + 1 + (p_k(z) - 1)B}. \tag{13}$$

According to [4,19,20], it is noted that

1. $0 - \mathcal{S}_q[A, B] = \mathcal{S}_q[A, B]$.
2. $0 - \mathcal{K}_q[A, B] = \mathcal{K}_q[A, B]$.
3. $\lim_{q \to 1^-} k - \mathcal{S}_q[A, B] = k - \mathcal{S}[A, B]$.
4. $\lim_{q \to 1^-} k - \mathcal{K}_q[A, B] = k - \mathcal{K}[A, B]$.
5. $\lim_{q \to 1^-} 0 - \mathcal{S}_q[A, B] = \mathcal{S}[A, B]$.
6. $\lim_{q \to 1^-} 0 - \mathcal{K}_q[A, B] = \mathcal{K}[A, B]$.

Hence, from Equations (12) and (13) we can write

$$f \in k - \mathcal{K}_q[A,B] \Longleftrightarrow zD_q f \in k - \mathcal{S}_q[A,B].$$

As far as we know, there is minimal work on q-calculus related with conic domain in the literatures. The major objective of this work is to define a new subclass of q-starlike functions associated with the conic type domain. We find a number of useful results for our define function class and present some special cases of our results, in form of corollaries and remarks.

2. Main Results

In this section, we assume $\theta \in [0, 2\pi)$, $k \geq 0$, $0 < q < 1$, whereas $-1 \leq B < A \leq 1$.

Theorem 1. *For any analytic function, $f \in k - \mathcal{S}_q[A,B]$ iff*

$$\frac{1}{z}\left[\frac{z - \mathcal{L}qz^2}{(1-z)(1-qz)} * f(z)\right] \neq 0, \qquad (z \in \mathcal{U}), \tag{14}$$

for all $\mathcal{L} = \mathcal{L}_\theta = \dfrac{(A+1)p_k(e^{i\theta}) - (A-1)}{(A-B)(p_k(e^{i\theta}) - 1)}$, *also* $\mathcal{L} = 1$.

Proof. If $f \in k - \mathcal{S}_q[A,B]$, then

$$\frac{zD_q f(z)}{f(z)} \prec \frac{p_k(z) + 1 + (p_k(z) - 1)A}{p_k(z) + 1 + (p_k(z) - 1)B}. \tag{15}$$

The LHS of Equation (15) in $\mathcal{U}$ is holomorphic, which follows $f(z) \neq 0$ and $z \in \mathcal{U}^* = \mathcal{U}\backslash\{0\}$. This means $\frac{1}{z}f(z) \neq 0$, because Equation (14) holds for $\mathcal{L} = 1$. From Equation (15) we can say that there must exist a function $w(z)$ in $\mathcal{U}$, which should be analytic having $|w(z)| < 1$ with $w(0) = 0$. This is because of the property of subordination between the two holomorphic functions, such that

$$\frac{zD_q f(z)}{f(z)} = \frac{[p_k(w(z))](1+A) - (A-1)}{[p_k(w(z))](1+B) - (B-1)}, \qquad (z \in \mathcal{U}),$$

and is equivalent to

$$\frac{zD_q f(z)}{f(z)} \neq \frac{(p_k(e^{i\theta}) - 1)A + (p_k(e^{i\theta}) + 1)}{(p_k(e^{i\theta}) - 1)B + (p_k(e^{i\theta}) + 1)}, \qquad (z \in \mathcal{U} : 0 \leq \theta < 2\pi, k \geq 0) \tag{16}$$

or

$$\frac{1}{z}\left[\{(p_k(e^{i\theta}) - 1)B + (p_k(e^{i\theta}) + 1)\}zD_q f(z) - \{(p_k(e^{i\theta}) - 1)A + (p_k(e^{i\theta}) + 1)\}f(z)\right] \neq 0. \tag{17}$$

Since,

$$f(z) = \frac{z}{1-z} * f(z) \quad \text{and} \quad \frac{z}{(1-z)(1-zq)} * f(z) = zD_q f(z). \tag{18}$$

From Equation (18), we may write Equation (17) in the form

$$\frac{1}{z}\left[f(z) * \left(\frac{z\{(p_k(e^{i\theta}) - 1)B + (p_k(e^{i\theta}) + 1)\}}{(1-z)(1-zq)} - \frac{z\{(p_k(e^{i\theta}) - 1)A + (p_k(e^{i\theta}) + 1)\}}{(1-z)}\right)\right]$$
$$= \frac{(B-A)(p_k(e^{i\theta}) - 1)}{z}\left[f(z) * \frac{z - \left(\frac{(p_k(e^{i\theta})-1)A + (p_k(e^{i\theta})+1)}{(A-B)(p_k(e^{i\theta})-1)}\right)qz^2}{(1-z)(1-zq)}\right] \neq 0,$$
$$(z \in \mathcal{U} : 0 \leq \theta < 2\pi, k \geq 0),$$

which gives similar result as presented in Equation (14). This proves the necessary part of Theorem 1.

Conversely: As we know that for $L = 1$, Equation (14) holds, it obeys the condition that $\frac{1}{z}f(z)$ is not equal to zero for all $z \in \mathcal{U}$. Therefore, the function $\psi(z) = \frac{zD_qf(z)}{f(z)}$ is analytic in $\mathcal{U}$. In the earlier part of our proof, it was shown that our supposition Equation (14) can also be written in the form of Equation (16). So

$$\frac{zD_qf(z)}{f(z)} \neq \frac{(p_k(e^{i\theta})-1)A+(p_k(e^{i\theta})+1)}{(p_k(e^{i\theta})-1)B+(p_k(e^{i\theta})+1)}, \quad (z \in \mathcal{U} : 0 \leq \theta < 2\pi, k \geq 0). \tag{19}$$

If we write

$$\zeta(z) = \frac{(p_k(e^{i\theta})-1)A+(p_k(e^{i\theta})+1)}{(p_k(e^{i\theta})-1)B+(p_k(e^{i\theta})+1)}, \quad (z \in \mathcal{U}),$$

then Equation (19) shows that $\psi(z) \cap \zeta(z) = \emptyset$. Thus, the connected component of $\mathbb{C}\backslash\zeta(\partial\mathcal{U})$ includes the simply-connected domain $\psi(\mathcal{U})$. As we know that $\psi(0) = \zeta(0)$ along with univalence of function giving $\psi(z) \prec \zeta(z)$, as mentioned in the subordination fact (Equation (15)), i.e., $f \in k - \mathcal{S}_q[A,B]$, which gives the desired result. □

Corollary 1 ([20])**.** *For any function f represented by Equation (1), $f \in \mathcal{S}_q[A,B]$ iff*

$$\frac{1}{z}\left[\frac{z-\mathcal{L}qz^2}{(qz-1)(z-1)} * f(z)\right] \neq 0, \quad (z \in \mathcal{U}),$$

$\forall\, \mathcal{L} = \mathcal{L}_\theta = \frac{e^{-i\theta}+A}{A-B}$, *also* $\mathcal{L} = 1$.

Corollary 2 ([20])**.** *For any function f represented by Equation (1), $f \in \mathcal{S}[A,B]$ iff*

$$\frac{1}{z}\left[\frac{z-\mathcal{L}z^2}{(1-z)^2} * f(z)\right] \neq 0, \quad (z \in \mathcal{U}),$$

$\forall\, \mathcal{L} = \frac{e^{-i\theta}+A}{A-B} = \mathcal{L}_\theta$, *as well as* $\mathcal{L} = 1$.

Corollary 3 ([20])**.** *For any function f represented by Equation (1), $f \in \mathcal{S}_q(\alpha)$ iff*

$$\frac{1}{z}\left[\frac{z-\mathcal{M}qz^2}{(1-z)(1-qz)} * f(z)\right] \neq 0, \quad (z \in \mathcal{U}),$$

$\forall\, \mathcal{M} = \frac{e^{-i\theta}+1-2\alpha}{2(1-\alpha)} = \mathcal{M}_\theta$ *with* $0 \leq \alpha < 1$, *as well as* $\mathcal{M} = 1$.

By putting $q \to 1^-$ in corollary 2, we will get the desired corollary.

Theorem 2. *For any function f represented by Equation (1). The function $f \in k - \mathcal{K}_q[A,B]$ iff*

$$\frac{1}{z}\left[\frac{z+[1-(1+q)\mathcal{L}]zq^2}{(1-z)(1-qz)(1-q^2z)} * f(z)\right] \neq 0, \quad (z \in \mathcal{U}),$$

$\mathcal{L} = \mathcal{L}_\theta = \frac{(1+A)p_k(e^{i\theta})-(A-1)}{(A-B)(p_k(e^{i\theta})-1)}$, *also* $\mathcal{L} = 1$.

Proof. Let us consider

$$g(z) = \frac{z-\mathcal{L}zq^2}{(1-z)(1-qz)}$$

and

$$zD_q g(z) = \frac{z + [1 - (1+q)\mathcal{L}]zq^2}{(1-z)(1-qz)(1-q^2z)}.$$

From identity $zD_q f(z) * g(z) = f(z) * zD_q g(z)$. By using the relation

$$f \in k - \mathcal{K}_q[A, B] \Longleftrightarrow zD_q f(z) \in k - \mathcal{S}_q[A, B].$$

Theorem 1 also gives this result.

By setting $k = 0$ in Theorem 2, one may get the result obtained by Aouf and Seoudy, presented in [20]. □

Corollary 4 ([20]). *For any function f represented by (1), $f \in \mathcal{K}_q[A, B]$ iff*

$$\frac{1}{z}\left[\frac{z + [1 - (1+q)\mathcal{L}]zq^2}{(1-z)(1-qz)(1-q^2z)} * f(z)\right] \neq 0, \qquad (z \in \mathcal{U}),$$

$\forall\, \mathcal{L} = \mathcal{L}_\theta = \dfrac{e^{-i\theta} + A}{A - B}$, *also* $\mathcal{L} = 1$.

Substituting $k = 0$ in Theorem 2, one may get the desired corollary.

Corollary 5 ([20]). *For any function f represented by Equation (1), $f \in \mathcal{K}[A, B]$ iff*

$$\frac{1}{z}\left[\frac{z + [1 - 2\mathcal{L} * f(z)]z^2}{(1-z)^3}\right] \neq 0, \qquad (z \in \mathcal{U}),$$

$\forall\, \mathcal{L} = \dfrac{e^{-i\theta} + A}{A - B} = \mathcal{L}_\theta$, *as well as* $\mathcal{L} = 1$.

Corollary 6 ([20]). *For any function f represented by Equation (1), $f \in \mathcal{K}_q(\alpha)$ with $(0 \leq \alpha < 1)$ iff*

$$\frac{1}{z}\left[\frac{z + [1 - (1+q)\mathcal{M}]qz^2}{(1-z)(1-qz)(1-q^2z)} * f(z)\right] \neq 0, \qquad (z \in \mathcal{U}),$$

$\forall\, \mathcal{M} = \mathcal{M}_\theta = \dfrac{e^{-i\theta} + 1 - 2\alpha}{2(1-\alpha)}$ *and* $\mathcal{M} = 1$.

By taking $k = 0$ and $q \to 1^-$ in corollary 5, we will get the desired corollary.

Theorem 3. *For any function f represented by Equation (1), $f \in k - \mathcal{S}_q[A, B]$ has a necessary and sufficient condition*

$$1 - \sum_{n=2}^{\infty} \frac{[n]_q\{(p_k(e^{i\theta}) - 1)B + (p_k(e^{i\theta}) + 1)\} - \{(p_k(e^{i\theta}) - 1)A + (p_k(e^{i\theta}) + 1)\}}{(A - B)(p_k(e^{i\theta}) - 1)} a_n z^{n-1} \neq 0. \quad (20)$$

Proof. Keeping in view Theorem 1, we could found $f \in k - \mathcal{S}_q[A, B]$ iff

$$\frac{1}{z}\left[\frac{z - \mathcal{L}qz^2}{(1-z)(1-qz)} * f(z)\right] \neq 0, \qquad (z \in \mathcal{U}), \quad (21)$$

for all $\mathcal{L} = \mathcal{L}_\theta = \dfrac{(1+A)p_k(e^{i\theta}) - (A-1)}{(A-B)(p_k(e^{i\theta}) - 1)}$ and $\mathcal{L} = 1$.

Using Equation (21) we may write

$$\begin{aligned}\frac{1}{z}\left[\frac{z}{(1-z)(1-qz)} * f(z) - \frac{\mathcal{L}qz^2}{(1-z)(1-qz)}\right] &= \frac{1}{z}\{zD_q f(z) - \mathcal{L}[zD_q f(z) - f(z)]\} \\ &= 1 - \sum_{n=2}^{\infty}([n]_q(\mathcal{L} - 1) - \mathcal{L})a_n z^{n-1},\end{aligned}$$

which completes the desired proof. □

Corollary 7 ([20])**.** *For any function f represented by Equation (1), $f \in \mathcal{S}_q[A,B]$ has a necessary and sufficient condition*

$$1-\sum_{n=2}^{\infty}[n]_q\left[\frac{(B+e^{-i\theta})-A-e^{-i\theta}}{(A-B)}\right]a_n z^{n-1}\neq 0, \qquad (z\in\mathcal{U}).$$

Substituting $k=0$ in Theorem 3, one may get the desired corollary.

Corollary 8 ([20])**.** *For any function f represented by Equation (1), $f \in \mathcal{S}[A,B]$ has a necessary and sufficient condition*

$$1-\sum_{n=2}^{\infty}[n]_q\left[\frac{(B+e^{-i\theta})-A-e^{-i\theta}}{(A-B)}\right]a_n z^{n-1}\neq 0, \qquad (z\in\mathcal{U}).$$

Corollary 9 ([20])**.** *For any function f represented by Equation (1), $f \in \mathcal{S}_q(\alpha)$ has a necessary and sufficient condition*

$$1-\sum_{n=2}^{\infty}[n]_q\left[\frac{(e^{-i\theta}-1)-1-e^{-i\theta}+2\alpha}{2(1-\alpha)}\right]a_n z^{n-1}\neq 0, \qquad (z\in\mathcal{U}).$$

By taking $k=0$ and $q\to 1^-$ in Theorem 3, one may get the desired corollary.

Theorem 4. *For any function f represented by Equation (1). The function $f\in k-\mathcal{K}_q[A,B]$ has a necessary and sufficient condition*

$$1-\sum_{n=2}^{\infty}\frac{[n]_q\{(1+B)p_k(e^{i\theta})-(B-1)\}-(1+A)p_k(e^{i\theta})+(A-1)}{(A-B)(p_k(e^{i\theta})-1)}a_n z^{n-1}\neq 0,\ (z\in\mathcal{U}). \tag{22}$$

Proof. From Theorem 2, we could found $f\in k-\mathcal{K}_q[A,B]$ iff

$$\frac{1}{z}\left[\frac{z+[1-(1+q)\mathcal{L}]qz^2}{(1-z)(1-qz)(1-q^2z)}*f(z)\right]\neq 0, \qquad (z\in\mathcal{U}), \tag{23}$$

$\forall\ \mathcal{L}=\mathcal{L}_\theta=\dfrac{(1+A)p_k(e^{i\theta})-(A-1)}{(A-B)(p_k(e^{i\theta})-1)}$ and $\mathcal{L}=1$.

After simplification, the LHS of Equation (23) may takes the form

$$\begin{aligned}\frac{1}{z}\left[\frac{z+[1-(1+q)\mathcal{L}]qz^2}{(1-z)(1-qz)(1-q^2z)}*f(z)\right] &= 1-\sum_{n=2}^{\infty}[n]_q\big(q[n-1]_q(\mathcal{L}-1)-1\big)a_n z^{n-1}\\ &= 1-\sum_{n=2}^{\infty}\big([n]_q(\mathcal{L}-1)-\mathcal{L}\big)a_n z^{n-1}.\end{aligned}$$

This proves our required result. □

Corollary 10 ([20])**.** *For any function f represented by Equation (1), $f \in \mathcal{K}_q[A,B]$ has a necessary and sufficient condition*

$$1-\sum_{n=2}^{\infty}[n]_q\left[\frac{[n]_q(B+e^{-i\theta})-A-e^{-i\theta}}{(A-B)}\right]a_n z^{n-1}\neq 0, \qquad (z\in\mathcal{U}).$$

Substituting $k=0$ in Theorem 4, one may get the desired corollary.

Corollary 11 ([20]). *For any function f represented by Equation (1), $f \in \mathcal{K}[A,B]$ has a necessary and sufficient condition*

$$1-\sum_{n=2}^{\infty} n\left[\frac{n(B+e^{-i\theta})-A-e^{-i\theta}}{(A-B)}\right]a_n z^{n-1} \neq 0, \quad (z \in \mathcal{U}).$$

Corollary 12 ([20]). *For any function f represented by Equation (1), $f \in \mathcal{K}_q(\alpha)$ has a necessary and sufficient condition*

$$1-\sum_{n=2}^{\infty} [n]_q\left[\frac{[n]_q(e^{-i\theta}-1)-1-e^{-i\theta}+2\alpha}{2(1-\alpha)}\right]a_n z^{n-1} \neq 0, \quad (z \in \mathcal{U}).$$

As an application of Theorems 3 and 4, one may determine inclusion property and coefficient estimates for a function of the form Equation (1) in subclasses defined by $k-\mathcal{S}_q[A,B]$ and $k-\mathcal{K}_q[A,B]$.

Theorem 5. *If any function f represented by Equation (1) satisfies*

$$|B-A| > \sum_{n=2}^{\infty}\{2(1+k)([n]_q-1)+|[n]_q(1+B)-(1+A)|\}|a_n|. \tag{24}$$

Then $f \in k-\mathcal{S}_q[A,B]$.

Proof. Assume that Equation (24) holds, it is sufficient that

$$k\left|\frac{\frac{zD_qf(z)}{f(z)}(B-1)-(A-1)}{\frac{zD_qf(z)}{f(z)}(1+B)-(1+A)}-1\right|-\Re\left(\frac{\frac{zD_qf(z)}{f(z)}(B-1)-(A-1)}{\frac{zD_qf(z)}{f(z)}(1+B)-(1+A)}-1\right)<1$$

Now let us consider

$$\begin{aligned} & k\left|\frac{\frac{zD_qf(z)}{f(z)}(B-1)-(A-1)}{\frac{zD_qf(z)}{f(z)}(1+B)-(1+A)}-1\right|-\Re\left(\frac{\frac{zD_qf(z)}{f(z)}(B-1)-(A-1)}{\frac{zD_qf(z)}{f(z)}(1+B)-(1+A)}-1\right) \\ & \leq 2(1+k)\left|\frac{zD_qf(z)-f(z)}{(1+B)zD_qf(z)-(1+A)f(z)}\right| \\ & = 2(1+k)\left|\frac{\sum_{n=2}^{\infty}(1+[k]_q)a_n z^n}{(B-A)z+\sum_{n=2}^{\infty}\{[n]_q(1+B)-(1+A)\}a_n z^n}\right| \\ & \leq 2(1+k)\left(\frac{\sum_{n=2}^{\infty}|[n]_q+1||a_n|}{|B-A|+\sum_{n=2}^{\infty}|[n]_q(1+B)-(1+A)||a_n|}\right), \end{aligned}$$

which is bounded by 1 if

$$|B-A| > \sum_{n=2}^{\infty}\{2(1+k)([n]_q-1)+|[n]_q(1+B)-(1+A)|\}|a_n|.$$

□

Corollary 13 ([20]). *If any function f represented by Equation (1) satisfies*

$$|B-A| > \sum_{n=2}^{\infty}\{[n]_q(1-B)+A-1\}|a_n|.$$

Then $f \in \mathcal{S}_q[A,B]$.

Substituting $k=0$ in Theorem 5, we will get the desired corollary.

Theorem 6. *If any function f represented by Equation (1) satisfies*

$$|B-A| > \sum_{n=2}^{\infty}[n]_q\{2(1+k)([n]_q-1)+|[n]_q(1+B)-(1+A)|\}|a_n|.$$

Then $f \in k-\mathcal{K}_q[A,B]$.

Theorem 5 and Equation (12) give the immediate proof of the desired theorem.

Corollary 14 ([20]). *If any function f represented by Equation (1) satisfies*

$$|B-A| > \sum_{n=2}^{\infty}[n]_q\{[n]_q(1-B)+A-1\}|a_n|.$$

Then $f \in \mathcal{K}_q[A,B]$.

Substituting $k=0$ in Theorem 6, one may get the desired corollary.

Corollary 15 ([19]). *If any function f represented by Equation (1) satisfies*

$$|B-A| > \sum_{n=2}^{\infty}n\{2(1+k)(n-1)+|n(1+B)-(1+A)|\}|a_n|.$$

Then $f \in k-\mathcal{K}[A,B]$.

3. Conclusions

In this work, we have introduced two subclasses of analytic functions $k-\mathcal{S}_q[A,B]$ and $k-\mathcal{K}_q[A,B]$ associated with q-difference operator in an open unit disk. The necessary and sufficient conditions of these newly introduced subclasses are investigated, whereas certain results and properties are studied by applying the q-difference operator in detail. The obtained results are compared to the previous known work with corollaries.

In concluding our present investigation, we draw the attention of interested readers toward the prospect that the results presented in this paper can be obtain for other subclasses of analytic functions. One may attempt to produce the same results for the class of q-symmetric starlike functions involving the Janowski functions and conic domains.

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