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HERMITE WAVELET METHOD FOR APPROXIMATE SOLUTION OF HIGHER ORDER BOUNDARY VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

In this paper, Hermite wavelet method (HWM) is considered for numerical solution of 12- and 13-order boundary value problems (BVPs) of ordinary differential equations (ODEs). The proposed algorithm for HWM developed in Maple software converts the ODEs into an algebraic systems of equations. These algebraic equations are then solved by evaluating the unknown constants present in the system of equations and the approximate solution of the problem is obtained. Test problems are considered and their solutions are investigated using HWM-based algorithm. The obtained results from the test problems are compared with exact solution, and with other numerical methods solution in the existing literature. Results comparison are presented both graphically and in tabular form showing close agreement with exact solution, and greater accuracy than homotopy perturbation method (HPM) and differential transform method (DTM).

Keywords: Wavelet; Hermite Wavelet Method; Ordinary Differential Equations; Boundary Value Problems; Numerical Solutions.

1. INTRODUCTION

A wavelet is a wave that starts at zero, rises and falls and then come back to zero one or many times. The equivalent French word for wavelet is "ondelette" which means "small wave" was initially used by Haar in 1990 but later on used by Morlet and Grossmann in 1980.[1] The theory of wavelet is applicable to many interesting areas of science and technology. Some of them are as follows: it is used in music,[2] optics,[3] signal and image processing,[4] radar,[5] nuclear engineering,[6] earthquake-prediction,[7] physics,[8] geology,[9] astronomy,[10] etc. In the field of mathematics particularly in the area of numerical analysis it is used to investigate the approximate solutions to those problems that cannot be solved using analytical techniques.[11] Wavelets can be classified into two types, that is, continuous wavelet and discrete wavelet. A wavelet is a mathematical function that is generated by a family of functions which are continuous in its components. A wavelet transform is the representation of a function which is based on wavelet algorithm. A wavelet is basically defined by a function which is known as the "mother wavelet" and is dilated and shifted to create the wavelets.

The fundamental concept of representing a complicated function by a series of summation of functions which was established by Fourier in the 1800 is the foundation of wavelets. Comparing Wavelet analysis to Fourier analysis we can say that both allow a function to be defined as a sum of basis functions. The trigonometric function of sine is used as the basis functions in Fourier analysis. But there are some limitations of Fourier study which includes that its basis function is a nonlocal sine function, ranging from negative to positive infinity because of which it does not perform in a better way to a problem having the estimation of localized and sharp irregularities. Otherwise, Fourier analysis gives us good results when the function being approximated is generally smooth and periodic. To solve this problem another study was investigated which is known as Windowed Fourier analysis.[12] But still Windowed Fourier analysis taking help of the sine function as its basis, having infinite domain. For defining functions with discontinuities and strong peaks, as well as correctly, wavelet transforms perform better than classical Fourier transforms.[13] Wavelet analysis takes help of similar, orthonormal basis functions that are defined in time and space allowing the implementation of wavelet at various areas. Therefore, the concept of wavelets was introduced that uses localized basis functions in finite domains, making them better for identifying both sharp irregularities and smooth perturbation and give us better approximation as compared to other numerical techniques.

Shah *et al.*[14] investigated approximate solution of time-fractional order telegraph equations having Dirichlet boundary conditions using an efficient operational matrix method based on Fibonacci wavelet procedure. Ahmad *et al.*[15] discussed biorthogonal wavelets on the spectrum and showed that wavelets can generate Reisz bases under mild conditions on the scaling functions and on wavelets attached with nonuniform multiresoluation analysis. The study of the controllability results analysis for Hilfer neutral fractional derivative with non-dense domain is carried out in the research studies.[16,17] For other studies related to controllability results for various kind of differential equations, we refer to Refs. 18–22. Pennes bioheat transfer equation has been solved with an efficient Fibonacci polynomials-based wavelet method by Irfan *et al*.[23] Kumar *et al.*[24] utilized Bernstein wavelets for solving fractional order SIR model. Fractional order Lotka–Volterra system in the Caputo sense has been inspected using Haar wavelet and Adams-Bashforth–Moulton methods.[25] Kumar *et al.*[26] presented solution of nonlinear fractional two species predator–prey biological model by applying Euler's and Bernstein wavelet methods. Nisar and Shah[27] studied fractional order relaxation-oscillation equations using Gegenbauer wavelet-based numerical scheme. The solution of fractional order population growth model in a closed system has been obtained with the help of Gegenbauer wavelets-based collocation method combined with the quasi-linearization procedure.[28]

In this work, Hermite wavelet approach[29] is investigated for approximate solutions of boundary value problems (BVPs) of ordinary differential equations (ODEs) of order 12 and 13. Hermite wavelet method (HWM) is defined by the Hermite polynomials which are the basis functions. The solution function is approximated with the help of Hermite wavelet which allows the use of collocation points. By generating collocation points, we obtain an algebraic system of equations for the given differential equation which are then approximated to obtain solution of the considered BVP. The algorithm for the solution procedure is implemented in Maple software. Test problems are solved with HWM-based algorithm to validate the applicability of the proposed method. Moreover, the results obtained are compared with exact solutions, and with other numerical results available in the literature for accuracy of the presented HWM-based algorithm.

The paper is organized as follows. Section 2 consists of basics of HWM. Section 3 contains procedure for solution function approximation while Sec. 4 contains test problems solutions and its comparison with the existing results in the literature. The conclusion of this work is presented in Sec. 5.

2. PRELIMINARIES

Hermite wavelets are introduced by a French mathematician Hermite. Hermite wavelets are a family of continuous functions which are formed from dilation and shifting of a single function known as analyzing wavelet.

2.1. Hermite Wavelet

Hermite wavelets are defined as

$$\phi_{s,j}(x) = \begin{cases} \dfrac{2^{\frac{r+1}{2}}}{\sqrt{\pi}} N_j(2^r x - 2s + 1), & \dfrac{s-1}{2^{r-1}} \le x < \dfrac{s}{2^{r-1}}, \\ 0, & \text{otherwise}, \end{cases}$$

where $j = 0, 1, 2, \ldots$.

2.2. Hermite Polynomial

Hermite polynomial $N_j(s)$ of degree j is defined on the real line $\mathbb{R}$ that satisfies the following recurrence

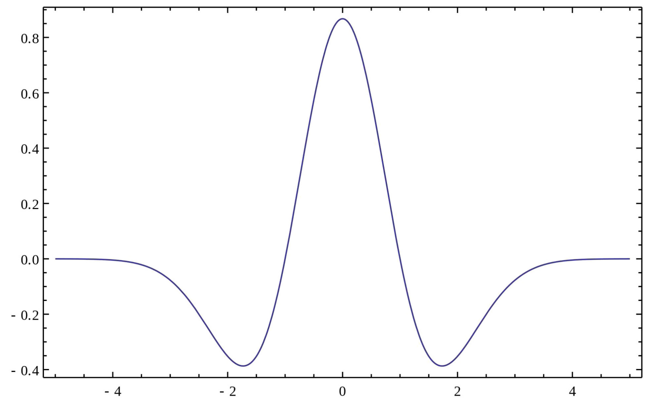

Fig. 1 Graph of Mexican hat wavelets.

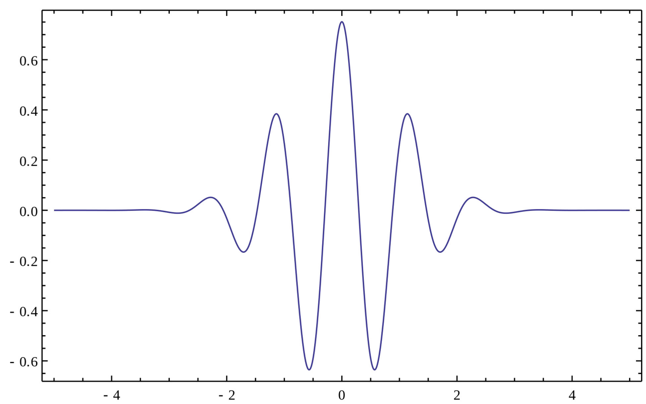

Fig. 2 Graph of Morlet wavelets.

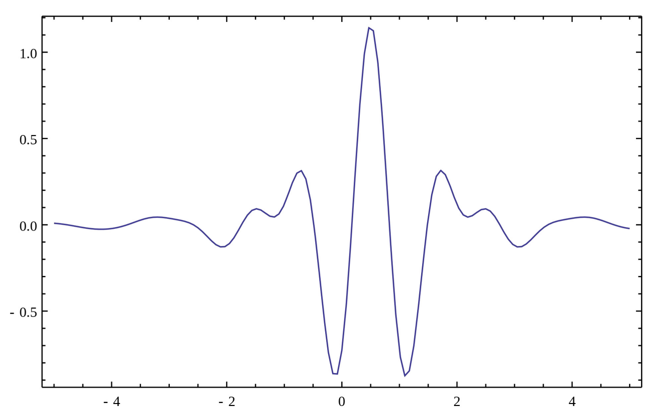

Fig. 3 Graph of Meyer wavelets.

formula:

$$N_0(x) = 1,$$

$$N_1(x) = 2x.$$

$$\vdots$$

$$N_{j+2}(x) = 2xN_{j+1}(x) - 2(j+1)N_j(x),$$

where $j = 0, 1, 2, \ldots$.

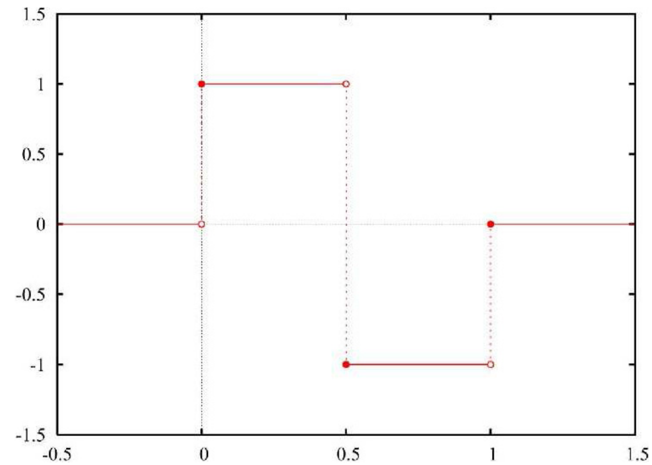

Fig. 4 Graph of Haar wavelets.

2.3. Continuous Wavelets

The wavelet is said to be continuous wavelet if the dilation parameter α and translation parameter β are changing continuously as

$$\phi_{\alpha,\beta}(x) = |\alpha|^{\frac{-1}{2}}\phi\left(\frac{x-b}{\alpha}\right),$$

for all $\alpha \in \mathbb{R}, \beta \in \mathbb{R} - \{0\}$.

For example, continuous wavelets contain Beta wavelet,[30] Meyer wavelets,[31] Morlet wavelets,[32] Hermite wavelet,[29] Hermitian hat wavelet,[33] Mexican hat wavelet,[34] Spline wavelet,[35] etc. Graphical representations of some wavelets are given in Figs. 1–4.

2.4. Discrete Wavelets

The wavelet is said to be discrete wavelet if we apply restrictions on the parameters α and β to discrete values

$$\alpha = \alpha_0^{-r}, \beta = s\beta_0\alpha_0^{-r}, \quad \alpha_0 > 1, \quad \beta_0 > 0,$$

then the family of discrete wavelets is

$$\phi_{r,s}(x) = |\alpha_0^{-r}|^{\frac{1}{2}}\phi(\alpha_0^r x - s\beta_0),$$

for all $\alpha \in \mathbb{R}, \beta \in \mathbb{R} - \{0\}$, where $\phi_{r,s}(x)$ forms a wavelet basis for $L^2(\mathbb{R})$.

For example, discrete wavelets constitute Haar wavelet,[36] Legendre wavelet,[37] Villasenor wavelet (VW),[38] Cohen–Daubechies–Feauveau wavelet,[39] Daubechies wavelet,[40] etc.

2.5. Orthonormal Wavelets

Orthonormal wavelet is a function $\zeta \in L^2(\mathbb{R})$ which can be used to define basis that is a complete orthonormal system. The basis is formed as a functions family $\{\zeta_{r,s} : r, s \in \mathbb{Z}\}$ by shifting and dilations of ζ

$$\zeta_{r,s}(x) = 2^{\frac{r}{2}}\zeta(2^{rx} - s),$$

for $r, s \in \mathbb{Z}$.

2.6. Dual Wavelets

If a wavelet has the property that for $\phi \in L^2(\mathbb{R})$ there exists a function $\widetilde{\phi} \in L^2(\mathbb{R})$ such that

$$\widetilde{\phi}_{r,s} = \phi^{rs},$$

for $r, s \in \mathbb{Z}$, where

$$\widetilde{\phi}_{r,s} = 2^{\frac{r}{2}}\phi(2^r x - s),$$

then $\widetilde{\phi}$ is called dual wavelet or the wavelet dual to ϕ.

Theorem 1 (Ref. 41). *Let $y(x) \in H^2[0,1)$ Hilbert space, such that $y(x) < M$, where $M \in H^2[0,1)$ is continuous function and $0 \le x < 1$. Then $y(x)$ under Hermite wavelet converges to it.*

Theorem 2 (Ref. 41). *Let $\phi_{s,j}(x)$ be a function defined by Hermite wavelets. Then $\phi_{s,j}(x)$ are continuous uniformly for all $x \in I$, where $I = (0,1)$.*

Theorem 3 (Ref. 41). *Suppose $\phi_{s,j}(x)$ is defined under Hermite wavelets. Then $\phi_{s,j}(x)$ is continuous for all $x \in I$, where $I = (0,1)$.*

Theorem 4 (Ref. 41). *The series solution obtained by the approximation of Hermite wavelet algorithm defined as $y(x) = \sum_{s=1}^{\infty}\sum_{j=0}^{\infty} G_{s,j}\phi(x)$ converges to $y(x)$.*

Proof. Let $H^2(\mathbb{R})$ be a Hilbert space of the infinite dimension. Define

$$\phi_{s,j}(x) = \begin{cases} \dfrac{2^{\frac{r+1}{2}}}{\sqrt{\pi}}N_j(2^r x - 2s + 1), & \dfrac{s-1}{2^{r-1}} \le x < \dfrac{s}{2^{r-1}}, \\ 0, & \text{otherwise}, \end{cases}$$

where $j = 0, 1, 2, \ldots$, which forms an orthonormal basis. Let us have the truncated series as

$$y(x) \approx \sum_{s=1}^{2^{r-1}}\sum_{j=0}^{J-1} G_{s,j}\phi_{s,j}(x) = G^T\phi(x),$$

where G and $\phi(x)$ are $2^{r-1}\times$ J-1 matrix and $r, J \in \mathbb{Z}^+$. Let us define the sequence of partial sums S_k of $\{G_{s,j}\phi_{s,j}(x)\}$ and suppose we have partial sums for some k, l, that is, S_k and S_l such that $k \ge l$. Now, we prove that S_k is Cauchy sequence in $H^2(\mathbb{R})$. Consider

$$S_k = \sum_{j=0}^{k} G_{s,j}\phi_{s,j}(x),$$

and

$$\langle y(x), S_k\rangle = \left\langle y(x), \sum_{j=0}^{k} G_{s,j}\phi_{s,j}(x)\right\rangle$$

$$= \sum_{j=l+1}^{k} |G_{s,j}|^2,$$

which implies

$$\|S_k - S_l\|^2 = \sum_{j=l+1}^{k} |G_{s,j}|^2,$$

for all $k > l$, therefore, we have

$$\left\|\sum_{j=l+1}^{k} G_{s,j}\phi_{s,j}(x)\right\|^2$$

$$= \left\langle \sum_{j=l+1}^{k} G_{s,j}\phi_{s,j}(x), \sum_{j=l+1}^{k} G_{s,j}\phi_{s,j}(x)\right\rangle$$

$$= \sum_{j=l+1}^{k} |G_{s,j}|^2,$$

which implies

$$\left\|\sum_{j=l+1}^{k} G_{s,j}\phi_{s,j}(x)\right\|^2 = \sum_{j=l+1}^{k} |G_{s,j}|^2.$$

Using Bessel's inequality, we have

$$\sum_{j=l+1}^{k} |G_{s,j}|^2 \le \| y(x) \|^2,$$

which implies that

$$\sum_{j=l+1}^{k} |G_{s,j}|^2,$$

is bounded. Therefore,

$$\left\|\sum_{j=l+1}^{k} G_{s,j}\phi_{s,j}(x)\right\|^2 \to 0,$$

as $k \to \infty, l \to \infty$.

Hence, we have

$$\left\|\sum_{j=l+1}^{k} G_{s,j}\phi_{s,j}(x)\right\| \to 0$$

which means that S_k is a Cauchy sequence.

Let S_k converge to P. To show that $y(x)=P$, we consider

$$\langle P - y(x), \phi_{s,j}(x)\rangle = \langle P, \phi_{s,j}(x)\rangle - \langle y(x), \phi_{s,j}(x)\rangle.$$

By applying limit k tends to ∞, we have

$$\langle P - y(x), \phi_{s,j}(x)\rangle = \langle P, \phi_{s,j}(x)\rangle - \lim_{k\to\infty} S_k, \phi_{s,j}(x)\rangle = 0,$$

hence

$$\langle P - y(x), \phi_{s,j}(x)\rangle = 0.$$

So $y(x) = P$ and the series solution $y(x) = \sum_{s=1}^{\infty}\sum_{j=0}^{\infty} G_{s,j}\phi(x)$ converges to $y(x)$ as $n \longrightarrow \infty$. Hence, it is proved. □

2.7. Convergence Analysis of Hermite Wavelets

Let us suppose $\xi(x)$ is a function such that $\xi'(x)$ exists with

$$\xi(x) \le L,$$

for all $x \in (a, b)$, $L \in \mathbb{R}^+$.

Hermite wavelet approximation for the function $\xi(x)$ is given by

$$\xi_j(x) = \sum_{i=1}^{2j} \alpha_i \chi(x).$$

Babolian and Shahsavaran[42] have already shown that error norm square approximation for wavelet is given by

$$\|\xi(x) - \xi_j(x)\| = \frac{L^3}{3(2J)^2}.$$

Therefore,

$$\|\xi(x) - \xi_j(x)\| = O\left(\frac{1}{J}\right). \tag{1}$$

From Eq. (1), it is obviously shown that the error is inversely proportional to the resolution level of Hermite wavelet. It means that the rate of convergence of approximation of Hermite wavelets is increased as the number of J is increased.

3. NUMERICAL PROCEDURE BASED ON HWM FOR FUNCTION APPROXIMATION

Consider an ODE of order s

$$y^{(s)}(x) = y^{(j)}(x) + y(x) + g(x), \tag{2}$$

where $j < s,\ s, j \in \mathbb{Z}^+$, with boundary conditions

$$y(0) = w_0, \quad y(r) = w_r, \quad y^{(s)}(0) = y_0, \quad y^{(s)}(r) = y_r. \tag{3}$$

For the numerical solution function $y(x)$ by HWM, we take

$$y(x) = \sum_{s=1}^{\infty}\sum_{j=0}^{\infty} G_{s,j}\phi(x), \tag{4}$$

where $\phi_{s,j}(x)$ is given by

$$\phi_{s,j}(x) = \begin{cases} \dfrac{2^{\frac{r+1}{2}}}{\sqrt{\pi}} N_j(2^r x - 2s + 1), & \dfrac{s-1}{2^{r-1}} \le x < \dfrac{s}{2^{r-1}}, \\ 0, & \text{otherwise}, \end{cases}$$

where $j = 0, 1, 2, \ldots,$.

To approximate $y(x)$ we truncate Eq. (4) up to some finite limit values. That is,

$$y(x) \approx \sum_{s=1}^{2^{r-1}}\sum_{j=0}^{J-1} G_{s,j}\phi(x) = G^T\phi(x), \tag{5}$$

where G and $\phi(x)$ are $2^{r-1}\times$ J-1 matrix and $r, J \in \mathbb{Z}^+$,

$$G^T = [g_{1,0}, \ldots, g_{1,J-1}, g_{2,0}, \ldots, g_{2,J-1}, \ldots, g_{2^{r-1},0}, \ldots, g_{2^{r-1},J-1}], \tag{6}$$

$$\phi(y) = [\phi_{1,0}, \ldots, \phi_{1,J-1}, \phi_{2,0}, \ldots, \phi_{2,J-1}, \ldots, \phi_{2^{r-1},0}, \ldots, \phi_{2^{r-1},J-1}]^T. \tag{7}$$

Therefore, Eq. (2) is approximated by using Eq. (5) as

$$\frac{d^s}{dx^s}\sum_{s=1}^{2^{r-1}}\sum_{j=0}^{J-1} G_{s,j}\phi(x) = \frac{d^j}{dx^j}\sum_{s=1}^{2^{r-1}}\sum_{j=0}^{J-1} G_{s,j}\phi(x) + \sum_{s=1}^{2^{r-1}}\sum_{j=0}^{J-1} G_{s,j}\phi(x) + g(x). \tag{8}$$

Now, using the subjected conditions of Eqs. (3) and (5), we have

$$\begin{aligned}
\sum_{s=1}^{2^{r-1}} \sum_{j=0}^{J-1} G_{s,j}\phi(0) &= w_0, \\
\frac{d^s}{dx^s} \sum_{s=1}^{2^{r-1}} \sum_{j=0}^{J-1} G_{s,j}\phi(0) &= z_0, \\
\sum_{s=1}^{2^{r-1}} \sum_{j=0}^{J-1} G_{s,j}\phi(r) &= z_0, \\
\frac{d^s}{dx^s} \sum_{s=1}^{2^{r-1}} \sum_{j=0}^{J-1} G_{s,j}\phi(r) &= w_0.
\end{aligned} \tag{9}$$

Then, solving Eqs. (8) and (9) with the help of Maple software, we obtain the values of the following coefficient constants as

$$g_{1,0}, \ldots, g_{1,J-1}, g_{2,0}, \ldots, g_{2,J-1}, \ldots, g_{2^{r-1},0}, \ldots, g_{2^{r-1},J-1}, \tag{10}$$

$$\phi_{1,0}, \ldots, \phi_{1,J-1}, \phi_{2,0}, \ldots, \phi_{2,J-1}, \ldots, \phi_{2^{r-1},0}, \ldots, \phi_{2^{r-1},J-1}. \tag{11}$$

We achieve the approximation by substituting the unknown constants in Eqs. (10) and (11) into Eqs. (8) and (9). Hence Eqs. (4)–(11) constitute the HWM-based algorithm numerical procedure for problem described in Eq. (2) with conditions given in Eq. (3).

4. TEST PROBLEMS

Here, in this section, we applied the numerical procedure described in Sec. 3 to obtain solutions to BVPs of ODEs. The results obtained are analyzed and compared with exact solutions and with solutions of homotopy perturbation method (HPM) and differential transform method (DTM) methods.

4.1. Example

Consider ODE of order 12.[42]

$$y^{(12)}(x) = 2e^x y^{(2)}(x) + y^{(3)}(x), \quad 0 < x < 1, \tag{12}$$

with boundary conditions

$$\begin{gathered}
y(0) = 0, \quad y(1) = e^{-1}, \quad y''(0) = 1, \\
y''(1) = e^{-1}, \quad y^{(4)}(0) = 1, \quad y^{(4)}(1) = e^{-1}, \\
y^{(6)}(0) = 1, \quad y^{6}(1) = e^{-1}, \quad y^{(8)}(0) = 1, \\
y^{(8)}(1) = e^{-1}, \quad y^{(10)}(0) = 1, \quad y^{(10)}(1) = e^{-1}.
\end{gathered} \tag{13}$$

The exact solution of the problem is given as

$$y = e^{-x}. \tag{14}$$

Applying the proposed HWM-based procedure, we consider

$$y(x) = \sum_{s=1}^{\infty} \sum_{j=0}^{\infty} G_{s,j}\phi(x), \tag{15}$$

and truncate the series up to r and J to approximate $y(x)$ using Eq. (5) as

$$y(x) \approx \sum_{s=1}^{2^{r-1}} \sum_{j=0}^{J-1} G_{s,j}\phi(x) = G^T\phi(x). \tag{16}$$

By putting the values of $r = 1$ and $J = 15$ for better approximations of Eqs. (12) and (16) becomes

$$y(x) \approx \sum_{j=0}^{15-1} G_{s,j}\phi(x) = G^T\phi(x), \tag{17}$$

where G and $\phi(x)$ are matrices of appropriate dimensions

$$\begin{aligned}
G^T = [&g_{1,0}, g_{1,1}, g_{1,2}, g_{1,3}, g_{1,4}, g_{1,5}, g_{1,6}, g_{1,7}, \\
&g_{1,8}, g_{1,9}, g_{1,10}, g_{1,11}, g_{1,12}, g_{1,13}, g_{1,14}, g_{1,15}],
\end{aligned} \tag{18}$$

$$\begin{aligned}
\phi(y) = [&\phi_{1,0}, \phi_{1,1}, \phi_{1,2}, \phi_{1,3}, \phi_{1,4}, \phi_{1,5}, \phi_{1,6}, \phi_{1,7}, \\
&\phi_{1,8}, \phi_{1,9}, \phi_{1,10}, \phi_{1,11}, \phi_{1,12}, \\
&\phi_{1,13}, \phi_{1,14}, \phi_{1,15}]^T.
\end{aligned} \tag{19}$$

To find the values of the unknown in Eq. (18), we have to make a system of 16 equations. But the equations which can be formed by Eq. (13) are given below

$$2.592524841\times10^{7}g_{1,15}-7.47943176\times10^{6}g_{1,14}-4.217840344\times10^{5}g_{1,13}+3.467916596\times10^{5}g_{1,12}-12354.51297g_{1,11}-16104.13166g_{1,10}+2475.709791g_{1,9}+697.044505g_{1,8}-276.4142004g_{1,7}-12.0180087g_{1,6}+30.04502177g_{1,5}-6.00900436g_{1,4}-3.004502177g_{1,3}+3.004502178g_{1,2}-1.502251089g_{1,1}+0.7511255444g_{1,0}=1,$$

$$4.47157602\times10^{8}g_{1,15}+9.533230616\times10^{8}g_{1,14}-9.267672972\times10^{7}g_{1,13}-2.616887746\times10^{7}g_{1,12}+6.141683170\times10^{6}g_{1,11}+4.464930614\times10^{5}g_{1,10}-3.368888198\times10^{5}g_{1,9}+22882.2886g_{1,8}+15575.33928g_{1,7}-4326.483136g_{1,6}-192.2881394g_{1,5}+432.6483134g_{1,4}-144.2161045g_{1,3}+24.03601742g_{1,2}=1,$$

$$-4.637672400\times10^{11}g_{1,15}-5.989022962\times10^{10}g_{1,14}+1.841314306\times10^{10}g_{1,13}+1.583838922\times10^{8}g_{1,12}-6.135991428\times10^{8}g_{1,11}+6.202446234\times10^{7}g_{1,10}+1.498924505\times10^{7}g_{1,9}-4.522617036\times10^{6}g_{1,8}+92298.30696g_{1,7}+1.845966137\times10^{5}g_{1,6}-46149.15346g_{1,5}+4614.915346g_{1,4}=1,$$

$$6.946590918\times10^{13}g_{1,15}-1.170773753\times10^{12}g_{1,14}-1.567269547\times10^{12}g_{1,13}+1.719318096\times10^{11}g_{1,12}+2.307309998\times10^{10}g_{1,11}-7.070788692\times10^{9}g_{1,10}+2.480978492\times10^{8}g_{1,9}+1.550611556\times10^{8}g_{1,8}-3.101223110\times10^{7}g_{1,7}+2.215159366\times10^{6}g_{1,6}=1,$$

$$-5.354054784\times10^{15}g_{1,15}+5.858126098\times10^{14}g_{1,14}+5.187428306\times10^{13}g_{1,13}-1.571947970\times10^{13}g_{1,12}+5.954348378\times10^{11}g_{1,11}+2.143565414\times10^{11}g_{1,10}-3.572609026\times10^{10}g_{1,9}+1.984782792\times10^{9}g_{1,8}=1,$$

$$-6.287791884\times10^{13}g_{1,11}+4.401454318\times10^{14}g_{1,12}+1.760581728\times10^{15}g_{1,13}-4.740995078\times10^{16}g_{1,14}+1.602129374\times10^{17}g_{1,15}+2.858087220\times10^{12}g_{1,10}=1,$$

$$-2.5925174\times10^{7}g_{1,15}-7.4794773\times10^{6}g_{1,14}+4.2179801\times10^{5}g_{1,13}+3.4679204\times10^{5}g_{1,12}+12354.696g_{1,11}-16104.1323g_{1,10}-2475.711g_{1,9}+697.0445g_{1,8}+276.414193g_{1,7}-12.0180097g_{1,6}-30.045022g_{1,5}-6.0090044g_{1,4}+3.00450218g_{1,3}+3.004502178g_{1,2}+1.502251089g_{1,1}+0.7511255444g_{1,0}=e^{-1},$$

$$-4.47164\times10^{8}g_{1,15}+9.533213\times10^{8}g_{1,14}+9.267717328\times10^{7}g_{1,13}-2.6168866\times10^{7}g_{1,12}-6.141680\times10^{6}g_{1,11}+4.4649291\times10^{5}g_{1,10}$$

$$
\begin{aligned}
&+3.3688879\times 10^5 g_{1,9}\\
&+22882.2886 g_{1,8}-15575.33882 g_{1,7}\\
&-4326.48317 g_{1,6}+192.28814 g_{1,5}\\
&+432.648314 g_{1,4}+144.2161045 g_{1,3}\\
&+24.03601742 g_{1,2}=e^{-1},
\end{aligned}
$$

$$
\begin{aligned}
&4.6376651\times 10^{11} g_{1,15}-5.989025\times 10^1 0 g_{1,14}\\
&-1.84131368\times 10^{10} g_{1,13}\\
&+1.583838922\times 10^8 g_{1,12}\\
&+6.1359915\times 10^8 g_{1,11}+6.202445\times 10^7 g_{1,10}\\
&-1.49892460\times 10^7 g_{1,9}\\
&-4.52261708\times 10^6 g_{1,8}\\
&-92298.30304 g_{1,7}+1.84596614\times 10^5 g_{1,6}\\
&+46149.1544 g_{1,5}+4614.915346 g_{1,4}=e^{-1},
\end{aligned}
$$

$$
\begin{aligned}
&-6.9465950\times 10^{13} g_{1,15}-1.170772153\times 10^1 2 g_{1,14}\\
&+1.5672697\times 10^{12} g_{1,13}\\
&+1.7193181\times 10^{11} g_{1,12}\\
&-2.3073101\times 10^{10} g_{1,11}\\
&-7.0707888\times 10^9 g_{1,10}\\
&-2.4809786\times 10^8 g_{1,9}\\
&+1.550611560\times 10^8 g_{1,8}\\
&+3.101223112\times 10^7 g_{1,7}\\
&+2.215159366\times 10^6 g_{1,6}=e^{-1},
\end{aligned}
$$

$$
\begin{aligned}
&5.3540536\times 10^{15} g_{1,15}+5.858126898\times 10^{14} g_{1,14}\\
&-5.1874282\times 10^{13} g_{1,13}\\
&-1.571947960\times 10^{13} g_{1,12}\\
&-5.954348422\times 10^{11} g_{1,11}\\
&+2.14356541\times 10^{11} g_{1,10}\\
&+3.572609024\times 10^{10} g_{1,9}\\
&+1.984782792\times 10^9 g_{1,8}=e^{-1},
\end{aligned}
$$

$$
\begin{aligned}
&6.287791886\times 10^{13} g_{1,11}+4.40145433\times 10^{14} g_{1,12}\\
&-1.76058171\times 10^{15} g_{1,13}\\
&-4.7409949\times 10^{16} g_{1,14}\\
&-1.6021297\times 10^{17} g_{1,15}\\
&+2.858087220\times 10^{12} g_{1,10}=e^{-1}.
\end{aligned}
$$

These equations are not sufficient to obtain the values of the unknowns, therefore, further equations are obtained by substituting Eq. (12) in Eq. (17) as

$$
\begin{aligned}
&y^{(12)}\sum_{j=0}^{15-1} g_{1,j}\phi(x)=2e^x y^{(2)}\\
&\sum_{j=0}^{15-1} g_{1,j}\phi(x)+y^{(3)\sum_{j=0}^{15-1} g_{1,j}}\phi(x).
\end{aligned}
\tag{20}
$$

We collocate Eq. (20) by limit points of the following sequence:

$$
\{x_i\}=\left\{\frac{1}{2}\left(1+cos\frac{(i-1)\pi}{9}\right)\right\},
$$

where $i=2,3,\ldots$.

Hence, we get

$$
\begin{aligned}
&y^{(12)}\sum_{j=0}^{15-1} g_{1,j}\phi(x_i)=2e^{x_i}xy^{(2)}\\
&\sum_{j=0}^{15-1} g_{1,j}\phi(x_i)+y^{(3)}\sum_{j=0}^{15-1} g_{1,j}\phi(x_i).
\end{aligned}
\tag{21}
$$

Solving Eq. (21) for the collocation points x_i, we get the required remaining system of equations in the following manner:

$$
\begin{aligned}
&-1.065666380\times 10^{19} g_{1,15}\\
&-5.211007408\times 10^{17} g_{1,14}\\
&+6.86626867\times 10^{16} g_{1,13}\\
&+6.036280060\times 10^{15} g_{1,12}\\
&+1.441781715\times 10^7 g_{1,11}\\
&+6.767984708\times 10^6 g_{1,10}\\
&-1.92319454\times 10^5 g_{1,9}\\
&-3.42235624\times 10^5 g_{1,8}\\
&-19430.90525 g_{1,7}+18602.4897 g_{1,6}\\
&+3699.162264 g_{1,5}-870.8038717 g_{1,4}\\
&-547.3554128 g_{1,3}-98.63741100 g_{1,2}=0,
\end{aligned}
$$

$$
\begin{aligned}
&-1.215651435\times 10^1 9 g_{1,15}\\
&-2.464499915\times 10^{17} g_{1,14}\\
&+8.82805971\times 10^{16} g_{1,13}\\
&+6.036280125\times 10^{15} g_{1,12}\\
&+3.536510539\times 10^7 g_{1,11}
\end{aligned}
$$

$$
\begin{aligned}
&+5.733201954\times 10^{6} g_{1,10}\\
&-1.092059464\times 10^{6} g_{1,9}\\
&-3.7595790\times 10^{5} g_{1,8}\\
&+18515.69181 g_{1,7}+24869.3013 g_{1,6}\\
&+2568.169984 g_{1,5}-1465.287063 g_{1,4}\\
&-642.8037580 g_{1,3}-104.9989775 g_{1,2}=0,\\
-&1.249768220\times 10^{19} g_{1,15}\\
&+9.686344083\times 10^{16} g_{1,14}\\
&+1.078985075\times 10^{17} g_{1,13}\\
&+6.036280259\times 10^{15} g_{1,12}\\
&+5.136054550\times 10^{7} g_{1,11}\\
&+2.198425942\times 10^{6} g_{1,10}\\
&-2.037454014\times 10^{6} g_{1,9}\\
&-3.2390559\times 10^{5} g_{1,8}\\
&+68234.83460 g_{1,7}+29390.2687 g_{1,6}\\
&+635.066166 g_{1,5}-2183.652522 g_{1,4}\\
&-749.4868812 g_{1,3}-111.7708297 g_{1,2}=0,\\
-&1.142268207\times 10^{19} g_{1,15}\\
&+5.088395615\times 10^{17} g_{1,14}\\
&+1.275164185\times 10^{17} g_{1,13}\\
&+6.036280434\times 10^{15} g_{1,12}\\
&+5.505546956\times 10^{7} g_{1,11}\\
&-3.844331894\times 10^{6} g_{1,10}\\
&-2.815203204\times 10^{6} g_{1,9}\\
&-1.6126351\times 10^{5} g_{1,8}\\
&+1.266294191\times 10^{5} g_{1,7}+30861.2104 g_{1,6}\\
&-2300.355780 g_{1,5}-3046.013116 g_{1,4}\\
&-868.4569248 g_{1,3}-118.9794289 g_{1,2}=0.
\end{aligned}
$$

Combining the obtained system of equations and solving for the unknown constants with the help of Maple software, we get

$$
\begin{aligned}
g_{1,0}&=0.8074963266,\\
g_{1,1}&=-0.2061596801,\\
g_{1,2}&=0.2603935908e^{-1},\\
g_{1,3}&=-0.2183499593e^{-2},\\
g_{1,4}&=0.1370415228e^{-3},\\
g_{1,5}&=-0.6872030296e^{-5},\\
g_{1,6}&=2.86843870\times 10^{-7},\\
g_{1,7}&=-1.02667381\times 10^{-8},\\
g_{1,8}&=3.225944\times 10^{-10},\\
g_{1,9}&=-8.90845\times 10^{-12},\\
g_{1,10}&=2.128620\times 10^{-13},\\
g_{1,11}&=-4.56594\times 10^{-15},\\
g_{1,12}&=1.0065188\times 10^{-16},\\
g_{1,13}&=1.3393745\times 10^{-17},\\
g_{1,14}&=-6.593572\times 10^{-19},\\
g_{1,15}&=3.359710\times 10^{-20}.
\end{aligned}
\tag{22}
$$

By putting the values of the unknown constants available in Eq. (22) in Eq. (21), we obtain the numerical solution of our problem Eqs. (12) and (13)

$$
\begin{aligned}
y(x)=&\,2.709656457\times 10^{-11}x^{15}\\
&-3.361695959\times 10^{-10}x^{14}\\
&+2.008819871\times 10^{-9}x^{13}\\
&-4.385976284\times 10^{-9}x^{12}\\
&-1.855534054\times 10^{-8}x^{11}\\
&+2.755731921\times 10^{-7}x^{10}\\
&-0.2732877519e^{-5}x^{9}\\
&+0.2480158730e^{-4}x^{8}\\
&-0.1986637645e^{-3}x^{7}\\
&+0.1388888889e^{-2}x^{6}\\
&-0.8332187922e^{-2}x^{5}\\
&+0.4166666669e^{-1}x^{4}\\
&+0.9999999988-0.1666690251x^{3}\\
&+0.4999999998x^{2}-0.9999985593x.
\end{aligned}
$$

The results for Example 4.1 are provided in Figs. 5–7 and Tables 1–3.

4.2. Example

Consider ODE of order 13 along with boundary conditions as[43]

$$
y^{(13)}=11(\cos x-\sin x),\quad 0<x<1, \tag{23}
$$

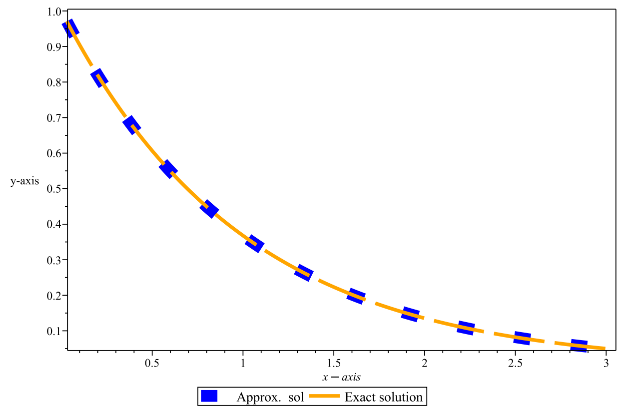

Fig. 5 Comparison graphs between exact solution and approximate solution obtained by HWM for Example 4.1.

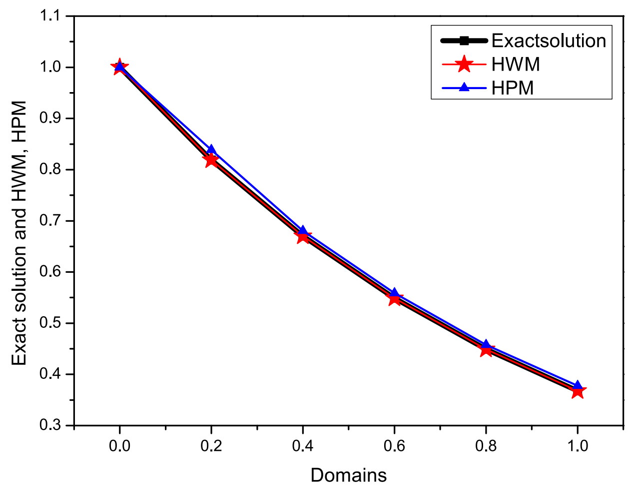

Fig. 6 Comparison graphs among exact solution, approximate solution obtained by HWM, and with approximate results by HPM[42] for Example 4.1.

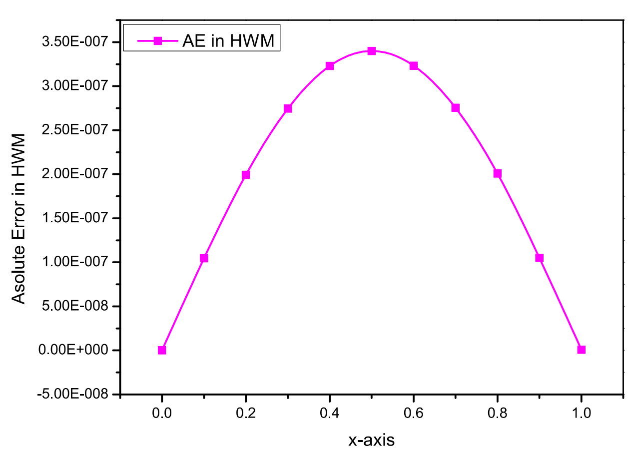

Fig. 7 Absolute error graph by HWM for Example 4.1.

Table 1 Comparison Between Exact Solution and Approximate Solution Obtained by HWM for Example 4.1.

x	Exact Solution $y(x) = e^{-x}$	Approximate Solution by HWM
0.0	1.000000000000000	1.000000000000000
0.1	0.904837418000000	0.904837522700000
0.2	0.818730753100000	0.818730952500000
0.3	0.740818220700000	0.740818495200000
0.4	0.670320046000000	0.670320369000000
0.5	0.606530659700000	0.606530999700000
0.6	0.548811636100000	0.548811959400000
0.7	0.496585303800000	0.496585579200000
0.8	0.449328964100000	0.449329164900000
0.9	0.406569659700000	0.406569764700000
1.0	0.367879441200000	0.367879440400000

Table 2 Comparison Between Approximate Solution of HWM and Approximate Solution of HPM[42] for Example 4.1.

x	Approximate Solution by HPM	Approximate Solution by HWM
0.0	$10.00000000 \times 10^{-01}$	$10.000000000 \times 10^{-01}$
0.2	$8.187308703 \times 10^{-01}$	$8.1873095250 \times 10^{-01}$
0.4	$6.703208540 \times 10^{-01}$	$6.7032036900 \times 10^{-01}$
0.6	$5.488114451 \times 10^{-01}$	$5.4881195940 \times 10^{-01}$
0.8	$4.493289646 \times 10^{-01}$	$0.4493291649 \times 10^{-01}$
1.0	$3.678794453 \times 10^{-01}$	$0.3678794404 \times 10^{-01}$

Table 3 Absolute Error of HWM for Example 4.1.

x	Exact Solution	Absolute Error by HWM
0.0	1.000000000000000	0.0000000000E+00
0.1	0.904837418000000	1.0470000000E−07
0.2	0.818730753100000	1.9940000000E−07
0.3	0.740818220700000	2.7450000000E−07
0.4	0.670320046000000	3.2300000000E−07
0.5	0.606530659700000	3.4000000000E−07
0.6	0.548811636100000	3.2330000000E−07
0.7	0.496585303800000	2.7540000000E−07
0.8	0.449328964100000	2.0080000000E−07
0.9	0.406569659700000	1.0500000000E−07
1.0	0.367879441200000	8.0000000000E−10

$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -1,$$
$$y'''(0) = -1, \quad y^{(4)}(0) = 1, \quad y^{(5)}(0) = 1,$$
$$y^{(6)}(0) = -1,$$
$$y(1) = \sin 1 + \cos 1, \quad y'(1) = -\sin 1 + \cos 1,$$
$$y''(1) = -\sin 1 - \cos 1, \quad y'''(1) = \sin 1 - \cos 1,$$
$$y^{(4)}(1) = \sin 1 + \cos 1, \quad y^{(5)}(1) = -\sin 1 + \cos 1. \tag{24}$$

Analytical result of the problem Eqs. (23) and (24) is given by

$$y(x) = \cos x + \sin x. \tag{25}$$

As we know from Hermite wavelet algorithm

$$y(x) \approx \sum_{s=1}^{2^{r-1}} \sum_{j=0}^{J-1} G_{s,j}\phi(x) = G^T\phi(x), \tag{26}$$

For approximating Eq. (23) in a better way by the proposed method (HWM), we choose $r = 1$ and $J = 25$ to truncate the series as

$$y(x) \approx \sum_{j=0}^{25-1} G_{1,j}\phi(x) = G^T\phi(x), \tag{27}$$

where G and $\phi(x)$ are matrices of specific order. These are represented as

$$G^T = [g_{1,0}, g_{1,1}, g_{1,2}, g_{1,3}, \ldots, g_{1,25}], \tag{28}$$

and

$$\phi(x) = [\phi_{1,0}, \phi_{1,1}, \phi_{1,2}, \ldots, \phi_{1,25}]^T. \tag{29}$$

Applying algorithm Eq. (27) to Eq. (23) which converts the given differential equation to a system of algebraic equations, we have

$$y^{(13)} \sum_{j=0}^{25-1} G_{1,j}\phi(x) = 11(\cos x - \sin x). \tag{30}$$

Computing Eq. (30) with the help of Maple software, we obtained a system of algebraic equations which contains some unknowns. For the solution of unknown constant values we have to take help from the boundary conditions (24). To get better approximation to Eq. (23) the given boundary conditions are not sufficient to find the unknowns, therefore taking help of Hermite wavelet algorithm we generate more boundary conditions which help us in finding the unknown constants in the system of equations which will lead us to approximate the given problem Eq. (23) with minimum possible error. The unknowns are given as

$$g_{1,0}, g_{1,1}, g_{1,2}, g_{1,3}, g_{1,4}, g_{1,5}, g_{1,6}, g_{1,7}, g_{1,8}, g_{1,9}, g_{1,10},$$
$$g_{1,11}, g_{1,12}, g_{1,13}, g_{1,14}, g_{1,15}, g_{1,16}, g_{1,17}, g_{1,18},$$
$$g_{1,19}, g_{1,20}, g_{1,21}, g_{1,22}, g_{1,23}, g_{1,24}, g_{1,25}.$$

To generate the other boundary conditions, we collocate equation (30) by the limit points of the following sequence to obtain the remaining equations:

$$\{x_i\} = \left\{\frac{1}{2}\left(1 + \cos\frac{(i-1)\pi}{9}\right)\right\},$$

where $i = 2, 3, \ldots$.

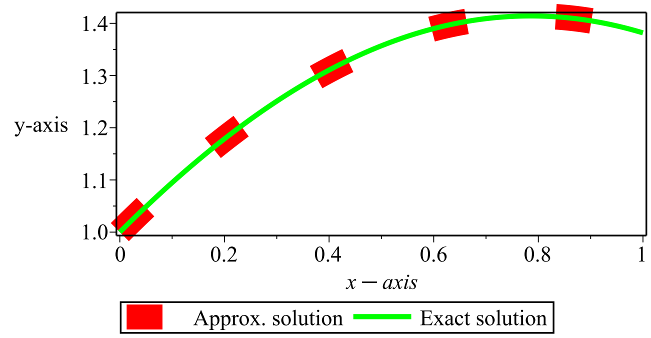

Fig. 8 Comparison graphs between exact and approximate solution by HWM: Example 4.2.

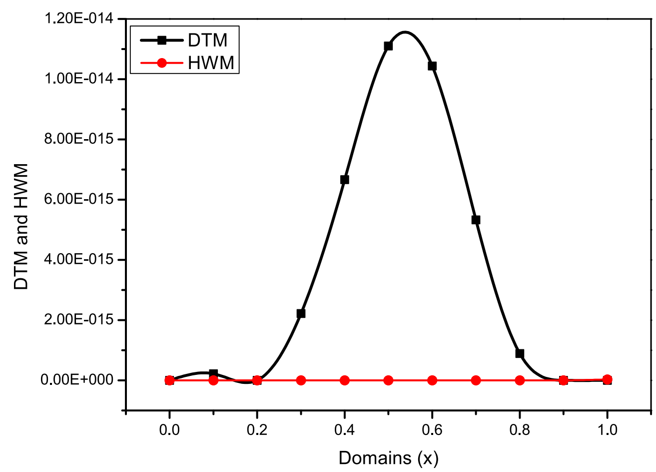

Fig. 9 Comparison graphs between absolute error in DTM[43] and absolute error in HWM: Example 4.2.

Table 4 Comparison Between Exact Solution, Approximate Solution Obtained by HWM, and DTM Solution[43]: Example 4.2.

x	Exact Solution	Approximate Solution by HWM	Approximate Solution by DTM
0.0	1.000000000000000	1.00000000000000	1.0000
0.1	1.094837582000000	1.094837582000000	1.09484
0.2	1.178735909000000	1.178735909000000	1.17874
0.3	1.250856696000000	1.250856696000000	1.25086
0.4	1.310479336000000	1.310479336000000	1.31048
0.5	1.357008100000000	1.357008100000000	1.35701
0.6	1.389978088000000	1.389978088000000	1.38998
0.7	1.409059874000000	1.409059874000000	1.40906
0.8	1.414062800000000	1.414062800000000	1.41406
0.9	1.404936878000000	1.404936878000000	1.40494
1.0	1.381773291000000	1.381773291000000	1.38177

Then, Eq. (30) becomes

$$y^{(13)} \sum_{j=0}^{25-1} G_{1,j}\phi(x_i) = 11(\cos x_i - \sin x_i). \tag{31}$$

Getting more boundary conditions by computation of Eq. (31), we obtain more equations with the help of Maple software which will become sufficient to find the values of the unknown constants. Then solving the system of equations for the unknown constants, we obtain

$$\begin{aligned}
g_{1,0} &= 1.806632873, \quad g_{1,1} = 0.1298102999,\\
g_{1,2} &= -0.5472917253e^{-1},\\
g_{1,3} &= -0.132978747e^{-2},\\
g_{1,4} &= 0.2820825646e^{-3}, \quad g_{1,5} = 0.4126596e^{-5},\\
g_{1,6} &= -5.83654818 \times 10^{-7},\\
g_{1,7} &= -6.015224 \times 10^{-9},\\
g_{1,8} &= 7.332573 \times 10^{-10},\\
g_{1,9} &= 9.97879 \times 10^{-12},\\
g_{1,10} &= 2.4350520 \times 10^{-12},\\
g_{1,11} &= 6.902986 \times 10^{-14},\\
g_{1,12} &= 3.3806785 \times 10^{-14},\\
g_{1,13} &= 1.298506361 \times 10^{-17},\\
g_{1,14} &= -7.930078551 \times 10^{-19},\\
g_{1,15} &= -1.01147937 \times 10^{-20},\\
g_{1,16} &= 2.6101406 \times 10^{-22},\\
g_{1,17} &= -1.2377814 \times 10^{-22},\\
g_{1,18} &= -5.880727 \times 10^{-25},\\
g_{1,19} &= -1.8541620 \times 10^{-24},\\
g_{1,20} &= -2.273798 \times 10^{-27},\\
g_{1,21} &= -1.8075282 \times 10^{-26},\\
g_{1,22} &= 5.963925 \times 10^{-29},\\
g_{1,23} &= -9.8233263 \times 10^{-29},\\
g_{1,24} &= 3.5794574 \times 10^{-31},\\
g_{1,25} &= -2.19869295 \times 10^{-31}.
\end{aligned} \tag{32}$$

Putting the values of the unknown constants from Eq. (32) in Eq. (31) and then solving with the help of Maple software we obtained the numerical result for the BVP Eqs. (23) and (24) as

$$\begin{aligned}
y(x) = &-1.154870275 \times 10^{-10}x^{14}\\
&+ 1.753435560 \times 10^{-9}x^{13}\\
&+ 4.194565852 \times 10^{-7}x^{12}\\
&+ 4.301262830 \times 10^{-14}x^{22}\\
&- 1.859417435 \times 10^{-16}x^{25}\\
&+ 3.30814066 \times 10^{-12}x^{16}\\
&- 2.231030132 \times 10^{-13}x^{18}\\
&- 1.503677694 \times 10^{-11}x^{15}\\
&- 0.5000000002x^2 + 0.4166666667e^{-1}x^4\\
&- 0.1666666667x^3 - 0.1388888888e^{-2}x^6\\
&+ 0.8333333330e^{-2}x^5 - 0.2965534651e^{-5}x^9\\
&+ 0.2804415707e^{-4}x^8 - 0.1991607224e^{-3}x^7\\
&- 0.2332892658e^{-5}x^{11}\\
&+ 0.4838952945e^{-5}x^{10}\\
&+ 1.442303306 \times 10^{-13}x^{19}\\
&+ 2.966319514 \times 10^{-14}x^{20}\\
&- 7.356039012 \times 10^{-14}x^{21}\\
&+ 2.399949771 \times 10^{-15}x^{24}\\
&- 1.363097025 \times 10^{-14}x^{23}\\
&- 4.61766534 \times 10^{-13}x^{17}\\
&+ 0.9999999997 + 1.000000000x.
\end{aligned}$$

Table 5 **Comparison Between the Absolute Error by HWM and Absolute Error by DTM[43]: Example 4.2.**

x	Absolute Error in DTM	Absolute Error by HWM
0.0	0.000000E+00	0.0000000000E+00
0.1	2.220450E−16	0.0000000000E+00
0.2	0.000000E+00	0.0000000000E+00
0.3	2.220450E−15	0.0000000000E+00
0.4	6.661340E−15	0.0000000000E+00
0.5	1.110220E−14	2.0000000000E−19
0.6	1.043610E−14	0.0000000000E+00
0.7	5.329070E−15	0.0000000000E+00
0.8	8.881780E−16	1.0000000000E−19
0.9	0.000000E+00	2.0000000000E−19
1.0	0.000000E+00	3.0000000000E−17

The results obtained for Example 4.2 are depicted as in Figs. 8 and 9 and Tables 4 and 5.

5. CONCLUSIONS

In this paper, HWM is investigated for studying the numerical solution of order 12- and 13-order ODEs with boundary conditions. The algorithm based on HWM is proposed by utilizing Maple software and applied on test problems. The algorithm makes the conversions of differential equations into a system of algebraic equations in a straightforward manner. The obtained system of equations has been solved and the approximate solution of the problem is obtained in a short interval of time. The results obtained from the proposed HWM-based algorithm are accurate, valid and very close to the exact solution, and are better than other numerical methods solutions present in the existing literature. Moreover, we found that the accuracy of the numerical results can be increased by enhancing the numbers of J that is increasing the order of approximation.

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