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A Generalized Convexity and Inequalities Involving the Unified Mittag–Leffler Function

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Abstract: This article aims to obtain inequalities containing the unified Mittag–Leffler function which give bounds of integral operators for a generalized convexity. These findings provide generalizations and refinements of many inequalities. By setting values of monotone functions, it is possible to reproduce results for classical convexities. The Hadamard-type inequalities for several classes related to convex functions are identified in remarks, and some of them are also presented in last section.

Keywords: integral operators; fractional integral operators; convex functions; bounds

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1. Introduction

Having fascinating properties of a geometric and analytic nature, convex functions are of significant importance in mathematics, graph theory, optimization theory, statistics and economics. Convex functions with smoothness properties can be characterized in various ways. For applications of convex functions in different areas of research, we refer the readers to [1–4]. In the study of discrete as well as integral inequalities, convex functions are equally important, and many classical inequalities are direct consequences of convex functions. Extensive investigations have been made to demonstrate a significant relationship between convex functions and inequality theory, see [5–9]. In this paper, we aim to give inequalities for integral operators applied to a general class of several convexities. The bounds of operators having Mittag–Leffler functions in kernels are established in different forms.

First, we give definition of $(\alpha, h-m)$-convex functions with respect to a strictly monotonically increasing function along with important consequences. Then, we give the Hadamard inequality for convex functions. Henceforth, there is given a brief introduction of the Mittag–Leffler function, and associated integral operators are discussed.

Definition 1 (see [10]). *A function φ is called $(\alpha, h-m)$-convex with respect to a strictly monotone function ψ, if $\varphi \circ \psi^{-1}$ is $(\alpha, h-m)$-convex, i.e., we have*

$$\varphi \circ \psi^{-1}(t\varpi + m(1-t)y) \leq h(t^{\alpha})\varphi \circ \psi^{-1}(\varpi) + mh(1-t^{\alpha})\varphi \circ \psi^{-1}(y), \quad (1)$$

for $(\alpha, m) \in [0,1]^2$, $\varpi, y \in Image(\psi)$, $t \in (0,1)$ with $Image(\psi) \subseteq Domain(\varphi)$.

From the above definition, by setting $\psi(x) = x^p;\ p \in \mathbb{R} - \{0\},\ x \in (0,\infty)$, one can obtain the definition of $(\alpha, h-m)$-p-convexity, $(\alpha, h-m)$-convexity can be obtained by taking $\psi(x) = x$, and (p,h)-convexity can be obtained by taking $\alpha = m = 1$ along with $\psi(x) = x^p;\ p \in \mathbb{R} - \{0\},\ x \in (0,\infty)$. Furthermore, definitions of (s,m)-convexity and $(\alpha, m) - HA$-convexity, along with almost all kinds of, convexities can be obtained by convenient substitutions; for further detail, one can see [10]. The celebrated Hadamard inequality is an equivalent representation of a convex function on $[\sigma_1, \sigma_2] \subset \mathbb{R}$, given as follows:

$$\varphi\left(\frac{\varpi + y}{2}\right) \leq \frac{1}{y - \varpi}\int_{\varpi}^{y} \varphi(\xi)d\xi \leq \frac{\varphi(\varpi) + \varphi(y)}{2} \tag{2}$$

where $\varpi, y \in [\sigma_1, \sigma_2]$, $\varpi < y$. Both of the inequalities in (2) will hold in reverse order if φ is concave function. Several variants of the above inequality have been published by analyzing various kinds of convexities; for instance, the reader can see [11–16] and the references therein.

The well-known Mittag–Leffler function is the generalization of some important special functions, including the exponential function. The role of the Mittag–Leffler function in solutions of fractional differential equations is as vital as the role of the exponential function in solving ordinary differential equations. In the theory of fractional calculus, integrals and derivatives of fractional order are evaluated by fractional derivative/integral operators, and the Mittag–Leffler function is also used in defining fractional integrals; for detailed studies, we refer the readers to [17–21].

Next, we give a definition of the unified Mittag–Leffler function, suppose that the convergence conditions are satisfied as they are given in [22], and skip them.

Definition 2 ([22])**.** *The unified Mittag–Leffler function is given by;*

$$M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(z; \underline{a}, \underline{b}, \underline{c}, p') = \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{n} B_{p'}(b_i, a_i)(\lambda)_{\rho l}(\theta)_{kl} z^l}{\prod_{i=1}^{n} B(c_i, a_i)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l + \beta)}, \tag{3}$$

where $\underline{a} = (a_1, a_2, \ldots, a_n)$, $\underline{b} = (b_1, b_2, \ldots, b_n)$, $\underline{c} = (c_1, c_2, \ldots, c_n)$, $a_i, b_i, c_i \in \mathbb{C}; i = 1,2,3,\ldots,n$, $\Gamma(\mu) = \int_0^{\infty} e^{-z} z^{\mu-1} dz$, $(\theta)_{kl} = \frac{\Gamma(\theta+lk)}{\Gamma(\theta)}$ *and* $\beta_{p'}$ *is the extended beta function, defined by;*

$$\beta_{p'}(\varpi, y) = \int_0^1 \varpi^{\varpi-1}(1-\varpi)^{y-1} e^{-\left(\frac{p'}{\varpi(1-\varpi)}\right)} d\varpi. \tag{4}$$

Throughout the paper, the unified Mittag–Leffler function is considered to be valid. The upcoming definitions are very useful for establishing the results of this paper.

Definition 3 (see [22])**.** *Let* $\varphi \in L_1[\sigma_1, \sigma_2]$. *Then, for* $\varpi \in [\sigma_1, \sigma_2]$, *the fractional integral operator containing the unified Mittag–Leffler function* $M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(z; \underline{a}, \underline{b}, \underline{c}, p')$ *satisfying all the convergence conditions is defined as follows:*

$$\left(\Upsilon^{\omega,\lambda,\rho,k,n}_{\sigma_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi\right)(\varpi; \underline{a}, \underline{b}, \underline{c}, p') = \int_{\sigma_1}^{\varpi} (\varpi - \tau)^{\alpha-1} M^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\varpi-\tau)^{\mu}; \underline{a}, \underline{b}, \underline{c}, p')\varphi(\tau)d\tau, \tag{5}$$

$$\left(\Upsilon^{\omega,\lambda,\rho,k,n}_{\sigma_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi\right)(\varpi; \underline{a}, \underline{b}, \underline{c}, p') = \int_{\varpi}^{\sigma_2} (\tau - \varpi)^{\alpha-1} M^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\varpi-\tau)^{\mu}; \underline{a}, \underline{b}, \underline{c}, p')\varphi(\tau)d\tau. \tag{6}$$

One can obtain the integral operator corresponding to generalized Q function by setting $a_i = l$, $p' = 0$ and $\Re(p') > 0$ in Definition 3 as follows (see [23]):

$$\left({}_{Q}\Upsilon^{\omega,\lambda,\rho,k,n}_{\sigma_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi\right)(\varpi;\underline{a},\underline{b}) = \int_{\sigma_1}^{\varpi}(\varpi-\tau)^{\alpha-1}Q^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\varpi-\tau)^{\mu};\underline{a},\underline{b})\varphi(\tau)d\tau, \quad (7)$$

$$\left({}_{Q}\Upsilon^{\omega,\lambda,\rho,k,n}_{\sigma_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi\right)(\varpi;\underline{a},\underline{b}) = \int_{\varpi}^{\sigma_2}(\tau-\varpi)^{\alpha-1}Q^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\tau-\varpi)^{\mu};\underline{a},\underline{b})\varphi(\tau)d\tau, \quad (8)$$

where

$$Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(z;\underline{a},\underline{b}) = \sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}\beta(b_i,l)(\lambda)_{\rho l}(\theta)_{kl}z^l}{\prod_{i=1}^{n}\beta(a_i,l)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l+\beta)}$$

is a general form of Q function given in [24].

A further extended and general version of integral operators defined in the aforementioned definitions is given in the following definition.

Definition 4 ([25]). *Let $\phi \in L_1[\sigma_1,\sigma_2]$, $0 < \sigma_1,\sigma_2 < \infty$ be a positive function, and let $\Psi : [\sigma_1,\sigma_2] \to \mathbb{R}$ be a differentiable and strictly increasing function. Furthermore, let $\frac{\phi}{\varpi}$ be an increasing function on $[\sigma_1,\infty)$ and $\varpi \in [\sigma_1,\sigma_2]$. The unified integral operator is given by:*

$$({}^{\phi}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\sigma_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi)(\varpi;p') = \int_{u}^{\varpi}\varphi(\tau)\Lambda^{\tau}_{\varpi}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\Psi;\phi)d(\Psi(\tau)), \quad (9)$$

$$({}^{\phi}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\sigma_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi)(\varpi;p') = \int_{\varpi}^{v}\varphi(\tau)\Lambda^{\varpi}_{\tau}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\Psi;\phi)d(\Psi(\tau)), \quad (10)$$

where

$$\Lambda^{\tau}_{\varpi}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu},\Psi;\phi) = \frac{\phi(\Psi(\varpi)-\Psi(\tau))}{\Psi(\varpi)-\Psi(\tau)}M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\Psi(\varpi)-\Psi(\tau))^{\mu};\underline{a},\underline{b},\underline{c},p'). \quad (11)$$

Definition 5 ([23]). *By setting $a_i = l$, $p' = 0$ and $\Re(p') > 0$ in (9) and (10), we obtain the fractional integral operator associated with generalized Q function:*

$$\left({}^{\Psi}_{Q}\Upsilon^{\phi,\omega,\lambda,\rho,\theta,k,n}_{\sigma_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi\right)(\varpi;\underline{a},\underline{b}) = \int_{\sigma_1}^{\varpi}\varphi(\tau)\Lambda^{y}_{\varpi}(Q^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\mu,\nu},\Psi;\phi)d(\Psi(\tau)), \quad (12)$$

$$\left({}^{\Psi}_{Q}\Upsilon^{\phi,\omega,\lambda,\rho,\theta,k,n}_{\sigma_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\varphi\right)(\varpi;\underline{a},\underline{b}) = \int_{\varpi}^{b}\varphi(\tau)\Lambda^{y}_{\varpi}(Q^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\mu,\nu},\Psi;\phi)d(\Psi(\tau)), \quad (13)$$

where $\Lambda^{\tau}_{\varpi}(Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu},\Psi;\phi) = \dfrac{\phi(\Psi(\varpi)-\Psi(\tau))}{\Psi(\varpi)-\Psi(\tau)}Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\Psi(\varpi)-\Psi(\tau))^{\mu},\underline{a},\underline{b},p')$.

It can be noted that, if Ψ and $\frac{\phi}{\varpi}$ are increasing, then, for $u < \tau < v, u, v \in [\sigma_1,\sigma_2]$, the kernel $\Lambda^{u}_{\tau}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu},\Psi;\phi)$ satisfies the upcoming inequality

$$\Lambda^{u}_{\tau}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu},\Psi;\phi)\Psi'(\tau) \leq \Lambda^{u}_{v}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu},\Psi;\phi)\Psi'(\tau). \quad (14)$$

Keeping in view (14), the following inequalities hold for the kernel of unified operator of Definition 5:

$$\Lambda^{\tau}_{\varpi}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)\Psi'(\tau) \leq \Lambda^{\sigma_1}_{\varpi}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)\Psi'(\tau),\ \tau \in (\sigma_1,\varpi) \quad (15)$$

$$\Lambda_{\tau}^{\varpi}(M_{\vartheta,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)\Psi'(\tau)\leq\Lambda_{\sigma_2}^{\varpi}(M_{\vartheta,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)\Psi'(\tau),\ \tau\in(\varpi,\sigma_2) \tag{16}$$

$$\Lambda_{\varpi}^{\sigma_1}(M_{\vartheta,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)\Psi'(\varpi)\leq\Lambda_{\sigma_2}^{\sigma_1}(M_{\vartheta,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)\Psi'(\varpi),\ \ \varpi\in(\sigma_1,\sigma_2) \tag{17}$$

$$\Lambda_{\sigma_2}^{\varpi}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)\Psi'(\varpi)\leq\Lambda_{\sigma_2}^{\sigma_1}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)\Psi'(\varpi),\ \ \varpi\in(\sigma_1,\sigma_2). \tag{18}$$

The paper is organized in the following way: The upcoming section contains the bounds of integral operators having the unified Mittag–Leffler function in their kernels by using a generalized convexity. The Hadamard-type inequality is proven, which generates plenty of such inequalities in particular cases. The definition of $(\alpha,h-m)$-convexity with respect to a strictly monotone function is analyzed to obtain the desired inequalities. The established results provide generalizations of several inequalities published in current years.

2. Main Results

Theorem 1. *Let I,J be intervals in $\mathbb{R}$ and $\varphi:I\to\mathbb{R}$ be an $(\alpha,h-m)$-convex function with respect to ψ, where $\psi:J\to\mathbb{R}$ is a strictly monotonic function and Image(ψ)$\subset I$. If $h(\varpi)h(y)\leq h(\varpi+y)$; then, for $(\alpha,m)\in(0,1]^2$, we have the following inequality for integral operators (12) and (13):*

$$\begin{aligned}&\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\sigma_1^+}\varphi\circ\psi^{-1}\right)(\varpi;p')+\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\gamma,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\sigma_2^-}\varphi\circ\psi^{-1}\right)(\varpi;p')\\&\leq\Lambda_{\varpi}^{\sigma_1}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)(\varpi-\sigma_1)\Big(\phi(\sigma_1)\zeta_{\varpi}^{\sigma_1}(r^{\alpha},h;\Psi')+m\varphi\Big(\frac{\varpi}{m}\Big)\zeta_{\varpi}^{\sigma_1}(1-r^{\alpha},h;\Psi')\Big)\\&+\Lambda_{\sigma_2}^{\varpi}(M_{\vartheta,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)(\sigma_2-\varpi)\Big(\varphi(\sigma_2)\zeta_{\varpi}^{\sigma_2}(r^{\alpha},h;\Psi')+m\varphi\Big(\frac{\varpi}{m}\Big)\zeta_{\varpi}^{\sigma_2}(1-r^{\alpha},h;\Psi')\Big),\end{aligned} \tag{19}$$

while $\zeta_{\varpi}^{\sigma_1}(r^{\alpha},h;\Psi')=\int_0^1 h(r^{\alpha})\Psi'(\varpi-r(\varpi-\sigma_1))dr$ and $\zeta_{\varpi}^{\sigma_1}(1-r^{\alpha},h;\Psi')=\int_0^1 h(1-r^{\alpha})\Psi'(\varpi-r(\varpi-\sigma_1))dr$.

Proof. Since φ is $(\alpha,h-m)$-convex with respect to a monotonic function, one can have

$$\varphi\circ\psi^{-1}(\tau)\leq h\Big(\frac{\varpi-\tau}{\varpi-\sigma_1}\Big)^{\alpha}\varphi\circ\psi^{-1}(\sigma_1)+mh\Big(1-\Big(\frac{\varpi-\tau}{\varpi-\sigma_1}\Big)^{\alpha}\Big)\varphi\circ\psi^{-1}\Big(\frac{\varpi}{m}\Big), \tag{20}$$

$$\varphi\circ\psi^{-1}(\tau)\leq h\Big(\frac{\tau-\varpi}{\sigma_2-\varpi}\Big)^{\alpha}\varphi\circ\psi^{-1}(\sigma_2)+mh\Big(1-\Big(\frac{\tau-\varpi}{\sigma_2-\varpi}\Big)^{\alpha}\Big)\varphi\circ\psi^{-1}\Big(\frac{\varpi}{m}\Big). \tag{21}$$

From (15) and (20), one can obtain the incoming inequality:

$$\begin{aligned}&\int_{\sigma_1}^{\varpi}\Lambda_{\varpi}^{\tau}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)\varphi\circ\psi^{-1}(\tau)d(\Psi(\tau))\leq\Lambda_{\varpi}^{\sigma_1}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)\Big(\varphi\circ\psi^{-1}(\sigma_1)\\&\times\int_{\sigma_1}^{\varpi}h\Big(\frac{\varpi-\tau}{\varpi-\sigma_1}\Big)^{\alpha}d(\Psi(\tau))+m\varphi\circ\psi^{-1}\Big(\frac{\varpi}{m}\Big)\int_{\sigma_1}^{\varpi}h\Big(1-\Big(\frac{\varpi-\tau}{\varpi-\sigma_1}\Big)^{\alpha}\Big)d(\Psi(\tau))\Big).\end{aligned} \tag{22}$$

By putting $r=\dfrac{\varpi-\tau}{\varpi-\sigma_1}$ and using Definition 4, from (22) one can obtain

$$\begin{aligned}&\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\sigma_1^+}\varphi\circ\psi^{-1}\right)(\varpi;p')\leq\Lambda_{\varpi}^{\sigma_1}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)(\varpi-\sigma_1)\Big(\varphi\circ\psi^{-1}(\sigma_1)\\&\times\int_0^1 h(r^{\alpha})\Psi'(\varpi-r(\varpi-\sigma_1))dr+m\varphi\circ\psi^{-1}\Big(\frac{\varpi}{m}\Big)\int_0^1 h(1-r^{\alpha})\Psi'(\varpi-r(\varpi-\sigma_1))dr\Big).\end{aligned} \tag{23}$$

From the inequality (23), one can yield

$$\left({}_{\Lambda}^{\phi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\sigma_1^+}\varphi\circ\psi^{-1}\right)(\varpi;p') \le \Lambda^{\sigma_1}_{\varpi}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)(\varpi-\sigma_1) \times \left(\varphi\circ\psi^{-1}(\sigma_1)\zeta^{\sigma_1}_{\varpi}(r^\alpha,h;\Psi') + m\varphi\circ\psi^{-1}\left(\frac{\varpi}{m}\right)\zeta^{\sigma_1}_{\varpi}(1-r^\alpha,h;\Psi')\right). \tag{24}$$

On the other hand, multiplying (16) and (21), and adopting the same pattern as we did for (15) and (20), the following inequality holds true:

$$\left({}_{\Lambda}^{\phi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\sigma_2^-}\varphi\circ\psi^{-1}\right)(\varpi;p') \le \Lambda^{\varpi}_{\sigma_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)(\sigma_2-\varpi)\Bigg(\varphi\circ\psi^{-1}(\sigma_2) \times \int_0^1 h(r^\alpha)\Psi'(\varpi-r(\varpi-\sigma_2))dr + m\varphi\circ\psi^{-1}\left(\frac{\varpi}{m}\right)\int_0^1 h(1-r^\alpha)\Psi'(\varpi-r(\varpi-\sigma_2))dr\Bigg). \tag{25}$$

From the inequality (25), one can yield

$$\left({}_{\Lambda}^{\phi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\sigma_2^-}\varphi\circ\psi^{-1}\right)(\varpi;p') \le \Lambda^{\varpi}_{\sigma_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)(\sigma_2-\varpi) \times \left(\varphi\circ\psi^{-1}(\sigma_2)\zeta^{\sigma_2}_{\varpi}(r^\alpha,h;\Psi') + m\varphi\circ\psi^{-1}\left(\frac{\varpi}{m}\right)\zeta^{\sigma_2}_{\varpi}(1-r^\alpha,h;\Psi')\right). \tag{26}$$

By adding (24) and (26), (19) can be achieved. □

Remark 1. *(i) If $\psi(\varpi)=\varpi$ in (19), then the result for $(\alpha,h-m)$-convex function is obtained;*
(ii) If $\psi(\varpi)=\varpi$, $\kappa=\vartheta$ and $h(\tau)=\tau$ in (19), then [26] (Theorem 2) is obtained;
(iii) If $\psi(\varpi)=\varpi$, $\alpha=1$, $\kappa=\vartheta$ and $h(\tau)=\tau^s$ in (19), then the result for (s,m)-convex function is obtained.

The following lemma is required for the next theorem.

Lemma 1. *Let I,J be intervals in $\mathbb{R}$ and $\varphi: I\to\mathbb{R}$ be a function, also let $\psi: J\to\mathbb{R}$ be strictly monotonic function and Image(ψ)$\subset I$ such that $\varphi\in L_1[\sigma_1,\sigma_2]$. Furthermore, φ is $(\alpha,h-m)$-convex function with respect to ψ where $(\alpha,m)\in(0,1]^2$; then. if*

$$\varphi\circ\psi^{-1}(\varpi)=\varphi\circ\psi^{-1}\left(\frac{\sigma_1+\sigma_2-\varpi}{m}\right), \tag{27}$$

we have the following inequality:

$$\varphi\circ\psi^{-1}\left(\frac{\sigma_1+\sigma_2}{2}\right)\le\left(h\left(\frac{1}{2^\alpha}\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\right)\varphi\circ\psi^{-1}(\varpi). \tag{28}$$

Proof. Since φ is $(\alpha,h-m)$-convex function with respect to ψ, we have

$$\varphi\circ\psi^{-1}\left(\frac{\sigma_1+\sigma_2}{2}\right)\le h\left(\frac{1}{2^\alpha}\right)\varphi\circ\psi^{-1}\left(\frac{\varpi-\sigma_1}{\sigma_2-\sigma_1}\sigma_2+\frac{\sigma_2-\varpi}{\sigma_2-\sigma_1}\sigma_1\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)$$
$$\varphi\circ\psi^{-1}\left(\frac{\frac{\varpi-\sigma_1}{\sigma_2-\sigma_1}\sigma_2+\frac{\sigma_2-\varpi}{\sigma_2-\sigma_1}\sigma_1}{m}\right)=h\left(\frac{1}{2^\alpha}\right)\varphi\left((\varpi)^{\frac{1}{p}}\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\varphi\left(\frac{-\varpi+\sigma_1+\sigma_2}{m}\right)$$

using the condition (27) in the above inequality, one can obtain the required inequality (28). □

The subsequent result gives the Hadamard-type inequality.

Theorem 2. *Under the assumptions of Theorem 1, if (27) holds, then we have*

$$\frac{\varphi\circ\psi^{-1}\left(\frac{\sigma_1+\sigma_2}{2}\right)}{h\left(\frac{1}{2^\alpha}\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)}\left(\left({}^{\phi}_{\Lambda}Y^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma\mu,\nu,\sigma_2^-}1\right)(\sigma_1;p')+\left({}^{\phi}_{\Lambda}Y^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma\mu,\nu,\sigma_1^+}1\right)(\sigma_2;p')\right) \tag{29}$$
$$\leq\left({}^{\phi}_{\Lambda}Y^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\sigma_1^+}\varphi\circ\psi^{-1}\right)(\sigma_2;p')+\left({}^{\phi}_{\Lambda}Y^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\sigma_2^-}\varphi\circ\psi^{-1}\right)(\sigma_1;p')$$
$$\leq(\sigma_2-\sigma_1)(\Lambda^{\sigma_1}_{\sigma_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)+\Lambda^{\sigma_1}_{\sigma_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi))\Big(\varphi\circ\psi^{-1}(\sigma_2)\zeta^{\sigma_1}_{\sigma_2}(r^\alpha,h;\Psi')$$
$$+m\varphi\circ\psi^{-1}\left(\frac{\sigma_1}{m}\right)\zeta^{\sigma_1}_{\sigma_2}(1-r^\alpha,h;\Psi')\Big).$$

Proof. Since φ is $(\alpha,h-m)$-convex function with respect to strictly monotonic function, one can have

$$\varphi\circ\psi^{-1}(\varpi)\leq h\left(\frac{\varpi-\sigma_1}{\sigma_2-\xi 1}\right)^\alpha\varphi\circ\psi^{-1}(\sigma_2)+mh\left(1-\left(\frac{\varpi-\sigma_1}{\sigma_2-\xi 1}\right)^\alpha\right)\varphi\circ\psi^{-1}\left(\frac{\sigma_1}{m}\right). \tag{30}$$

Multiplying (17) with (30) and integrating the resulting inequality over $[\sigma_1,\sigma_2]$, we obtain:

$$\int_{\sigma_1}^{\sigma_2}\Lambda^{\sigma_1}_{\varpi}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)\varphi\circ\psi^{-1}((\varpi))d(\Psi(\varpi))\leq\Lambda^{\sigma_1}_{\sigma_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)\Big(\varphi\circ\psi^{-1}(\sigma_2)$$
$$\times\int_{\sigma_1}^{\sigma_2}h\left(\frac{\varpi-\sigma_1}{\sigma_2-\xi 1}\right)^\alpha d(\Psi(\varpi))+m\varphi\circ\psi^{-1}\left(\frac{\sigma_1}{m}\right)\int_{\sigma_1}^{\sigma_2}h\left(1-\left(\frac{\varpi-\sigma_1}{\sigma_2-\xi 1}\right)^\alpha\right)d(\Psi(\varpi))\Big).$$

By setting $r=\left(\frac{\varpi-\sigma_1}{\sigma_2-\xi 1}\right)$ on right hand side and using Definition (4) on left hand side of the aforementioned inequality, one can have

$$\left({}^{\phi}_{\Lambda}Y^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\sigma_2^-}\varphi\circ\psi^{-1}\right)(\sigma_1;p')\leq\Lambda^{\sigma_1}_{\sigma_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)(\sigma_2-\sigma_1)\Big(\varphi\circ\psi^{-1}(\sigma_2) \tag{31}$$
$$\times\int_0^1 h(r^\alpha)\Psi'(a+r(\sigma_2-\sigma_1))dr+m\varphi\circ\psi^{-1}\left(\frac{\sigma_1}{m}\right)\int_0^1 h(1-r^\alpha)\Psi'(a+r(\sigma_2-\sigma_1))dr\Big).$$

From the inequality (31), one can yield

$$\left({}^{\phi}_{\Lambda}Y^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\sigma_2^-}\varphi\circ\psi^{-1}\right)(\sigma_1;p')\leq\Lambda^{\sigma_1}_{\sigma_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)(\sigma_2-\sigma_1) \tag{32}$$
$$\times\Big(\varphi\circ\psi^{-1}(\sigma_2)\zeta^{\sigma_1}_{\sigma_2}(r^\alpha,h;\Psi')+m\varphi\circ\psi^{-1}\left(\frac{\sigma_1}{m}\right)\zeta^{\sigma_1}_{\sigma_2}(1-r^\alpha,h;\Psi')\Big).$$

As we treated with (17) and (30), one can obtain the following inequality from (18) and (30):

$$\left({}^{\phi}_{\Lambda}Y^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\sigma_1^+}\varphi\circ\psi^{-1}\right)(\sigma_1;p')\leq\Lambda^{\sigma_1}_{\sigma_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)(\sigma_2-\sigma_1) \tag{33}$$
$$\times\Big(\varphi\circ\psi^{-1}(\sigma_2)\zeta^{\sigma_1}_{\sigma_2}(r^\alpha,h;\Psi')+m\varphi\circ\psi^{-1}\left(\frac{\sigma_1}{m}\right)\zeta^{\sigma_1}_{\sigma_2}(1-r^\alpha,h;\Psi')\Big).$$

By summing the inequalities (32) and (33), the following inequality can be obtained:

$$\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\sigma_1^+}\varphi\circ\psi^{-1}\right)(\sigma_2;p')+\left({}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\sigma_2^-}\varphi\circ\psi^{-1}\right)(\sigma_1;p')\leq(\sigma_2-\sigma_1) \tag{34}$$
$$\times\left((\Lambda^{\sigma_1}_{\sigma_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)+\Lambda^{\sigma_1}_{\sigma_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)\right)\left(\varphi\circ\psi^{-1}(\sigma_2)\zeta^{\sigma_1}_{\sigma_2}(r^{\alpha},h;\Psi')\right.$$
$$\left.+m\varphi\circ\psi^{-1}\left(\frac{\sigma_1}{m}\right)\zeta^{\sigma_1}_{\sigma_2}(1-r^{\alpha},h;\Psi')\right).$$

Multiplying both sides of (28) by $\Lambda^{\sigma_1}_{\varpi}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\alpha,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)\Psi'(\varpi)$, and integrating over $[\sigma_1,\sigma_2]$, one can obtain

$$\varphi\circ\psi^{-1}\left(\frac{\sigma_1+\sigma_2}{2}\right)\int_{\sigma_1}^{\sigma_2}\Lambda^{\sigma_1}_{\varpi}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\alpha,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)d(\Psi(\varpi))$$
$$\leq\left(h\left(\frac{1}{2^{\alpha}}\right)+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\right)\int_{\sigma_1}^{\sigma_2}\Lambda^{\sigma_1}_{\varpi}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\alpha,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)\varphi\circ\psi^{-1}(\varpi)d(\Psi(\varpi)).$$

Definition 4, along with the above inequality, gives the upcoming inequality:

$$\frac{1}{h\left(\frac{1}{2^{\alpha}}\right)+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)}\varphi\circ\psi^{-1}\left(\frac{\sigma_1+\sigma_2}{2}\right)\left({}^{\phi}_{\Lambda}\Upsilon^{\lambda,\rho,\theta,k,n}_{\kappa,\alpha,\beta,\gamma,\delta,\mu,\nu,\sigma_2^-}1\right)(\sigma_1;p') \tag{35}$$
$$\leq\left({}^{\phi}_{\Lambda}\Upsilon^{\lambda,\rho,\theta,k,n}_{\kappa,\alpha,\beta,\gamma,\delta,\mu,\nu,\sigma_2^-}\varphi\circ\psi^{-1}\right)(\sigma_1;p').$$

Now, multiplying by $\Lambda^{\varpi}_{\sigma_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)\Psi'(\varpi)$ on both sides of (28), then integrating over $[\sigma_1,\sigma_2]$, we obtain

$$\frac{1}{h\left(\frac{1}{2^{\alpha}}\right)+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)}\varphi\circ\psi^{-1}\left(\frac{\sigma_1+\sigma_2}{2}\right)\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\alpha,\beta,\gamma,\delta,\mu,\nu,\sigma_1^+}1\right)(\sigma_2;p') \tag{36}$$
$$\leq\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\sigma_1^+}\varphi\circ\psi^{-1}\right)(\sigma_2;p').$$

From (34)–(36), inequality (29) can be achieved. □

Remark 2. *(i) If $\psi(\varpi)=\varpi$ in (29), then then the result for $(\alpha,h-m)$-convex function is obtained;*

(ii) If $\psi(\varpi)=\varpi$, $\kappa=\vartheta$ and $h(\tau)=\tau$ in (29), then [26] Theorem 1 is obtained;

(iii) If $\psi(\varpi)=\varpi$, $\alpha=1$, $\kappa=\vartheta$ and $h(\tau)=\tau^{s}$ in (29), then the result for (s,m)-convex function is obtained.

Theorem 3. *Let I,J be intervals in $\mathbb{R}$ and $\varphi: I\to\mathbb{R}$ be differentiable and $|\varphi'|$ an $(\alpha,h-m)$-convex function with respect to ψ, where $\psi: J\to\mathbb{R}$ is strictly monotonic function and Image(ψ)$\subset I$. If $h(\varpi)h(y)\leq h(\varpi+y)$, then for $(\alpha,m)\in(0,1]^2$, we have the following inequality for integral operators (12) and (13):*

$$\left|\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\sigma_1^+}(\varphi\circ\psi^{-1}*\Psi)\right)(\varpi;p')+\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\mu,\eta,l,\sigma_2^-}(\varphi\circ\psi^{-1}*\Psi)\right)(\varpi;p')\right| \tag{37}$$
$$\leq \Lambda^{\sigma_1}_{\varpi}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)(\varpi-\sigma_1)\left(|\varphi'(\psi^{-1}(\sigma_1))|\zeta^{\sigma_1}_{\varpi}(r^{\alpha},h;\Psi')+m\left|\varphi'\left(\psi^{-1}\left(\frac{\varpi}{m}\right)\right)\right|\right.$$
$$\left.\times\,\zeta^{\sigma_1}_{\varpi}(1-r^{\alpha},h;\Psi')\right)+\Lambda^{\varpi}_{\sigma_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)(\sigma_2-\varpi)\left(|\varphi'(\psi^{-1}(\sigma_2))|\zeta^{\sigma_2}_{\varpi}(r^{\alpha},h;\Psi')\right.$$
$$\left.+\,m\left|\varphi'\left(\psi^{-1}\left(\frac{\varpi}{m}\right)\right)\right|\zeta^{\sigma_2}_{\varpi}(1-r^{\alpha},h;\Psi')\right),$$

where

$$\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\sigma_1^+}(\varphi\circ\psi^{-1}*\Psi)\right)(\varpi;p'):=\int_{\sigma_1}^{\varpi}\Lambda^{\tau}_{\varpi}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,},\Psi;\phi)\varphi'(\psi^{-1}(\tau))d(\Psi(\tau)),$$
$$\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\sigma_2^-}(\varphi\circ\psi^{-1}*\Psi)\right)(\varpi;p'):=\int_{\varpi}^{\sigma_2}\Lambda^{\varpi}_{\tau}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,},\Psi;\phi)\varphi'(\psi^{-1}(\tau)d(\Psi(\tau)).$$

Proof. Since $|\varphi'|$ is $(\alpha, h-m)$-convex function with respect to strictly monotonic function, one can have

$$|\varphi'(\psi^{-1}(\tau))|\leq h\left(\frac{\varpi-\tau}{\varpi-\sigma_1}\right)^{\alpha}|\varphi'(\psi^{-1}(\sigma_1))|+mh\left(1-\left(\frac{\varpi-\tau}{\varpi-\sigma_1}\right)^{\alpha}\right)\left|\varphi'(\psi^{-1}\left(\frac{\varpi}{m}\right)\right|. \tag{38}$$

Inequality (38) can takes the following form:

$$-\left(h\left(\frac{\varpi-\tau}{\varpi-\sigma_1}\right)^{\alpha}|\varphi'(\psi^{-1}(\sigma_1))|+mh\left(1-\left(\frac{\varpi-\tau}{\varpi-\sigma_1}\right)^{\alpha}\right)\left|\varphi'\left(\psi^{-1}\left(\frac{\varpi}{m}\right)\right)\right|\right)\leq\varphi'(\psi^{-1}(\tau)) \tag{39}$$
$$\leq\left(h\left(\frac{\varpi-\tau}{\varpi-\sigma_1}\right)^{\alpha}|\varphi'(\psi^{-1}(\sigma_1))|+mh\left(1-\left(\frac{\varpi-\tau}{\varpi-\sigma_1}\right)^{\alpha}\right)\left|\varphi'\left(\psi^{-1}\left(\frac{\varpi}{m}\right)\right)\right|\right).$$

From inequality (39), we have

$$\varphi'(\psi^{-1}(\tau))\leq h\left(\frac{\varpi-\tau}{\varpi-\sigma_1}\right)^{\alpha}|\varphi'(\psi^{-1}(\sigma_1))|+mh\left(1-\left(\frac{\varpi-\tau}{\varpi-\sigma_1}\right)^{\alpha}\right)\left|\varphi'\left(\psi^{-1}\left(\frac{\varpi}{m}\right)\right)\right|. \tag{40}$$

Multiplying (15) with (40) and integrating over $[a,x]$, we obtain:

$$\int_a^{\varpi}\Lambda^{\tau}_{\varpi}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)\varphi'(\psi^{-1}(\tau))d(\Psi(\tau))\leq\Lambda^{\sigma_1}_{\varpi}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)\left(|\varphi'(\psi^{-1}(\sigma_1))|\right.$$
$$\left.\times\int_{\sigma_1}^{\varpi}h\left(\frac{\varpi-\tau}{\varpi-\sigma_1}\right)^{\alpha}d(\Psi(\tau))+m\left|\varphi'\left(\psi^{-1}\left(\frac{\varpi}{m}\right)\right)\right|\int_{\sigma_1}^{\varpi}h\left(1-\left(\frac{\varpi-\tau}{\varpi-\sigma_1}\right)^{\alpha}\right)d(\Psi(\tau))\right).$$

This gives

$$\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\sigma_1^+}(\varphi(\psi^{-1})*\Psi)\right)(\varpi;p')\leq\Lambda^{\sigma_1}_{\varpi}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)(\varpi-\sigma_1) \tag{41}$$
$$\times\left(|\varphi'(\psi^{-1}(\sigma_1))|\zeta^{\sigma_1}_{\varpi}(r^{\alpha},h;\Psi')+m\left|\varphi'\left(\psi^{-1}\left(\frac{\varpi}{m}\right)\right)\right|\zeta^{\sigma_1}_{\varpi}(1-r^{\alpha},h;\Psi')\right).$$

By utilizing the second inequality of (39), working on the same pattern as we did for the right hand inequality, one can obtain

$$\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\sigma_1^+}((\varphi\circ\psi^{-1})*\Psi)\right)(\varpi;p') \geq -\Lambda^{\sigma_1}_{\varpi}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)(\varpi-\sigma_1) \quad (42)$$
$$\times\left(|\varphi'(\psi^{-1}(\sigma_1))|\zeta^{\sigma_1}_{\varpi}(r^{\alpha},h;\Psi') + m\left|\varphi'\left(\psi^{-1}\left(\frac{\varpi}{m}\right)\right)\right|\zeta^{\sigma_1}_{\varpi}(1-r^{\alpha},h;\Psi')\right).$$

From (41) and (42), following inequality is observed:

$$\left|\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\sigma_1^+}(\varphi\circ\psi^{-1}*\Psi)\right)(\varpi;p')\right| \leq \Lambda^{\sigma_1}_{\varpi}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)(\varpi-\sigma_1) \quad (43)$$
$$\times\left(|\varphi'(\psi^{-1}(\sigma_1))|\zeta^{\sigma_1}_{\varpi}(r^{\alpha},h;\Psi') + m\left|\varphi'\left(\psi^{-1}\left(\frac{\varpi}{m}\right)\right)\right|\zeta^{\sigma_1}_{\varpi}(1-r^{\alpha},h;\Psi')\right).$$

By applying $(\alpha, h-m)$-convexity of function $|\varphi'|$ with respect to strictly monotonic function, one can obtain

$$|\varphi'(\psi^{-1}(\tau))| \leq h\left(\frac{\tau-\varpi}{\sigma_2-\varpi}\right)^{\alpha}|\varphi'(\psi^{-1}(\sigma_2))| + mh\left(1-\left(\frac{\tau-\varpi}{\sigma_2-\varpi}\right)^{\alpha}\right)\left|\varphi'\left(\psi^{-1}\left(\frac{\varpi}{m}\right)\right)\right|. \quad (44)$$

Now on the same lines as we worked for (15) and (38), from (16) and (44), one can have the following inequality:

$$\left|\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\sigma_2^-}(\varphi\circ\psi^{-1}*\Psi)\right)(\varpi;p')\right| \leq \Lambda^{\varpi}_{\sigma_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)(\sigma_2-\varpi) \quad (45)$$
$$\times\left(|\varphi'(\psi^{-1}(\sigma_2))|\zeta^{\sigma_2}_{\varpi}(r^{\alpha},h;\Psi') + m\left|\varphi'\left(\psi^{-1}\left(\frac{\varpi}{m}\right)\right)\right|\zeta^{\sigma_2}_{\varpi}(1-r^{\alpha},h;\Psi')\right).$$

By adding (43) and (45), inequality (37) can be achieved. □

Remark 3. *(i) If $\psi(\varpi) = \varpi$ in (37), then then the result for $(\alpha, h-m)$-convex function is obtained;*
(ii) If $\psi(\varpi) = \varpi$, $\kappa = \vartheta$ and $h(\tau) = \tau$ in (37), then [26] Theorem 3 is obtained;
(iii) If $(\alpha, p) = (1,1)$, $\kappa = \vartheta$ and $h(\tau) = \tau^s$ in (37), then the result for (s,m)-convex function is obtained.

3. Applications in the Form of Hadamard-Type Inequalities

Here, we give Hadamard-type inequalities deducible from Theorem 2.

Corollary 1. *Under the assumption of Theorem 2, the following inequality holds for (h,m)-convex function φ with respect to strictly monotone function ψ:*

$$\frac{\varphi\left(\psi^{-1}\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)}{h\left(\frac{1}{2}\right)(1+m)}\left(\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\sigma_2^-}1\right)(\sigma_1;p') + \left({}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\sigma_1^+}1\right)(\sigma_2;p')\right)$$
$$\leq \left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\sigma_1^+}\varphi\circ\psi^{-1}\right)(\sigma_2;p') + \left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\sigma_2^-}\varphi\circ\psi^{-1}\right)(\sigma_1;p')$$
$$\leq (\sigma_2-\sigma_1)(\Lambda^{\sigma_1}_{\sigma_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi) + \Lambda^{\sigma_1}_{\sigma_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi))$$
$$\times\left(\varphi(\psi^{-1}(\sigma_2))\zeta^{\sigma_1}_{\sigma_2}(r,h;\Psi') + m\varphi\left(\psi^{-1}\left(\frac{\sigma_1}{m}\right)\right)\zeta^{\sigma_1}_{\sigma_2}(1-r,h;\Psi')\right).$$

Proof. By setting $\psi(x) = x^p$, $p \in \mathbb{R}-\{0\}$, and $\alpha = 1$, from (29), one can obtain the required inequality. □

Corollary 2. *Under the assumption of Theorem 2, the following inequality holds for (α, h)-convex function φ with respect to strictly monotone function ψ:*

$$\frac{\varphi\left(\psi^{-1}\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)}{h\left(\frac{1}{2^\alpha}\right)+h\left(\frac{2^\alpha-1}{2^\alpha}\right)}\left(\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\sigma_2^-}1\right)(\sigma_1;p')+\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\sigma_1^+}1\right)(\sigma_2;p')\right)$$
$$\leq\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\sigma_1^+}\varphi\circ\psi^{-1}\right)(\sigma_2;p')+\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\sigma_2^-}\varphi\circ\psi^{-1}\right)(\sigma_1;p')$$
$$\leq(\sigma_2-\sigma_1)(\Lambda^{\sigma_1}_{\sigma_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,},\Psi;\phi)+\Lambda^{\sigma_1}_{\sigma_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,},\Psi;\phi))$$
$$\times\left(\varphi(\psi^{-1}(\sigma_2))\zeta^{\sigma_1}_{\sigma_2}(r^\alpha,h;\Psi')+\varphi(\psi^{-1}(\sigma_1))\zeta^{\sigma_1}_{\sigma_2}(1-r^\alpha,h;\Psi')\right).$$

Proof. By setting $m=1$, from (29), one can obtain the required inequality. □

Corollary 3. *Under the assumption of Theorem 2, the following inequality holds for (α, m)-convex function φ, with respect to strictly monotone function ψ:*

$$\frac{2^\alpha\varphi\left(\psi^{-1}\left(\frac{\sigma_1+\sigma_2}{2}\right)\right)}{(1+m(2^\alpha-1))}\left(\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\sigma_2^-}1\right)(\sigma_1;p')+\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\sigma_1^+}1\right)(\sigma_2;p')\right)$$
$$\leq\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\sigma_2^-}\varphi\circ\psi^{-1}\right)(\sigma_1;p')+\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\sigma_1^+}\varphi\circ\psi^{-1}\right)(\sigma_2;p')$$
$$\leq(\Lambda^{\sigma_1}_{\sigma_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,},\Psi;\phi))\left(\left(\varphi(\psi^{-1}(\sigma_2))\Psi(\sigma_2)-m\varphi\left(\psi^{-1}\left(\frac{\sigma_1}{m}\right)\right)\Psi((\sigma_1))\right)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{(\sigma_2-\sigma_1)^\alpha}\left(\varphi(\psi^{-1}(\sigma_2))-m\varphi\left(\psi^{-1}\left(\frac{\sigma_1}{m}\right)\right)\right){}^{\alpha}I_{\sigma_2^-}\Lambda(a)\right).$$

Proof. By setting $h(\tau)=\tau$, from (29), one can obtain the required inequality. □

Remark 4. *Moreover, by setting $\psi(x)=x^p$, $p\geq 1$, the above theorems hold for $(h,m)-p$-convex, $(\alpha,h)-p$-convex and $(\alpha,m)-p$-convex functions.*

4. Concluding Remarks

This article investigated the bounds of fractional integral operators containing the unified Mittag–Leffler function via a generalized class of functions named as $(\alpha, h-m)$-convex functions, with respect to strictly increasing functions. The established results generalized many integral inequalities that have been provided in currently published articles. At the end, the Hadamard-type inequalities for some new classes of convex functions that are special cases of our main results are presented.

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