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Article

On Bounds of k-Fractional Integral Operators with Mittag-Leffler Kernels for Several Types of Exponentially Convexities

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Abstract: This paper aims to study the bounds of k-integral operators with the Mittag-Leffler kernel in a unified form. To achieve these bounds, the definition of exponentially $(\alpha, h-m)-p$-convexity is utilized frequently. In addition, a fractional Hadamard type inequality which shows the upper and lower bounds of k-integral operators simultaneously is presented. The results are directly linked with the results of many published articles.

Keywords: convex function; exponentially $(\alpha, h-m)-p$-convex function; Mittag-Leffler function; generalized integral operators

MSC: 26A51; 26A33; 33E12

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1. Introduction

Special functions, including trigonometric, hyperbolic, exponential, gamma, beta, and many others, have fascinating and unique characteristics. They play very important role in the fields of mathematical analysis, complex analysis, geometric function theory, physics, and statistics. The well known Mittag-Leffler function introduced in [1] represents a vital contribution to the class of special functions. It is very frequently used in applied sciences in regard to the generalization and extension of classical concepts; for further details, readers are referred to [2–5].

The Mittag-Leffler function is frequently utilized in the formation of generalizations of fractional integral operators. Fractional integral operators lead to the theory of fractional calculus, fractional analysis, fractional differential equations, and fractional dynamic systems; see [6–8]. The aim of this paper is to estimate fractional integral operators containing a specific Mittag-Leffler function via various types of exponential convexities.

The Mittag-Leffler function is a generalization of exponential, trigonometric, and hyperbolic functions, and is defined with the help of the gamma function. Likewise, the beta function can be utilized to extend the Mittag-Leffler function. In the following, we provide the definitions of the gamma function, beta function, ϖ-beta function, and pochhammer symbol.

Definition 1 ([5])**.** *The gamma function for $\Phi > 0$ is defined by:*

$$\Gamma(\Phi) = \int_0^\infty e^{-w} w^{\Phi-1} dw. \tag{1}$$

Definition 2 ([5])**.** *The beta function is defined by:*

$$\beta(\Psi, \Phi) = \int_0^1 w^{\Psi-1}(1-w)^{\Phi-1} dw,$$

where $\Re(\Psi), \Re(\Phi) > 0$.

Definition 3 ([9])**.** *The definition of the ϖ-beta function is defined by:*

$$\beta_\varpi(\Psi, \Phi) = \int_0^1 w^{\Psi-1}(1-w)^{\Phi-1} e^{-\frac{\varpi}{w(1-w)}} dw,$$

where $\min\{\Re(\Psi), \Re(\Phi)\} > 0$ *and* $\Re R(\varpi) > 0$.

Definition 4 ([5])**.** *The pochhammer symbol for $r \in \mathbb{C}$ is defined by:*

$$(r)_{n\lambda} = \frac{\Gamma(r+n\lambda)}{\Gamma(r)}. \tag{2}$$

By introducing additional parameters, almost all special functions can be extended and generalized. For more detailed study, we refer readers to [9–11]. By using these extended special functions, fractional integrals have been extended to k-fractional integral operators. For instance, in [11], k-analogues of Riemann–Liouville fractional integrals were defined, while in [12] k-analogues of Liouville–Caputo fractional derivatives were defined. For more detailed study on further such extensions, see [13–15]. The $k-$analogue of the gamma function [10] is defined by:

$$\Gamma_k(\Phi) = \int_0^\infty w^{\Phi-1} e^{\frac{-w^k}{k}} dw, \tag{3}$$

where $z \in \mathbb{C}$ with $\Re(\Phi) > 0$ and $k > 0$.

Fractional integral operators ($\mathcal{IO}$s) are very frequently used when generalizing integral inequalities. Almost all classical integral inequalities have been published for different integral operators. Due to the importance of $\mathcal{IO}$s, many researchers have defined the several $\mathcal{IO}$s by adopting different approaches. By applying the Mittag-Leffler function (4), Zhang et al. [15] defined the following generalized k-$\mathcal{IO}$ which is directly linked with many well-known $\mathcal{IO}$s:

Definition 5. *Let $\mathcal{Y}$ be a positive and integrable function and let $\mathcal{Z}$ be a differentiable and strictly increasing function such that $\mathcal{Y}$, $\mathcal{Z} : [\mu, \nu] \to \mathbb{R}$ with $0 < \mu < \nu$. Additionally, let $\varsigma, \varphi, \rho, \Psi, r, \eta \in \mathbb{C}$, $\varsigma, \epsilon, \Phi \geq 0$ with $k > 0$ and $0 < \lambda \leq \epsilon + \varsigma$. Then, for $\delta \in [\mu, \nu]$, we have:*

$$\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\mu^+}\mathcal{Y}\right)(\varpi;\Phi) = \int_\mu^\varpi (\mathcal{Z}(\varpi) - \mathcal{Z}(\delta))^{\frac{\eta}{k}-1} E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\left(\Psi(\mathcal{Z}(\varpi) - \mathcal{Z}(\delta))^{\frac{\varsigma}{k}};\Phi\right)\mathcal{Y}(\delta) d(\mathcal{Z}(\delta)), \tag{4}$$

$$\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\nu^-}\mathcal{Y}\right)(\varpi;\Phi) = \int_\varpi^\nu (\mathcal{Z}(\delta) - \mathcal{Z}(\varpi))^{\frac{\eta}{k}-1} E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\left(\Psi(\mathcal{Z}(\delta) - \mathcal{Z}(\varpi))^{\frac{\varsigma}{k}};\Phi\right)\mathcal{Y}(\delta) d(\mathcal{Z}(\delta)) \tag{5}$$

which are called generalized k-$\mathcal{IO}$s, where the Mittag-Leffler function is provided by:

$$E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}(\delta;\Phi) = \sum_{n=0}^\infty \frac{\beta_p(\rho+n\lambda, r-\rho)}{\beta(\rho, r-\rho)} \frac{(r)_{n\lambda}}{k\Gamma_k(\varsigma n+\eta)} \frac{\delta^n}{(\varphi)_{n\epsilon}}. \tag{6}$$

Remark 1. *The $\mathcal{IO}$s (4) and (5) can reproduce several well-known $\mathcal{IO}$s which already exist in the literature. For example, for $k = 1$, the $\mathcal{IO}$s defined in [16] are obtained. For $k = 1$ and $\mathcal{Z}(\varpi) = \varpi$, the $\mathcal{IO}$s defined in [17] are obtained. For $k = 1$, $\mathcal{Z}(\varpi) = \varpi$ and $\Phi = 0$, the $\mathcal{IO}$s defined in [18] are obtained. For $k = 1$, $\mathcal{Z}(\varpi) = \varpi$ and $\epsilon = \Psi = 1$, the $\mathcal{IO}$s defined in [19] are obtained. For $k = 1$, $\mathcal{Z}(\varpi) = \varpi$, $\Phi = 0$, and $\epsilon = \Psi = 1$, the $\mathcal{IO}$s defined in [20] are obtained. For $k = 1$, $\mathcal{Z}(\varpi) = \varpi$, $\Phi = 0$, and $\lambda = \epsilon = \Psi = 1$, the $\mathcal{IO}$s defined in [21] are obtained. For $k = 1$, $\mathcal{Z}(\varpi) = \frac{\varpi^\eta}{\eta}$, $\eta > 0$, and $\Psi = \Phi = 0$, the $\mathcal{IO}$s defined in [22] are obtained. For $k = 1$, $\mathcal{Z}(\varpi) = ln\varpi$ and $\Psi = \Phi = 0$, the $\mathcal{IO}$s defined in [23] are obtained. For $\mathcal{Z}(\varpi) = \frac{\varpi^{\eta+1}}{\eta+1}$ and $\Psi = \Phi = 0$, the $\mathcal{IO}$s defined in [14] are obtained. For $k = 1$, $\mathcal{Z}(\varpi) = \frac{\varpi^{\eta+\zeta}}{\eta+\zeta}$ and $\Psi = \Phi = 0$, the $\mathcal{IO}$s defined in [24] are obtained. For $\mathcal{Z}(\varpi) = \frac{(\varpi-\mu)^\eta}{\eta}$, $\eta > 0$ in (4), and $\mathcal{Z}(\varpi) = -\frac{(\nu-\varpi)^\eta}{\eta}$, $\eta > 0$ in (5) with $\Psi = \Phi = 0$, the $\mathcal{IO}$s defined in [13] are obtained. For $\mathcal{Z}(\varpi) = \frac{(\varpi-\mu)^\eta}{\eta}$, $\eta > 0$ in (4), and $\mathcal{Z}(\varpi) = -\frac{(\nu-\varpi)^\eta}{\eta}$ $\eta > 0$ in (5) with $k = 1$ and $\Psi = \Phi = 0$, the $\mathcal{IO}$s defined in [25] are obtained. For $\Psi = \Phi = 0$, the $\mathcal{IO}$s defined in [26] are obtained. For $\Psi = \Phi = 0$ and $\mathcal{Z}(\varpi) = \varpi$, the $\mathcal{IO}$s defined in [11] are obtained. For $\Psi = \Phi = 0$, $\mathcal{Z}(\varpi) = \varpi$, and $k = 1$, the classical Riemann–Liouville $\mathcal{IO}$s are obtained.*

From k-$\mathcal{IO}$s (4) and (5), for constant function we can write:

$$\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\mu^+}1\right)(\varpi;\Phi) = k(\mathcal{Z}(\varpi) - \mathcal{Z}(\mu))^{\frac{\eta}{k}} E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta+k,\varphi,k}\left(\Psi(\mathcal{Z}(\varpi) - \mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right) := \mathcal{F}_{\frac{\eta}{k},\mu^+}(\varpi;\Phi), \tag{7}$$

$$\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\nu^-}1\right)(\varpi;\Phi) = k(\mathcal{Z}(\nu) - \mathcal{Z}(\varpi))^{\frac{\eta}{k}} E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta+k,\varphi,k}\left(\Psi(\mathcal{Z}(\nu) - \mathcal{Z}(\varpi))^{\frac{\varsigma}{k}};\Phi\right) := \mathcal{F}_{\frac{\eta}{k},\nu^-}(\varpi;\Phi). \tag{8}$$

Next, we provide the definition of newly defined functions, namely, exponentially $(\alpha, h - m) - p$-convex functions, as follows:

Definition 6 ([27])**.** *A function $\mathcal{Y} : (0, \nu] \to \mathbb{R}$ is said to be exponentially $(\alpha, h - m) - p$-convex functions if $\mathcal{Y}$ is positive and*

$$\mathcal{Y}\left((\delta\mu^p + (1-\delta)\nu^p)^{\frac{1}{p}}\right) \leq h(\delta^\alpha)\frac{\mathcal{Y}(\mu)}{e^{\omega\mu}} + mh(1-\delta^\alpha)\frac{\mathcal{Y}(\nu)}{e^{\omega\nu}} \tag{9}$$

is valid, while $J \subseteq \mathbb{R}$ is an interval involving $(0, 1)$ and $h : J \to \mathbb{R}$ is a positive function with $(\delta\mu^p + (1-\delta)\nu^p)^{\frac{1}{p}} \in (0, \nu]$, $(\alpha, m) \in [0, 1]^2$, $0 < \delta < 1$, and $\omega \in \mathbb{R}$.

Remark 2. *The function satisfying (9) produces various kinds of convex functions. For example, for $p = 1$, exponentially $(\alpha, h - m)$-convex functions are obtained. For $h(\delta) = \delta$, exponentially $(\alpha, m) - p$-convex functions are obtained. For $m = 1$, exponentially $(\alpha, h) - p$-convex functions are obtained. For $\alpha = 1$, exponentially $(h - m) - p$-convex functions are obtained. For $p = m = 1$, exponentially (α, h)-convex functions are obtained. For $\alpha = m = 1$, exponentially (h, p)-convex functions are obtained. For $p = \alpha = m = 1$, exponentially h-convex functions are obtained. For $h(\delta) = \delta$, and $p = 1$, exponentially (α, m)-convex functions are obtained. For $h(\delta) = \delta$ and $m = 1$, exponentially (α, p)-convex functions are obtained. For $h(\delta) = \delta$ and $\alpha = 1$, exponentially (m, p)-convex functions are obtained. For $h(\delta) = \delta$ and $p = m = 1$, exponentially α-convex functions are obtained. For $h(\delta) = \delta$ and $p = \alpha = 1$, exponentially m-convex functions are obtained. For $h(\delta) = \delta$ and $\alpha = m = 1$, exponentially p-convex functions are obtained. For $h(\delta) = \delta$ and $p = \alpha = m = 1$, exponentially convex functions are obtained.*

In recent years, many authors have derived the bounds of several $\mathcal{IO}$s for different kinds of convex functions. For example, Farid [28] established the bounds of Riemann–Liouville $\mathcal{IO}$s using convex functions. Mehmood and Farid [29] provided the bounds of generalized Riemann–Liouville k-$\mathcal{IO}$s via m-convex functions. Yu et al. [30] proved the bounds of generalized $\mathcal{IO}$s involving the Mittag-Leffler function in their kernels via strongly exponentially $(\alpha, h-m)$-convex functions. Inspired by these previous works, the aim of this paper is to derive the bounds of generalized k-$\mathcal{IO}$s for exponentially $(\alpha, h-m)-p$-convex functions.

In the upcoming section, we first derive the bounds of the k-$\mathcal{IO}$s provided in (4) and (5) for functions satisfying (9) and derive a modulus inequality for these operators. Further, an identity is proved to derive the Hadamard type inequality for k-$\mathcal{IO}$s via exponentially $(\alpha, h-m)-p$-convex functions. In particular cases, the presented results provide bounds of various $\mathcal{IO}$s.

2. Main Results

First, we provide the bounds of k-$\mathcal{IO}$s via exponentially $(\alpha, h-m)-p$-convex functions:

Theorem 1. *Let $\mathcal{Y}: [\mu, \nu] \longrightarrow \mathbb{R}$ be a positive, integrable, and exponentially $(\alpha, h-m)-p$-convex functions with $m \in (0,1]$. In addition, let $\mathcal{Z}$ be differentiable and strictly increasing with $\mathcal{Z}' \in L_1[\mu, \nu]$. Then, for $\eta, \zeta \geq k$ and $\omega \in \mathbb{R}$, we have:*

$$\begin{aligned}
&\Big({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\mu^+}\mathcal{Y}\circ\mathcal{U}\Big)(\varpi;\Phi) + \Big({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,\Psi,\nu^-}\mathcal{Y}\circ\mathcal{U}\Big)(\varpi;\Phi) \\
&\leq (\varpi-\mu)\Big(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu)\Big)^{\frac{\eta}{k}-1} E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\Big(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\Big) \\
&\left[\frac{\mathcal{Y}((\mu)^{\frac{1}{p}})}{e^{\omega((\mu)^{\frac{1}{p}})}}\mathcal{T}^{\mu}_{\varpi}\Big(h,\mathcal{R}^{\alpha};\mathcal{Z}'\Big) + m\frac{\mathcal{Y}\Big((\frac{\varpi}{m})^{\frac{1}{p}}\Big)}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\mathcal{T}^{\mu}_{\varpi}\Big(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\Big)\right] \\
&+(\nu-\varpi)\Big(\mathcal{Z}(\nu)-\mathcal{Z}(\varpi)\Big)^{\frac{\zeta}{k}-1} E^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,k}\Big(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\varpi))^{\frac{\varsigma}{k}};\Phi\Big) \\
&\left[\frac{\mathcal{Y}((\nu)^{\frac{1}{p}})}{e^{\omega((\nu)^{\frac{1}{p}})}}\mathcal{T}^{\nu}_{\varpi}\Big(h,\mathcal{R}^{\alpha};\mathcal{Z}'\Big) + m\frac{\mathcal{Y}\Big((\frac{\varpi}{m})^{\frac{1}{p}}\Big)}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\mathcal{T}^{\nu}_{\varpi}\Big(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\Big)\right],
\end{aligned} \tag{10}$$

where $\mathcal{U}(\delta) = \delta^{\frac{1}{p}}$, $\mathcal{T}^{\mu}_{\varpi}\Big(h,\mathcal{R}^{\alpha};\mathcal{Z}'\Big) = \int_0^1 h(\mathcal{R}^{\alpha})\mathcal{Z}'(\varpi-\mathcal{R}(\varpi-\mu))d\mathcal{R}$, $\mathcal{T}^{\mu}_{\varpi}\Big(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\Big) = \int_0^1 h(1-\mathcal{R}^{\alpha})\mathcal{Z}'(\varpi-\mathcal{R}(\varpi-\mu))d\mathcal{R}$.

Proof. Under the given assumptions, the following inequalities hold:

$$\begin{aligned}
&(\mathcal{Z}(\varpi)-\mathcal{Z}(\delta))^{\frac{\eta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\Big(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\delta))^{\frac{\varsigma}{k}};\Phi\Big)\mathcal{Z}'(\delta) \\
&\leq (\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\eta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\Big(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\Big)\mathcal{Z}'(\delta), \quad \mu<\delta<\varpi,
\end{aligned} \tag{11}$$

$$\begin{aligned}
&(\mathcal{Z}(\delta)-\mathcal{Z}(\varpi))^{\frac{\zeta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,k}\Big(\Psi(\mathcal{Z}(\delta)-\mathcal{Z}(\varpi))^{\frac{\varsigma}{k}};\Phi\Big)\mathcal{Z}'(\delta) \\
&\leq (\mathcal{Z}(\nu)-\mathcal{Z}(\varpi))^{\frac{\zeta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,k}\Big(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\varpi))^{\frac{\varsigma}{k}};\Phi\Big)\mathcal{Z}'(\delta), \quad \varpi<\delta<\nu.
\end{aligned} \tag{12}$$

By utilizing the exponentially $(\alpha, h-m)-p$-convexity of $\mathcal{Y}$, we can obtain the following:

$$\mathcal{Y}((\delta)^{\frac{1}{p}}) \leq h\Big(\frac{\varpi-\delta}{\varpi-\mu}\Big)^{\alpha}\frac{\mathcal{Y}((\mu)^{\frac{1}{p}})}{e^{\omega((\mu)^{\frac{1}{p}})}} + mh\Big(1-\Big(\frac{\varpi-\delta}{\varpi-\mu}\Big)^{\alpha}\Big)\frac{\mathcal{Y}\Big((\frac{\varpi}{m})^{\frac{1}{p}}\Big)}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}, \tag{13}$$

$$\mathcal{Y}((\delta)^{\frac{1}{p}}) \leq h\left(\frac{\delta-\varpi}{\nu-\varpi}\right)^{\alpha} \frac{\mathcal{Y}((\nu)^{\frac{1}{p}})}{e^{\omega((\nu)^{\frac{1}{p}})}} + mh\left(1-\left(\frac{\delta-\varpi}{\nu-\varpi}\right)^{\alpha}\right) \frac{\mathcal{Y}\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}. \tag{14}$$

From inequalities (11) and (13), the following inequality is valid:

$$\begin{aligned}
&\int_{\mu}^{\varpi} (\mathcal{Z}(\varpi)-\mathcal{Z}(\delta))^{\frac{\eta}{k}-1} E_{\varsigma,\eta,\varphi,k}^{\rho,\epsilon,\lambda,r}\left(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\delta))^{\frac{\varsigma}{k}};\Phi\right)\mathcal{Z}'(\delta)\mathcal{Y}((\delta)^{\frac{1}{p}})d\delta \\
&\leq \left(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu)\right)^{\frac{\eta}{k}-1} E_{\varsigma,\eta,\varphi,k}^{\rho,\epsilon,\lambda,r}\left(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)\left[\frac{\mathcal{Y}((\mu)^{\frac{1}{p}})}{e^{\omega((\mu)^{\frac{1}{p}})}}\right. \\
&\left.\int_{\mu}^{\varpi} h\left(\frac{\varpi-\delta}{\varpi-\mu}\right)^{\alpha}\mathcal{Z}'(\delta)d\delta + \frac{m\mathcal{Y}\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\int_{\mu}^{\varpi} h\left(1-\left(\frac{\varpi-\delta}{\varpi-\mu}\right)^{\alpha}\right)\mathcal{Z}'(\delta)d\delta\right].
\end{aligned}$$

By utilizing the integral operator (4) on the left-hand side and making the substitution $\mathcal{R} = (\varpi-\delta)/(\varpi-\mu)$ on the right-hand side, we obtain:

$$\begin{aligned}
&\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\mu^+}\mathcal{Y}\circ\mathcal{U}\right)(\varpi;\Phi) \leq (\varpi-\mu)\left(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu)\right)^{\frac{\eta}{k}-1} \\
&E_{\varsigma,\eta,\varphi,k}^{\rho,\epsilon,\lambda,r}\left(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)\left[\frac{\mathcal{Y}((\mu)^{\frac{1}{p}})}{e^{\omega((\mu)^{\frac{1}{p}})}}\int_0^1 h(\mathcal{R}^{\alpha})\mathcal{Z}'(\varpi-\mathcal{R}(\varpi-\mu))d\mathcal{R}\right. \\
&\left.+\frac{m\mathcal{Y}\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\int_0^1 h(1-\mathcal{R}^{\alpha})\mathcal{Z}'(\varpi-\mathcal{R}(\varpi-\mu))d\mathcal{R}\right].
\end{aligned}$$

The above inequality can be written in the following form:

$$\begin{aligned}
&\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\mu^+}\mathcal{Y}\circ\mathcal{U}\right)(\varpi;\Phi) \\
&\leq (\varpi-\mu)\left(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu)\right)^{\frac{\eta}{k}-1} E_{\varsigma,\eta,\varphi,k}^{\rho,\epsilon,\lambda,r}\left(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right) \\
&\left[\frac{\mathcal{Y}((\mu)^{\frac{1}{p}})}{e^{\omega((\mu)^{\frac{1}{p}})}}\mathcal{T}_{\varpi}^{\mu}\left(h,\mathcal{R}^{\alpha};\mathcal{Z}'\right) + \frac{m\mathcal{Y}\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\mathcal{T}_{\varpi}^{\mu}\left(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\right)\right].
\end{aligned} \tag{15}$$

On the other hand, by multiplying (12) and (14) and adopting the same approach as we did for (11) and (13), the following inequality can be obtained:

$$\begin{aligned}
&\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,\Psi,\nu^-}\mathcal{Y}\circ\mathcal{U}\right)(\varpi;\Phi) \leq (\nu-\varpi)\left(\mathcal{Z}(\nu)-\mathcal{Z}(\varpi)\right)^{\frac{\zeta}{k}-1} \\
&E_{\varsigma,\zeta,\varphi,k}^{\rho,\epsilon,\lambda,r}\left(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\varpi))^{\frac{\varsigma}{k}};\Phi\right)\left[\frac{\mathcal{Y}((\nu)^{\frac{1}{p}})}{e^{\omega((\nu)^{\frac{1}{p}})}}\int_0^1 h(\mathcal{R}^{\alpha})\mathcal{Z}'(\varpi-\mathcal{R}(\varpi-\nu))d\mathcal{R}\right. \\
&\left.+\frac{m\mathcal{Y}\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\int_0^1 h(1-\mathcal{R}^{\alpha})\mathcal{Z}'(\varpi-\mathcal{R}(\varpi-\nu))d\mathcal{R}\right].
\end{aligned}$$

The above inequality takes the following form:

$$\begin{aligned}&\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,\Psi,\nu^-}\mathcal{Y}\circ\mathcal{U}\right)(\varpi;\Phi)\\&\leq(\nu-\varpi)\Big(\mathcal{Z}(\nu)-\mathcal{Z}(\varpi)\Big)^{\frac{\zeta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,k}\Big(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\varpi))^{\frac{\varsigma}{k}};\Phi\Big)\\&\left[\frac{\mathcal{Y}((\nu)^{\frac{1}{p}})}{e^{\omega((\nu)^{\frac{1}{p}})}}\mathcal{T}^{\nu}_{\varpi}\Big(h,\mathcal{R}^{\alpha};\mathcal{Z}'\Big)+m\frac{\mathcal{Y}\Big((\frac{\varpi}{m})^{\frac{1}{p}}\Big)}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\mathcal{T}^{\nu}_{\varpi}\Big(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\Big)\right].\end{aligned}\tag{16}$$

By adding the inequalities (15) and (16), the inequality (10) is obtained. □

Corollary 1. *For $\eta=\zeta$ in (10), the following inequality is valid:*

$$\begin{aligned}&\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\mu^+}\mathcal{Y}\circ\mathcal{U}\right)(\varpi;\Phi)+\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\nu^-}\mathcal{Y}\circ\mathcal{U}\right)(\varpi;\Phi)\\&\leq(\varpi-\mu)\Big(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu)\Big)^{\frac{\eta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\Big(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\Big)\\&\left[\frac{\mathcal{Y}((\mu)^{\frac{1}{p}})}{e^{\omega((\mu)^{\frac{1}{p}})}}\mathcal{T}^{\mu}_{\varpi}\Big(h,\mathcal{R}^{\alpha};\mathcal{Z}'\Big)+\frac{m\mathcal{Y}\Big((\frac{\varpi}{m})^{\frac{1}{p}}\Big)}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\mathcal{T}^{\mu}_{\varpi}\Big(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\Big)\right]\\&+(\nu-\varpi)\Big(\mathcal{Z}(\nu)-\mathcal{Z}(\varpi)\Big)^{\frac{\eta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\Big(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\varpi))^{\frac{\varsigma}{k}};\Phi\Big)\\&\left[\frac{\mathcal{Y}((\nu)^{\frac{1}{p}})}{e^{\omega((\nu)^{\frac{1}{p}})}}\mathcal{T}^{\nu}_{\varpi}\Big(h,\mathcal{R}^{\alpha};\mathcal{Z}'\Big)+\frac{m\mathcal{Y}\Big((\frac{\varpi}{m})^{\frac{1}{p}}\Big)}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\mathcal{T}^{\nu}_{\varpi}\Big(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\Big)\right].\end{aligned}\tag{17}$$

Remark 3. *From Theorem 1, a large number of new bounds for all kinds of $\mathcal{IO}$s (as identified in Remark 1) via all kinds of convex functions (as identified in Remark 2) can be obtained.*

In the following, we provide the modulus inequality for k-$\mathcal{IO}$s via exponentially $(\alpha,h-m)-p$-convex functions:

Theorem 2. *Let $\mathcal{Y}:[\mu,\nu]\longrightarrow\mathbb{R}$ be positive integrable and let $|\mathcal{Y}'|$ be an exponentially $(\alpha,h-m)-p$-convex function with $m\in(0,1]$. In addition, let $\mathcal{Z}$ be differentiable and strictly increasing with $\mathcal{Z}'\in L_1[\mu,\nu]$. Then, for $\eta,\zeta\geq k$ and $\omega\in\mathbb{R}$, we have:*

$$\begin{aligned}&\left|\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\mu^+}(\mathcal{Z}*\mathcal{Y})\circ\mathcal{U}\right)(\varpi,w;\Phi)+\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,\Psi,\nu^-}(\mathcal{Z}*\mathcal{Y})\circ\mathcal{U}\right)(\varpi,w;\Phi)\right|\\&\leq(\varpi-\mu)\Big(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu)\Big)^{\frac{\eta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\Big(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\Big)\\&\left[\frac{|\mathcal{Y}'((\mu)^{\frac{1}{p}})|}{e^{\omega((\mu)^{\frac{1}{p}})}}\mathcal{T}^{\mu}_{\varpi}\Big(h,\mathcal{R}^{\alpha};\mathcal{Z}'\Big)+\frac{m|\mathcal{Y}'\Big((\frac{\varpi}{m})^{\frac{1}{p}}\Big)|}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\mathcal{T}^{\mu}_{\varpi}\Big(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\Big)\right]\\&+(\nu-\varpi)\Big(\mathcal{Z}(\nu)-\mathcal{Z}(\varpi)\Big)^{\frac{\zeta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,k}\Big(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\varpi))^{\frac{\varsigma}{k}};\Phi\Big)\\&\left[\frac{|\mathcal{Y}'((\nu)^{\frac{1}{p}})|}{e^{\omega((\nu)^{\frac{1}{p}})}}\mathcal{T}^{\nu}_{\varpi}\Big(h,\mathcal{R}^{\alpha};\mathcal{Z}'\Big)+\frac{m|\mathcal{Y}'\Big((\frac{\varpi}{m})^{\frac{1}{p}}\Big)|}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\mathcal{T}^{\nu}_{\varpi}\Big(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\Big)\right],\end{aligned}\tag{18}$$

where

$$\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\mu^{+}}(\mathcal{Z}*\mathcal{Y})\circ\mathcal{U}\right)(\varpi,w;\Phi)$$
$$:=\int_{\mu}^{\varpi}(\mathcal{Z}(\varpi)-\mathcal{Z}(\delta))^{\frac{\eta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\left(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\delta))^{\frac{\varsigma}{k}};\Phi\right)\mathcal{Z}'(\delta)\mathcal{Y}'((\delta)^{\frac{1}{p}})d\delta$$

and

$$\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,\Psi,\nu^{-}}(\mathcal{Z}*\mathcal{Y})\circ\mathcal{U}\right)(\varpi,w;\Phi)$$
$$:=\int_{\varpi}^{\nu}(\mathcal{Z}(\delta)-\mathcal{Z}(\varpi))^{\frac{\zeta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,k}\left(\Psi(\mathcal{Z}(\delta)-\mathcal{Z}(\varpi))^{\frac{\varsigma}{k}};\Phi\right)\mathcal{Z}'(\delta)\mathcal{Y}'((\delta)^{\frac{1}{p}})d\delta.$$

Proof. By utilizing the strongly exponentially $(\alpha,h-m)-p$-convexity of $|\mathcal{Y}'|$, the following inequality holds:

$$|\mathcal{Y}'((\delta)^{\frac{1}{p}})|\leq h\left(\frac{\varpi-\delta}{\varpi-\mu}\right)^{\alpha}\frac{|\mathcal{Y}'((\mu)^{\frac{1}{p}})|}{e^{\omega((\mu)^{\frac{1}{p}})}}+mh\left(1-\left(\frac{\varpi-\delta}{\varpi-\mu}\right)^{\alpha}\right)\frac{|\mathcal{Y}'((\frac{\varpi}{m})^{\frac{1}{p}})|}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}. \tag{19}$$

The above inequality can be written in the following form:

$$-\left(h\left(\frac{\varpi-\delta}{\varpi-\mu}\right)^{\alpha}\frac{|\mathcal{Y}'((\mu)^{\frac{1}{p}})|}{e^{\omega((\mu)^{\frac{1}{p}})}}+mh\left(1-\left(\frac{\varpi-\delta}{\varpi-\mu}\right)^{\alpha}\right)\frac{|\mathcal{Y}'((\frac{\varpi}{m})^{\frac{1}{p}})|}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\right) \tag{20}$$
$$\mathcal{Y}'((\delta)^{\frac{1}{p}})\leq\left(h\left(\frac{\varpi-\delta}{\varpi-\mu}\right)^{\alpha}\frac{|\mathcal{Y}'((\mu)^{\frac{1}{p}})|}{e^{\omega((\mu)^{\frac{1}{p}})}}+mh\left(1-\left(\frac{\varpi-\delta}{\varpi-\mu}\right)^{\alpha}\right)\frac{|\mathcal{Y}'((\frac{\varpi}{m})^{\frac{1}{p}})|}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\right).$$

Now, by multiplying the inequality (11) with the first inequality of (20) and integrating over $[\mu,\varpi]$, we obtain the following:

$$\int_{\mu}^{\varpi}(\mathcal{Z}(\varpi)-\mathcal{Z}(\delta))^{\frac{\eta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\left(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\delta))^{\frac{\varsigma}{k}};\Phi\right)\mathcal{Z}'(\delta)\mathcal{Y}'((\delta)^{\frac{1}{p}})d\delta \tag{21}$$
$$\leq(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\eta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\left(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)\left(\frac{|\mathcal{Y}'((\mu)^{\frac{1}{p}})|}{e^{\omega((\mu)^{\frac{1}{p}})}}\int_{\mu}^{\varpi}h\left(\frac{\varpi-\delta}{\varpi-\mu}\right)^{\alpha}\right.$$
$$\left.\mathcal{Z}'(\delta)d\delta+\frac{m|\mathcal{Y}'((\frac{\varpi}{m})^{\frac{1}{p}})|}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\int_{\mu}^{\varpi}h\left(1-\left(\frac{\varpi-\delta}{\varpi-\mu}\right)^{\alpha}\right)\mathcal{Z}'(\delta)d\delta\right).$$

After simplifying the inequality (21), we have

$$\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\mu^{+}}(\mathcal{Z}*\mathcal{Y})\circ\mathcal{U}\right)(\varpi,w;\Phi) \tag{22}$$
$$\leq(\varpi-\mu)\left(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu)\right)^{\frac{\eta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\left(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)$$
$$\left[\frac{|\mathcal{Y}'((\mu)^{\frac{1}{p}})|}{e^{\omega((\mu)^{\frac{1}{p}})}}\mathcal{T}^{\mu}_{\varpi}\left(h,\mathcal{R}^{\alpha};\mathcal{Z}'\right)+\frac{m|\mathcal{Y}'\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)|}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\mathcal{T}^{\mu}_{\varpi}\left(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\right)\right].$$

By using the second inequality of (20) and following the same approach as we did for the first inequality, we can obtain the following:

$$\begin{aligned}&\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\mu^{+}}(\mathcal{Z}*\mathcal{Y})\circ\mathcal{U}\right)(\varpi,w;\Phi)\\&\geq-(\varpi-\mu)\left(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu)\right)^{\frac{\eta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\left(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)\\&\left[\frac{|\mathcal{Y}'((\mu)^{\frac{1}{p}})|}{e^{\omega((\mu)^{\frac{1}{p}})}}\mathcal{T}^{\mu}_{\varpi}\left(h,\mathcal{R}^{\alpha};\mathcal{Z}'\right)+\frac{m|\mathcal{Y}'\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)|}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\mathcal{T}^{\mu}_{\varpi}\left(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\right)\right].\end{aligned}\tag{23}$$

From inequalities (22) and (23), we have:

$$\begin{aligned}&\left|\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\mu^{+}}(\mathcal{Z}*\mathcal{Y})\circ\mathcal{U}\right)(\varpi,w;\Phi)\right|\\&\leq(\varpi-\mu)\left(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu)\right)^{\frac{\eta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\left(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)\\&\left[\frac{|\mathcal{Y}'((\mu)^{\frac{1}{p}})|}{e^{\omega((\mu)^{\frac{1}{p}})}}\mathcal{T}^{\mu}_{\varpi}\left(h,\mathcal{R}^{\alpha};\mathcal{Z}'\right)+\frac{m|\mathcal{Y}'\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)|}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\mathcal{T}^{\mu}_{\varpi}\left(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\right)\right].\end{aligned}\tag{24}$$

Again, by utilizing the strongly exponentially $(\alpha,h-m)-p$-convexity of $|\mathcal{Y}'|$, we have

$$|\mathcal{Y}'((\delta)^{\frac{1}{p}})|\leq h\left(\frac{\delta-\varpi}{\nu-\varpi}\right)^{\alpha}\frac{|\mathcal{Y}'((\nu)^{\frac{1}{p}})|}{e^{\omega((\nu)^{\frac{1}{p}})}}+mh\left(1-\left(\frac{\delta-\varpi}{\nu-\varpi}\right)^{\alpha}\right)\frac{|\mathcal{Y}'\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)|}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}.\tag{25}$$

By following the same approach as we did for (11) and (19), from (12) and (25) we can obtain:

$$\begin{aligned}&\left|\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,\Psi,\nu^{-}}(\mathcal{Z}*\mathcal{Y})\circ\mathcal{U}\right)(\varpi,w;\Phi)\right|\\&\leq(\nu-\varpi)\left(\mathcal{Z}(\nu)-\mathcal{Z}(\varpi)\right)^{\frac{\zeta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,k}\left(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\varpi))^{\frac{\varsigma}{k}};\Phi\right)\\&\left[\frac{|\mathcal{Y}'((\nu)^{\frac{1}{p}})|}{e^{\omega((\nu)^{\frac{1}{p}})}}\mathcal{T}^{\nu}_{\varpi}\left(h,\mathcal{R}^{\alpha};\mathcal{Z}'\right)+\frac{m|\mathcal{Y}'\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)|}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\mathcal{T}^{\nu}_{\varpi}\left(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\right)\right].\end{aligned}\tag{26}$$

By adding the inequalities (24) and (26), the required inequality (18) is obtained. □

Corollary 2. *For $\eta=\zeta$ in (18), the following inequality is valid:*

$$\begin{aligned}&\left|\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\mu^{+}}(\mathcal{Z}*\mathcal{Y})\circ\mathcal{U}\right)(\varpi,w;\Phi)+\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\nu^{-}}(\mathcal{Z}*\mathcal{Y})\circ\mathcal{U}\right)(\varpi,w;\Phi)\right|\\&\leq(\varpi-\mu)\left(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu)\right)^{\frac{\eta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\left(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)\\&\left[\frac{|\mathcal{Y}'((\mu)^{\frac{1}{p}})|}{e^{\omega((\mu)^{\frac{1}{p}})}}\mathcal{T}^{\mu}_{\varpi}\left(h,\mathcal{R}^{\alpha};\mathcal{Z}'\right)+\frac{m|\mathcal{Y}'\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)|}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\mathcal{T}^{\mu}_{\varpi}\left(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\right)\right]\\&+(\nu-\varpi)\left(\mathcal{Z}(\nu)-\mathcal{Z}(\varpi)\right)^{\frac{\zeta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\left(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\varpi))^{\frac{\varsigma}{k}};\Phi\right)\\&\left[\frac{|\mathcal{Y}'((\nu)^{\frac{1}{p}})|}{e^{\omega((\nu)^{\frac{1}{p}})}}\mathcal{T}^{\nu}_{\varpi}\left(h,\mathcal{R}^{\alpha};\mathcal{Z}'\right)+\frac{m|\mathcal{Y}'\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)|}{e^{\omega\left((\frac{\varpi}{m})^{\frac{1}{p}}\right)}}\mathcal{T}^{\nu}_{\varpi}\left(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\right)\right],\end{aligned}\tag{27}$$

Remark 4. *From Theorem 2, the bounds for all $\mathcal{IO}$s (provided in Remark 1) via all kinds of convex functions (provided in Remark 2) can be obtained.*

The following identity is useful to prove the Hadamard-type inequality:

Lemma 1. *Let $\mathcal{Y} : [\mu, m\nu] \to \mathbb{R}$, $\mu < m\nu$, be an exponentially $(\alpha, h - m) - p$-convex function. If the condition*

$$\frac{\mathcal{Y}((\varpi)^{\frac{1}{p}})}{e^{\omega((\varpi)^{\frac{1}{p}})}} = \frac{\mathcal{Y}\left(\frac{(\mu^p + m\nu^p - \varpi)^{\frac{1}{p}}}{m}\right)}{e^{\omega\left(\frac{(\mu^p + m\nu^p - \varpi)^{\frac{1}{p}}}{m}\right)}} \tag{28}$$

holds for $m \in (0, 1]$, then we have

$$\mathcal{Y}\left(\left(\frac{\mu^p + m\nu^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{\mathcal{Y}((\varpi)^{\frac{1}{p}})}{e^{\omega((\varpi)^{\frac{1}{p}})}}\left(h\left(\frac{1}{2^\alpha}\right) + mh\left(1 - \frac{1}{2^\alpha}\right)\right). \tag{29}$$

Proof. We use the following identity:

$$\left(\frac{\mu^p + m\nu^p}{2}\right)^{\frac{1}{p}} = \left[\frac{1}{2}\left(\left(\frac{\varpi - \mu^p}{m\nu^p - \mu^p} m\nu^p + \frac{m\nu^p - \varpi}{m\nu^p - \mu^p}\mu^p\right)^{\frac{1}{p}}\right)^p\right.$$
$$\left. + m\left(1 - \frac{1}{2}\right)\left(\left(\frac{\frac{\varpi - \mu^p}{m\nu^p - \mu^p}\mu^p + \frac{m\nu^p - \varpi}{m\nu^p - \mu^p} m\nu^p}{m}\right)^{\frac{1}{p}}\right)^p\right]^{\frac{1}{p}}. \tag{30}$$

By applying the exponentially $(\alpha, h - m) - p$-convexity of $\mathcal{Y}$, we obtain:

$$\mathcal{Y}\left(\left(\frac{\mu^p + m\nu^p}{2}\right)^{\frac{1}{p}}\right) \leq h\left(\frac{1}{2^\alpha}\right)\frac{\mathcal{Y}\left(\left(\frac{\varpi - \mu^p}{m\nu^p - \mu^p} m\nu^p + \frac{m\nu^p - \varpi}{m\nu^p - \mu^p}\mu^p\right)^{\frac{1}{p}}\right)}{e^{\omega\left(\left(\frac{\varpi - \mu^p}{m\nu^p - \mu^p} m\nu^p + \frac{m\nu^p - \varpi}{m\nu^p - \mu^p}\mu^p\right)^{\frac{1}{p}}\right)}} \tag{31}$$
$$+ mh\left(1 - \frac{1}{2^\alpha}\right)\frac{\mathcal{Y}\left(\left(\frac{\frac{\varpi - \mu^p}{m\nu^p - \mu^p}\mu^p + \frac{m\nu^p - \varpi}{m\nu^p - \mu^p} m\nu^p}{m}\right)^{\frac{1}{p}}\right)}{e^{\omega\left(\left(\frac{\frac{\varpi - \mu^p}{m\nu^p - \mu^p}\mu^p + \frac{m\nu^p - \varpi}{m\nu^p - \mu^p} m\nu^p}{m}\right)^{\frac{1}{p}}\right)}}$$
$$= h\left(\frac{1}{2^\alpha}\right)\frac{\mathcal{Y}((\varpi)^{\frac{1}{p}})}{e^{\omega((\varpi)^{\frac{1}{p}})}} + mh\left(1 - \frac{1}{2^\alpha}\right)\frac{\mathcal{Y}\left(\frac{(\mu^p + m\nu^p - \varpi)^{\frac{1}{p}}}{m}\right)}{e^{\omega\left(\frac{(\mu^p + m\nu^p - \varpi)^{\frac{1}{p}}}{m}\right)}}.$$

By using the assumption provided in (28), the required inequality (29) is obtained. □

In the following, we provide the Hadamarad type inequality for k-$\mathcal{IO}$s via exponentially $(\alpha, h - m) - p$-convex functions:

Theorem 3. *With the same conditions on $\mathcal{Y}$, $\mathcal{Z}$, and h as in Theorem 1, and additionally if (28) is satisfied, then we have:*

$$\frac{\mathcal{Q}(\omega)}{\left(h\left(\frac{1}{2^{\alpha}}\right)+mh\left(1-\frac{1}{2^{\alpha}}\right)\right)}\left[\mathcal{Y}\left(\left(\frac{\mu^{p}+m\nu^{p}}{2}\right)^{\frac{1}{p}}\right)\left(\mathcal{F}_{\frac{\eta}{k},\mu^{+}}(\nu;\Phi)+\mathcal{F}_{\frac{\zeta}{k},\nu^{-}}(\mu;\Phi)\right)\right] \tag{32}$$
$$\leq\left({}_{\mathcal{Z}}^{k}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\mu^{+}}\mathcal{Y}\circ\mathcal{U}\right)(\nu;\Phi)+\left({}_{\mathcal{Z}}^{k}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,\Psi,\nu^{-}}\mathcal{Y}\circ\mathcal{U}\right)(\mu;\Phi)$$
$$\leq(\nu-\mu)\left((\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\eta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\left(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)\right.$$
$$\left.+(\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\zeta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,k}\left(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)\right)$$
$$\left[\frac{\mathcal{Y}((\nu)^{\frac{1}{p}})}{e^{\omega((\nu)^{\frac{1}{p}})}}\mathcal{T}^{\mu}_{\nu}\left(h,\mathcal{R}^{\alpha};\mathcal{Z}'\right)+\frac{m\mathcal{Y}\left((\frac{\mu}{m})^{\frac{1}{p}}\right)}{e^{\omega(\frac{\mu}{m})^{\frac{1}{p}}}}\mathcal{T}^{\mu}_{\nu}\left(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\right)\right].$$

where $\mathcal{U}(\delta)=\delta^{\frac{1}{p}}$ *and* $\mathcal{Q}(\omega)=e^{\omega\mu}$ *for* $\omega\geq 0$; $\mathcal{Q}(\omega)=e^{\omega\nu}$ *for* $\omega<0$.

Proof. Under the given assumptions, the following inequalities hold:

$$(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\zeta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,k}\left(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)\mathcal{Z}'(\varpi) \tag{33}$$
$$\leq(\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\zeta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,k}\left(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)\mathcal{Z}'(\varpi),\quad \varpi\in[\mu,\nu],$$

$$(\mathcal{Z}(\nu)-\mathcal{Z}(\varpi))^{\frac{\eta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\left(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\varpi))^{\frac{\varsigma}{k}};\Phi\right)\mathcal{Z}'(\varpi) \tag{34}$$
$$\leq(\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\eta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\left(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)\mathcal{Z}'(\varpi)\quad \varpi\in[\mu,\nu].$$

By utilizing the strongly exponential $(\alpha,h-m)-p$-convexity of $\mathcal{Y}$, we obtain:

$$\mathcal{Y}((\varpi)^{\frac{1}{p}})\leq h\left(\frac{\varpi-\mu}{\nu-\mu}\right)^{\alpha}\frac{\mathcal{Y}((\nu)^{\frac{1}{p}})}{e^{\omega((\nu)^{\frac{1}{p}})}}+mh\left(1-\left(\frac{\varpi-\mu}{\nu-\mu}\right)^{\alpha}\right)\frac{\mathcal{Y}\left((\frac{\mu}{m})^{\frac{1}{p}}\right)}{e^{\omega(\frac{\mu}{m})^{\frac{1}{p}}}}. \tag{35}$$

From inequalities (33) and (35), the following inequality holds:

$$\int_{\mu}^{\nu}(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\zeta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,k}\left(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)\mathcal{Z}'(\varpi)\mathcal{Y}((\varpi)^{\frac{1}{p}})d\varpi$$
$$\leq(\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\zeta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,k}\left(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)\left[\frac{\mathcal{Y}((\nu)^{\frac{1}{p}})}{e^{\omega((\nu)^{\frac{1}{p}})}}\right.$$
$$\left.\int_{\mu}^{\nu}h\left(\frac{\varpi-\mu}{\nu-\mu}\right)^{\alpha}\mathcal{Z}'(\varpi)d\varpi+\frac{m\mathcal{Y}\left((\frac{\mu}{m})^{\frac{1}{p}}\right)}{e^{\omega(\frac{\mu}{m})^{\frac{1}{p}}}}\int_{\mu}^{\nu}h\left(1-\left(\frac{\varpi-\mu}{\nu-\mu}\right)^{\alpha}\right)\mathcal{Z}'(\varpi)d\varpi\right].$$

By utilizing the integral operator (5) on the left-hand side and making the substitution $\mathcal{R} = (\varpi - \mu)/(\nu - \mu)$ on the right-hand side, we obtain the following:

$$\left({}_{\mathcal{Z}}^{k}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,\Psi,\nu^-}\mathcal{Y}\circ\mathcal{U} \right)(\mu;\Phi) \leq (\nu-\mu)(\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\zeta}{k}-1} E^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,k}\left(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)\left[\frac{\mathcal{Y}((\nu)^{\frac{1}{p}})}{e^{\omega((\nu)^{\frac{1}{p}})}}\int_0^1 h(\mathcal{R}^{\alpha})\mathcal{Z}'(\mu+\mathcal{R}(\nu-\mu))d\mathcal{R} + \frac{m\mathcal{Y}\left((\frac{\mu}{m})^{\frac{1}{p}}\right)}{e^{\omega(\frac{\mu}{m})^{\frac{1}{p}}}}\int_0^1 h(1-\mathcal{R}^{\alpha})\mathcal{Z}'(\mu+\mathcal{R}(\nu-\mu))d\mathcal{R}\right]. \tag{36}$$

The above inequality takes the following form:

$$\left({}_{\mathcal{Z}}^{k}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,\Psi,\nu^-}\mathcal{Y}\circ\mathcal{U} \right)(\mu;\Phi) \leq (\nu-\mu)(\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\zeta}{k}-1} E^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,k}\left(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)\left[\frac{\mathcal{Y}((\nu)^{\frac{1}{p}})}{e^{\omega((\nu)^{\frac{1}{p}})}}\mathcal{T}^{\mu}_{\nu}\left(h,\mathcal{R}^{\alpha};\mathcal{Z}'\right) + \frac{m\mathcal{Y}\left((\frac{\mu}{m})^{\frac{1}{p}}\right)}{e^{\omega(\frac{\mu}{m})^{\frac{1}{p}}}}\mathcal{T}^{\mu}_{\nu}\left(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\right)\right]. \tag{37}$$

Similarly, from inequalities (34) and (35), after simplification, the following inequality is obtained:

$$\left({}_{\mathcal{Z}}^{k}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\mu^+}\mathcal{Y}\circ\mathcal{U} \right)(\nu;\Phi) \leq (\nu-\mu)(\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\eta}{k}-1} E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\left(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)\left[\frac{\mathcal{Y}((\nu)^{\frac{1}{p}})}{e^{\omega((\nu)^{\frac{1}{p}})}}\mathcal{T}^{\mu}_{\nu}\left(h,\mathcal{R}^{\alpha};\mathcal{Z}'\right) + \frac{m\mathcal{Y}\left((\frac{\mu}{m})^{\frac{1}{p}}\right)}{e^{\omega(\frac{\mu}{m})^{\frac{1}{p}}}}\mathcal{T}^{\mu}_{\nu}\left(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\right)\right]. \tag{38}$$

By adding the inequalities (37) and (38), we obtain:

$$\left({}_{\mathcal{Z}}^{k}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\mu^+}\mathcal{Y}\circ\mathcal{U} \right)(\nu;\Phi) + \left({}_{\mathcal{Z}}^{k}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,\Psi,\nu^-}\mathcal{Y}\circ\mathcal{U} \right)(\mu;\Phi) \leq (\nu-\mu)\left((\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\eta}{k}-1} E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\left(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right) + (\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\zeta}{k}-1} E^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,k}\left(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)\right)\left[\frac{\mathcal{Y}((\nu)^{\frac{1}{p}})}{e^{\omega((\nu)^{\frac{1}{p}})}}\mathcal{T}^{\mu}_{\nu}\left(h,\mathcal{R}^{\alpha};\mathcal{Z}'\right) + \frac{m\mathcal{Y}\left((\frac{\mu}{m})^{\frac{1}{p}}\right)}{e^{\omega(\frac{\mu}{m})^{\frac{1}{p}}}}\mathcal{T}^{\mu}_{\nu}\left(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\right)\right]. \tag{39}$$

Now, multiplying the inequality (29) with $(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\zeta}{k}-1} E^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,k}\left(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)$ $\mathcal{Z}'(\varpi)$ and integrating over $[\mu,\nu]$, we have:

$$\mathcal{Y}\left(\left(\frac{\mu^p+m\nu^p}{2}\right)^{\frac{1}{p}}\right)\int_{\mu}^{\nu}(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\zeta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,k}\left(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)\mathcal{Z}'(\varpi)d\varpi \tag{40}$$
$$\leq\left(h\left(\frac{1}{2^{\alpha}}\right)+mh\left(1-\frac{1}{2^{\alpha}}\right)\right)\int_{\mu}^{\nu}(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\zeta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,k}\left(\Psi(\mathcal{Z}(\varpi)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)$$
$$\times\,\mathcal{Z}'(\varpi)\frac{\mathcal{Y}((\varpi)^{\frac{1}{p}})}{e^{\omega((\varpi)^{\frac{1}{p}})}}d\varpi.$$

By utilizing the integral operator from (5) and (8), we obtain:

$$\frac{\mathcal{Q}(\omega)}{\left(h\left(\frac{1}{2^{\alpha}}\right)+mh\left(1-\frac{1}{2^{\alpha}}\right)\right)}\left(\mathcal{Y}\left(\left(\frac{\mu^p+m\nu^p}{2}\right)^{\frac{1}{p}}\right)\mathcal{F}_{\frac{\zeta}{k},\nu^-}(\mu;\Phi)\right)\leq\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,\Psi,\nu^-}\mathcal{Y}\circ\mathcal{U}\right)(\mu;\Phi). \tag{41}$$

Similarly, by multiplying the inequality (29) with $(\mathcal{Z}(\nu)-\mathcal{Z}(\varpi))^{\frac{\eta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\left(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\varpi))^{\frac{\varsigma}{k}};\Phi\right)\mathcal{Z}'(\varpi)$ and integrating over $[\mu,\nu]$, then utilizing integral (4) and (7), we obtain:

$$\frac{\mathcal{Q}(\omega)}{\left(h\left(\frac{1}{2^{\alpha}}\right)+mh\left(1-\frac{1}{2^{\alpha}}\right)\right)}\left(\mathcal{Y}\left(\left(\frac{\mu^p+m\nu^p}{2}\right)^{\frac{1}{p}}\right)\mathcal{F}_{\frac{\eta}{k},\mu^+}(\nu;\Phi)\right)\leq\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\mu^+}\mathcal{Y}\circ\mathcal{U}\right)(\nu;\Phi). \tag{42}$$

By adding the inequalities (41) and (42), we obtain:

$$\frac{\mathcal{Q}(\omega)}{\left(h\left(\frac{1}{2^{\alpha}}\right)+mh\left(1-\frac{1}{2^{\alpha}}\right)\right)}\left[\mathcal{Y}\left(\left(\frac{\mu^p+m\nu^p}{2}\right)^{\frac{1}{p}}\right)(\mathcal{F}_{\frac{\eta}{k},\mu^+}(\nu;\Phi)+\mathcal{F}_{\frac{\zeta}{k},\nu^-}(\mu;\Phi))\right] \tag{43}$$
$$\leq\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\mu^+}\mathcal{Y}\circ\mathcal{U}\right)(\nu;\Phi)+\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\zeta,\varphi,\Psi,\nu^-}\mathcal{Y}\circ\mathcal{U}\right)(\mu;\Phi).$$

From inequalities (39) and (43), the required inequality (32) is obtained. □

Corollary 3. *For $\eta=\zeta$ in (32), the following inequality is valid:*

$$\frac{\mathcal{Q}(\omega)}{\left(h\left(\frac{1}{2^{\alpha}}\right)+mh\left(1-\frac{1}{2^{\alpha}}\right)\right)}\left[\mathcal{Y}\left(\left(\frac{\mu^p+m\nu^p}{2}\right)^{\frac{1}{p}}\right)(\mathcal{F}_{\frac{\eta}{k},\mu^+}(\nu;\Phi)+\mathcal{F}_{\frac{\eta}{k},\nu^-}(\mu;\Phi))\right] \tag{44}$$
$$\leq\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\mu^+}\mathcal{Y}\circ\mathcal{U}\right)(\nu;\Phi)+\left({}^{k}_{\mathcal{Z}}\mathcal{D}^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,\Psi,\nu^-}\mathcal{Y}\circ\mathcal{U}\right)(\mu;\Phi)$$
$$\leq(\nu-\mu)(\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\eta}{k}-1}E^{\rho,\epsilon,\lambda,r}_{\varsigma,\eta,\varphi,k}\left(\Psi(\mathcal{Z}(\nu)-\mathcal{Z}(\mu))^{\frac{\varsigma}{k}};\Phi\right)$$
$$\left[\frac{\mathcal{Y}((\nu)^{\frac{1}{p}})}{e^{\omega((\nu)^{\frac{1}{p}})}}\mathcal{T}^{\mu}_{\nu}\left(h,\mathcal{R}^{\alpha};\mathcal{Z}'\right)+\frac{m\mathcal{Y}\left((\frac{\mu}{m})^{\frac{1}{p}}\right)}{e^{\omega(\frac{\mu}{m})^{\frac{1}{p}}}}\mathcal{T}^{\mu}_{\nu}\left(h,1-\mathcal{R}^{\alpha};\mathcal{Z}'\right)\right].$$

Remark 5. *From Theorem 3, Hadamard-type inequalities for all kinds of $\mathcal{IO}$s (provided in Remark 1) for all kinds of convex functions (provided in Remark 2) can be obtained.*

3. Concluding Remarks

In this article, we have investigated the bounds of k-$\mathcal{IO}$s. These bounds were achieved by applying thedefinition of exponentially $(\alpha,h-m)-p$-convex functions. The presented results provide a large number of new bounds of several $\mathcal{IO}$s for various kinds of convexities using convenient substitutions. Further, an identity is established to prove the Hadamard-type inequality for k-$\mathcal{IO}$s via exponentially $(\alpha, h-m)-p$-convex functions.

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