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Starlike Functions Associated with Bernoulli's Numbers of Second Kind

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Abstract: The aim of this paper is to introduce a class of starlike functions that are related to Bernoulli's numbers of the second kind. Let $\varphi_{BS}(\xi) = \left(\frac{\xi}{e^{\xi}-1}\right)^2 = \sum_{n=0}^{\infty} \frac{\xi^n B_n^2}{n!}$, where the coefficients of B_n^2 are Bernoulli numbers of the second kind. Then, we introduce a subclass of starlike functions $\digamma$ such that $\frac{\xi\digamma'(\xi)}{\digamma(\xi)} \prec \varphi_{BS}(\xi)$. We found out the coefficient bounds, several radii problems, structural formulas, and inclusion relations. We also found sharp Hankel determinant problems of this class.

Keywords: starlike functions; subordination; Bernoulli's number of second kind; radii problems; inclusion results; coefficient bounds; Hankel determinants

MSC: 30C45; 30C50

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1. Introduction and Preliminaries

The Bernoulli numbers first appeared in the posthumous publications of Jakob Bernoulli in (1713), and they were independently discovered by the Japanese mathematician Seki Takakazu in 1712 [1]. We define the Bernoulli numbers of the k kind as follows:

$$\varphi_{BS}(\xi) = \left(\frac{\xi}{e^{\xi}-1}\right)^k = \sum_{n=0}^{\infty} \frac{\xi^n B_n^k}{n!}. \quad (1)$$

Bernoulli numbers of the k kind are denoted by B_n^k. The function defined in (1) for $k = 1$ is known as the Bernoulli function. The convexity of the function φ_{BS} given in (1), as well as its reciprocal function $\left(e^{\xi}-1\right)/\xi$ are studied in [2,3]; see also [4].

Let $\mathbf{H}$ denote a class of analytic functions in $\mathbf{E} = [\xi \in \mathbb{C} : |\xi| < 1]$. Let $\mathbf{A}_n \subset \mathbf{H}$ represent the functions $\digamma$ having the series expansion $\digamma(\xi) = \xi + d_{n+1}\xi^{n+1} + d_{n+2}\xi^{n+2} + \cdots$ in $\mathbf{E}$. The class $\mathbf{A}_1 = \mathbf{A}$ represents the function $\digamma$ with a power series representation:

$$\digamma(\xi) = \xi + \sum_{n=2}^{\infty} d_n \xi^n, \quad \xi \in \mathbf{E}. \quad (2)$$

The class $\mathbf{S} \subset \mathbf{A}$ contains the univalent function $\digamma$ (i.e., $\digamma(\xi_1) = \digamma(\xi_2)$, which implies that $\xi_1 = \xi_2$ in $\mathbf{E}$). Let $\digamma \in \mathbf{A}$. Then, $\digamma$ is in the $\mathbf{S}^*$ of univalent starlike functions if, and only if

$$Re\left\{\frac{\xi\digamma'(\xi)}{\digamma(\xi)}\right\} > 0, \quad \xi \in \mathbf{E}.$$

Let $\mathbf{B} \subset \mathbf{H}$ represent a class of self maps ϖ (Schwarz functions) in $\mathbf{E}$ with $\varpi(0) = 0$. Assume that F and g are analytic (holomorphic) in $\mathbf{E}$. Then, $F \prec g$ and reads as F, which is subordinated by g such that $F(\xi) = g(\varpi(\xi))$ for $\xi \in \mathbf{E}$ and $\varpi \in \mathbf{B}$ if the subordinating function g is univalent. Then,

$$F(0) = g(0) \Leftrightarrow F(\mathbf{E}) \subseteq g(\mathbf{E}).$$

In [5], the authors have introduced a subclass of $\mathbf{S}^*$ defined by

$$\mathbf{S}^*(\varphi) = \left\{ F \in \mathbf{A} : \frac{\xi F'(\xi)}{F(\xi)} \prec \varphi(\xi) \right\}.$$

The function φ is one-to-one in $\mathbf{E}$, and maps $\mathbf{E}$ onto a starlike domain with respect to $\varphi(0) = 1$, with $\varphi'(0) > 0$ being symmetric about the real axis . We obtain subclasses of $\mathbf{S}^*$ by taking particular φ. The functions in class $\mathbf{S}^*[a,b] := \mathbf{S}^*((1+a\xi)/(1+b\xi))$ are Janowski starlike functions [6]. Furthermore, $\mathbf{S}^*(\lambda) := \mathbf{S}^*[1-2\lambda, -1]$ represents starlike functions of order $\lambda \in [0,1)$, whereas $\mathbf{S}^*(0) = \mathbf{S}^*$. The class

$$\mathbf{SS}^*(\beta) := \mathbf{S}^*[(1+\xi)/(1-\xi)]^{\beta} = \{F \in \mathbf{A} : \left|\arg\left(\xi F'(\xi)/F(\xi)\right)\right| < \beta\pi/2\},\ \beta \in (0,1]$$

represents strongly starlike functions in $\mathbf{E}$. The class $\mathbf{SL}^* := \mathbf{S}^*\left(\sqrt{1+\xi}\right)$ contains starlike functions related with a lemniscate of the Bernoulli; see [7]. The classes

$$\mathbf{S}^*_{\mathbf{RL}} := \mathbf{S}^*\left(\sqrt{2} - \left(\sqrt{2}-1\right)\left((1-\xi)/\left(1+2\left(\sqrt{2}-1\right)\xi\right)\right)^{1/2}\right)$$

and $\mathbf{S}^*_e := \mathbf{S}^*(e^{\xi})$ were studied in [8,9]. The class $\mathbf{S}^*_C := \mathbf{S}^*\left(1+4\xi/3+2\xi^2/3\right)$ represents starlike functions related with a cardioid [10]. The classes $\mathbf{S}^*_s := \mathbf{S}^*(1+\sin\xi)$ and $\mathbf{S}^*_{\cos} := \mathbf{S}^*(\cos\xi)$ are related with sine and cosine functions, respecitvely; see [11] and [12] respectively. The class $\mathbf{S}^*_{\Delta} = \mathbf{S}^*\left(\xi + \sqrt{1+\xi^2}\right)$ is related with the lune, see [13], whereas the class $\mathbf{BS}^*(\lambda) := \mathbf{S}^*\left(1+\xi/\left(1-\lambda\xi^2\right)\right)$, $\lambda \in [0,1]$ is related with the Booth lemniscate; see [14]. The class $\mathbf{S}^*_{\mathbf{B}} := \mathbf{S}^*(e^{e^{\xi}} - 1)$ is related to the Bell numbers; see [15]. The class $\mathbf{S}^*_{\mathbf{T}} = \mathbf{S}^*(e^{(\xi+\mu\frac{\xi^2}{2})})$ is related to telephone numbers; see [16]. The class $\mathbf{S}^*_{\mathbf{BF}} = \mathbf{S}^*\left(\xi/e^{\xi}-1\right)$ contains starlike functions related with Bernoulli functions' see [17].

For some recent work, we refer to [18–23] and the references therein.

We now define the class $\mathbf{S}^*_{BS}$ associated with the Bernoulli numbers of the second kind.

Definition 1. *Let $F \in \mathbf{A}$. Then, $F \in \mathbf{S}^*_{BS}$ if and only if*

$$\frac{\xi F'(\xi)}{F(\xi)} \prec \left(\frac{\xi}{e^{\xi}-1}\right)^2 = \varphi_{BS}(\xi), \quad \xi \in \mathbf{E}.$$

In other words, a function $F \in \mathbf{S}^*_{BS}$ can be written as

$$F(\xi) = \xi \exp\left(\int_0^{\xi} \frac{\varphi(s)-1}{s} ds\right), \tag{3}$$

where φ is analytic and satisfies $\varphi(\xi) \prec \varphi_{BS}(\xi) = (\frac{\xi}{e^{\xi}-1})^2$ $(\xi \in \mathbf{E})$.

To give some examples of functions in the class $\mathbf{S}^*_{BS}$, consider

$$\varphi_1(\xi) = 1 + \frac{\xi}{4}, \quad \varphi_2(\xi) = \frac{5+2\xi}{5+\xi}, \quad \varphi_3(\xi) = \frac{3+\xi e^{\xi}}{3}, \quad \varphi_4(\xi) = 1 + \frac{\xi\cos(\xi)}{4}.$$

The function $\varphi_0(\xi) = (\frac{\xi}{e^{\xi}-1})^2$ is univalent in $\mathbf{E}$, $\varphi_i(0) = \varphi_0(0)$ $(i = 1,2,3,4)$ and $\varphi_i(\mathbf{E}) \subset \varphi_0(\mathbf{E})$; it is easy to conclude that $\varphi_i(\xi) \prec \varphi_0(\xi)$. The functions $F_i \in \mathbf{S}^*_{BS}$ corresponding to every φ_i. respectively, are given as follows:

$$\begin{aligned} F_1(\xi) &= \xi e^{\xi/4}, \quad F_2(\xi) = \xi\left(1+\frac{\xi}{5}\right), \\ F_3(\xi) &= \xi \exp\left(\frac{e^{\xi}-1}{3}\right), \quad F_4(\xi) = \xi \exp\left(\frac{\sin(\xi)}{4}\right). \end{aligned}$$

In particular, if $\varphi_0(\xi) = \left(\frac{\xi}{e^{\xi}-1}\right)^2$, then (3) takes the form

$$\begin{aligned} F_0(\xi) &= \xi \exp\left(\int_0^{\xi} \frac{\varphi_0(s)-1}{s} ds\right) = \xi - \xi^2 + \frac{17\xi^3}{24} - \frac{29\xi^4}{72} \\ &\quad + \frac{377\xi^5}{1920} - \frac{11\xi^6}{120} + \cdots . \end{aligned} \tag{4}$$

The above function acts as an extremal function for $\mathbf{S}^*_{BS}$.

The following theorem gives the sharp estimates for φ_{BS}:

Lemma 1. *The function* $\varphi_{BS}(\xi) = \left(\frac{\xi}{e^{\xi}-1}\right)^2$ *satisfies*

$$\begin{aligned} \min_{|\xi|=\ell} Re\, \varphi_{BS}(\xi) &= \varphi_{BS}(\ell) = \min_{|\xi|=\ell} |\varphi_{BS}(\xi)| \\ \max_{|\xi|=\ell} Re\, \varphi_{BS}(\xi) &= \varphi_{BS}(-\ell) = \max_{|\xi|=\ell} |\varphi_{BS}(\xi)|, \end{aligned}$$

whenever $\ell \in (0,1)$.

2. Inclusion and Radius Problems

Theorem 1. *The class* $\mathbf{S}^*_{BS}$ *satisfies the following inclusion relations:*

1. *If* $0 \le \lambda \le \frac{1}{(e-1)^2}$, *then* $\mathbf{S}^*_{BS} \subset \mathbf{S}^*(\lambda)$.
2. *If* $\beta \ge (\frac{e}{e-1})^2$, *then* $\mathbf{S}^*_{BS} \subset \mathbf{RS}^*(1/\beta) \subset \mathbf{M}(\beta)$.
3. $\mathbf{S}^*_{BS} \subset \mathbf{SS}^*(\beta)$, *where* $\beta_0 \le \beta \le 1$, *wherein* $\beta_0 = 2\mathsf{h}(y_2)/\pi \approx 0.6454469651m$ *and* h *is defined in* (5).

Proof. (1) If $F \in \mathbf{S}^*_{BS}$, then $\frac{\xi F'(\xi)}{F(\xi)} \prec (\frac{\xi}{e^{\xi}-1})^2$. According to Lemma 1, we have

$$\min_{|\xi|=1} Re\left(\frac{\xi}{e^{\xi}-1}\right)^2 < Re\frac{\xi F'(\xi)}{F(\xi)} < \max_{|\xi|=1} Re\left(\frac{\xi}{e^{\xi}-1}\right)^2;$$

therefore,

$$Re\frac{\xi F'(\xi)}{F(\xi)} > \frac{1}{(e-1)^2}.$$

(2) Similarly,

$$Re\frac{\xi F'(\xi)}{F(\xi)} < \frac{e^2}{(e-1)^2}.$$

Thus, $F \in \mathbf{S}^*\left(\frac{1}{(e-1)^2}\right) \cap \mathbf{M}\left(\frac{e^2}{(e-1)^2}\right)$. Now, we have the following:

$$Re\frac{F(\xi)}{\xi F'(\xi)} > \left(\frac{e-1}{e}\right)^2.$$

This implies that $\digamma \in \mathbf{RS}^*(\beta)$ for $\beta \leq \left(\frac{e-1}{e}\right)^2$. Identically, $\digamma \in \mathbf{RS}^*(1/\beta)$ for $\beta \geq \left(\frac{e}{e-1}\right)^2$. Also, $\digamma \in \mathbf{RS}^*(1/\beta)$ if and only if

$$\left|\frac{\xi\digamma'(\xi)}{\digamma(\xi)} - \frac{\beta}{2}\right| < \frac{\beta}{2},$$

which leads to $Re(\xi\digamma'(\xi)/\digamma(\xi)) < \beta$. Therefore, $\mathbf{S}^*_{BS} \subset \mathbf{RS}^*(1/\beta) \subset \mathbf{M}(\beta)$ whenever $\beta \geq \left(\frac{e}{e-1}\right)^2$.

(3) If $\digamma \in \mathbf{S}^*_{BS}$, then

$$\begin{aligned}\left|\arg\frac{\xi\digamma'(\xi)}{\digamma(\xi)}\right| &< \max_{|\xi|=1}\arg\left(\frac{\xi}{e^{\xi}-1}\right)^2 \\ &= \max_{0\leq y<2\pi}\arctan\left(\frac{V}{U}\right).\end{aligned}$$

Let

$$h(y) = \arctan\left(\frac{V}{U}\right), \tag{5}$$

where U and V are given as

$$\begin{aligned} U &= \cos(2y)\begin{pmatrix} \left(e^{\cos(y)}\right)^2(\cos(\sin(y)))^2 - 2\,e^{\cos(y)}\cos(\sin(y)) + 1 \\ -\left(e^{\cos(y)}\right)^2(\sin(\sin(y)))^2 \end{pmatrix} \\ &\quad + \sin(2y)\left(2\left(e^{\cos(y)}\right)^2\cos(\sin(y))\sin(\sin(y)) - 2\,e^{\cos(y)}\sin(\sin(y))\right), \\ V &= \sin(2y)\begin{pmatrix} \left(e^{\cos(y)}\right)^2(\cos(\sin(y)))^2 - 2e^{\cos(y)}\cos(\sin(y)) + 1 \\ -\left(e^{\cos(y)}\right)^2(\sin(\sin(y)))^2 \end{pmatrix} \\ &\quad - \cos(2y)\left(2\left(e^{\cos(y)}\right)^2\cos(\sin(y))\sin(\sin(y)) - 2e^{\cos(y)}\sin(\sin(y))\right). \end{aligned}$$

Here, $h'(y) = 0$ has $y_1 \approx 1.409746460$ and $y_2 \approx 4.873438847$ roots in $[0, 2\pi]$. In addition, $h''(y_2) \approx -1.0988577$. Hence, $\max_{0\leq y<2\pi} h(y) = h(y_2) \approx 1.013865722$, and $\left|\arg\frac{\xi\digamma'(\xi)}{\digamma(\xi)}\right| \leq \frac{\pi\beta}{2}$; that is, $\beta \geq 0.645186552$. This implies that $\mathbf{S}^*_{BS} \subset \mathbf{SS}^*_{\beta}$. $\square$

Now, we discuss some radii problems for the class $\mathbf{S}^*_{BS}$. The following definitions and lemmas are needed to establish the results. The class $\mathbf{P}$ represents the functions p of the form

$$p(\xi) = 1 + \sum_{n=1}^{\infty} p_n\xi^n \tag{6}$$

that are analytic in $\mathbf{E}$ such that $Rep(\xi) > 0$, $\xi \in \mathbf{E}$. Let

$$\mathbf{P}_n[a,b] := \left\{p(\xi) = 1 + \sum_{k=n}^{\infty} c_n\xi^n : p(\xi) \prec \frac{1+a\xi}{1+b\xi},\ -1 \leq b < a \leq 1\right\}.$$

In particular, $\mathbf{P}_n(\lambda) := \mathbf{P}_n[1-2\lambda, -1]$, and $\mathbf{P}_n := \mathbf{P}_n(0)$. Let $\mathbf{S}^*_n[a,b] = \mathbf{A}_n \cap \mathbf{S}^*[a,b]$, and $\mathbf{S}^*_n(\lambda) := \mathbf{S}^*_n[1-2\lambda, -1]$. Also, let

$$\mathbf{S}^*_{BS,n} := \mathbf{A}_n \cap \mathbf{S}^*_{BS}, \quad \mathbf{S}^*_n(\lambda) := \mathbf{A}_n \cap \mathbf{S}^*(\lambda), \quad \mathbf{S}^*_{L,n} := \mathbf{A}_n \cap \mathbf{S}^*_L.$$

Additionally,

$$\mathbf{S}_n := \{\digamma \in \mathbf{A}_n : \digamma(\xi)/\xi \in \mathbf{P}_n\},$$

and

$$\mathbf{CS}_n(\lambda) := \left\{ F \in \mathbf{A}_n : \frac{F(\xi)}{g(\xi)} \in \mathbf{P}_n,\ g \in \mathbf{S}_n^*(\lambda) \right\};$$

see [24].

Lemma 2 ([25])**.** *If $p \in \mathbf{P}_n(\lambda)$, then for $|\xi| = \ell$,*

$$\left| \frac{\xi p'(\xi)}{p(\xi)} \right| \leq \frac{2(1-\lambda)n\ell^n}{(1-\ell^n)(1+(1-2\lambda)\ell^n)}.$$

Lemma 3 ([26])**.** *Let $p \in \mathbf{P}$. Then,*

$$|jp_1^3 - kp_1p_2 + lp_3| \leq 2|j| + 2|k-2j| + 2|j-k+l|.$$

Lemma 4 ([27])**.** *If $p \in \mathbf{P}_n[a,b]$, then for $|\xi| = \ell$,*

$$\left| p(\xi) - \frac{1-ab\ell^{2n}}{1-b^2\ell^{2n}} \right| \leq \frac{(a-b)\ell^n}{1-b^2\ell^{2n}}.$$

If $p \in \mathbf{P}_n(\lambda)$, then for $|\xi| = \ell$,

$$\left| p(\xi) - \frac{(1+(1-2\lambda))\ell^{2n}}{1-\ell^{2n}} \right| \leq \frac{2(1-\lambda)\ell^n}{1-\ell^{2n}}.$$

In the following lemmas, we find disks centered at $(\nu, 0)$ and $(1, 0)$ of the largest and the smallest radii, respectively, such that $\mho_{BS} := \varphi_{BS}(\mathbf{E})$ lies in the disk with the smallest radius and contains the largest disk.

Lemma 5. *Let $\left(\frac{1}{e-1}\right)^2 \leq \nu \leq \left(\frac{e}{e-1}\right)^2$. Then,*

$$\{w \in \mathbb{C} : |w-\nu| < \ell_\nu\} \subset \mho_{BS} \subset \left\{ w \in \mathbb{C} : |w-1| < \left(\frac{e}{e-1}\right)^2 \right\},$$

where

$$\ell_\nu = \begin{cases} \nu - \left(\frac{1}{e-1}\right)^2, & \frac{1}{(e-1)^2} \leq \nu \leq \frac{e^2+1}{2(e-1)^2}, \\ \left(\frac{e}{e-1}\right)^2 - \nu, & \frac{e^2+1}{2(e-1)^2} \leq \nu \leq \left(\frac{e}{e-1}\right)^2. \end{cases}$$

Proof. Let $\cos(y) = \varrho$ and $\sin(y) = \varsigma$. Then, the square of the distance from the boundary $\mho_{BS}$ to the point $(\nu, 0)$ is given by

$$\psi(y) = \left(\frac{\mathbf{A}}{(1-2e^\varrho cos(\varsigma)+(e^\varrho)^2)^2} - \nu \right)^2 + \left(\frac{\mathbf{B}}{(1-2e^\varrho cos(\varsigma)+(e^\varrho)^2)^2} \right)^2,$$

where

$$\begin{aligned} \mathbf{A} &= \cos(2y)\left\{1-2e^\varrho\cos(\varsigma)+(e^\varrho)^2\cos(2\varsigma)\right\} + 2e^\varrho\{e^\varrho\cos(\varsigma)-1\}\sin(\varsigma)\sin(2y), \\ \mathbf{B} &= \sin(2y)\left\{1-2e^\varrho\cos(\varsigma)+(e^\varrho)^2\cos(2\varsigma)\right\} - 2e^\varrho\{e^\varrho\cos(\varsigma)-1\}\sin(\varsigma)\cos(2y). \end{aligned}$$

To show that $|w-\nu| < \ell_\nu$ is largest disk contained in $\mho_{BS}$, it is enough to show that the $\min\limits_{0\leq\leq 2\pi} \psi(y) = \ell_\nu$. Since $\psi(y) = \psi(-y)$, it is enough to take the range $0 \leq y \leq \pi$.

Case 1: When $\frac{1}{(e-1)^2} \leq \nu < \frac{e^2}{(2e^2-8e+9)(e-1)^2}$, then $\psi'(y) = 0$ has 0 and π roots. In addition, $\psi'(y) > 0$ for $y \in (0, \pi)$. Thus,

$$\min_{0 \leq \ell \leq \pi} \psi(y) = \min\{\psi(0),\ \psi(\pi)\} = \psi(0).$$

Hence,

$$\ell_\nu = \min_{0 \leq y \leq \pi} \sqrt{\psi(y)} = \sqrt{\psi(0)} = \frac{1}{(e-1)^2} - \nu.$$

Case 2: When $\frac{e^2}{(2e^2-8e+9)(e-1)^2} < \nu \leq \frac{e^2}{(e-1)^2}$, then $\psi'(y) = 0$ has 0, y_ν, and π roots, where y_ν depends on ν. In addition, $\psi'(y) > 0$ for $y \in (0, y_\nu)$, and $\psi'(y) < 0$ when $y \in (y_\nu, \pi)$. Therefore, $\psi(y)$ has minima at 0 or π. We also see that $\psi(0) < \psi(\pi)$ for $\frac{e^2}{(2e^2-8e+9)(e-1)^2} < \nu \leq \frac{e^2+1}{2(e-1)^2}$ and $\psi(0) > \psi(\pi)$ for $\frac{e^2+1}{2(e-1)^2} < \nu \leq \left(\frac{e}{e-1}\right)^2$.

Thus, the first part of the proof is completed.

Now, for the smallest disc that contains $\mho_{BS}$, the function $\psi(y)$ for $\nu = 1$ attains its maximum value at π. Thus, the disk with the smallest radius that contains $\mho_{BS}$ has a radius of $\left(\frac{e}{e-1}\right)^2$. □

Theorem 2. *The sharp* $R_{\mathbf{S}^*_{BS,n}}$ *for* $\mathbf{S}_n$ *is*

$$R_{\mathbf{S}^*_{BS,n}}(\mathbf{S}_n) = \left(\frac{e^2 - 2e}{\sqrt{n^2(e-1)^4 + (e^2 - 2e)} + n(e-1)^2}\right)^{1/n}.$$

Proof. Consider a function $\hbar(\xi) \in \mathbf{P}_n$ such that $\hbar(\xi) = F(\xi)/\xi$. Now, we have the following:

$$\frac{\xi F'(\xi)}{F(\xi)} - 1 = \frac{\xi \hbar'(\xi)}{\hbar(\xi)}.$$

From Lemma 2, we have

$$\left|\frac{\xi F'(\xi)}{F(\xi)} - 1\right| = \left|\frac{\xi \hbar'(\xi)}{\hbar(\xi)}\right| \leq \frac{2n\ell^n}{1 - \ell^{2n}}.$$

From Lemma 4, the map of $|\xi| \leq \ell$ under $\xi F'/F$ lies in the $\mho_{BS}$ if the following is satisfied:

$$\frac{2n\ell^n}{1 - \ell^{2n}} \leq 1 - \frac{1}{(e-1)^2}.$$

This is equivalently written as

$$((e-1)^2 - 1)\ell^{2n} + 2n(e-1)^2\ell^n + 1 - (e-1)^2 \leq 0.$$

Thus, the $\mathbf{S}^*_{BS,n}$-radius of the $\mathbf{S}_n$ is the root $\ell \in (0,1)$ of

$$((e-1)^2 - 1)\ell^{2n} + 2n(e-1)^2\ell^n + 1 - (e-1)^2 = 0;$$

that is,

$$R_{\mathbf{S}^*_{BS,n}}(\mathbf{S}_n) = \left(\frac{e(e-2)}{n(e-1)^2 + \sqrt{1 + (n^2+1)(e-1)^4 - 2(e-1)^2}}\right)^{1/n}.$$

Consider $F_0(\xi) = \xi(1+\xi^n)/1 - \xi^n$. Then, $\hbar_0(\xi) = F_0(\xi)/\xi = (1+\xi^n)/(1-\xi^n) > 0$. Thus, $F_0 \in \mathbf{S}_n$, and $\xi F_0'(\xi)/F_0(\xi) = 1 + 2n\xi^n/(1-\xi^{2n})$. This is beacuse at $\xi = R_{\mathbf{S}^*_{BS,n}}$, we have

$$\frac{\xi F_0'(\xi)}{F_0(\xi)} - 1 = \frac{2n\xi^n}{1 - \xi^{2n}} = 1 - \frac{1}{(e-1)^2}.$$

Therefore, $\digamma_0$ gives a sharp result. Hence, the proof is completed. □

Theorem 3. *Let*

$$R_1 = \left(\frac{4e-e^2-1}{2(e-1)^2(1-2\lambda)+e^2+1}\right)^{\frac{1}{2n}},$$
$$R_2 = \left(\frac{(e-1)^2-1}{(1+n-\lambda)(e-1)^2+\sqrt{1+(2n(1-\lambda)+\lambda^2+n^2)(e-1)^4-2(e-1)^2\lambda}}\right)^{\frac{1}{n}},$$
$$R_3 = \left(\frac{e^2}{(1+n-\lambda)(e-1)^2+\sqrt{1+(1+n-\lambda)^2(e-1)^4-(1-2\lambda)e^4}}\right)^{\frac{1}{n}}.$$

Then, a sharp $\mathbf{S}^*_{BS,n}$*-radius for the class* $\mathbf{CS}_n(\lambda)$ *is*

$$R_{\mathbf{S}^*_{BS,n}}(\mathbf{CS}_n(\lambda)) = \begin{cases} R_2, & if \;\; R_2 \le R_1, \\ R_3, & if \;\; R_2 > R_1. \end{cases}$$

Proof. Define a function $\hbar(\xi) = \digamma(\xi)/g(\xi)$, where $g \in \mathbf{S}^*_n(\lambda)$. Then, $\hbar \in \mathbf{P}_n$, and $\xi g'(\xi)/g(\xi) \in P_n(\lambda)$. From the definition of $\hbar$, we have

$$\frac{\xi \digamma'(\xi)}{\digamma(\xi)} = \frac{\xi g'(\xi)}{g(\xi)} + \frac{\xi \hbar'(\xi)}{\hbar(\xi)}.$$

From Lemmas 2 and 3, we see that

$$\left|\frac{\xi \digamma'(\xi)}{\digamma(\xi)} - \frac{1+(1-2\lambda)\ell^{2n}}{1-\ell^{2n}}\right| \le \frac{2(1+n-\lambda)\ell^n}{1-\ell^{2n}}. \tag{7}$$

Now, we find the values R_1, R_2 and R_3 for $0<\ell<1$ and $0 \le \lambda < 1$. Firstly, we find R_1. For $\ell \le R_1$, this can be found if and only if

$$\frac{1+(1-2\lambda)\ell^{2n}}{1-\ell^{2n}} \le \frac{e^2+1}{2(e-1)^2}.$$

This implies that

$$\ell \le \left(\frac{4e-e^2-1}{2(e-1)^2(1-2\lambda)+e^2+1}\right)^{\frac{1}{2n}}.$$

Now, we obtain R_2. For this, we must have

$$\frac{2(1+n-\lambda)\ell^n}{1-\ell^{2n}} \le \frac{1+(1-2\lambda)\ell^{2n}}{1-\ell^{2n}} - \frac{1}{(e-1)^2}.$$

This implies that

$$\ell \le \frac{(e-1)^2-1}{(1+n-\lambda)(e-1)^2+\sqrt{1+(-2n\lambda+\lambda^2+2n+n^2)(e-1)^4-2(e-1)^2\lambda}}.$$

For R_3, we have

$$\frac{2(1+n-\lambda)\ell^n}{1-\ell^{2n}} \le \left(\frac{e}{e-1}\right)^2 - \frac{1+(1-2\lambda)\ell^{2n}}{1-\ell^{2n}}.$$

This implies that

$$\ell \le \frac{e^2}{(1+n-\lambda)(e-1)^2+\sqrt{1+(1+n-\lambda)^2(e-1)^4-(1-2\lambda)e^4}}.$$

□

Theorem 4. *The $\mathbf{S}^*_{BS,n}$-radius for $\mathbf{S}^*_n[a,b]$ is*

$$R_{\mathbf{S}^*_{BS,n}}(\mathbf{S}^*_n[a,b]) = \begin{cases} \min\{1;\ell_1\}, & -1 \leq b \leq 0 < a \leq 1, \\ \min\{1;\ell_2\}, & 0 < b < a \leq 1, \end{cases}$$

where

$$\ell_1 = \left(\frac{2e-1}{(e-1)^2 a - be^2}\right)^{1/n},$$

and

$$\ell_2 = \left(\frac{e(e-2)}{a(e-1)^2 - b}\right)^{1/n}.$$

Proof. Let $F \in \mathbf{S}^*_n[a,b]$. Then, from Lemma 3, we can write

$$\left|\frac{\xi F'(\xi)}{F(\xi)} - C\right| \leq \frac{(a-b)\ell^n}{1-b^2\ell^{2n}}, \tag{8}$$

where

$$C = \frac{1-ab\ell^{2n}}{1-b^2\ell^{2n}}, \ |\xi| = \ell.$$

For $b < 0$, we see that $C \geq 1$. Also by using Lemma 4, $F \in \mathbf{S}^*_{BS,n}$ if

$$\frac{1+(a-b)\ell^n - ab\ell^{2n}}{1-b^2\ell^{2n}} \leq \frac{e^2}{(e-1)^2},$$

which is equivalent to

$$\ell \leq \left(\frac{2e-1}{(e-1)^2 a - be^2}\right)^{1/n} = \ell_1.$$

Furthermore, if $b = 0$, then $C = 1$. From (8), we have

$$\left|\frac{\xi F'(\xi)}{F(\xi)} - 1\right| \leq a\ell^n, \quad (0 < a \leq 1).$$

By using Lemma 4 with $a = 1$, this gives $\ell \leq \left(\frac{e(e-2)}{a(e-1)^2}\right)^{1/n}$ for $F \in \mathbf{S}^*_{BS,n}$. We see that $C < 1$ for $0 < b < a \leq 1$. Thus, from Lemma 4 and (8), we have $F \in \mathbf{S}^*_{BS,n}$ if

$$\frac{(a-b)\ell^n}{1-b^2\ell^{2n}} \leq \frac{1-ab\ell^{2n}}{1-b^2\ell^{2n}} - \frac{1}{(e-1)^2},$$

or, equivalently, if

$$\ell \leq \left(\frac{e(e-2)}{a(e-1)^2 - b}\right)^{1/n} = \ell_2.$$

This completes the result. □

Theorem 5. *Let $-1 < b < a \leq 1$. If either*

(a) $(1-b) \leq (e-1)^2(1-a)$ *and* $2(1-b^2) \leq (1-ab)(e-1)^2 < (1-b^2)(1+e^2)$ *or if*

(b) $(a+1)(e-1)^2 \leq e^2(1+b)$ *and* $(1-b^2)(1+e^2) \leq 2(1-ab)(e-1)^2 \leq 2e^2(1-b^2)$ *hold, then* $\mathbf{S}^*_n[a,b] \subset \mathbf{S}^*_{BS,n}$.

Proof. (a) Let $p(\xi) = \xi F'(\xi)/F(\xi)$. From Lemma 3, $F \in \mathbf{S}_n^*[a,b]$ if

$$\left| p(\xi) - \frac{1-ab}{1-b^2} \right| \leq \frac{a-b}{1-b^2}.$$

In connection with Lemma 4, $F \in \mathbf{S}_n^*[a,b]$ if

$$\frac{a-b}{1-b^2} \leq \frac{1-ab}{1-b^2} - \frac{1}{(e-1)^2},$$

and

$$\frac{1}{(e-1)^2} \leq \frac{1-ab}{1-b^2} \leq \frac{1}{2}\frac{1+e^2}{(e-1)^2},$$

which, upon simplification, reduce to (a).

(b) Let $p(\xi) = \xi F'(\xi)/F(\xi)$. Since $F \in \mathbf{S}_n^*[a,b]$, thus, in the view of Lemma 3,

$$\left| p(\xi) - \frac{1-ab}{1-b^2} \right| \leq \frac{a-b}{1-b^2}.$$

By using Lemma 4, we note that $F \in \mathbf{S}_n^*[a,b]$ if the following is satisfied:

$$\frac{a-b}{1-b^2} \leq \frac{e^2}{(e-1)^2} - \frac{1-ab}{1-b^2},$$

and

$$\frac{1}{2}\left(\frac{1+e^2}{(e-1)^2}\right) \leq \frac{1-ab}{1-b^2} \leq \frac{e^2}{(e-1)^2},$$

which reduced to the conditions (b). □

Theorem 6. *The sharp radii for* $\mathbf{S}_{\mathbf{L}}^*$, $\mathbf{S}_{\mathbf{RL}}^*$, $\mathbf{S}_{\mathbf{e}}^*$, *and* $\mathbf{S}_{\mathbf{lim}}^*$ *are*

$R_{\mathbf{S}_{BS}^*}(\mathbf{S}_{\mathbf{L}}^*) = \frac{(e-1)^4-1}{(e-1)^4} \approx 0.889$,

$R_{\mathbf{S}_{BS}^*}(\mathbf{S}_{\mathbf{RL}}^*) = \frac{(5+4\sqrt{2})(e-1)^4+(-6\sqrt{2}-8)(e-1)^2+3+2\sqrt{2}}{(5+4\sqrt{2})(e-1)^4+(8+4\sqrt{2})(e-1)^2+2+2\sqrt{2}} \approx 0.87193$,

$R_{\mathbf{S}_{BS}^*}(\mathbf{S}_{\mathbf{e}}^*) = 2 - 2\ln(e-1) \approx 0.917350$,

$R_{\mathbf{S}_{BS}^*}(\mathbf{S}_{\mathbf{lim}}^*) = \frac{\sqrt{2}(e-2)}{e-1} \approx 0.591174$.

Proof. (1) For $F \in \mathbf{S}_{\mathbf{L}}^*$, we have

$$\frac{\xi F'(\xi)}{F(\xi)} = \sqrt{1+\omega(\xi)}.$$

By the Schwarz Lemma $|\omega(\xi)| \leq |\xi|$, we thus have $\left|\sqrt{1+\omega(\xi)} - 1\right| \leq 1 - \sqrt{1-\ell}$. Thus, for $|\xi| = \ell$, we have

$$\left| \frac{\xi F'(\xi)}{F(\xi)} - 1 \right| \leq 1 - \sqrt{1-\ell}.$$

By Lemma 4, we have $1 - \sqrt{1-\ell} \leq 1 - \frac{1}{(e-1)^2}$. Consider $F_0(\xi) = \frac{4\xi \exp\{2(\sqrt{1+\xi}-1)\}}{(1+\sqrt{1+\xi})^2}$, which is in $\mathbf{S}_{\mathbf{L}}^*$ and $\frac{\xi F_0'(\xi)}{F_0(\xi)} = \sqrt{1+\xi} = \frac{1}{(e-1)^2}$ at $R_{\mathbf{S}_{BS}^*}(\mathbf{S}_{\mathbf{L}}^*)$. Hence, the sharpness is verified.

(2) Let $F \in \mathbf{S}^*_{\mathbf{RL}}$. Then, for $|\xi| = \ell$, we have

$$\left|\frac{\xi F'(\xi)}{F(\xi)} - 1\right| \leq 1 - \sqrt{2} + (\sqrt{2} - 1)\sqrt{\frac{1+\ell}{1-2(\sqrt{2}-1)\ell}}$$
$$\leq 1 - \frac{1}{(e-1)^2}$$

provided that

$$\ell \leq \frac{(5+4\sqrt{2})(e-1)^4 + (-6\sqrt{2}-8)(e-1)^2 + 3 + 2\sqrt{2}}{(5+4\sqrt{2})(e-1)^4 + (8+4\sqrt{2})(e-1)^2 + 2 + 2\sqrt{2}} = R_{\mathbf{S}^*_{BS}}(\mathbf{S}^*_{\mathbf{RL}}).$$

Consider the function F_1 defined by

$$F_1(\xi) = \xi \exp\left(\int_0^{\xi} \frac{\varphi_0(t) - 1}{t} dt\right),$$

where

$$\varphi_0(\xi) = \sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1-\xi}{1+2(\sqrt{2}-1)\xi}}.$$

At $\xi = R_{\mathbf{S}^*_{BS}}(\mathbf{S}^*_{\mathbf{RL}})$, we have

$$\frac{\xi F_1'(\xi)}{F_1(\xi)} = \sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1-\xi}{1+2(\sqrt{2}-1)\xi}} = \frac{1}{(e-1)^2}.$$

Hence, the sharpness is verified.

(3) $F \in \mathbf{S}^*_e$, so we have

$$\left|\frac{\xi F'(\xi)}{F(\xi)} - 1\right| \leq e^{\xi} - 1 \leq \frac{e^2}{(e-1)^2} - 1.$$

The result is sharp for F_2 such that $\frac{\xi F_2'(\xi)}{F_2(\xi)} = e^{\xi}$.

(4) Suppose that $F \in (\mathbf{S}^*_{\mathbf{lim}})$; then $\frac{\xi F'(\xi)}{F(\xi)} \prec 1 + \sqrt{2}\xi + \frac{\xi^2}{2}$. Thus, for $|\xi| = \ell$, we can it write as

$$\left|\frac{\xi F'(\xi)}{F(\xi)} - 1\right| = \left|1 + \sqrt{2}\xi + \frac{\xi^2}{2} - 1\right| \leq \sqrt{2}\ell - \frac{\ell^2}{2} \leq 1 - \frac{1}{(e-1)^2},$$

which is satisfied for $\ell \leq \frac{\sqrt{2}(e-2)}{e-1}$. Consider

$$F_3(\xi) = \xi \exp \frac{4\sqrt{2}\xi + \xi^2}{4}.$$

Since $\frac{\xi F_3'(\xi)}{F_3(\xi)} = 1 + \sqrt{2}\xi + \frac{\xi^2}{2}$, it follows that $F_3 \in (\mathbf{S}^*_{\mathbf{lim}})$ and at $\xi = R_{\mathbf{S}^*_{BS}}(\mathbf{S}^*_{\mathbf{lim}})$, so we have $\frac{\xi F_3'(\xi)}{F_3(\xi)} = \frac{1}{(e-1)^2}$. □

Consider the families:

$$\mathbf{F}_1 := \left\{F \in \mathbf{A}_n : Re\left(\frac{F(\xi)}{g(\xi)}\right) > 0 \text{ and } Re\left(\frac{g(\xi)}{\xi}\right) > 0,\ g \in \mathbf{A}_n\right\},$$

$$\mathbf{F}_2 := \left\{F \in \mathbf{A}_n : Re\left(\frac{F(\xi)}{g(\xi)}\right) > 0 \text{ and } Re\left(\frac{g(\xi)}{\xi}\right) > 1/2,\ g \in \mathbf{A}_n\right\},$$

and

$$\mathbf{F}_3 := \left\{ \digamma \in \mathbf{A}_n : \left| \frac{\digamma(\xi)}{g(\xi)} - 1 \right| < 1 \quad \text{and} \quad Re\left(\frac{g(\xi)}{\xi} \right) > 0,\ g \in \mathbf{A}_n \right\}.$$

Theorem 7. *The sharp radii for functions in the families* $\mathbf{F}_1$, $\mathbf{F}_2$, *and* $\mathbf{F}_3$ *respectively, are:*

$$R_{\mathbf{S}^*_{BS,n}}(\mathbf{F}_1) = \left(\frac{e(e-2)}{2n(e-1)^2 + \sqrt{1+(4n^2+1)(e-1)^4 - 2(e-1)^2}} \right)^{1/n},$$
$$R_{\mathbf{S}^*_{BS,n}}(\mathbf{F}_2) = \left(\frac{2e(e-2)}{3n(e-1)^2 + \sqrt{(9n^2+4n+4)(e-1)^4 - 4(n+2)(e-1)^2 + 4}} \right)^{1/n},$$
$$R_{\mathbf{S}^*_{BS,n}}(\mathbf{F}_3) = \left(\frac{2e(e-2)}{3n(e-1)^2 + \sqrt{(9n^2+4n+4)(e-1)^4 - 4(n+2)(e-1)^2 + 4}} \right)^{1/n}.$$

Proof. (1) Let $\digamma \in \mathbf{F}_1$ and define $p,\ \hbar : \mathbf{E} \to \mathbb{C}$ by $p(\xi) = \frac{g(\xi)}{\xi}$ and $\hbar(\xi) = \frac{\digamma(\xi)}{g(\xi)}$. Then, clearly, $p,\ \hbar \in \mathbf{P}_n$, since $\digamma(\xi) = \xi p(\xi)\hbar(\xi)$. By Lemma 2, and by combining the above inequalities, we have

$$\left| \frac{\xi \digamma'(\xi)}{\digamma(\xi)} - 1 \right| \leq \frac{4n\ell^n}{1-\ell^{2n}} \leq 1 - \frac{1}{(e-1)^2}.$$

After some simplification, we arrive at

$$\ell \leq \left(\frac{e(e-2)}{2n(e-1)^2 + \sqrt{1 + (4n^2+1)(e-1)^4 - 2(e-1)^2}} \right)^{1/n} = R_{\mathbf{S}^*_{BS,n}}(\mathbf{F}_1).$$

To verify the sharpness of result, consider the functions defined by

$$\digamma_4(\xi) = \xi \left(\frac{1+\xi^n}{1-\xi^n} \right)^2 \text{ and } g_0(\xi) = \xi \left(\frac{1+\xi^n}{1-\xi^n} \right).$$

Then, clearly $Re\left(\frac{\digamma_4(\xi)}{g_0(\xi)}\right) > 0$, and $Re\left(\frac{g_0(\xi)}{\xi}\right) > 0$. Hence, $\digamma_0 \in \mathbf{F}_1$. We see that at $\xi = R_{\mathbf{S}^*_{BS,n}}(\mathbf{F}_1)e^{i\pi/n}$ as follows:

$$\frac{\xi \digamma_4'(\xi)}{\digamma_4(\xi)} = 1 + \frac{4n\xi^n}{1-\xi^{2n}} = \frac{1}{(e-1)^2}.$$

Hence, the sharpness is satisfied.

(2) Let $\digamma \in \mathbf{F}_2$. Define $p,\ \hbar : \mathbf{E} \to \mathbb{C}$ by $p(\xi) = \frac{g(\xi)}{\xi}$ and $\hbar(\xi) = \frac{\digamma(\xi)}{g(\xi)}$. Then, $p \in \mathbf{P}_n$, and $\hbar \in \mathbf{P}_n(1/2)$. Since $\digamma(\xi) = \xi p(\xi)\hbar(\xi)$, then according to Lemma 2, we have

$$\left| \frac{\xi \digamma'(\xi)}{\digamma(\xi)} - 1 \right| \leq \frac{3n\ell^n + n\ell^{2n}}{1-\ell^{2n}} \leq 1 - \frac{1}{(e-1)^2},$$

which implies that

$$\ell \leq \left(\frac{2e(e-2)}{3n(e-1)^2 + \sqrt{9n^2(e-1)^4 + 4[(n+1)(e-1)^2 - 1][e(e-2)]}} \right)^{1/n} = R_{\mathbf{S}^*_{BS,n}}(\mathbf{F}_2).$$

Thus, $\digamma \in \mathbf{S}^*_{BS,n}$ for $\ell \leq R_{\mathbf{S}^*_{BS,n}}(\mathbf{F}_2)$.

For sharpness, consider the following:

$$\digamma_5(\xi) = \frac{\xi(1+\xi^n)}{(1-\xi^n)^2} \text{ and } g_1(\xi) = \frac{\xi}{1-\xi^n}.$$

Then clearly $Re\left(\frac{F_5(\xi)}{g_1(\xi)}\right) > 0$, and $Re\left(\frac{g_1(\xi)}{\xi}\right) > \frac{1}{2}$. Hence, $F \in \mathbf{F}_2$. Now, at $\xi = R_{\mathbf{S}^*_{BS,n}}(\mathbf{F}_2)$

$$\frac{\xi F_5'(\xi)}{F_5(\xi)} = 1 + \frac{3n\xi^n + n\xi^{2n}}{1-\xi^{2n}} = \frac{1}{(e-1)^2}.$$

Hence, the sharpness is satisfied.

(3) Let $F \in \mathbf{F}_3$. Define $p, \hbar : \mathbf{E} \to \mathbb{C}$ by $p(\xi) = \frac{g(\xi)}{\xi}$ and $\hbar(\xi) = \frac{g(\xi)}{F(\xi)}$. Then, $p \in \mathbf{P}_n$ and

$$\left|\frac{1}{\hbar(\xi)} - 1\right| < 1 \iff Re(\hbar(\xi)) > 1/2;$$

therefore, $\hbar \in \mathbf{P}_n(1/2)$. Since $F(\xi)\hbar(\xi) = \xi p(\xi)$, then according to Lemma 2, we have

$$\left|\frac{\xi F'(\xi)}{F(\xi)} - 1\right| \le \frac{3n\ell^n + n\ell^{2n}}{1-\ell^{2n}} \le 1 - \frac{1}{(e-1)^2}.$$

This implies that

$$\ell \le \left(\frac{2e(e-2)}{3n(e-1)^2 + \sqrt{(9n^2+4n+4)(e-1)^4 - 4(n+2)(e-1)^2 + 4}}\right)^{1/n} = R_{\mathbf{S}^*_{BS,n}}(\mathbf{F}_3).$$

Thus, $F \in \mathbf{S}^*_{BS,n}$ for $\ell \le R_{\mathbf{S}^*_{BS,n}}(\mathbf{F}_3)$. For sharpness, consider the following:

$$F_6(\xi) = \frac{\xi(1+\xi^n)^2}{1-\xi^n} \text{ and } g_2(\xi) = \frac{\xi(1+\xi^n)}{1-\xi^n}.$$

We see that

$$Re\left(\frac{g_2(\xi)}{F_6(\xi)}\right) = Re\left(\frac{1}{1+\xi^n}\right) > \frac{1}{2},$$

and

$$Re\left(\frac{F_6(\xi)}{\xi}\right) = Re\left(\frac{1+\xi^n}{1-\xi^n}\right) > 0.$$

Therefore, $F_6 \in \mathbf{F}_3$. A computation shows that at $\xi = R_{\mathbf{S}^*_{BS,n}}(\mathbf{F}_3)e^{i\pi/n}$, which comes out to

$$\frac{\xi F_6'(\xi)}{F_6(\xi)} - 1 = \frac{3n\xi^n + n\xi^{2n}}{1-\xi^{2n}} = 1 - \frac{1}{(e-1)^2}.$$

Hence, the sharpness is satisfied. □

3. Coefficient and Hankel Determinant Problems for the Class $\mathbf{S}^*_{BS}$

Pommerenke [28] was the first to introduce the qth Hankel determinant for analytic functions, and it is stated as follows:

$$H_{q,n}(F) := \begin{vmatrix} d_n & d_{n+1} & \dots & d_{n+q-1} \\ d_{n+1} & d_{n+2} & \dots & d_{n+q} \\ \vdots & \vdots & \dots & \vdots \\ d_{n+q-1} & d_{n+q} & \dots & d_{n+2q-2} \end{vmatrix}, \tag{9}$$

where $n, q \in \mathbb{N}$. We note that

$$H_{2,1}(F) = d_3 - d_2^2, \quad H_{2,2}(F) = d_2 d_4 - d_3^2.$$

In this section, we focus on obtaining sharp coefficient bounds and bounds on $H_{2,1}(F)$ and $H_{2,2}(F)$.

We will use the following results related to the class $\mathbf{P}$.

Lemma 6 ([5])**.** *Let $p \in \mathbf{P}$ and be of the form (6). Then for v, a complex number*

$$|p_2 - vp_1^2| \leq 2\max(1, |2v-1|).$$

Lemma 7 ([29,30])**.** *Let $p \in \mathbf{P}$ and be of the form (6) such that $|\rho| \leq 1$, and $|\eta| \leq 1$. Then,*

$$2p_2 = p_1^2 + \rho(4 - p_1^2), \tag{10}$$
$$4p_3 = p_1^3 + 2(4-p_1^2)p_1\rho - (4-p_1^2)p_1^2\rho + 2(4-p_1^2)(1-|\rho|^2)\eta, \tag{11}$$

Lemma 8 ([31])**.** *Let $\varpi \in \mathbf{B}$ be given by $\varpi(z) = \sum_{n=0}^{\infty} c_n\xi^n$, and thus*

$$\psi(u,v) = \left|c_3 + \mu_1 c_1 c_2 + \mu_2 c_1^3\right|.$$

Then, $\psi(u,v) \leq |v|$ if $(u,v) \in D_6$, where

$$D_6 = \left\{(u,v) : 2 \leq |\mu| \leq 4, \quad \nu \geq \frac{1}{12}\left(\mu^2 + 8\right)\right\}.$$

Lemma 9 ([32])**.** *Let $\overline{\mathbf{E}} := \{\rho \in \mathbb{C} : |\rho| \leq 1\}$, and, for j, k, and $l \in \mathbb{R}$, let*

$$Y(j,k,l) := \max\left\{|j + k\rho + l\rho^2| + 1 - |\rho|^2 : \rho \in \overline{\mathbf{E}}\right\}. \tag{12}$$

If $jl \geq 0$, then

$$Y(j,k,l) = \begin{cases} |j| + |k| + |l|, & |k| \geq 2(1-|l|), \\ 1 + |j| + \dfrac{k^2}{4(1-|l|)}, & |k| < 2(1-|l|). \end{cases}$$

If $jl < 0$, then

$$Y(j,k,l) = \begin{cases} 1 - |j| + \dfrac{k^2}{4(1-|l|)}, & -4jl\left(l^{-2}-1\right) \leq k^2 \wedge |k| < 2(1-|l|), \\ 1 + |j| + \dfrac{k^2}{4(1+|l|)}, & k^2 < \min\{4(1+|l|)^2, -4jl\left(l^{-2}-1\right)\}. \\ R(j,k,l), & \text{otherwise}. \end{cases}$$

In such as case,

$$R(j,k,l) = \begin{cases} |j| + |k| - |l|, & |l|(|k| + 4|j|) \leq |jk|, \\ 1 + |j| + \dfrac{k^2}{4(1+|l|)}, & |jk| < |l|(|k| - 4|j|) \leq |jk|. \\ |l| + |j|\sqrt{1 - \dfrac{k^2}{4jl}}, & \text{otherwise}. \end{cases}$$

Theorem 8. *Let $\digamma \in \mathbf{S}^*_{BS}$ and be of the form (2). Then,*

$$|d_2| \leq 1, \quad |d_3| \leq \frac{17}{24}, \quad |d_4| \leq \frac{29}{72}.$$

These bounds are the best possible.

Proof. If $\digamma \in \mathbf{S}^*_{BS}$, then

$$\frac{\xi\digamma'(\xi)}{\digamma(\xi)} = \left(\frac{\varpi(\xi)}{e^{\varpi(\xi)} - 1}\right)^2,$$

where $\varpi \in \mathbf{B}$. The class $\mathbf{B}$ consists of Schwarz functions ϖ that are analytic in $\mathbf{E}$, with $\varpi(0) = 0$, and $|\varpi(\xi)| \leq |\xi|$. Let p be of the form (6). Then,

$$\varpi(\xi) = \frac{p(\xi) - 1}{p(\xi) + 1}.$$

Now by using (2), we can write out the following:

$$\frac{\xi F'(\xi)}{F(\xi)} = 1 + d_2\xi + (2d_3 - d_2^2)\xi^2 + (3d_4 - 3d_2d_3 + d_2^3)\xi^3 + (4d_5 - 2d_3^2 - 4d_2d_4 + 4d_2^2d_3 - d_2^4)\xi^4 + \cdots. \tag{13}$$

In addition,

$$\left(\frac{\mu(\xi)}{e^{\mu(\xi)} - 1}\right)^2 = 1 - \frac{1}{2}p_1\xi + \left(\frac{29}{96}p_1^2 - \frac{1}{4}p_2\right)\xi^2 + \left(\frac{-109}{576}p_1^3 + \frac{13}{36}p_1p_2 - \frac{1}{6}p_3\right)\xi^3 + \left(\frac{11011}{92160}p_1^4 - \frac{215}{576}p_2p_1^2 + \frac{25}{96}p_1p_3 + \frac{23}{192}p_2^2 - \frac{1}{8}p_4\right)\xi^4 + \cdots. \tag{14}$$

From (13) and (14), we obtain

$$d_2 = \frac{1}{2}p_1, \tag{15}$$

$$d_3 = \frac{29}{96}p_1^2 - \frac{1}{4}p_2, \tag{16}$$

$$d_4 = \frac{-109}{576}p_1^3 + \frac{13}{36}p_1p_2 - \frac{1}{6}p_3. \tag{17}$$

From (15), we have $|d_2| = \frac{1}{2}|p_1| \leq 1$. From (16), we can write out the following:

$$|d_3| = \frac{1}{4}\left|p_2 - \frac{29}{24}p_1^2\right|.$$

An application of Lemma 6 for $v = \frac{29}{24}$ gives the required bound.

The function $\varpi \in \mathbf{B}$ can be written as a power series:

$$\varpi(z) = \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbf{E}. \tag{18}$$

Since $p \in \mathbf{P}$, therefore,

$$p(z) = \frac{1 + \varpi(z)}{1 - \varpi(z)}.$$

By comparing the coefficients at powers of z in

$$[1 - \varpi(z)]p(z) = 1 + \varpi(z),$$

we obtain

$$p_1 = 2c_1, \quad p_2 = 2c_2 + 2c_1^2, \quad p_3 = 2c_3 + 4c_1c_2 + 2c_1^3.$$

By putting these values in (17), we obtain

$$d_4 = -\frac{1}{3}\left(c_3 + \mu_1c_1c_2 + \mu_2c_1^3\right),$$

where $\mu_1 = -\frac{7}{3}$, and $\mu_2 = \frac{29}{24}$. Now, by using Lemma 8, we have $2 \le |\mu_1| \le 4$, and $\mu_1 - \frac{1}{12}(\mu_2 + 8) = \frac{19}{216}$; therefore,

$$|d_4| \le \frac{1}{3}|\mu_2| = \frac{29}{72}.$$

The equalities in each coefficient $|d_2|$, $|d_3|$, and $|d_4|$ are respectively obtained by taking the following:

$$\digamma_1(\xi) = \xi \exp\left(\int_0^{\xi} \frac{(\frac{t}{e^t-1})^2 - 1}{t} dt\right) = \xi - \xi^2 + \frac{17}{24}\xi^3 - \frac{29}{72}\xi^4 + \cdots. \quad (19)$$

□

Theorem 9. *Let $\digamma \in \mathbf{S}^*_{BS}$ and have the series representation given in (2). Then,*

$$|d_3 - d_2^2| \le \frac{1}{2}. \quad (20)$$

Theorem 10. *Let $\digamma \in \mathbf{S}^*_{BS}$ and have the series representation given in (2). Then,*

$$H_{2,2}(\digamma) = |d_2 d_4 - d_3^2| \le \frac{521}{576}. \quad (21)$$

The equality is obtained by the $\digamma_1$ given in (19).

Proof. Using (15)–(17), we obtain

$$H_{2,2}(\digamma) = d_2 d_4 - d_3^2 = -\frac{571}{3072}p_1^4 + \frac{191}{576}p_1^2 p_2 - \frac{1}{12}p_1 p_3 - \frac{1}{16}p_2^2. \quad (22)$$

As we can see that the functional $H_{2,2}(\digamma)$ and the class $\mathbf{S}^*_{BS}$ are rotationally invariant, we may therefore take $p := p_1$ such that $p \in [0, 2]$. Then, by using Lemma 7, and after some computations, we may write out the following:

$$H_{2,2}(\digamma) = -\frac{571}{9216}p^4 + \frac{107}{1152}p^2(4 - p^2)\rho - \frac{1}{192}(4 - p^2)\left(12 - 7p^2\right)\rho^2 - \frac{1}{24}p(4 - p^2)\left(1 - |\rho|^2\right)\eta,$$

where ρ and η satisfy the relation $|\rho| \le 1$ and $|\eta| \le 1$.

Firstly, we consider the case when $p = 0$. Then, $|H_{2,2}(\digamma)| = \left|-\frac{1}{4}\rho^2\right| \le \frac{1}{4}$. Next, we assume that $p = 2$; then, $|H_{2,2}(\digamma)| = \frac{521}{576}$. Now suppose that $p \in (0, 2)$; then,

$$|H_{2,2}(\digamma)| \le \frac{1}{24}p(4 - p^2)\Phi(j, k, l),$$

where

$$\Phi(j, k, l) = \left|j + k\rho + l\rho^2\right| + 1 - |\rho|^2, \qquad \rho \in \overline{\mathbf{E}},$$

with $j = \frac{521p^3}{384(4 - p^2)}$, $k = -\frac{107p}{48}$, and $l = \frac{(12 - 7p^2)}{8p}$; then clearly,

$$jl = \frac{521p^2(12 - 7p^2)}{3072(4 - p^2)} \ge 0, \qquad \text{for} \quad p \in \left[\sqrt{\frac{12}{7}}, 2\right). \quad (23)$$

In addition,

$$|k| - 2(1 - |l|) = \frac{23p^2 - 96p + 144}{48p} > 0 \qquad p \in \left[\sqrt{\frac{12}{7}}, 2\right)$$

so that $|k| > 2(1 - |l|)$, and by applying Lemma 9, we can obtain

$$|H_{2,2}(F)| \leq \frac{1}{24} p(4 - p^2)(|j| + |k| + |l|) = g(p),$$

where

$$g(p) = \frac{1}{9216} p^4 + \frac{24}{288} p^2 + \frac{1}{4}. \tag{24}$$

Clearly, $g'(p) > 0$, and so

$$\max g(p) = g(2) = \frac{521}{576}.$$

We also see from (23) that

$$jl = \frac{521p^2(12 - 7p^2)}{3072(4 - p^2)} < 0, \qquad \text{for} \quad p \in \left(0, \sqrt{\frac{12}{7}}\right).$$

Thus,

$$k^2 - 4jl\left(l^{-2} - 1\right) = \frac{1}{576} \frac{p^2(889p^2 - 20280)}{7p^2 - 12} < 0, \quad p \in \left(0, \sqrt{\frac{12}{7}}\right).$$

This shows that $-4jl\left(l^{-2} - 1\right) \leq k^2 \wedge |k| < 2(1 - |l|)$. In addition,

$$\begin{aligned} \Phi(p) &= 4(1 + |l|)^2 + 4jl\left(l^{-2} - 1\right) \\ &= \frac{(7p - 6)\left(1295p^5 - 4266p^4 + 2688p^3 + 2073p^2 + 2304p - 13824\right)}{p^2(12 - 7p^2)}. \end{aligned}$$

We see that $\Phi(p) > 0$ for $p \in (0, 0.76032) \cup \left(\frac{6}{7}, \sqrt{\frac{12}{7}}\right)$, and $\Phi(p) < 0$ for $p \in \left(0.76032, \frac{6}{7}\right)$. Hence, we conclude that

$$\min\left\{4(1 + |l|)^2, -4jl\left(l^{-2} - 1\right)\right\} = \begin{cases} -4jl\left(l^{-2} - 1\right), & p \in (0, 0.76032) \cup \left(\frac{6}{7}, \sqrt{\frac{12}{7}}\right), \\ 4(1 + |l|)^2 & \left(0.76032, \frac{6}{7}\right). \end{cases}$$

As a result,

$$k^2 - 4(1 + |l|)^2 = \frac{\left(23p^2 - 96p + 144\right)\left(191p^2 + 96p - 144\right)}{2304} > 0 \text{ for } \left(0.76032, \frac{6}{7}\right).$$

In addition,

$$k^2 + 4(1 + |l|)^2 = \frac{p^2\left(78365p^2 - 96828\right)}{1152(7p^2 - 12)} < 0 \text{ for } \left(\sqrt{\frac{96828}{78365}}, \sqrt{\frac{12}{7}}\right).$$

This shows that $k^2 < \min\left\{4(1 + |l|)^2, -4jl\left(l^{-2} - 1\right)\right\}$ hold for $p \in \left(\sqrt{\frac{96828}{78365}}, \sqrt{\frac{12}{7}}\right)$. By applying Lemma 9, we arrive at the following:

$$|H_{2,2}(F)| \leq \frac{1}{24} p(4 - p^2)\left(1 + |j| + \frac{k^2}{4(1 + |l|)}\right) = g_1(p),$$

where

$$g_1(p) = \frac{p\left(127p^4 - 1364p^3 - 1728p^2 - 8064p + 6912\right)}{6912(6-7p)}.$$

This attains its maxima at $p = \sqrt{\frac{12}{7}}$. Hence,

$$|H_{2,2}(F)| \leq \frac{\sqrt{21}\left(-4413 + 3034\sqrt{21}\right)}{-148176 + 49392\sqrt{21}} < \frac{521}{576}.$$

We are left with the case $p \in \left(0, \sqrt{\frac{96828}{78365}}\right)$. We also see that

$$|l|(|k| + 4|j|) - |jk| = \frac{4171p^4 - 55392p^2 + 246528}{18432(p^2-4)} < 0 \qquad p \in \left(0, \sqrt{\frac{96828}{78365}}\right).$$

We conclude that $|l|(|k| + 4|j|) < |jk|$. By applying Lemma 9, we arrive at the following:

$$|H_{2,2}(F)| \leq \frac{1}{24}p(4-p^2)(|j| + |k| - |l|) = g(p),$$

where g is given in (24), this giving us the required result. The function given in (19) belongs to the $\mathbf{S}^*_{BS}$, as $d_2 = -1$, $d_4 = -29/72$, and $d_3 = 17/24$, which yields the sharpness of (21). Hence, the proof is done. □

4. Conclusions

We have introduced a subclass of S^* associated with Bernoulli numbers of the second kind and studied some geometrical properties of the introduced class. These results include radii problems, inclusion problems, coefficient bounds, and Hankel determinants. The new defined class can further be studied for determining the bounds of Hankel and Toeplitz determinants, and the same can also be found for logarithmic coefficients and for the coefficients of inverse functions.

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