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Applications of q-Hermite Polynomials to Subclasses of Analytic and Bi-Univalent Functions

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Abstract: In mathematics, physics, and engineering, orthogonal polynomials and special functions play a vital role in the development of numerical and analytical approaches. This field of study has received a lot of attention in recent decades, and it is gaining traction in current fields, including computational fluid dynamics, computational probability, data assimilation, statistics, numerical analysis, and image and signal processing. In this paper, using q-Hermite polynomials, we define a new subclass of bi-univalent functions. We then obtain a number of important results such as bonds for the initial coefficients of $|a_2|$, $|a_3|$, and $|a_4|$, results related to Fekete–Szegö functional, and the upper bounds of the second Hankel determinant for our defined functions class.

Keywords: Fekete–Szegö functional; coefficients bounds; orthogonal polynomials; q-Hermite polynomials; bi-univalent functions; q-derivatives

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1. Introduction and Background Review

Let $\mathcal{H}(\mathbb{U})$ denote the class of functions which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Let the subclass of functions $\mathcal{H}(\mathbb{U})$ be denoted by $\mathcal{A}$, which fulfills the following normalization condition

$$f(0) = f'(0) - 1 = 0,$$

In other words, a function having the following series form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in \mathbb{U}). \tag{1}$$

Additionally, let $\mathcal{S}$ be the class of functions in $\mathcal{A}$, which are univalent in $\mathbb{U}$. It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),$$

and

$$f^{-1}(f(\omega)) = \omega \qquad \left(|\omega| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right)$$

where

$$f^{-1}(\omega) = g(\omega) = \omega - a_2\omega^2 + \left(2a_2^2 - a_3\right)\omega^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)\omega^4 + \cdots. \tag{2}$$

A function is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent function in $\mathbb{U}$ given by (2).

Moreover, a bi-univalent functions class Σ was studied by Lewin [1]. He showed that $|b_2| < 1.51$. Subsequently, Brannan and Clunie [2] conjectured that $|b_2| < \sqrt{2}$. Netanyahu [3], on the other hand, showed that

$$\max_{f\in\Sigma}|b_2| = \frac{4}{3}.$$

Similar to the subclasses $\mathcal{K}(\zeta)$ and $\mathcal{S}^*(\zeta)$ of convex and starlike functions, certain subclasses of the bi-univalent function class Σ were given by Brannan and Taha [4]. They called it $\mathcal{S}^*_\Sigma(\zeta)$ and $\mathcal{K}_\Sigma(\zeta)$ of bi-starlike functions and bi-convex functions of order $\zeta(0 \leq \zeta < 1)$, respectively. In each of the functions classes $\mathcal{S}^*_\Sigma(\zeta)$ and $\mathcal{K}_\Sigma(\zeta)$, it was shown that the first two Taylor–Maclaurin coefficients $|b_2|$ and $|b_3|$ are non-sharp.

Moreover, for two analytic functions s_1 and s_2, the function s_1 is called subordinated to the function s_2 denoted as

$$s_1(z) \prec s_2(z) \qquad (z \in \mathbb{U}),$$

if for a function w with the properties

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1$$

exist, such that

$$s_1(z) = s_2(w(z)).$$

If the function s_2 is univalent in $\mathbb{U}$, then

$$s_1(z) \prec s_2(z) \Leftrightarrow s_1(0) = s_2(0) \quad \text{and} \quad s_1(\mathbb{U}) \subset s_2(\mathbb{U}).$$

The q-derivative operator $\mathfrak{D}_q$ was for the first time given by Jackson [5] as follows:

$$\mathfrak{D}_q f(z) = \frac{f(z) - f(qz)}{z(1-q)} = \frac{1}{z}\left\{z + \sum_{k=2}^{\infty}[k]_q a_k z^k\right\} \tag{3}$$

and $\mathfrak{D}_q f(0) = f'(0)$. In case $f(z) = z^k$ for k is a positive integer, the q-derivative of $f(z)$ is given by

$$\mathfrak{D}_q z^k = \frac{z^k - (zq)^k}{z(1-q)} = [k]_q z^{k-1}, \tag{4}$$

where

$$[k]_q = 1 + q + q^2 + \cdots + q^{k+1} = \frac{q^k - 1}{q - 1}. \tag{5}$$

We see that

$$\lim_{q\longrightarrow 1-}[k]_q = \lim_{q\longrightarrow 1-}\frac{1-q^k}{1-q} = k, \qquad (z \neq 0,\ q \neq 0). \tag{6}$$

For the usage of the q-derivatives in geometric function theory of complex analysis, we may refer the readers to [6].

In recent years, the quantum (or q-) calculus has been used as a powerful tool in many different areas of sciences. In analytic function theory, its usage is quite significant. Historically speaking, it was Srivastava [7] who used the basic (or q-) calculus in the context of univalent functions first. Due to certain applications in many diverse disciplines, a substantial number of authors have inspired this and they studied a number of applications. For example, some subclasses of multivalent q-starlike functions were studied and investigated by Khan et al. [8]. In [6], using certain q-Poisson distribution, some new subclasses of analytic functions were developed, and for each of the defined functions classes, the authors derived some useful results, such as necessary and sufficient conditions. Additionally, Shi et al. [9] utilized the q-derivative operator to construct an innovative subclass of Janowski-type multivalent q-starlike functions. Moreover, in both papers [8,9] the authors derived many sufficient conditions and some of their consequences.

The q-calculus has been studied by many different authors and a variety of its applications have been investigated. Furthermore, Srivastava's [10] recently released survey-cum-expository review study is valuable for researchers and scholars working on these subjects (see, for example, [11]).

The q-Hermite polynomial was first introduced by Rogers [12] (see also [13,14]) and is usually defined by means of their generating function as follows:

$$N_k(s|q) = \sum_{k=0}^{\infty} H_k(x;q)\frac{t^k}{(q;q)_k} = \prod_{k=0}^{\infty} \frac{1}{1-2xtq^k+t^2q^{2k}} \qquad (0<q<1).$$

We have the q-derivative of q-Hermite polynomial as follows:

$$\mathfrak{D}_q\{N_{k+1}(s|q)\} = [k]_q N_k(s|q). \tag{7}$$

Additionally, Ismail et al. [12] were able to define the recursion relation as

$$tN_k(s|q) = N_{k+1}(s|q) + [k]_q N_{k-1}(s|q) \tag{8}$$

with

$$N_0(s|q) = 1 \quad \text{and} \quad N_{-1}(s|q) = 0.$$

Remark 1. *It is clear that*

$$N_k(s|q=1) = N_{e_k}(s)$$

is the Hermite polynomials. Additionally, when

$$N_k(s|q=0) = U_k(s/2),$$

we have Chebyshev polynomials of the first kind and they are defined by the recursion relation

$$2sU_k(s) = U_{k-1}(s) + U_{k+1}(s) \tag{9}$$

with

$$U_0(s) = 1 \quad \text{and} \quad U_{-1}(s) = 0.$$

Definition 1. *Let $Q(z,s,q)$ be defined as follows:*

$$Q(z,s,q) = \sum_{k=2}^{\infty} N_k(s|q) z^k. \tag{10}$$

A function $f \in \Sigma$ given by (1) is said to be in the class $\Lambda_{\Sigma}^{q}(s)$, if the following conditions are satisfied:

$$(\mathfrak{D}_q f)(z) \prec Q(z,s,q) \qquad \left(s \in \left(\frac{1}{2},1\right), 0<q<1,\ z \in \mathbb{U}\right) \tag{11}$$

and

$$(\mathfrak{D}_q g)(\omega) \prec Q(\omega, s, q) \qquad \left(s \in \left(\frac{1}{2}, 1\right), 0 < q < 1,\ \omega \in \mathbb{U}\right). \tag{12}$$

We can say that

$$Q(z, s, q) = 1 + N_1(s|q)z + N_2(s|q)z^2 + N_3(s|q)z^3 + \cdots \tag{13}$$

where $z \in \mathbb{U}$ *and* $-1 < s < 1$.

Additionally from (8), we have

$$N_1(s|q) = s$$
$$N_2(s|q) = s^2 - 1$$
$$N_3(s|q) = s^3 - (2+q)s$$
$$N_4(s|q) = s^4 - (3 + 2q + q^2)s^2 + (1 + q + q^2).$$

According to Pommerenke [15], the Hankel determinant of $f(z)$ for $m \geq 1$ and $n \geq 1$ is defined as

$$H_{m,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+m-1} & a_{n+m} & \cdots & a_{n+2m-2} \end{vmatrix}, \quad (a_1 = 1). \tag{14}$$

Clearly, $H_{1,1}(f)$ becomes the Fekete–Szegö functional $a_3 - \rho a_2^2$ with $\rho = 1$ [16]. Additionally, for $m = 2$ and $n = 2$ functional,

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2 a_4 - a_3^2|, \tag{15}$$

is known as the second Hankel determinant, obtained for various subclasses of univalent and multivalent holomorphic functions.

In particular, sharp bonds for the functional $H_{2,2}(f)$ for each of the class of starlike functions (S^*) and the class of convex functions (C) were investigated by Janteng et al. [17,18]. Additionally, Krishna et al. [19] obtained the sharp estimates of $|H_{2,2}(f)|$ for the set of Bazilevic functions.

As far as we know, there is no study linked with bi-univalent functions in the literature for the q-Hermite polynomials. The major purpose of this study is to begin an investigation into the properties of bi-univalent functions linked with q-Hermite polynomials. We use the q-Hermite polynomials expansions to determine the Fekete–Szegö problem, initial coefficient estimates, and estimate of $|\mathcal{H}_2(2)|$ the class $\Lambda_\Sigma^q(s)$.

2. A Set of Lemmas

Lemma 1 ([20])**.** *Let* $\varphi(z) \in \mathcal{P}$, *then*

$$|p_j| \leq 2 \qquad (j \in \mathcal{N}).$$

Lemma 2 ([21])**.** *Let* $\varphi(z) \in \mathcal{P}$, *then*

$$2p_2 = p_1^2 + x\left(4 - p_1^2\right)$$
$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z$$

for some complex number satisfying x, z, $|x| \leq 1$ *and* $|z| \leq 1$.

3. Coefficient Estimates for the Class $\Lambda_{\Sigma}^{q}(s)$

Theorem 1. *Let $f \in \Lambda_{\Sigma}^{q}(s)$. Then*

$$|a_2| \leq \sqrt{\Omega_1(s,q)}, \tag{16}$$

$$|a_3| \leq \frac{s^2}{[2]_q^2} + \frac{s}{[3]_q}, \tag{17}$$

and

$$|a_4| \leq \frac{5s^2}{2[2]_q[3]_q} + \frac{s}{[4]_q} + \frac{2(s^2-s-1)}{[4]_q} + \frac{s^3-2x^2-x-2qs-2}{[4]_q},$$

where

$$\Omega_1(s,q) = \frac{s^3}{|[3]_q s^2 - [2]_q(s^2-s-1)|}. \tag{18}$$

Proof. Let $f \in \Sigma$ given by (1) be in the class $\Lambda_{\Sigma}^{q}(s)$. Then

$$\mathfrak{D}_q f(z) = Q(d(z),s,q) \tag{19}$$

and

$$\mathfrak{D}_q g(\omega) = Q(\varpi(\omega),s,q), \tag{20}$$

where $p, y \in \mathcal{P}$ and defined by

$$p(z) = \frac{1+d(z)}{1-d(z)} = 1 + p_1(z) + p_2 z^2 + p_3 z^3 + \cdots \Rightarrow d(z) = \frac{p(z)-1}{p(z)+1}, \quad (z \in \mathbb{U}) \tag{21}$$

and

$$y(\omega) = \frac{1+\varpi(\omega)}{1-\varpi(\omega)} = 1 + y_1(\omega) + y_2\omega^2 + y_3\omega^3 + \cdots \Rightarrow \varpi(\omega) = \frac{y(\omega)-1}{y(\omega)+1}, \quad (\omega \in \mathbb{U}). \tag{22}$$

It follows that from (21) and (22) that

$$d(z) = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right)z^3 + \cdots\right] \tag{23}$$

and

$$\varpi(\omega) = \frac{1}{2}\left[y_1\omega + \left(y_2 - \frac{y_1^2}{2}\right)\omega^2 + \left(y_3 - y_1y_2 + \frac{y_1^3}{4}\right)\omega^3 + \cdots\right]. \tag{24}$$

From (23) and (24), applying $Q(z,s,q)$ as given in (10), we see that

$$Q(d(z),s,q) = 1 + \frac{N_1(s|q)}{2}p_1 z + \left[\frac{N_1(s|q)}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{N_2(s|q)}{4}p_1^2\right]z^2 + \left[\frac{N_1(s|q)}{2}\left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right) + \frac{N_2(s|q)}{2}p_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{N_3(s|q)}{8}p_1^3\right]z^3 + \cdots$$

and

$$Q(\varpi(\omega),s,q) = 1 + \frac{N_1(s|q)}{2}y_1\omega + \left[\frac{N_1(s|q)}{2}\left(y_2 - \frac{y_1^2}{2}\right) + \frac{N_2(s|q)}{4}y_1^2\right]\omega^2 + \left[\frac{N_1(s|q)}{2}\left(y_3 - y_1y_2 + \frac{y_1^3}{4}\right) + \frac{N_2(s|q)}{2}y_1\left(y_2 - \frac{y_1^2}{2}\right) + \frac{N_3(s|q)}{8}y_1^3\right]\omega^3 + \cdots. \tag{25}$$

It follows from (19), (20), and(25) that we have

$$[2]_q a_2 = \frac{N_1(s|q)}{2} p_1, \tag{26}$$

$$[3]_q a_3 = \frac{N_1(s|q)}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{N_2(s|q)}{4} p_1^2, \tag{27}$$

$$\begin{aligned}[4]_q a_4 \quad &= \quad \frac{N_1(s|q)}{2}\left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right) \\ &\quad + \frac{N_2(s|q)}{2} p_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{N_3(s|q)}{8} p_1^3, \end{aligned} \tag{28}$$

$$-[2]_q a_2 = \frac{N_1(s|q)}{2} y_1, \tag{29}$$

$$[3]_q(2a_2^2 - a_3) = \frac{N_1(s|q)}{2}\left(y_2 - \frac{y_1^2}{2}\right) + \frac{N_2(s|q)}{4} y_1^2, \tag{30}$$

$$\begin{aligned}-[4]_q(5a_2^3 - 5a_2 a_3 + a_4) &= \frac{N_1(s|q)}{2}\left(y_3 - y_1 y_2 + \frac{y_1^3}{4}\right) \\ &\quad + \frac{N_2(s|q)}{2} y_1\left(y_2 - \frac{y_1^2}{2}\right) + \frac{N_3(s|q)}{8} y_1^3. \end{aligned} \tag{31}$$

Adding (26) and (29), we have

$$p_1 = -y_1, \quad p_1^2 = y_1^2 \quad \text{and} \quad p_1^3 = -y_1^3 \tag{32}$$

and

$$a_2^2 = \frac{(N_1(s|q))^2(p_1^2 + y_1^2)}{8[2]_q^2}. \tag{33}$$

Additionally, adding (27), (30), and applying (32) yields

$$4[3]_q a_2^2 = N_1(s|q)(p_2 + y_2) - y_1^2(N_1(s|q) - N_2(s|q)). \tag{34}$$

Applying (32) in (33) gives

$$y_1^2 = \frac{4[2]_q^2 a_2^2}{(N_1(s|q))^2}. \tag{35}$$

Inputting (35) into (34), and with some calculations, we have

$$|a_2|^2 = \left|\frac{(N_1(s|q))^3(p_2 + y_2)}{4[3]_q(N_1(s|q))^2 + 4[2]_q^2(N_1(s|q) - N_2(s|q))}\right|.$$

Now, by using the trigonometric inequalities in conjunction with Lemma 1, we have

$$|a_2| \le \sqrt{\Omega_1(s,q)}. \tag{36}$$

Subtracting (30) from (27) and with some calculations, we have

$$a_3 = a_2^2 + \frac{N_1(s|q)[p_2 - y_2]}{4[3]_q} \tag{37}$$

$$a_3 = \frac{(N_1(s|q))^2 p_1^2}{4[2]_q^2} + \frac{N_1(s|q)[p_2 - y_2]}{4[3]_q}. \tag{38}$$

Applying triangular inequality and Lemma 1, we have

$$|a_3| \leq \frac{s^2}{[2]_q^2} + \frac{s}{[3]_q}. \tag{39}$$

Subtracting (31) from (28), we have

$$\begin{aligned} 2[4]_q a_4 = & \frac{5[4]_q (N_1(s|q))^2 p_1(p_2 - y_2)}{8[2]_q[3]_q} \\ & + \frac{N_1(s|q)(p_3 - y_3)}{2} + \frac{[N_2(s|q) - N_1(s|q)]p_1(p_2 + y_2)}{2} \\ & + \frac{(N_1(s|q) - 2N_2(s|q) + N_3(s|q))p_1^3}{4}. \end{aligned} \tag{40}$$

By using the trigonometric inequalities in conjunction with Lemma 1, we have

$$|a_4| \leq \frac{5s^2}{2[2]_q[3]_q} + \frac{s}{[4]_q} + \frac{2(s^2 - s - 1)}{[4]_q} + \frac{s^3 - 2x^2 - x - 2qs - 2}{[4]_q}.$$

□

4. Fekete–Szegö Inequalities for the Function Class $\Lambda_\Sigma^q(s)$

In this section, we aim to determine the upper bonds of the coefficient functional $|a_3 - \delta a_2^2|$ for the function class $\Lambda_\Sigma^q(s)$.

Theorem 2. *Let $f \in \Lambda_\Sigma^q(s)$. Then, and for some $\delta \in \mathbb{R}$,*

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} 2|1-\delta|\Omega_1(s,q) & \left(|1-\delta|\Omega_1(s,q) \geq \frac{s}{[3]_q}\right) \\ \frac{2s}{[3]_q} & \left(|1-\delta|\Omega_1(s,q) \leq \frac{s}{[3]_q}\right), \end{cases}$$

where

$$\Omega_1(s,q) = \frac{s^3}{|[3]_q s^2 - [2]_q(s^2 - s - 1)|}. \tag{41}$$

Proof. From (37), we have

$$\begin{aligned} a_3 - \delta a_2^2 &= a_2^2 + \frac{N_1(s|q)[p_2 - y_2]}{4[3]_q} - \delta a_2^2 \\ &= \frac{s(p_2 - y_2)}{4[3]_q} + (1-\delta)\left[\frac{s^3(p_2 + y_2)}{4s^2[3]_q - 4[2]_q^2(s^2 - s - 1)}\right]. \end{aligned}$$

By triangular inequality, we have

$$|a_3 - \delta a_2^2| \leq \frac{s}{[3]_q} + |1-\delta|\Omega_1(s,q). \tag{42}$$

Suppose

$$|1-\delta|\Omega_1(s,q) \geq \frac{s}{[3]_q}$$

then, we have

$$|a_3 - \delta a_2^2| \leq 2|1-\delta|\Omega_1(s,q) \tag{43}$$

where

$$|1-\delta| \geq \frac{s}{[3]_q \Omega_1(s,q)}$$

and suppose

$$|1-\delta|\Omega_1(s,q) \leq \frac{s}{[3]_q},$$

then, we have

$$|a_3 - \delta a_2^2| \leq \frac{2s}{[3]_q}$$

where

$$|1-\delta| \leq \frac{s}{[3]_q \Omega_1(s,q)}$$

and $\Omega_1(s,q)$ is given in (41). □

5. $\mathcal{H}_2(2)$ of $\Lambda_\Sigma^q(s)$

In this section, we aim to determine the upper bonds of the second Hankel determinant for the function class $\Lambda_\Sigma^q(s)$.

Theorem 3. *Let the function $f \in \Lambda_\Sigma^q(s)$. Then*

$$H_2(2) = \left|a_2 a_4 - a_3^2\right| \leq \begin{cases} T(2,s) & (R_1 \geq 0 \quad and \quad R_2 \geq 0) \\ \max\left\{\frac{s^2}{[3]_q^2}, T(2,s)\right\} & (R_1 > 0 \quad and \quad R_2 < 0) \\ \frac{s^2}{[3]_q^2} & (R_1 \leq 0 \quad and \quad R_2 \leq 0) \\ \max\{T(m_0,s), T(2,s)\} & (R_1 < 0 \quad and \quad R_2 > 0). \end{cases}$$

where

$$T(2,t) = \frac{N_1(s|q)[N_1(s|q) - 2N_2(s|q) + N_3(s|q)]}{[2]_q[4]_q} + \frac{2N_1(s|q)[N_2(s|q) + N_1(s|q)]}{[2]_q[4]_q} + \frac{(N_1(s|q))^2}{[2]_q[4]_q} + \frac{(N_1(s|q))^4}{[2]_q^4} - \frac{(N_1(s|q))^2}{[3]_q^2},$$

$$T(m_0,t) = \frac{(N_1(s|q))^2}{[3]_q^2} + \frac{R_2^4[2]_q^4}{2[4]_q[3]_q^2 R_1^3} - \frac{[2]_q^2 R_2^3}{2[4]_q[3]_q^2 R_1^2},$$

$$R_1 = N_1(s|q)\Big[2[2]_q^3[3]_q^3[N_1(s|q) - 2N_2(s|q) + N_3(s|q)] + 2(N_1(s|q))^3[4]_q[3]_q^3 - 4N_1(s|q)[2]_q^3[3]_q^3 + 2N_1(s|q)2]_q^4[4]_q - (N_1(s|q))^2[2]_q^2[3]_q[4]_q\Big]$$

and

$$R_2 = N_1(s|q)\Big[4[N_2(s|q) - N_1(s|q)][2]_q[3]_q^2 - 4N_1(s|q)[2]_q^2[4]_q + 6N_1(s|q)[2]_q[3]_q^2 + (N_1(s|q))^2[3]_q[4]_q\Big].$$

Proof. From (26) and (40), we have

$$a_2a_4 = \frac{5(N_1(s|q))^3(p_2-y_2)}{32[2]_q^2[3]_q}p_1^2 + \frac{(N_1(s|q))^2(p_3-y_3)}{8[2]_q[4]_q}p_1 + \frac{N_1(s|q)[N_2(s|q)+N_1(s|q)](p_2+y_2)}{8[2]_q[4]_q}p_1^2 + \frac{N_1(s|q)[N_1(s|q)-2N_2(s|q)+N_3(s|q)]}{16[2]_q[4]_q}p_1^4.$$

With some calculations, we have

$$a_2a_4 - a_3^2 = \frac{5(N_1(s|q))^3(p_2-y_2)}{32[2]_q^2[3]_q}p_1^2 + \frac{(N_1(s|q))^2(p_3-y_3)}{8[2]_q[4]_q}p_1 + \frac{N_1(s|q)[N_2(s|q)+N_1(s|q)](p_2+y_2)}{8[2]_q[4]_q}p_1^2 + \frac{N_1(s|q)[N_1(s|q)-2N_2(s|q)+N_3(s|q)]}{16[2]_q[4]_q}p_1^4 - \frac{(N_1(s|q))^4}{16[2]_q^4}p_1^4 - \frac{(N_1(s|q))^3(p_2-y_2)}{8[2]_q^2[3]_q}p_1^2 - \frac{(N_1(s|q))^2(p_2-y_2)^2}{16[3]_q^2}. \quad (44)$$

By using Lemma 2,

$$p_2 - y_2 = \frac{4-p_1^2}{2}(x-h) \quad (45)$$

$$p_2 + y_2 = p_1^2 + \frac{4-p_1^2}{2}(x+h) \quad (46)$$

and

$$p_3 - y_3 = \frac{p_1^3}{2} + \frac{4-p_1^2}{2}p_1(x+h) - \frac{4-p_1^2}{4}p_1(x^2+h^2) + \frac{4-p_1^2}{2}\Big[(1-|x|^2z) - (1-|h|^2)w\Big] \quad (47)$$

for some x, h, z, w with $|x| \leq 1,\ \ |h| \leq 1,\ \ |z| \leq 1,\ \ |w| \leq 1$, $|p_1| \in [0,2]$ and substituting $(p_2+y_2),\ \ (p_2-y_2)$, and (p_3-y_3), and after some straightforward simplifications, we have

$$a_2a_4 - a_3^2 = \frac{(N_1(s|q))^3(4-p_1^2)(x-h)}{64[2]_q^2[3]_q}p_1^2 + \frac{N_1(s|q)[N_2(s|q)+N_1(s|q)]}{8[2]_q[4]_q}p_1^4 + \frac{(N_1(s|q))^2}{16[2]_q[4]_q}p_1^4 + \frac{N_1(s|q)[N_2(s|q)+N_1(s|q)](4-p_1^2)(x+h)}{16[2]_q[4]_q}p_1^2 + \frac{(N_1(s|q))^2(4-p_1^2)(x+h)}{16[2]_q[4]_q}p_1^2 - \frac{(N_1(s|q))^2(4-p_1^2)(x^2+h^2)}{32[2]_q[4]_q}p_1^2 + \frac{(N_1(s|q))^2(4-p_1^2)[(1-|x|^2)z-(1-|h|^2)w]}{16[2]_q[4]_q}p_1 - \frac{(N_1(s|q))^2(4-p_1^2)^2(x-h)^2}{64[3]_q^2} - \frac{(N_1(s|q))^4}{16[2]_q^4}p_1^4 + \frac{N_1(s|q)[N_1(s|q)-2N_2(s|q)+N_3(s|q)]}{16[2]_q[4]_q}p_1^4.$$

Let $m = p_1$, assume that $m \in [0,2]$, $\lambda_1 = |x| \leq 1$, $\lambda_2 = |h| \leq 1$ and applying triangular inequality, we have

$$|a_2a_4 - a_3^2| \leq \left\{ \frac{N_1(s|q)[N_1(s|q) - 2N_2(s|q) + N_3(s|q)]}{16[2]_q[4]_q} m^4 + \frac{N_1(s|q)[N_2(s|q) + N_1(s|q)]}{8[2]_q[4]_q} m^4 + \frac{(N_1(s|q))^2}{16[2]_q[4]_q} m^4 + \frac{(N_1(s|q))^2(4-m^2)}{8[2]_q[4]_q} m + \frac{(N_1(s|q))^4}{16[2]_q^4} m^4 \right\} + \left\{ \frac{N_1(s|q)[N_2(s|q) + N_1(s|q)](4-m^2)}{16[2]_q[4]_q} m^2 + \frac{(N_1(s|q))^3(4-m^2)}{64[2]_q^2[3]_q} m^2 + \frac{(N_1(s|q))^2(4-m^2)}{16[2]_q[4]_q} m^2 \right\}(\lambda_1 + \lambda_2) + \left\{ \frac{(N_1(s|q))^2(4-m^2)}{32[2]_q[4]_q} m^2 - \frac{(N_1(s|q))^2(4-m^2)}{16[2]_q[4]_q} m \right\}(\lambda_1^2 + \lambda_2^2) + \frac{(N_1(s|q))^2(4-m^2)^2(\lambda_1+\lambda_2)^2}{64[3]_q^2}$$

and equivalently, we have

$$|a_2a_4 - a_3^2| \leq L_1(s,m) + L_2(s,m)(\lambda_1 + \lambda_2) + L_3(s,m)(\lambda_1^2 + \lambda_2^2) + L_4(s,m)(\lambda_1 + \lambda_2)^2 = Z(\lambda_1, \lambda_2) \tag{48}$$

where

$$L_1(s,m) = \left\{ \frac{N_1(s|q)[N_1(s|q) - 2N_2(s|q) + N_3(s|q)]}{16[2]_q[4]_q} m^4 + \frac{N_1(s|q)[N_2(s|q) + N_1(s|q)]}{8[2]_q[4]_q} m^4 + \frac{(N_1(s|q))^2}{16[2]_q[4]_q} m^4 + \frac{(N_1(s|q))^2(4-m^2)}{8[2]_q[4]_q} m + \frac{(N_1(s|q))^4}{16[2]_q^4} m^4 \right\} \geq 0$$

$$L_2(s,m) = \left\{ \frac{N_1(s|q)[N_2(s|q) + N_1(s|q)](4-m^2)}{16[2]_q[4]_q} m^2 + \frac{(N_1(s|q))^3(4-m^2)}{64[2]_q^2[3]_q} m^2 + \frac{(N_1(s|q))^2(4-m^2)}{16[2]_q[4]_q} m^2 \right\} \geq 0$$

$$L_3(s,m) = \left\{ \frac{(N_1(s|q))^2(4-m^2)}{32[2]_q[4]_q} m^2 - \frac{(N_1(s|q))^2(4-m^2)}{16[2]_q[4]_q} m \right\} \leq 0$$

$$L_4(s,m) = \frac{(N_1(s|q))^2(4-m^2)^2}{64[3]_q^2} \geq 0$$

where $m \in [0,2]$. Now, we maximize the function $Z(\lambda_1, \lambda_2)$ in the closed square

$$\Delta = \{(\lambda_1, \lambda_2) : \lambda_1 \in [0,1], \lambda_2 \in [0,1]\} \quad for \quad m \in [0,2].$$

For a fixed value of s, the coefficients of the function $Z(\lambda_1, \lambda_2)$ in (48) are dependent on m; therefore, the maximum value of $Z(\lambda_1, \lambda_2)$ needs to be investigated; for this, we take the cases when $m = 0$, $m = 2$, and $m \in (0, 2)$.

First Case:

If $m = 0$,

$$Z(\lambda_1, \lambda_2) = L_4(s, 0) = \frac{(N_1(s|q))^2}{4[3]_q^2}(\lambda_1 + \lambda_2)^2.$$

It is obvious that the function $Z(\lambda_1, \lambda_2)$ reaches its maximum at (λ_1, λ_2) and

$$max\{Z(\lambda_1, \lambda_2) : \lambda_1, \lambda_2 \in [0, 1]\} = Z(1, 1) = \frac{(N_1(s|q))^2}{[3]_q^2}. \tag{49}$$

Second Case:

When $m = 2$, the function $Z(\lambda_1, \lambda_2)$ is constant with respect to m; therefore, we have

$$Z(\lambda_1, \lambda_2) = N_1(s, 2) = \left\{ \frac{N_1(s|q)[N_1(s|q) - 2N_2(s|q) + N_3(s|q)]}{[2]_q[4]_q} + \frac{2N_1(s|q)[N_2(s|q) + N_1(s|q)]}{[2]_q[4]_q} + \frac{(N_1(s|q))^2}{[2]_q[4]_q} + \frac{(N_1(s|q))^4}{[2]_q^4} \right\}.$$

Third Case:

When $m \in (0, 2)$, let $\lambda_1 + \lambda_2 = s$ and $\lambda_1 \cdot \lambda_2 = l$, then (48) can be of the form

$$Z(\lambda_1, \lambda_2) = L_1(s, m) + L_2(s, m)c + (L_3(s, m) + L_4(s, m))c^2 - 2L_3(s, m)l = V(c, l) \tag{50}$$

where $c \in [0, 2]$ *and* $l \in [0, 1]$. Now, we need to investigate the maximum of

$$V(c, l) \in \Lambda = \{(c, l) : c \in [0, 2], l \in [0, 1]\}. \tag{51}$$

By differentiating $V(c, l)$ partially, we have

$$\frac{\partial V}{\partial c} = L_2(s, m) + 2(L_3(s, m) + L_4(s, m))c = 0$$
$$\frac{\partial V}{\partial l} = -2L_3(s, m) = 0.$$

The above results show that there is no critical point in Λ for $V(c, l)$ and therefore $Z(\lambda_1, \lambda_2)$ does not have a critical point in the square Δ.

From the above observation, we see that $Z(\lambda_1, \lambda_2)$ does not have the maximum value in the interior of Δ. Therefore, we next investigate the maximum of $Z(\lambda_1, \lambda_2)$ on the boundary of the square Δ.

For $\lambda_1 = 0$, $\lambda_2 \in [0, 1]$ (also, for $\lambda_2 = 0, \lambda_1 \in [0, 1]$) and

$$Z(0, \lambda_2) = L_1(s, m) + L_2\beta_2 + (L_3(s, m) + L_4(s, m))\lambda_2^2 = Q(\lambda_2). \tag{52}$$

Now, since $L_3(s, m) + L_4(s, m) \geq 0$, we have

$$Q'(\lambda_2) = L_2(s, m) + 2[L_3(s, m) + L_4(s, m)]\lambda_2 > 0$$

showing that the function $Q(\lambda_2)$ is an increasing. Therefore, for $m \in [0, 2)$ and $s \in (1/2, 1]$, the maximum occurs at $\lambda_2 = 1$. Thus, from (52),

$$\begin{aligned} \max\{G(0, \lambda_2) : \lambda_2 \in [0, 1]\} &= Z(0, 1) \\ &= L_1(s, m) + L_2(s, m) + L_3(s, m) + L_4(s, m). \end{aligned} \tag{53}$$

For $\lambda_1 = 1, \lambda_2 \in [0,1]$ (also, for $\lambda_2 = 1$, $\lambda_1 \in [0,1]$) and

$$Z(1,\lambda_2) = L_1(s,m) + L_2(s,m) + L_3(s,m) + L_4(s,m) + [L_2(s,m) + 2L_4(s,m)]\lambda_2 + [L_3(s,m) + L_4(s,m)]\lambda_2^2 = D(\lambda_2) \tag{54}$$

$$D'(\lambda_2) = [L_2(s) + 2L_4(s)] + 2[L_3(s) + L_4(s)]\lambda_2. \tag{55}$$

We know that $L_3(s) + L_4(s) \geq 0$, then

$$D'(\lambda_2) = [L_2(s) + 2L_4(s)] + 2[L_3(s) + L_4(s)]\lambda_2 > 0.$$

Therefore, the maximum occurs at $\lambda_2 = 1$ because the function $D(\lambda_2)$ is increasing. From (54), we have

$$\max\{Z(1,\lambda_2) : \lambda_2 \in [0,1]\} = Z(1,1) = L_1(s,m) + 2[L_2(s,m) + L_3(s,m)] + 4L_4(s,m). \tag{56}$$

Now for $m \in (0,2)$ and from (53) and (56), we have

$$L_1(s,m) + 2[L_2(s,m) + L_3(s,m)] + 4L_4(s,m) > L_1(s,m) + L_2(s,m) + L_3(s,m) + L_4(s,m).$$

Therefore,

$$\max\{Z(\lambda_1,\lambda_2) : \lambda_1 \in [0,1], \lambda_2 \in [0,1]\} = L_1(s,m) + 2[L_2(s,m) + L_3(s,m)] + 4L_4(s,m).$$

Since

$$Q(1) \leq D(1) \quad for \quad m \in [0,2] \quad and \quad s \in [1,1],$$

then

$$\max\{Z(\lambda_1,\lambda_2)\} = Z(1,1)$$

occurs on the boundary of square Δ.

Let $T : (0,2) \to \mathbb{R}$ be defined by

$$T(m,s) = \max\{Z(\lambda_1,\lambda_2)\} = Z(1,1) = L_1(s,m) + 2L_2(s,m) + 2L_3(s,m) + 4L_4(s,m). \tag{57}$$

Now, inserting the values of $L_1(s,m), L_2(s,m), L_3(s,m)$, and $L_4(s,m)$ into (57) and with some calculations, we have

$$\begin{aligned} T(m,s) = &\left\{ \frac{N_1(s|q)[N_1(s|q) - 2N_2(s|q) + N_3(s|q)]}{16[2]_q[4]_q} m^4 \right. \\ &+ \frac{N_1(s|q)[N_2(s|q) + N_1(s|q)]}{8[2]_q[4]_q} m^4 + \frac{(N_1(s|q))^2}{16[2]_q[4]_q} m^4 \\ &\left. + \frac{(N_1(s|q))^2(4-m^2)}{8[2]_q[4]_q} m + \frac{(N_1(s|q))^4\gamma^4}{16[2]_q^4} m^4 \right\} \\ &+ \left\{ \frac{N_1(s|q)[N_2(s|q) + N_1(s|q)](4-m^2)}{8[2]_q[4]_q} m^2 + \frac{(N_1(s|q))^3(4-m^2)}{32[2]_q^2[3]_q} m^2 \right. \\ &\left. + \frac{(N_1(s|q))^2(4-m^2)}{8[2]_q[4]_q} m^2 \right\} + \left\{ \frac{(N_1(s|q))^2(4-m^2)}{16[2]_q[4]_q} m^2 \right. \\ &\left. - \frac{(N_1(s|q))^2(4-m^2)}{8[2]_q[4]_q} m \right\} + \frac{(N_1(s|q))^2(4-m^2)^2}{16[3]_q^2}. \end{aligned}$$

By simplifying, we have

$$T(m,s) = \frac{(N_1(s|q))^2}{[3]_q} + \frac{R_1}{32[2]_q^4[4]_q[3]_q^2}m^4 + \frac{R_2}{8[2]_q^2[4]_q[3]_q^2}m^2,$$

where

$$R_1 = N_1(s|q)\bigg[2[2]_q^3[3]_q^3[N_1(s|q) - 2N_2(s|q) + N_3(s|q)] + 2(N_1(s|q))^3[4]_q[3]_q^3 - 4N_1(s|q)[2]_q^3[3]_q^3 + 2N_1(s|q)2]_q^4[4]_q - (N_1(s|q))^2[2]_q^2[3]_q[4]_q\bigg]$$

$$R_2 = N_1(s|q)\bigg[4[N_2(s|q) - N_1(s|q)][2]_q[3]_q^2 - 4N_1(s|q)[2]_q^2[4]_q + 6N_1(s|q)[2]_q[3]_q^2 + (N_1(s|q))^2[3]_q[4]_q\bigg].$$

If the maximum value of $T(m,s)$ is in the interior of $m \in [0,2]$, then we have

$$T'(m,s) = \frac{R_1}{8[2]_q^4[4]_q[3]_q^2}m^3 + \frac{R_2}{4[2]_q^2[4]_q[3]_q^2}m.$$

Now, we need the following cases:

First Result:

Let $R_1 \geq 0$ and $R_2 \geq 0$, then $T'(m,s) \geq 0$. This observation shows that the function $T(m,s)$ is increasing on the boundary of $m \in [0,2]$ that is $m = 2$. Therefore, we have

$$\begin{aligned}\max\{T(m,s) : m \in (0,2)\} = &\frac{N_1(s|q)[N_1(s|q) - 2N_2(s|q) + N_3(s|q)]}{[2]_q[4]_q} \\ &+ \frac{2N_1(s|q)[N_2(s|q) + N_1(s|q)]}{[2]_q[4]_q} \\ &+ \frac{(N_1(s|q))^2}{[2]_q[4]_q} + \frac{(N_1(s|q))^4}{[2]_q^4} - \frac{(N_1(s|q))^2}{[3]_q^2}.\end{aligned}$$

Second Result:

If $R_1 > 0$ and $R_2 < 0$, then

$$T'(m,s) = \frac{R_1m^3 + 2[2]_q^2mR_2}{8[2]_q^4[4]_q[3]_q^2}m^3 = 0 \tag{58}$$

at critical point

$$m_0 = \sqrt{\frac{-2[2]_q^2R_2}{R_1}} \tag{59}$$

is a critical point of the function $T(m,t)$. Now,

$$T''(m_0) = \frac{-3R_2}{4[2]_q^2[4]_q[3]_q^2} + \frac{R_2}{4[2]_q^2[4]_q[3]_q^2} > 0.$$

Therefore, m_0 is the minimum point of the function $T(m,s)$. Hence, $T(m,s)$ can not have a maximum.

Third Result:

If $R_1 \leq 0$ and $R_2 \leq 0$, then

$$T'(m,s) \leq 0.$$

Therefore, the function $T(m,s)$ is decreasing on the $(0,2)$ interval; hence,

$$\max\{T(m,s) : m \in (0,2)\} = T(0) = \frac{(N_1(s|q))^2}{[3]_q^2}. \quad (60)$$

Fourth Result:

If $R_1 < 0$ and $R_2 > 0$

$$T''(m_0,s) = \frac{-3R_2}{4[2]_q^2[4]_q[3]_q^2} + \frac{R_2}{4[2]_q^2[4]_q[3]_q} < 0.$$

Therefore, $T''(m,s) < 0$. Hence, m_0 is the maximum point of the function $T(m,s)$ and the maximum value occurs at $m = m_0$. Thus,

$$max\{T(m,s) : m \in (0,2)\} = T(m_0,s)$$

$$T(m_0,t) = \frac{(N_1(s|q))^2}{[3]_q^2} + \frac{R_2^4[2]_q^4}{2[4]_q[3]_q^2R_1^3} - \frac{[2]_q^2R_2^3}{2[4]_q[3]_q^2R_1^2}.$$

□

6. Conclusions

During the past decades, in mathematics, physics, engineering, and in other branches of sciences the orthogonal polynomials and special functions have played an incredible role, as highlighted in the Section 1. In our present investigation, we were essentially motivated by the recent research occurring as cited in our first section. We used q-Hermite polynomials and first defined a new subclass of bi-univalent functions systematically. We then obtained a number of important results such as bonds for the initial coefficients of $|a_2|$, $|a_3|$, and $|a_4|$, results related to Fekete–Szegö functional, and the upper bounds of the second Hankel determinant for our defined functions class.

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