axioms

MDPI

Article

# Zalcman Functional and Majorization Results for Certain Subfamilies of Holomorphic Functions

**Muhammad Ghafar Khan [1], Bilal Khan [2,*], Ferdous M. O. Tawfiq [3] and Jong-Suk Ro [4,5,*]**

[1] Institute of Numerical Sciences, Kohat University of Sciences and Technology, Kohat 26000, Pakistan; ghaffarkhan020@gmail.com
[2] School of Mathematical Sciences and Shanghai Key Laboratory of PMMP, East China Normal University, 500 Dongchuan Road, Shanghai 200241, China
[3] Department of Mathematics, College of Science, King Saud University, P.O. Box 22452, Riyadh 11495, Saudi Arabia; ftoufic@ksu.edu.sa
[4] School of Electrical and Electronics Engineering, Chung-Ang University, Dongjak-gu, Seoul 06974, Republic of Korea
[5] Department of Intelligent Energy and Industry, Chung-Ang University, Dongjak-gu, Seoul 06974, Republic of Korea
* Correspondence: bilalmaths789@gmail.com (B.K.); jongsukro@gmail.com (J.-S.R.)

**Abstract:** In this paper, we investigate sharp coefficient functionals, like initial four sharp coefficient bounds, sharp Fekete–Szegö functionals, and, for $n = 1$ and 2, sharp Zalcman functionals are evaluated for class of functions associated with tangent functions. Furthermore, we provide some majorization results for some non-vanishing holomorphic functions, whose ratios are related to various domains in the open unit disk.

**Keywords:** olomorphic functions; tangent domain; Zalcman functional; majorization

**MSC:** 30C45; 30C50

check for updates

**Citation:** Khan, M.G.; Khan, B.; Tawfiq, F.M.O.; Ro, J.-S. Zalcman Functional and Majorization Results for Certain Subfamilies of Holomorphic Functions. *Axioms* **2023**, *12*, 868. https://doi.org/10.3390/axioms12090868

Academic Editor: Miodrag Mateljevic

Received: 28 July 2023
Revised: 31 August 2023
Accepted: 6 September 2023
Published: 8 September 2023



## 1. Introduction

Here, in this section, we provide some fundamental and significant concepts for a better comprehension of the primary results. Starting with the most fundamental definition, for which we use the symbol $\mathcal{A}$. Let $\mathcal{A}$ stand for the family of analytic functions with the following series representation:

$$\omega(\kappa) = \kappa + a_2\kappa^2 + a_3\kappa^3 + a_4\kappa^4 + \cdots = \kappa + \sum_{\ell=2}^{\infty} d_n\kappa^n, \tag{1}$$

where
$$\omega(0) = \omega'(0) - 1 = 0 \quad \text{and} \quad z \in \Omega := \{\kappa : \kappa \in \mathbb{C} \quad \text{and} \quad |\kappa| < 1\},$$

$\mathbb{C}$ being the set of complex numbers. Additionally, all univalent functions of family $\mathcal{A}$ are included in the separate family $\mathcal{S}$. In 1916, Biberbach [1] made the coefficient conjecture, which helped the topic gain popularity as a potential area for further study, although the concept of function theory was first developed in 1851. This conjuncture was proven in 1985 by De-Branges [2]. Many of the top academics in the world tried to support or refute the Bieberbach conjucture between 1916 and 1985. As a result, they discovered several subfamilies of $\mathcal{S}$ family of normalised univalent functions that are connected to various image domains. The most fundamental and important subclasses of the functions class $\mathcal{S}$ are the $\mathcal{S}^*$ and $\mathcal{K}$, known as families of starlike and convex functions, respectively. Analytically,

$$\mathcal{S}^* = \left\{\omega \in \mathcal{S} : \mathrm{Re}\left(\frac{\kappa\omega'(\kappa)}{\omega(\kappa)}\right) > 0\right\}$$

and

$$\mathcal{K} = \left\{ \omega \in \mathcal{S} : \mathrm{Re}\left( \frac{(\kappa\omega'(\kappa))'}{\omega'(\kappa)} \right) > 0 \right\}.$$

The concept of quasi-subordination between holomorphic functions was first introduced by Robertson [3] in 1970. Let $g_1$ and $g_2$ are two members of the family $\mathcal{A}$, are connected to quasi-subordination relationship, mathematically demonstrated by $g_1 \prec_q g_2$, if there occurs functions $\Phi, u \in \mathcal{A}$, so that $\frac{\kappa\omega'(\kappa)}{\Phi(\kappa)}$ is holomorphic in $\Omega$ with properties

$$|\Phi(\kappa)| \leq 1, \ |u(\kappa)| \leq |\kappa| \text{ and } u(0) = 0,$$

satisfying the relationship

$$g_1(\kappa) = \Phi(\kappa) g_2(u(\kappa)), \ \kappa \in \ \Omega.$$

Moreover, by choosing

$$u(\kappa) = \kappa \text{ and } \Phi(\kappa) = 1,$$

we get, at the most, helpful ideas in GFT, known as subordination between holomorphic functions. Actually, if $g_2 \in \mathcal{S}$, then for $g_1, g_2 \in \mathcal{A}$, the relationship of subordination has

$$g_1(\kappa) \prec g_2(\kappa) \Leftrightarrow [g_1(\Omega) \subset g_2(\Omega) \text{ with } g_1(0) = g_2(0)].$$

By assuming that $u(\kappa) = \kappa$, the above definition reduced to the majorization between holomorphic functions and is given by

$$g_1(\kappa) \ll g_2(\kappa), \ g_1(\kappa), g_2(\kappa) \in \mathcal{A}.$$

So, $g_1(\kappa) \ll g_2(\kappa)$, if the function $\Phi(\kappa) \in \mathcal{A}$ meets the requirement $|\Phi(\kappa)| \leq 1$, such that

$$g_1(\kappa) = \Phi(\kappa) g_2(\kappa), \ \kappa \in \ \Omega. \tag{2}$$

In 1967, MacGregor [4] first proposed this concept. This concept has appeared in numerous articles. For some recent research on this subject, we refer the readers to see [5–9].

In 1992, Ma and Minda defined [10]

$$\mathcal{S}^*(\phi) = \left\{ \omega \in \mathcal{A} : \frac{\kappa\omega'(\kappa)}{\omega(\kappa)} \prec \phi(\kappa) \right\}. \tag{3}$$

with $\Re(\phi) > 0$ in $\Omega$, additionally, the function $\phi$ maps $\Omega$ onto a star-shaped region, and the image domain is symmetric about the real axis and starlike, with respect to $\phi(0) = 1$ with $\phi'(0) > 0$. The set $\mathcal{S}^*(\phi)$ generalizes a number of subfamilies of the function class $\mathcal{A}$, including, for instance:

1. If
$$\phi(\kappa) = \frac{1 + L\kappa}{1 + M\kappa},$$
with $-1 \leq M < L \leq 1$, then
$$\mathcal{S}^*[L, M] \equiv \mathcal{S}^*\left( \frac{1 + L\kappa}{1 + M\kappa} \right),$$
where $\mathcal{S}^*[L, M]$ is the class of Janowski starlike functions, see [11].
2. Choose $\phi(\kappa) = \sqrt{1 + \kappa}$, we receive the family $\mathcal{S}^*_{\mathcal{L}}$, defined and investigated by Sokol et al. [12].
3. For the function
$$\phi(\kappa) = 1 + \sinh^{-1} \kappa,$$

we receive the class $\mathcal{S}^*_\rho$, introduced by kumar and Arora [13].

4. If $\phi(\kappa) = e^\kappa$, then the class $\mathcal{S}^*(\phi)$ becomes $\mathcal{S}^*_e$, which is defined and studied by Mendiratta [14].
5. For $\phi(\kappa) = 1 + \sin(\kappa)$, the class $\mathcal{S}^*(\phi)$ reduces to the class $\mathcal{S}^*_{\sin}$. The family $\mathcal{S}^*_s$ was introduced by Cho et al. [15] as:

$$\mathcal{S}^*_{\sin} = \left\{ \omega \in \mathcal{A} : \frac{\kappa\omega'(\kappa)}{\omega(\kappa)} \prec 1 + \sin(\kappa), \ \ (\kappa \in \Omega) \right\}, \tag{4}$$

which means that $\frac{\kappa\omega'(\kappa)}{\omega(\kappa)}$ lies in an eight shaped region.

6. If we pick $\phi(\kappa) = \cos(\kappa)$, we receive the family $\mathcal{S}^*_{\cos}$, initiated by Bano and Raza [16].
7. If we select $\phi(\kappa) = \sec h(\kappa)$, we obtain a family $\mathcal{S}^*_{\sec h}$ introduced by Al-Shbeil et al. [17].

In this paper, we define the following subfamily of holomorphic functions

$$\mathcal{S}^*_{\tan} = \left\{ \omega \in \mathcal{A} : \frac{\kappa\omega'(\kappa)}{\omega(\kappa)} \prec \frac{2 + \tan(\kappa)}{2} \right\} \quad (\kappa \in \Omega). \tag{5}$$

## 2. Set of Lemma's

The following are some useful lemma, which we use in our main finding.

Let $\mathcal{P}$ denote the family of all holomorphic functions $p$ with a positive real part, having the following series representation:

$$p(\kappa) = 1 + \sum_{n=1}^{\infty} c_n \kappa^n, \ \kappa \in \Omega. \tag{6}$$

**Lemma 1.** *If $p \in \mathcal{P}$, then the following estimates are valid:*

$$|c_k| \leq 2, \ k \geq 1, \tag{7}$$

$$|c_{k+n} - \mu c_k c_n| < 2, \ \ 0 < \mu \leq 1, \tag{8}$$

and, for $\eta \in \mathbb{C}$, we have

$$\left| c_2 - \eta c_1^2 \right| < 2 \max\{1, |2\eta - 1|\}. \tag{9}$$

For the inequalities (7) and (8), see [18], and (9) is given in [19].

**Lemma 2** ([20]). *If $p \in \mathcal{P}$ and has the form (6), then*

$$|\alpha_1 c_1^3 - \alpha_2 c_1 c_2 + \alpha_3 c_3| \leq 2|\alpha_1| + 2|\alpha_2 - 2\alpha_1| + 2|\alpha_1 - \alpha_2 + \alpha_3|, \tag{10}$$

*where $\alpha_1, \alpha_2$, and $\alpha_3$ are real numbers.*

**Lemma 3** ([21]). *Let $m_1, n_1, l_1$, and $r_1$ satisfy the inequalities $m_1, r_1 \in (0,1)$ and*

$$\begin{aligned} &8r_1(1 - r_1)\left[ (m_1 n_1 - 2l_1)^2 + (m_1(r_1 + m_1) - n_1)^2 \right] \\ &+ m_1(1 - m_1)(n_1 - 2r_1 m_1)^2 \\ &\leq 4m_1^2(1 - m_1)^2 r_1(1 - r_1). \end{aligned}$$

*If $h \in \mathcal{P}$ and is of the form (6), then*

$$\left| l_1 c_1^4 + r_1 c_2^2 + 2m_1 c_1 c_3 - \frac{3}{2} n_1 c_1^2 c_2 - c_4 \right| \leq 2.$$

## 3. Coefficient Bounds and Zalcman Functional for the Family $\mathcal{S}^*_{\tan}$

One of the established conjectures in the study of Geometric function theory, posited by Lawrence Zalcman in 1960, states that the coefficients of class $\mathcal{S}$ satisfy the inequality

$$\left|d_n^2 - d_{2n-1}\right| \leq (n-1)^2.$$

The equality in the above inequality holds only for the renowned Koebe function $k(\kappa) = \frac{\kappa}{(1-\kappa)^2}$ and its rotations. The famous Fekete–Szegö inequality holds for $n = 2$. Zalcman's conjecture in its original setting, i.e., for the entire class $\mathcal{S}$, has been proved by Krushkal for $n = 3, 4, 5$, and 6 in [22]. This important result is now presented also in the books [23,24]. For detailed studies on Zalcman functionals, see the articles [25–30].

**Theorem 1.** *Let $\omega \in \mathcal{S}^*_{\tan}$. Then, the following estimates hold:*

$$\begin{aligned} |d_2| &\leq \frac{1}{2}, \\ |d_3| &\leq \frac{1}{4}, \\ |d_4| &\leq \frac{1}{6}, \\ |d_5| &\leq \frac{1}{8}. \end{aligned}$$

*These estimates are sharp for the function*

$$\omega_n(\kappa) = \kappa + \frac{1}{2n}\kappa^{n+1} + \cdots, \quad n = 1, 2, 3, \ldots. \tag{11}$$

**Proof.** Let $\omega \in \mathcal{S}^*_{\tan}$. Then, by property of the Schwarz function $u(\kappa)$, so that $u(0) = 0$ and $|u(\kappa)| \leq |\kappa|$, we have

$$\frac{\kappa\omega'(\kappa)}{\omega(\kappa)} = \frac{2 + \tan(u(\kappa))}{2}.$$

As there is one–one correspondence between $u(\kappa)$ and the function with the positive real part $p(\kappa)$, we can write

$$u(\kappa) = \frac{p(\kappa) - 1}{p(\kappa) + 1}.$$

Now,

$$\begin{aligned} \frac{2 + \tan(u(\kappa))}{2} = {} & 1 + \frac{1}{4}c_1\kappa + \left(\frac{1}{4}c_2 - \frac{1}{8}c_1^2\right)\kappa^2 + \left(\frac{1}{12}c_1^3 - \frac{1}{4}c_2c_1 + \frac{1}{4}c_3\right)\kappa^3 \\ & \left(-\frac{1}{16}c_1^4 + \frac{1}{4}c_1^2c_2 - \frac{1}{4}c_3c_1 - \frac{1}{8}c_2^2 + \frac{1}{4}c_4\right)\kappa^4 + \cdots. \end{aligned} \tag{12}$$

And,

$$\begin{aligned} \frac{\kappa\omega'(\kappa)}{\omega(\kappa)} = {} & 1 + d_2\kappa + \left(2d_3 - d^2\right)\kappa^2 + \left(3d_4 - 3d_2d_3 - d_2^3\right)\kappa^3 + \\ & \left(4d_5 - d_2^4 + 4d_2^2d_3 - 4d_2d_4 - 2d_3^2\right) + \cdots. \end{aligned} \tag{13}$$

On comparing (12) and (13), we have

$$d_2 = \frac{1}{4}c_1, \tag{14}$$

$$d_3 = \frac{1}{8}c_2 - \frac{1}{16}c_1^2, \tag{15}$$

$$d_4 = \frac{1}{144}c_1^3 - \frac{5}{96}c_2c_1 + \frac{1}{12}c_3, \tag{16}$$

$$d_5 = -\frac{65}{9216}c_1^4 + \frac{13}{384}c_1^2c_2 - \frac{1}{24}c_3c_1 - \frac{3}{128}c_2^2 + \frac{1}{16}c_4. \tag{17}$$

Applying (7) to (14), we have

$$|d_2| \leq \frac{1}{2}.$$

To find the bound of $d_3$, apply (8) to (15), and then we have

$$|d_3| \leq \frac{1}{4}.$$

Applying (10) to (16), we get

$$|d_4| \leq \frac{1}{6}.$$

And,

$$|d_5| = \frac{1}{16}\left|\frac{65}{576}c_1^4 - \frac{13}{24}c_1^2c_2 + \frac{2}{3}c_3c_1 + \frac{3}{8}c_2^2 - c_4\right|.$$

Now, using Lemma 3 with $l_1 = \frac{65}{576}$, $r_1 = \frac{3}{8}$, $m_1 = \frac{1}{3}$, and $n_1 = \frac{13}{36}$, we have

$$|d_5| \leq \frac{1}{8}.$$

□

**Theorem 2.** *Let $\omega \in \mathcal{S}^*_{\tan}$. Then, for a complex number $\lambda$, we have*

$$\left|d_3 - \lambda d_2^2\right| \leq \frac{1}{4}\max\{1, |\lambda|\}.$$

*The estimate is sharp for function $\omega_1$, defined by (11).*

**Proof.** From (14) and (15), we have

$$\left|d_3 - \lambda d_2^2\right| = \frac{1}{8}\left|c_2 - \frac{1+\lambda}{2}c_1^2\right|.$$

The desired result is achieved by applying (9) to the above equation. □

Set $\lambda = 1$ in the above Theorem 2, and then we receive the following corollary, which is a special case of Zalcman functional when $n = 2$.

**Corollary 1.** *Let $\omega \in \mathcal{S}^*_{\tan}$. Then,*

$$\left|d_3 - d_2^2\right| \leq \frac{1}{4}.$$

*A sharp result is achieved for the function $\omega_1$, defined by (11).*

**Proof.** From (15) and (17), we have

$$\begin{aligned}\left|d_3^2-d_5\right| &= \left|\frac{101}{9216}c_1^4-\frac{19}{384}c_1^2c_2+\frac{1}{24}c_3c_1+\frac{5}{128}c_2^2-\frac{1}{16}c_4\right| \\ &\leq \frac{1}{8}\text{ (using Lemma 3).}\end{aligned}$$

Which is our desired proof. For sharpness, set $n=4$ in Equation (11), that is

$$\omega_4(\kappa)=\kappa+\frac{1}{8}\kappa^4+\cdots.$$

And,

$$\left|d_3^2-d_5\right|=|d_5|=\frac{1}{8}.$$

□

## 4. Majorization Results

First, we select the holomorphic nonvanishing functions $\Lambda_1$ and $\Lambda_2$ in open unit disc $\Omega$ with conditions

$$\Lambda_1(0)=1\text{ and }\Lambda_2(0)=1.$$

Following that, the families established in this article consist of functions $\omega\in\mathcal{A}$, whose ratios $\frac{\omega(\kappa)}{\kappa q(\kappa)}$ and $q(\kappa)$ are, respectively, subordinated to $\Lambda_1$ and $\Lambda_2$, for the certain holomorphic function $q$, with $q(0)=1$ as

$$\frac{\omega(\kappa)}{\kappa q(\kappa)}\prec\Lambda_1(\kappa)\text{ and }q(\kappa)\prec\Lambda_2(\kappa).$$

We will now choose a certain particular functions rather than $\Lambda_1$ and $\Lambda_2$. These options include

$$\Lambda_1(\kappa)=\cos(\kappa),$$

or

$$\Lambda_1(\kappa)=\sqrt{1+\kappa},$$

or

$$\Lambda_1(\kappa)=\sec h(\kappa),$$

and

$$\Lambda_2(\kappa)=\frac{2+\tan(\kappa)}{2}.$$

Using the previously discussed concepts, we now study the following new subfamilies:

$$\begin{aligned}\Upsilon_{\cos} &= \left\{\omega\in\mathcal{A}:\frac{\omega(\kappa)}{\kappa q(\kappa)}\prec\cos(\kappa)\text{ and }q(\kappa)\prec\frac{2+\tan(\kappa)}{2}\right\}, \\ \Upsilon_{\mathcal{L}} &= \left\{\omega\in\mathcal{A}:\frac{\omega(\kappa)}{\kappa q(\kappa)}\prec\sqrt{1+\kappa}\text{ and }q(\kappa)\prec\frac{2+\tan(\kappa)}{2}\right\}, \\ \Upsilon_{\sec h} &= \left\{\omega\in\mathcal{A}:\frac{\omega(\kappa)}{\kappa q(\kappa)}\prec\sec h(\kappa)\text{ and }q(\kappa)\prec\frac{2+\tan(\kappa)}{2}\right\}.\end{aligned}$$

Majorization results for each of the aforementioned families are covered in the current section.

**Lemma 4.** *Let $q(\kappa) \prec 1 + \frac{\tan(\kappa)}{2}$. Then, for $|\kappa| \leq s$, we have*

$$\left|\frac{\kappa q'(\kappa)}{q(\kappa)}\right| \leq \frac{s\sec^2(s)}{(1-s^2)(2-\tan(s))}.$$

**Proof.** As $q(\kappa) \prec 1 + \frac{\tan(\kappa)}{2}$, by the properties of the Schwarz function $u(\kappa)$, we have

$$q(\kappa) = \frac{2+\tan(u(\kappa))}{2}.$$

Taking log differentiation on both sides, we have

$$\frac{\kappa q'(\kappa)}{q(\kappa)} = \frac{u'(\kappa)\sec^2(u(\kappa))}{2+\tan(u(\kappa))} \tag{18}$$

Let us assume that $u(\kappa) = \mathrm{Re}^{i\theta}$, $\theta \in [0, 2\pi]$, and $|\kappa| = R \leq s$. Consider

$$\begin{aligned} \left|\tan\left(\mathrm{Re}^{i\theta}\right)\right|^2 &= \left(\frac{\sin(R\cos\theta)\cos(R\cos\theta)}{\cos^2(R\cos\theta)+\sinh^2(R\cos\theta)}\right)^2 + \left(\frac{\sinh(R\sin\theta)\cosh(R\sin\theta)}{\cos^2(R\cos\theta)+\sinh^2(R\cos\theta)}\right) \\ &= Q(\theta)\ \text{(say)}. \end{aligned}$$

Clearly, $Q(\theta)$ is satisfied with the condition of the even function, which is $Q(-\theta) = Q(\theta)$, so we could consider interval $[0, \pi]$ instead of $[0, 2\pi]$. Moreover, $Q'(\theta) = 0$ has three roots in interval $0 \leq \theta \leq \pi$, namely, $\theta = 0$, $\frac{\pi}{2}$ and $\pi$. We notice that

$$Q\left(\frac{\pi}{2}\right) = \tanh^2 R \text{ and } Q(0) = \tan^2 R = Q(\pi).$$

Moreover , we see that

$$\begin{aligned} \max\left\{Q(0), Q\left(\frac{\pi}{2}\right), Q(\pi)\right\} &= \tan^2 R \leq \tan^2 s, \\ \min\left\{Q(0), Q\left(\frac{\pi}{2}\right), Q(\pi)\right\} &= \tanh^2 R \leq \tanh^2 s. \end{aligned}$$

Hence,

$$\tanh s \leq \left|\tan\left(\mathrm{Re}^{i\theta}\right)\right| \leq \tan s. \tag{19}$$

Also,

$$\begin{aligned} \left|\sec\left(\mathrm{Re}^{i\theta}\right)\right|^2 &= \left(\frac{\cos(R\cos\theta)\cosh(R\sin\theta)}{\cos^2(R\cos\theta)+\sinh^2(R\sin\theta)}\right)^2 + \left(\frac{\sin(R\cos\theta)\sinh(R\sin\theta)}{\cos^2(R\cos\theta)+\sinh^2(R\sin\theta)}\right)^2 \\ &= E(\theta)\ \text{(say)}. \end{aligned}$$

As $E(-\theta) = E(\theta)$, it is an even function; therefore, we have to consider interval $[0, \pi]$. Furthermore, $E'(\theta) = 0$ also has three roots: $\theta = 0$, $\frac{\pi}{2}$, and $\pi$,

$$E\left(\frac{\pi}{2}\right) = \sec h^2 R \text{ and } Q(0) = \sec^2 R = E(\pi).$$

Moreover, we see that

$$\begin{aligned} \max\left\{E(0), E\left(\frac{\pi}{2}\right), E(\pi)\right\} &= \sec^2 R \leq \sec^2 s, \\ \min\left\{E(0), E\left(\frac{\pi}{2}\right), E(\pi)\right\} &= \sec h^2 R \leq \sec h^2 s. \end{aligned}$$

Hence,

$$\sec h^2 s \leq \left|\sec\left(\mathrm{R}e^{i\theta}\right)\right|^2 \leq \sec^2 s. \tag{20}$$

Also, the Schwarz function $u(\kappa)$ satisfies the following condition

$$|u'(\kappa)| \leq \frac{1-|u(\kappa)|^2}{1-|\kappa|^2} = \frac{1-R^2}{1-s^2} \leq \frac{1}{1-s^2}. \tag{21}$$

Now, from (18), we have

$$\left|\frac{\kappa q'(\kappa)}{q(\kappa)}\right| \leq \frac{|\kappa||u'(\kappa)||\sec^2(u(\kappa))|}{2-|\tan(u(\kappa))|}. \tag{22}$$

Now, using (19)–(21) to above inequality (22), we receive the desired achievement. □

**Theorem 3.** *Let $\omega \in \mathcal{A}$, $g \in \Upsilon_{\cos}$ and also assume that $\omega(\kappa) \ll g(\kappa)$ in $\Omega$. Then, for $|\kappa| \leq s_1$*

$$|\omega'(\kappa)| \leq |g'(\kappa)|,$$

*where $s_1$ is the smallest positive root of the equation*

$$\left(\left(1-2s-s^2\right)\cos s - s\sinh s\right)(2-\tan s) - s\cos s\sec^2 s = 0.$$

**Proof.** Let $g \in \Upsilon_{\cos}$. Then, by the property of the Schwarz function $u(\kappa)$, we receive

$$\frac{g(\kappa)}{\kappa q(\kappa)} = \cos(u(\kappa)).$$

After some straightforward calculations, we receive

$$\frac{\kappa g'(\kappa)}{g(\kappa)} = 1 + \frac{\kappa q'(\kappa)}{q(\kappa)} - \frac{\kappa u'(\kappa)\sin(u(\kappa))}{\cos(u(\kappa))}.$$

Now, by using (19)–(21) along with Lemma 4, we have

$$\begin{aligned} \left|\frac{g(\kappa)}{g'(\kappa)}\right| &= \frac{|\kappa|}{\left|1+\frac{\kappa q'(\kappa)}{q(\kappa)} - \frac{\kappa u'(\kappa)\sin(u(\kappa))}{\cos(u(\kappa))}\right|} \\ &\leq \frac{s\left(1-s^2\right)(2-\tan s)\cos s}{(1-s^2)(2-\tan s)\cos s - s\sec^2 s\cos s - s\sinh s(2-\tan s)}. \end{aligned} \tag{23}$$

From (2), we can write

$$\omega(\kappa) = \Phi(\kappa)g(\kappa),$$

upon differentiating, we get

$$\begin{aligned} \omega'(\kappa) &= \Phi'(\kappa)g(\kappa) + \Phi(\kappa)g'(\kappa) \\ &= \left(\Phi'(\kappa)\frac{g(\kappa)}{g'(\kappa)} + \Phi(\kappa)\right)g'(\kappa). \end{aligned} \tag{24}$$

Also, the Schwarz function satisfies the following condition

$$|\Phi'(\kappa)| \leq \frac{1-|\Phi(\kappa)|^2}{1-|\kappa|^2} = \frac{1-|\Phi(\kappa)|^2}{1-s^2}. \tag{25}$$

Now, using (23) and (24) in (25), we obtain

$$\left|\omega'(\kappa)\right| \leq \left(\frac{s\left(1-|\Phi(\kappa)|^2\right)(2-\tan s)\cos s}{(1-s^2)(2-\tan s)\cos s - s\sec^2 s\cos s - s\sinh s(2-\tan s)} + |\Phi(\kappa)|\right)\left|g'(\kappa)\right|,$$

let us assume that $|\Phi(\kappa)| = \delta$, where $0 \leq \delta \leq 1$. Then, the above inequality becomes

$$\left|\omega'(\kappa)\right| \leq \Psi(\delta, s)\left|g'(\kappa)\right|,$$

where

$$\Psi(\delta, s) = \frac{s(1-\delta^2)(2-\tan s)\cos s}{(1-s^2)(2-\tan s)\cos s - s\sec^2 s\cos s - s\sinh s(2-\tan s)} + \delta.$$

To determine $s_1$, it is sufficient to choose

$$s_1 = \max(s \in [0,1] : \Psi(\delta, s) \leq 1, \text{ for all } \delta \in [0,1]),$$

or

$$s_1 = \max(s \in [0,1] : G(\delta, s) \geq 0, \text{ for all } \delta \in [0,1]),$$

where

$$G(\delta, s) = \left(\left(1-s^2-s(1+\delta)\right)\cos s - s\sinh s\right)(2-\tan s) - s\sec^2 s\cos s.$$

Clearly, if we select $\delta = 1$, then we can see that the function $G(\delta, s)$ gets its minimum value, which is

$$\min(G(\delta, s), s \in [0,1]) = G(\delta, 1) = H(s),$$

where

$$H(s) = \left(\left(1-s^2-2s\right)\cos s - s\sinh s\right)(2-\tan s) - s\sec^2 s\cos s.$$

Next, we have the following inequalities:

$$H(0) = 2 > 0 \text{ and } H(1) = -2.8492 < 0.$$

There is indeed a $s_1$, so that $H(s) \geq 0$, for every $s$ in $[0, s_1]$, $s_1$ is the smallest positive root of the equation

$$\left(\left(1-s^2-2s\right)\cos s - s\sinh s\right)(2-\tan s) - s\sec^2 s\cos s.$$

The proof is completed. □

**Theorem 4.** *Let $\omega \in \mathcal{A}$, $g \in \Upsilon_{\mathcal{L}}$ and also suppose that $\omega(\kappa) \ll g(\kappa)$ in $\Omega$. Then, for $|\kappa| \leq s_1$,*

$$\left|\omega'(\kappa)\right| \leq \left|g'(\kappa)\right|,$$

*where $s_1$ is the smallest positive root of the equation*

$$\left(2-5s-3s^2\right)(2-\tan s) - 2s\sec^2 s = 0.$$

**Proof.** If $g \in \Upsilon_{\mathcal{L}}$. Then, a holomorphic function $u(\kappa)$ such that $u(0) = 0$ and $|u(\kappa)| \leq |\kappa|$, so that

$$\frac{g(\kappa)}{\kappa q(\kappa)} = \sqrt{1+u(\kappa)}.$$

After, simple calculations, we attain

$$\frac{\kappa g'(\kappa)}{g(\kappa)} = 1 + \frac{\kappa q'(\kappa)}{q(\kappa)} - \frac{\kappa u'(\kappa)}{2(1+u(\kappa))}.$$

Now, by using (21), we have

$$\begin{aligned} \left|\frac{\kappa u'(\kappa)}{2(1+u(\kappa))}\right| &\leq \frac{|\kappa|(1+|u(\kappa)|)}{2(1-|u(\kappa)|)} \leq \frac{|\kappa|(1+|\kappa|)}{2\left(1-|\kappa|^2\right)} = \frac{|\kappa|}{2(1-|\kappa|)} \\ &\leq \frac{s}{2(1-s)}. \end{aligned} \tag{26}$$

In the light of (26) and Lemma 4, we have

$$\begin{aligned} \left|\frac{g(\kappa)}{g'(\kappa)}\right| &= \frac{|\kappa|}{\left|1+\frac{\kappa q'(\kappa)}{q(\kappa)} - \frac{\kappa u'(\kappa)}{2(1+u(\kappa))}\right|} \\ &\leq \frac{|\kappa|}{1-\left|\frac{\kappa q'(\kappa)}{q(\kappa)}\right| - \left|\frac{\kappa u'(\kappa)}{2(1+u(\kappa))}\right|} \\ &\leq \frac{2s\left(1-s^2\right)(2-\tan s)}{2(1-s^2)(2-\tan s) - 2s\sec^2 s - s(1+s)(2-\tan s)}. \end{aligned} \tag{27}$$

From (2), we can write

$$\omega(\kappa) = \Phi(\kappa) g(\kappa),$$

upon differentiating, we get

$$\begin{aligned} \omega'(\kappa) &= \Phi'(\kappa) g(\kappa) + \Phi(\kappa) g'(\kappa) \\ &= \left(\Phi'(\kappa)\frac{g(\kappa)}{g'(\kappa)} + \Phi(\kappa)\right) g'(\kappa). \end{aligned} \tag{28}$$

Also, Schwarz function satisfies the following condition

$$|\Phi'(\kappa)| \leq \frac{1-|\Phi(\kappa)|^2}{1-|\kappa|^2} = \frac{1-|\Phi(\kappa)|^2}{1-s^2}. \tag{29}$$

Now, using (27) and (29) in (28), we obtain

$$|\omega'(\kappa)| \leq \left(\frac{2s\left(1-|\Phi(\kappa)|^2\right)(2-\tan s)}{2(1-s^2)(2-\tan s) - 2s\sec^2 s - s(1+s)(2-\tan s)} + |\Phi(\kappa)|\right)|g'(\kappa)|,$$

By using the same calculations as in Theorem 4, the desired outcomes are attained. □

**Theorem 5.** *Let $\omega \in \mathcal{A}$, $g \in \Upsilon_{\sec h}$ and also presume that $\omega(\kappa) \ll g(\kappa)$ in $\Omega$. Then, for $|\kappa| \leq s_1$,*

$$|\omega'(\kappa)| \leq |g'(\kappa)|,$$

*where $s_1$ is the smallest positive root of the equation*

$$\left(1-s^2-s(2+\tan s)\right)(2-\tan s) - s\sec^2 s = 0.$$

**Proof.** If $g \in \Upsilon_{\sec h}$, then a holomorphic function $u(\kappa)$is achieved, such that $u(0) = 0$, $|u(\kappa)| \leq |\kappa|$, and

$$\frac{g(\kappa)}{\kappa q(\kappa)} = \sec h(u(\kappa)).$$

After simple calculations, we attain

$$\frac{\kappa g'(\kappa)}{g(\kappa)} = 1 + \frac{\kappa q'(\kappa)}{q(\kappa)} - \kappa u'(\kappa)\tanh(u(\kappa)).$$

Let $u(\kappa) = \mathrm{R}e^{i\theta}$, $|\kappa| = s \leq s_1$. Then,

$$\begin{aligned}\left|\tanh\left(\mathrm{R}e^{i\theta}\right)\right|^2 &= \left(\frac{\sinh(R\cos\theta)\cosh(R\cos\theta)}{\sinh^2(R\cos\theta)+\cos^2(R\sin\theta)}\right)^2 + \left(\frac{\sin(R\sin\theta)\cos(R\sin\theta)}{\sinh^2(R\cos\theta)+\cos^2(R\sin\theta)}\right)^2 \\ &= F(\theta) \text{ (say)}.\end{aligned}$$

Clearly, $F(\theta)$ is even function, so it is enough to consider interval $[0,\pi]$ instead of $[0,\pi]$ for $\theta$. Also, after simple calculations, $F'(\theta) = 0$ has three roots, namely, $\theta = 0, \pi$ and $\frac{\pi}{2}$. We notice that

$$F\left(\frac{\pi}{2}\right) = \tan^2 R \text{ and } F(0) = \tanh^2 R = F(\pi).$$

Moreover, we see that

$$\max\left\{F(0), F\left(\frac{\pi}{2}\right), F(\pi)\right\} = \tan^2 R.$$

Hence,

$$\left|\tanh\left(\mathrm{R}e^{i\theta}\right)\right| \leq \tan R \leq \tan s. \tag{30}$$

In the light of (21), (30), and Lemma 4, we have

$$\begin{aligned}\left|\frac{g(\kappa)}{g'(\kappa)}\right| &= \frac{|\kappa|}{\left|1 + \frac{\kappa q'(\kappa)}{q(\kappa)} - \frac{\kappa u'(\kappa)}{2(1+u(\kappa))}\right|} \\ &\leq \frac{|\kappa|}{1 - \left|\frac{\kappa q'(\kappa)}{q(\kappa)}\right| - |\kappa u'(\kappa)\tanh(u(\kappa))|} \\ &\leq \frac{s\left(1-s^2\right)(2-\tan s)}{(1-s^2)(2-\tan s) - s\sec^2 s - s\tan s(2-\tan s)}.\end{aligned} \tag{31}$$

From (2), we can write

$$\omega(\kappa) = \Phi(\kappa)g(\kappa),$$

and upon differentiating, we receive

$$\begin{aligned}\omega'(\kappa) &= \Phi'(\kappa)g(\kappa) + \Phi(\kappa)g'(\kappa) \\ &= \left(\Phi'(\kappa)\frac{g(\kappa)}{g'(\kappa)} + \Phi(\kappa)\right)g'(\kappa).\end{aligned} \tag{32}$$

Now using (31) and (25) in (32), we obtain

$$|\omega'(\kappa)| \leq \left(\frac{s\left(1-|\Phi(\kappa)|^2\right)(2-\tan s)}{(1-s^2)(2-\tan s) - s\sec^2 s - s\tan s(2-\tan s)} + |\Phi(\kappa)|\right)|g'(\kappa)|.$$

The same calculations as in Theorem 4 are used to produce the required outcomes. □

## 5. Conclusions

In this article, we, first, derived sharp coefficient estimates, sharp Fekete–Szegö, and Zalcman functionals for the subfamily of holomorphic functions associated with tangent functions. Furthermore, we looked into the majorization results for a some families of holomorphic functions that are associated with various shapes domains. It is possible to extend these results into more subfamiles, including, for example, for the class of meromorphic functions and the families of harmonic functions. Furthermore, one may attempt to produce results for certain subclasses of $q$-starlike functions: for details, see [31–41].

**Author Contributions:** All authors equally contributed in this article. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work was supported by (1) The National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. NRF-2022R1A2C2004874). (2) The Korea Institute of Energy Technology Evaluation and Planning (KETEP) and the Ministry of Trade, Industry & Energy (MOTIE) of the Republic of Korea (No. 20214000000280). (3) The Researchers Supporting Project Number (RSP2023R440), King Saud University, Riyadh, Saudi Arabia.

**Data Availability Statement:** No data is used.

**Acknowledgments:** The authors would like to express their gratitude to the anonymous referees for many valuable suggestions regarding a previous version of this paper. This work was supported by (1) The National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. NRF-2022R1A2C2004874). (2) The Korea Institute of Energy Technology Evaluation and Planning (KETEP) and the Ministry of Trade, Industry & Energy (MOTIE) of the Republic of Korea (No. 20214000000280). (3) The Researchers Supporting Project Number (RSP2023R440), King Saud University, Riyadh, Saudi Arabia.

**Conflicts of Interest:** The authors declare no conflict of interest.

**Sample Availability:** Samples of the compounds are available from the authors.

## References

1. Bieberbach, L. Uber die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln. *Sitzungsberichte Preuss. Akad. Wiss.* **1916**, *138*, 940–955.
2. De Branges, L. A proof of the Bieberbach conjecture. *Acta Math.* **1985**, *154*, 137–152. [CrossRef]
3. Roberston, M.S. Quasi-subordination and coefficient conjectures. *Bull. Am. Math. Soc.* **1970**, *76*, 1–9.
4. MacGregor, T.H. Majorization by univalent functions. *Duke Math. J.* **1967**, *34*, 95–102. [CrossRef]
5. Tang, H.; Li, S.H.; Deng, G.T. Majorization properties for a new subclass of $\theta$-spiral functions of order $\gamma$. *Math. Slovaca* **2014**, *64*, 39–50. [CrossRef]
6. Tang, H.; Deng, G.T. Majorization problems for certain classes of multivalent analytic functions related with the Srivastava-Khairnar-More operator and exponential function. *Filomat* **2018**, *32*, 5319–5328. [CrossRef]
7. Tang, H.; Deng, G.T.; Li, S.H. Majorization properties for certain classes of analytic functions involving a generalized differential operator. *J. Math. Res. Appl.* **2013**, *33*, 578–586.
8. Tang, H.; Deng, G.T. Majorization problems for two subclasses of analytic functions connected with the Liu-Owa integral operator and exponential function. *J. Inequal. Appl.* **2018**, *2018*, 1–11. [CrossRef]
9. Tang, H.; Deng, G.T. Majorization problems for some subclasses of starlike functions. *J. Math. Res. Appl.* **2019**, *39*, 153–159.
10. Ma, W.C.; Minda, D. A unified treatment of some special classes of univalent functions. In *Proceedings of the Conference on Complex Analysis*; Conference Processing Lecture Notes Anal I; International Press Inc.: Somerville, MA, USA, 1992 ; p. 157169.
11. Janowski, W. Extremal problems for a family of functions with positive real part and for some related families. *Ann. Pol. Math.* **1971**, *23*, 159–177. [CrossRef]
12. Sokól, J.; Kanas, S. Radius of convexity of some subclasses of strongly starlike functions. *Zesz. Nauk. Politech. Rzeszowskiej Mat.* **1996**, *19*, 101–105.
13. Arora, K.; Kumar, S.S. Starlike functions associated with a petal shaped domain. *Bull. Korean Math. Soc.* **2022**, *59*, 993–1010.
14. Mendiratta, R.; Nagpal, S.; Ravichandran, V. On a subclass of strongly starlike functions associated with exponential function. *Bull. Malays. Math. Sci. Soc.* **2015**, *38*, 365–386. [CrossRef]
15. Cho, N.E.; Kumar, V.; Kumar, S.S.; Ravichandran, V. Radius problems for starlike functions associated with the sine function. *Bull. Iran. Math. Soc.* **2019**, *45*, 213–232. [CrossRef]
16. Bano, K.; Raza, M. Starlike Functions Associated with Cosine Functions. *Bull. Iran. Math. Soc.* **2021**, *47*, 1513–1532. [CrossRef]
17. Al-Shbeil, I.; Saliu, A.; C ataş, A.; Malik, S.N.; Oladejo, S.O. Some Geometrical Results Associated with Secant Hyperbolic Functions. *Mathematics* **2022**, *10*, 2697. [CrossRef]
18. Keogh, F.R.; Merkes, E.P. A coefficient inequality for certain classes of analytic functions. *Proc. Am. Math. Soc.* **1969**, *20*, 8–12. [CrossRef]
19. Sharma, K.; Jain, N.K.; Ravichandran, V. Starlike functions associated with a cardioid. *Afr. Math.* **2016**, *27*, 923–939. [CrossRef]
20. Arif, M.; Raza, M.; Tang, H.; Hussain, S.; Khan, H. Hankel determinant of order three for familiar subsets of analytic functions related with sine function. *Open Math.* **2019**, *17*, 1615–1630. [CrossRef]

21. Ravichandran, V.; Verma, S. Bound for the fifth coefficient of certain starlike functions. *Comptes Rendus Math.* **2015**, *353*, 505–510. [CrossRef]
22. Krushkal, S.L. Proof of the Zalcman conjecture for initial coefficients. *Georgian Math. J.* **2010**, *17*, 663–681; Corrigendum *Georgian Math. J.* **2012**, *19*, 777. [CrossRef]
23. Tomas, D.K.; Tuneski, N.; Vasudevarao, A. *Univalent Functions. A Primer*; Walter de Gruyter GmbH & Co KG: Berlin, Germany; Boston, MA, USA, 2018.
24. Hayman, W.K.; Lingham, E.F. *Research Problems in Function Theory—Fiftieth Aniversary Edition, Problems Books in Mathematics*; Springer: Berlin/Heidelberg, Germany, 2019.
25. Bansal, D.; Sokól, J. Zalcman conjecture for some subclass of analytic functions. *J. Fract. Calc. Appl.* **2017**, *8*, 1–5.
26. Brown, J.E.; Tsao, A. On the Zalcman conjecture for starlike and typically real functions. *Math. Z.* **1986**, *191*, 467–474. [CrossRef]
27. Ma, W.C. The Zalcman conjecture for close-to-convex functions. *Proc. Am. Math. Soc.* **1988**, *104*, 741–744. [CrossRef]
28. Allu, V.; Pandey, A. Proof of the generalized Zalcman conjecture for initial coefficients of univalent functions. *arXiv* **2022**, arXiv:2209.11231.
29. Efraimidis, I.; Vukotić, D. Application of Livingston-type inequalities to the generalized Zalcman functional. *Math. Nachr.* **2018**, *291*, 1502–1513. [CrossRef]
30. Obradović, M.; Tuneski, N. Certain provided a proof of this conjecture for univalent functions with real coefficients in. *arXiv* **2021**, arXiv:2112.15449v1.
31. Lyu, W.; Wang, Z. Global classical solutions for a class of reaction-diffusion system with density-suppressed motility. *Electron. Res. Arch.* **2022**, *30*, 995–1015. [CrossRef]
32. Lyu, W.; Wang, Z. Logistic damping effect in chemotaxis models with density-suppressed motility. *Adv. Nonlinear Anal.* **2023**, *12*, 1–12. [CrossRef]
33. Guo, C.; Hu, J. Fixed-Time Stabilization of High-Order Uncertain Nonlinear Systems: Output Feedback Control Design and Settling Time Analysis. *J. Syst. Sci. Complex.* **2023**, *2023*, 12–26. [CrossRef]
34. Chen, D.; Wang, Q.; Li, Y.; Li, Y.; Zhou, H.; Fan, Y. A general linear free energy relationship for predicting partition. *Chemosphere* **2020**, *247*, 125869. [CrossRef] [PubMed]
35. Cao, J.; Zhou, H.; Arjika, S. Generalized $q$ -difference equations for (q, c)-hypergeometric polynomials and some applications. *Ramanujan J.* **2023**, *60*, 1033–1067. [CrossRef]
36. Cao, J.; Huang, J. Mohammed Fadel and Sama Arjika, A Review of $q$-Difference Equations for Al-Salam-Carlitz Polynomials and Applications to $U(n+1)$ Type Generating Functions and Ramanujan's Integrals. *Mathematics* **2023**, *11*, 1655. [CrossRef]
37. Cao, J.; Qi, F.; Du, W. Closed-Form Formulas for the nth Derivative of the Power-Exponential Function $x^x$. *Symmetry* **2023**, *15*, 323. [CrossRef]
38. Khan, B.; Liu, Z.-G.; Shaba, T.G.; Araci, S.; Khan, N.; Khan, M.G. Applications of-Derivative Operator to the Subclass of Bi-Univalent Functions Involving $q$-Chebyshev Polynomials. *J. Math.* **2022**, *2022*, 8162182. [CrossRef]
39. Khan, B.; Aldawish, I.; Araci, S.; Khan, M.G. Third Hankel determinant for the logarithmic coefficients of starlike functions associated with sine function. *Fractal Fract.* **2022**, *6*, 261. [CrossRef]
40. Hu, Q.-X.; Srivastava, H.M.; Ahmad, B.; Khan, N.; Khan, M.G.; Mashwani, W.K.; Khan, B. A subclass of multivalent Janowski type $q$ -starlike functions and its consequences. *Symmetry* **2021**, *13*, 1275. [CrossRef]
41. Mateljevic, M.; Mutavdžić, N.; Örnek, B.N. Note on some classes of holomorphic functions related to Jack's and Schwarz's lemma. *Appl. Anal. Discret. Math.* **2022**, *16*, 111–131. [CrossRef]