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Coefficient Inequalities of *q*-Bi-Univalent Mappings Associated with *q*-Hyperbolic Tangent Function

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Abstract: The present study introduces a new family of analytic functions by utilizing the q-derivative operator and the q-version of the hyperbolic tangent function. We find certain inequalities, including the coefficient bounds, second Hankel determinants, and Fekete–Szegö inequalities, for this novel family of bi-univalent functions. It is worthy of note that almost all the results are sharp, and their corresponding extremal functions are presented. In addition, some special cases are demonstrated to show the validity of our findings.

Keywords: bi-univalent functions; q-fractional derivative; q-analogue of the hyperbolic tangent function; Hankel determinant; bounded turning functions; Fekete–Szegö inequality

MSC: 30C45; 30C50; 30C80; 11B65; 47B38

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1. Introduction and Preliminaries

In 1915, Alexander [1] introduced the first integral operator, this discovery played a crucial role in the examination of analytical functions. Since then, the main goal of current discovery in complex analysis (Geometric Function Theory) has revolved around this area, encompassing fractional derivative operators and derivatives that are often combined in various ways [2,3]. Recently published research, exemplified by [4], highlights the significance of integral fractional and differential operators in research.

Exciting advancements in the field of analytical functions and fractional calculus have emerged from different perspectives, including quantum calculus, which have proved useful in diverse areas of physics and mathematics. In a comprehensive review and survey study, Srivastava [5] highlights the intriguing real-world ramifications of utilizing these operator applications. Their uses in the axially symmetric potential theory, automated control, scattering theory, signal analysis, quantum mechanics, and absorption of radio waves in the ionospheric space environment and aeronomy [6,7] are also well known.

The versatility of q-calculus is evidenced by its numerous applications across disciplines such as quantum group theory, analytic number theory, special polynomials, fractional calculus, and numerical analysis. As a result, the expansive domain of fractional calculus has captured the attention of both mathematicians and physicists. The combination of analytical function theory with fractional calculus has led to the formulation of various mathematical models that employ fractional differential equations. These equations often compete with nonlinear differential equations as viable models in many contexts [8–11].

Consider an analytic function denoted by $f(\vartheta)$, which is defined within the open unit disk $\{E := \vartheta \in \mathbb{C} : |\vartheta| < 1\}$. Such functions belong to a class denoted as $\mathcal{A}$. Any function f belonging to $\mathcal{A}$ can be represented by the following series form:

$$f(\vartheta) = \vartheta + \sum_{k=2}^{\infty} a_k \vartheta^k. \tag{1}$$

The symbol $\mathcal{S}$ denotes the collection of univalent functions that adhere to the normalization conditions

$$f(0) = 0 \text{ and } f'(0) = 1. \tag{2}$$

An analytic function w that satisfies the conditions $|w(\vartheta)| < 1$ and $w(0) = 0$ in E is referred to as the Schwarz function. Letting $f, g \in \mathcal{A}$, we say f is subordinate to g, written as $f(\vartheta) \prec g(\vartheta)$, if and only if there exists a Schwarz function w such that $f(\vartheta) = g(w(\vartheta))$ for all $z \in E$.

The class $\mathcal{P}$ refers to the Carathéodory functions described by Miller [12], which meet the conditions: $p(0) = 1$, and $\operatorname{Re} p(\vartheta) > 0$ for all $\vartheta \in E$. These functions are known as the Carathéodory function.

Every polynomial function $p(\vartheta)$ belonging to the set $\mathcal{P}$ can be accurately represented using a Taylor series expansion in the specific format of

$$p(\vartheta) = 1 + c_1\vartheta + c_2\vartheta^2 + c_3\vartheta^3 + \cdots. \tag{3}$$

In other words, we say $p \in \mathcal{P}$, if and only if

$$p(\vartheta) \prec \frac{1+\vartheta}{1-\vartheta}, \quad \vartheta \in E.$$

In the realm of geometric function theory (GFT), significant novel subclasses of analytic functions have been constructed and explored, with a strong reliance on the principles of q-calculus. Credited with inaugurating q-calculus in 1909, Jackson [13,14] introduced the initial definitions of q-integrals and q-derivatives, marking a pivotal moment for this mathematical discipline. Jackson's contributions extend beyond these foundational concepts. In addition to suggesting the q-calculus operator and the q-difference operator ($\mathfrak{D}_q$), many q-special functions have been put forward. The q-calculus finds applications in diverse mathematical and scientific domains, including number theory, fundamental hypergeometric functions, physics, relativity, cybernetics, data analysis, and combinatorial mathematics.

Definition 1 ([13,14])**.** *The q-fractional derivative (denoted by $\mathfrak{D}_q f(\vartheta)$) of the function $f \in \mathcal{A}$ is defined as*

$$\mathfrak{D}_q f(\vartheta) = \begin{cases} \frac{f(\vartheta)-f(q\vartheta)}{(1-q)\vartheta} & \text{for } \vartheta \neq 0 \\ f'(0) & \text{for } \vartheta = 0. \end{cases} \tag{4}$$

The q-fractional derivative operator is applicable as the value of q approaches 1. Remarkably, as q approaches 1, the $\mathfrak{D}_q$ reduces to the classical derivative. For more details and recent applications of the q-fractional derivative, we refer the readers to [15–21] and the references therein.

Even though function theory was first introduced in 1851, Bieberbach's [22] conjecture in 1916 unveiled this topic and provided a fresh line of inquiry. De-Branges [23] validated the Bieberbach conjecture in 1985. A number of renowned scholars have made significant findings in this realm of mathematics, uncovering several novel subsets within the class $\mathcal{S}$ of normalized univalent functions that are linked with diverse geometrical characterizations.

In 1992, Ma and Minda [24] authored a notable and influential paper, which presented a remarkable contribution to the field, introducing a comprehensive definition for the subclasses of univalent functions as follows:

$$\mathcal{S}^*(\zeta) = \left\{ f(\vartheta) \in \mathcal{A} : \frac{zf'(\vartheta)}{f(\vartheta)} \prec \zeta(\vartheta) \right\}, \tag{5}$$

where ζ is an analytic function with the requirements $\zeta(0) > 0$, $\Re(\zeta(\vartheta)) > 0$ in E and $\zeta(E)$ is symmetric with respect to the real axis and starlike with respect to $\zeta(0)$. Supposing we take $\zeta(\vartheta) = \frac{1+\vartheta}{1-\vartheta}$ in (5), we then have the family of starlike functions, which is given as follows:

$$\mathcal{S}^* = \left\{ f(\vartheta) \in \mathcal{A} : \frac{zf'(\vartheta)}{f(\vartheta)} \prec \frac{1+\vartheta}{1-\vartheta}, \quad \vartheta \in E \right\}.$$

Recently, a class of starlike functions,

$$\mathcal{S}_s(n,q) = \left\{ f(\vartheta) \in \mathcal{S} : \frac{\mathfrak{D}_q^n f(\vartheta)}{f(\vartheta)} \prec 1 + \tanh(q\vartheta), \quad \vartheta \in E \right\}, \tag{6}$$

was introduced and studied by Swarup in [25].

Each function f belonging to the set $\mathcal{S}$ and defined by Equation (1) possesses a corresponding inverse function denoted as f^{-1}. This inverse function is determined through the utilization of the q-differential operator and q-version of the hyperbolic tangent function.

$$f^{-1}(f(\vartheta)) = \vartheta, \ \vartheta \in E, f^{-1}(f(w)) = w, w \in E_{t_0} = \{w \in \mathbb{C} : |w| < t_0(f)\}, \ 1/4 \le t_0(f)$$

and

$$f^{-1}(w) = l(w) = w + M_2 w^2 + M_3 w^3 + M_4 w^4 + \cdots, \ , \ w \in E_{t_0}, \tag{7}$$

where

$$M_2 = -a_2, \ M_3 = 2a_2^2 - a_3, \ M_4 = -5a_2^3 + 5a_2 a_3 - a_4.$$

A widely accepted truth is that a function $f(\vartheta) \in \mathcal{A}$ is classified as a bi-univalent function in E if both $f(\vartheta)$ and its inverse function $f^{-1}(w)$ are separately univalent in E and E_{t_0}, respectively. The collection of all these bi-univalent functions in E is denoted as Σ and has undergone thorough examination, accompanied by historical context and examples given in [26,27].

In the year 2016, Srivastava et al. [28] introduced a category of analytic and bi-univalent functions, defined in the subsequent manner:

$$f'(z) + \frac{1+e^{i\alpha}}{2} z f''(z) \prec \phi(z)$$

and

$$g'(w) + \frac{1+e^{i\alpha}}{2} w g''(w) \prec \phi(w).$$

and obtained some interesting results. In 2018, Yousef et al. [29] introduced a subclass of analytic and bi-univalent functions by means of Chebyshev polynomials, which is defined below:

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) + \mu z f''(z) \prec \frac{1}{1 - 2tz + z^2}$$

and

$$(1-\lambda)\frac{f(w)}{w} + \lambda f'(w) + \mu w f''(w) \prec \frac{1}{1 - 2tw + w^2}.$$

Mahzoon and Kargar [30] (2020) investigated the class

$$\Lambda(\delta,\gamma)=\left\{\Re\left(f'(z)+\frac{1+e^{i\delta}}{2}zf''(z)\right)>\gamma,\ \delta\in(-\pi,\pi],\gamma\in[0,1)\right\}$$

and some relevant results are presented in the article. In 2022, Lasode and Opoola [31] presented a category using the q-derivative, which is denoted as $\mathcal{E}_q(\beta,\delta)$. This category comprised analytic and univalent functions and was defined as follows:

$$\Lambda(\beta,\delta)=\left\{\Re\left(\mathcal{D}_qf(z)+\frac{1+e^{i\beta}}{2}z\mathcal{D}_q^2f(z)\right)>\delta,\ \beta\in(-\pi,\pi],\delta\in[0,1)\right\}.$$

In 2023, a recent development by Swarup [25] involves the utilization of the q-version of the hyperbolic tangent function along with a Salagean q-differential operator. This innovation led to the introduction of a fresh category of q-starlike functions, defined in the subsequent manner:

$$\mathcal{S}_s^*(l,q)=\left\{\frac{\mathcal{S}_q^lf(z)}{f(z)}\prec 1+\tanh(qz)\right\}.$$

The inspiration for introducing this new category of analytic and bi-univalent functions originated from the previously mentioned classes. Notably, the relationship between bi-univalent functions and the q-version of the hyperbolic tangent function remains largely unexplored in existing research. Previous studies have predominantly concentrated on analyzing the interplay between analytic functions and the q-version of the hyperbolic tangent function, as evidenced in the work of Swarup in 2023.

Furthermore, we were motivated to extend the analytic class of functions related to the q-version of the hyperbolic tangent function to bi-univalent functions since it has not been in the literature so far, which will also open up more research problems in the area of bi-univalent and analytic functions by generalizing the following newly defined class with some q-operators in Geometric Function Theory and also look into some other interesting properties.

The q-bi-univalent functions are used to study many interesting properties of the holomorphic functions. The main driving force behind our current research is the discovery of several distinctive and advantageous applications for q-derivatives in GFT (Geometric Function Theory). All those areas of applicable mathematics in which we deal with the complex transformations, such as robotics, computer added design (CAD) and computational geometry, demonstrate the applicability of q-bi-univalent functions. Specifically, we focus on exploring the bi-univalent function linked with the q-analogue of the tanh function, an area that has not yet been explored in the existing literature. In particular, the inspiration behind introducing this innovative class stems from the referenced articles [28–32]. Mustafa and Semra [32] introduced and studied a subclass of bi-univalent functions on the open unit disk in the complex plane. They also investigated similar problems as those studied in [31]. They also investigated the upper bound estimate for the second Hankel determinant and Fekete–Szegö inequality for the function belonging to this class. However, none of them or any other researcher has ever explored this dimension of the research. Furthermore, in a bid to push forward the ideas initially presented by Swarup in [25], we present a fresh subcategory involving analytic and bi-univalent functions. These functions are defined through q-derivatives and their connection to the q-analogue of the tanh function.

Definition 2. *For $f\in\mathcal{OB}^{\theta}_{\tau,\Sigma}(q)$, suppose the following conditions are met:*

$$(1-\tau)\left(\frac{f(\vartheta)}{\vartheta}\right)+\tau\mathfrak{D}_qf(\vartheta)+\frac{1+e^{i\theta}}{2}\vartheta\mathfrak{D}_q^2f(\vartheta)\prec\tanh(\vartheta q)+1$$

and

$$(1-\tau)\left(\frac{l(w)}{w}\right)+\tau\mathfrak{D}_q l(w)+\frac{1+e^{i\theta}}{2}w\mathfrak{D}_q^2 l(w)\prec \tanh(wq)+1$$

where $\tau\geq 1$, $\theta\in(-\pi,\pi]$, $\vartheta,w\in E$ *and* $l(w)$ *is given in* (7).

We consider a function $f_n(\vartheta)$ such that

$$(1-\tau)\left(\frac{f_n(\vartheta)}{\vartheta}\right)+\tau\mathfrak{D}_q f_n(\vartheta)+\frac{1+e^{i\theta}}{2}\vartheta\mathfrak{D}_q^2 f_n(\vartheta)=1+\tanh(q\vartheta^n) \tag{8}$$

and

$$(1-\tau)\left(\frac{l_n(w)}{w}\right)+\tau\mathfrak{D}_q l_n(w)+\frac{1+e^{i\theta}}{2}\vartheta\mathfrak{D}_q^2 l_n(w)=1+\tanh(qw^n),\quad (n=1,2,3). \tag{9}$$

Thus, we have

$$f_1(\vartheta)=\vartheta+\frac{2q}{2+2\tau([2]_q-1)+(e^{i\theta}+1)[2]_q}\vartheta^2-\frac{2q^3}{3[2+2\tau([4]_q-1)+(e^{i\theta}+1)[3]_q[4]_q]}\vartheta^4+\cdots, \tag{10}$$

$$f_2(\vartheta)=\vartheta+\frac{2q}{2+2\tau([3]_q-1)+(e^{i\theta}+1)[2]_q[3]_q}\vartheta^3-\frac{2q^3}{3[2+2\tau([7]_q-1)+(e^{i\theta}+1)[7]_q[6]_q]}\vartheta^7+\cdots, \tag{11}$$

$$f_3(\vartheta)=\vartheta+\frac{2q}{2+2\tau([4]_q-1)+(e^{i\theta}+1)[4]_q[3]_q}\vartheta^4-\frac{2q^3}{3[2+2\tau([10]_q-1)+(e^{i\theta}+1)[10]_q[9]_q]}\vartheta^{10}+\cdots. \tag{12}$$

Remark 1. *The class* $\mathcal{OB}^{\theta}_{\tau,\Sigma}(q)$ *is not empty. At least, the functions defined by* (10)–(12) *are univalent due to being extremal functions of the class of univalent functions. They all exist in the class* $\mathcal{OB}^{\theta}_{\tau,\Sigma}(q)$*. To show this, we proceed as follows.*

We start with $f_1(\vartheta)$*. Putting* $n=1$ *in* (8) *and* (9)*, we have the following:*

$$\frac{(1-\tau)f_1(\vartheta)}{\vartheta}=(1-\tau)+(1-\tau)\frac{2q}{\Theta_1}\vartheta-(1-\tau)\frac{2q^3}{3\Theta_3}\vartheta^3+\cdots, \tag{13}$$

$$\tau\mathfrak{D}_q f_1(\vartheta)=\tau+\tau\frac{2q[2]_q}{\Theta_1}\vartheta-\tau\frac{2q^3[3]_q}{3\Theta_3}\vartheta^3+\cdots, \tag{14}$$

$$\frac{1+e^{i\theta}}{2}\vartheta\mathfrak{D}_q^2 f_1(\vartheta)=\frac{q[2]_q(1+e^{i\theta})}{\Theta_1}\vartheta-\frac{q^3[3]_q[2]_q(1+e^{i\theta})}{3\Theta_3}\vartheta^3+\cdots. \tag{15}$$

where $\Theta_1=2+2\tau([2]_q-1)+(e^{i\theta}+1)[2]_q$ *and* $\Theta_3=2+2\tau([4]_q-1)+(e^{i\theta}+1)[4]_q[3]_q$.

Combining (13)–(15)*, we have*

$$(1-\tau)\left(\frac{f_1(\vartheta)}{\vartheta}\right)+\tau\mathfrak{D}_q f_1(\vartheta)+\frac{1+e^{i\theta}}{2}\vartheta\mathfrak{D}_q^2 f_1(\vartheta)=1+q\vartheta-\frac{q^3}{3}\vartheta^3+\cdots, \tag{16}$$

which gives us the series for the L.H.S of (8) *when* $n=1$.

Now, for the R.H.S of (8) *when* $n=1$*, we have*

$$1+\tanh(q\vartheta)=1+q\vartheta-\frac{q^3}{3}\vartheta^3+\cdots. \tag{17}$$

Comparing (16) *and* (17)*, we can clearly see that they are equal; therefore, we can say that* $f_1(\vartheta)$ *satisfies the first part of Definition* 2.

Now, we check if $f_1(\vartheta)$ satisfies the second part of the definition. Hence, we are going to find the inverse of (16) *to get the L.H.S of the second part of Definition* 2*, which is $l(w)$. Let*

$$w = (1-\tau)\left(\frac{f_1(\vartheta)}{\vartheta}\right) + \tau\mathfrak{D}_q f_1(\vartheta) + \frac{1+e^{i\theta}}{2}\vartheta\mathfrak{D}_q^2 f_1(\vartheta), \tag{18}$$

which implies

$$\vartheta = (1-\tau)\left(\frac{l_1(w)}{w}\right) + \tau\mathfrak{D}_q l_1(w) + \frac{1+e^{i\theta}}{2}w\mathfrak{D}_q^2 l_1(w) = w + A_2w^2 + A_3w^3 + A_4w^4 + \cdots \tag{19}$$

Now, substitute (19) *into* (18)*, which gives*

$$w = 1 + q[w + A_2w^2 + A_3w^3 + A_4w^4 + \cdots] - \frac{q^3}{3}[w + A_2w^2 + A_3w^3 + A_4w^4 + \cdots]^3 + \cdots \tag{20}$$

Substituting A_2 and A_3 into (19)*, we have*

$$\vartheta = (1-\tau)\left(\frac{l_1(w)}{w}\right) + \tau\mathfrak{D}_q l_1(w) + \frac{1+e^{i\theta}}{2}w\mathfrak{D}_q^2 l_1(w) = w + \frac{q^2}{3}w^3 + \cdots \tag{21}$$

for the R.H.S of (9) *when $n = 1$. The inverse of $1 + \tanh(q\vartheta)$ follows the same solving process as the L.H.S of* (9)*, since* (16) *and* (17) *are equal. That is*

$$\vartheta = [1 + \tanh(q\vartheta)]^{-1} = w + \frac{q^2}{3}w^3 + \cdots. \tag{22}$$

Comparing (21) *and* (22)*, we deduce that both sides are equal. Therefore, by applying the same process to $f_2(\vartheta)$ and $f_3(\vartheta)$, which gives more degrees, we conclude that they also satisfy both equations in Definition* 2*.*

Now, we can conclude that the extremal functions given in (10)*–*(12) *show that our defined class of analytic and bi-univalent function is not empty and also satisfies both the first and the second part of our Definition* 2 *related to $f(\vartheta)$ and $l(w)$.*

Remark 2. *The class $\mathcal{OB}_\Sigma^\theta(q)$, which satisfies the following criterion, is obtained by setting $\tau = 1$ in the preceding definition.*

$$f \in \mathcal{OB}_\Sigma^\theta(q) \Leftrightarrow \mathfrak{D}_q f(\vartheta) + \frac{1+e^{i\theta}}{2}\vartheta\mathfrak{D}_q^2 f(\vartheta) \prec \tanh(\vartheta q) + 1$$

and

$$\mathfrak{D}_q l(w) + \frac{1+e^{i\theta}}{2}w\mathfrak{D}_q^2 l(w) \prec \tanh(wq) + 1$$

where $\vartheta, w \in E$ and $l(w)$ is given in (7)*.*

Remark 3. *The class $\mathcal{OB}_\Sigma(q)$, which satisfies the following criterion, is obtained by setting $\tau = 1$ and $\theta = \pi$ in the preceding definition.*

$$f \in \mathcal{OB}_\Sigma(q) \Leftrightarrow \mathfrak{D}_q f(\vartheta) \prec \tanh(\vartheta q) + 1$$

and

$$\mathfrak{D}_q l(w) \prec \tanh(wq) + 1$$

where $\vartheta, w \in E$ and $l(w)$ is given in (7)*.*

Remark 4. *The class $\mathcal{OB}_\Sigma$, which satisfies the following criterion, is obtained by setting $\tau = 1$, $\theta = \pi$ and $q \longrightarrow 1$ in the preceding definition.*

$$f \in \mathcal{OB}_\Sigma \Leftrightarrow f'(\vartheta) \prec \tanh(\vartheta) + 1$$

and

$$l'(w) \prec \tanh(w) + 1$$

where $\vartheta, w \in E$ and $l(w)$ is given in (7).

This lemma plays a crucial role in establishing the validity of our main results.

Lemma 1 ([33,34])**.** *Let $\mathcal{P}$ represent the collection of all analytic functions $s(\vartheta)$ given in (3) such that $s(\vartheta)$ is analytic in the region $\mathcal{E}$ and has a real part greater than zero for all ϑ in that region. Additionally, the function satisfies the condition $s(0) = 1$. We then have the following mathematical statement:*

"For any natural number k, the absolute value of the coefficient s_k in the above representation of $s(\vartheta)$ is always less than or equal to 2. Furthermore, this inequality is the best possible choice for any value of k".

Lemma 2 ([33,34])**.** *Let $\mathcal{P}$ represent the collection of all analytic functions s given in (3) subject to the conditions: $\Re(s(\vartheta)) > 0, z \in \mathcal{E}$, and $s(0) = 1$.*

Then, we have the following two equations:

$$2s_2 = s_1^2 + (4 - s_1^2)e$$

$$4s_3 = s_1^3 + 2(4 - s_1^2)s_1 e - (4 - s_1^2)s_1 e^2 + 2(4 - s_1^2)(1 - |e|^2)\vartheta,$$

Here, e and ϑ are complex numbers satisfying $|e| \leq 1$ and $|\vartheta| \leq 1$.

Lemma 3 ([34,35])**.** *The Toeplitz determinants are satisfied if and only if*

$$\mathcal{H}_k = \begin{vmatrix} 2 & s_1 & s_2 & \dots & s_k \\ s_{-1} & 2 & s_1 & \dots & s_{k-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ s_{-k} & s_{-k+1} & s_{-k+2} & \dots & 2 \end{vmatrix}, \ k \in \mathbb{N}. \tag{23}$$

Given that $s_{-k} = \bar{_k}$, each of the power series composed of non-negative terms mentioned in Equation (3) converges within the region $\mathcal{E}$ to a function denoted as s, which belongs to the class $\mathcal{P}$. With the exception of the specific case given by

$$s(z) = \sum_{k=1}^{k} \rho_k s_0(s^{ix_k z}), \ \rho_k > 0, \ x_k \text{ real}.$$

If all pairs of variables, represented by x_k and x_j, are distinct (meaning x_k is not equal to x_j for any k and j combination), then all other situations will have positive values. Furthermore, in this specific situation, $\mathcal{H}_k$ will be greater than zero when k is less than $n - 1$, and $\mathcal{H}_k$ will be equal to zero when k is greater than or equal to n.

Notation 1. *Given that s belongs to the set $\mathcal{P}$, we can affirm that $\mathcal{H}_k$ is non-negative, and it holds true that $s{-}1 = \bar{s_1} \geq 0$, as mentioned in Lemma 3. This results in $\mathcal{H}_1 = \begin{vmatrix} 2 & s_1 \\ s_1 & 2 \end{vmatrix} \geq 0$ and let s_1 be a non-negative value such that $s_1 = \bar{s1} = s{-}1 \geq 0$. Consequently, we can deduce that $4 - s_1^2 \geq 0$, and s_1 falls within the range of $[0, 2]$. Based on these observations, we will proceed with the assumption that $|4 - s_1^2| = |4 - |s_1|^2| = 4 - |s_1|^2$ for s_1, which is the first coefficient in (3).*

2. Coefficients Bound Estimates

Theorem 1. *Let $f \in \mathcal{OB}^{\theta}_{\tau,\Sigma}(q)$. Then,*

$$|a_2| \leq \frac{2q}{\Theta_1(\tau,\theta,q)}, \quad |a_3| \leq \frac{2q}{\Theta_2(\tau,\theta,q)}, \quad |a_4| \leq \frac{2q}{\Theta_3(\tau,\theta,q)}$$

where

$$\begin{aligned}
\Theta_1(\tau,\theta,q) &= 2\tau([2]_q - 1) + (e^{i\theta}+1)[2]_q + 2,\\
\Theta_2(\tau,\theta,q) &= 2\tau([3]_q - 1) + (e^{i\theta}+1)[2]_q[3]_q + 2,\\
\Theta_3(\tau,\theta,q) &= 2\tau([4]_q - 1) + (e^{i\theta}+1)[3]_q[4]_q + 2.
\end{aligned}$$

The result obtained here are sharp.

Proof. Suppose f belongs to the class $\mathcal{OB}^{\theta}_{\tau,\Sigma}(q)$. In this case, there exist analytic functions m and v defined on E such that $m(0) = 0 = v(0)$, $|m(\vartheta)| \leq 1$, and $|v(w)| \leq 1$, meets the aforementioned requirements:

$$(1-\tau)\left(\frac{f(\vartheta)}{\vartheta}\right) + \tau\mathfrak{D}_q f(\vartheta) + \frac{1+e^{i\theta}}{2}\vartheta\mathfrak{D}_q^2 f(\vartheta) = \tanh(m(\vartheta)\cdot q) + 1 \tag{24}$$

and

$$(1-\tau)\left(\frac{l(w)}{w}\right) + \tau\mathfrak{D}_q l(w) + \frac{1+e^{i\theta}}{2}w\mathfrak{D}_q^2 l(w) = \tanh(v(w)\cdot q) + 1. \tag{25}$$

The functions $s, r \in \mathcal{P}$ are defined as follows:

$$s(\vartheta) = \frac{1+m(\vartheta)}{1-m(\vartheta)} = 1 + \sum_{k=1}^{\infty} s_k\vartheta^k, \quad \vartheta \in E.$$

and

$$r(w) = \frac{1+v(w)}{1-v(w)} = 1 + \sum_{k=1}^{\infty} r_k w^k, \quad w \in E.$$

Upon replacing the expressions for the functions $m(\vartheta)$ and $v(w)$ in Equations (24) and (25), we obtain

$$\begin{aligned}
(1-\tau)\left(\frac{f(\vartheta)}{\vartheta}\right) + \tau\mathfrak{D}_q f(\vartheta) + \frac{1+e^{i\theta}}{2}\vartheta\mathfrak{D}_q^2 f(\vartheta) &= 1 + \frac{qs_1}{2}\vartheta + \left(\frac{qs_2}{2} - \frac{qs_1^2}{4}\right)\vartheta^2\\
&+ \left(\frac{q}{2}s_3 - \frac{q}{2}s_1s_2 - \frac{(2q^2-3)}{24}s_1^3\right)\vartheta^3 + \cdots
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
(1-\tau)\left(\frac{l(w)}{w}\right) + \tau\mathfrak{D}_q l(w) + \frac{1+e^{i\theta}}{2}w\mathfrak{D}_q^2 l(w) &= 1 + \frac{qr_1}{2}w + \left(\frac{qr_2}{2} - \frac{qr_1^2}{4}\right)w^2\\
&+ \left(\frac{q}{2}r_3 - \frac{q}{2}r_1r_2 - \frac{(2q^2-3)}{24}r_1^3\right)w^3 + \cdots.
\end{aligned} \tag{27}$$

The Equations (28) and (29) produce expressions for variables of the same degree, namely a_2, a_3, and a_4, after performing specified operations and simplifications on their left-hand sides.

$$\frac{\Theta_1(\tau,\theta,q)}{2}a_2 = \frac{q}{2}s_1, \tag{28}$$
$$\frac{\Theta_2(\tau,\theta,q)}{2}a_3 = \frac{q}{2}s_2 - \frac{q}{4}s_1^2, \tag{29}$$
$$\frac{\Theta_3(\tau,\theta,q)}{2}a_3 = \frac{q}{2}s_3 - \frac{q}{2}s_1s_2 - \frac{q(2q^2-3)}{24}s_1^3, \tag{30}$$

and

$$-\frac{\Theta_1(\tau,\theta,q)}{2}a_2 = \frac{q}{2}r_1, \tag{31}$$
$$\Theta_2(\tau,\theta,q)a_2^2 - \frac{\Theta_2(\tau,\theta,q)}{2}a_3 = \frac{q}{2}r_2 - \frac{q}{4}r_1^2, \tag{32}$$
$$\frac{-5\Theta_3(\tau,\theta,q)}{2}a_2^3 + \frac{5\Theta_3(\tau,\theta,q)}{2}a_2a_3 - \frac{\Theta_3(\tau,\theta,q)}{2}a_4 = \frac{q}{2}r_3 - \frac{q}{2}r_1r_2 - \frac{q(2q^2-3)}{24}r_1^3. \tag{33}$$

Using Equations (28) and (31), we write

$$\frac{s_1}{\Theta_1(\tau,\theta,q)} = a_2 = -\frac{r_1}{\Theta_1(\tau,\theta,q)} \text{ and } s_1 = -r_1. \tag{34}$$

The first outcome of the theorem is obvious from this and Lemma 1.
Considering the equivalence $s_1 = -r_1$ and subtracting (32) from (29), we obtain

$$a_3 = a_2^2 + \frac{q[s_2 - r_2]}{2\Theta_2(\tau,\theta,q)}.$$

Furthermore,

$$a_3 = \frac{q^2 s_1^2}{\Theta_1^2(\tau,\theta,q)} + \frac{q[s_2 - r_2]}{2\Theta_2(\tau,\theta,q)}. \tag{35}$$

Moreover, after deducting the Equation labeled as (33) from Equation (30), and taking into account the equalities denoted as (34) and (35), we arrive at the subsequent outcome:

$$a_4 = \frac{q(3-2q^2)}{12\Theta_3(\tau,\theta,q)}s_1^3 + \frac{5q^2s_1(s_2-r_2)}{4\Theta_1(\tau,\theta,q)\Theta_2(\tau,\theta,q)} - \frac{qs_1(s_2+r_2)}{2\Theta_3(\tau,\theta,q)} + \frac{q(s_3-r_3)}{2\Theta_3(\tau,\theta,q)}. \tag{36}$$

Now, Lemma 2 says that because $s_1 = -r_1$, we can write

$$s_2 - r_2 = \frac{4-s_1^2}{2}(e-\mu), \quad s_2 + r_2 = s_1^2 + \frac{4-s_1^2}{2}(e+\mu) \tag{37}$$

and

$$\begin{aligned} s_3 - r_3 =& \frac{s_1^3}{2} + \frac{(4-s_1^2)(e+\mu)}{2}s_1 - \frac{(4-s_1^2)(e^2+\mu^2)}{4}s_1 \\ &+ \frac{4-s_1^2}{2}\left[\left(1-|e|^2\right)\vartheta - \left(1-|\mu|^2\right)w\right]. \end{aligned} \tag{38}$$

There exist values for e, w, ϑ, and μ such that their absolute values are less than or equal to 1. Specifically, $|e| \le 1$, $|w| \le 1$, $|\vartheta| \le 1$, and $|\mu| \le 1$.

To derive the coefficient a_3, we can achieve this by inserting the initial Equation (37) into (35).

$$a_3 = \frac{q^2 s_1^2}{\Theta_1^2(\tau,\theta,q)} + \frac{q(4-s_1^2)}{4\Theta_2(\tau,\theta,q)}(e-\mu).$$

Keep in mind that we can write $|4 - s_1^2| = |4 - |s_1|^2| = 4 - |s_1|^2 = |4 - c^2|$ for s_1 if we accept $|s_1| = c$. In other words, we can just suppose that $c \in [0,2]$. In such a situation, we can express the inequality for $|a_3|$ as

$$a_3 \leq \frac{q^2 c^2}{\Theta_1^2(\tau,\theta,q)} + \frac{q(4-c^2)}{4\Theta_2(\tau,\theta,q)}(\lambda+\sigma), \ \ (\lambda,\sigma) \in [0,1]^2,$$

by using a triangle inequality and the settings $|e| = \lambda$ and $|\mu| = \sigma$.

The function $\Omega : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is hereby defined in the following manner:

$$\Omega(\lambda,\sigma) = \frac{q^2 c^2}{\Theta_1^2(\tau,\theta,q)} + \frac{q(4-c^2)}{4\Theta_2(\tau,\theta,q)}(\lambda+\sigma), \ \ (\lambda,\sigma) \in [0,1]^2.$$

Maximizing the function Ω within the closed square $X = [(\lambda,\sigma) : (\lambda,\sigma) \in [0,1]^2]$ is essential.

The function Ω reaches its highest value at the edges of the square X that encloses it. By employing the parameter λ to derive the function $\Omega(\lambda,\sigma)$, we obtain the following.

$$\Omega_\lambda(\lambda,\sigma) = \frac{q(4-c^2)}{4\Theta_2(\tau,\theta,q)}.$$

Since $\Omega_\lambda(\lambda,\sigma)$ is non-negative, the function $\Omega(\lambda,\sigma)$ increases as λ increases and attains its highest value when λ is equal to 1.

$$\max\{\Omega(\lambda,\sigma) : \lambda \in [0,1]\} = \Omega(1,\sigma) = \frac{q^2 c^2}{\Theta_1^2(\tau,\theta,q)} + \frac{q(4-c^2)(1+\sigma)}{4\Theta_2(\tau,\theta,q)}$$

for every σ belonging to the interval from 0 to 1, and for every c belonging to the interval from 0 to 2.

Upon taking the derivative of the function $\Omega(1,\sigma)$, we obtain the following result.

$$\Omega'(1,\sigma) = \frac{q(4-c^2)}{4\Theta_2(\tau,\theta,q)}.$$

Because of the non-negativity of $\Omega'(1,\sigma)$, the function $\Omega(1,\sigma)$ becomes increasingly larger as σ increases and achieves its highest value when $\sigma = 1$. Consequently,

$$\max\{\Omega(1,1) : \lambda \in [0,1]\} = \Omega(1,1) = \frac{q^2 c^2}{\Theta_1^2(\tau,\theta,q)} + \frac{q(4-c^2)}{2\Theta_2(\tau,\theta,q)},$$

where $c \in [0,2]$.

Then, we get

$$\Omega(\lambda,\sigma) \leq \max\{\Omega(\lambda,\sigma) : (\lambda,\sigma) \in X\} = \Omega(1,1) = \frac{q^2 c^2}{\Theta_1^2(\tau,\theta,q)} + \frac{q(4-c^2)}{2\Theta_2(\tau,\theta,q)}.$$

Since $|a_3| \leq \Omega(\lambda,\sigma)$, we have

$$|a_3| \leq \Psi(q,\tau,\theta) \times c^2 + \frac{2q}{\Theta_2(\tau,\theta,q)}, \ \ c \in [0,2],$$

where

$$\Psi(q,\tau,\theta) = \frac{q^2}{\Theta_1^2(\tau,\theta,q)} - \frac{q}{2\Theta_2(\tau,\theta,q)}.$$

Let us now determine the function $H : \mathbb{R} \longrightarrow \mathbb{R}$ maximum value, which is defined as follows:

$$H(c) = \Psi(q,\tau,\theta) \times c^2 + \frac{2q}{\Theta_2(\tau,\theta,q)},$$

in the interval of $0 \le c \le 2$.

Moreover, upon taking the derivative of the function $H(c)$ with respect to c, we obtain $H'(c) = 2\Psi(q,\theta,q) \cdot c$, where c belongs to the interval $[0,2]$. It is known that $H'(c) \le 0$ when $\Psi(q,\theta,q) \le 0$. This indicates that the function $H(c)$ is decreasing, and its maximum value is achieved when $c = 0$. Thus, we have

$$\max\{H(c) : c \in [0,2]\} = H(0) = \frac{2q}{\Theta_2(\tau,\theta,q)}.$$

Furthermore, if $\Psi(q,\theta,q) \ge 0$, then $H'(c) \ge 0$. The function $H(c)$ is increasing, and its maximum value is achieved at $c = 2$. On the other hand, if we set $c = 0$, the function $H(c)$ becomes a decreasing function, and its maximum value still occurs at $c = 0$. Hence, we can conclude that

$$\max\{H(c) : c \in [0,2]\} = H(2) = \frac{4q^2}{\Theta_1^2(\tau,\theta,q)}.$$

As a result, we have derived the maximum possible value for $|a_3|$, as indicated in the following:

$$|a_3| \le \frac{2q}{\Theta_2(\tau,\theta,q)}.$$

We derive the following inequality for $|a_4|$ from (36), using (37), (38), and triangle inequality.

$$|a_4| \le y_1(c) + y_2(c)(\lambda+\sigma) + y_3(c)(\lambda^2+\sigma^2) := L(\lambda,\sigma)$$

where

$$\begin{aligned} y_1(c) &= \frac{q^3}{6\Theta_3(\tau,\theta,q)}c^3 + \frac{q(4-c^2)}{2\Theta_3(\tau,\theta,q)}, \\ y_2(c) &= \frac{5q^2(4-c^2)}{8\Theta_1(\tau,\theta,q)\Theta_2(\tau,\theta,q)}, \\ y_3(c) &= \frac{q(4-c^2)(c-2)}{8\Theta_3(\tau,\theta,q)}. \end{aligned}$$

For each $c \in [0,2]$, we must now maximize the function $L(\lambda,\sigma)$ on X.

As the coefficients $y_1(c)$, $y_2(c)$, and $y_3(c)$ of the function $H(\lambda,\sigma)$ rely on the parameter c, examining the highest value of the function $H(\lambda,\sigma)$ is essential across various values of c. Let $c = 0$, since $y_2(0) = 0$,

$$\begin{aligned} y_1(0) &= \frac{2q}{\Theta_3(\tau,\theta,q)} \text{ and} \\ y_3(0) &= -\frac{q}{\Theta_3(\tau,\theta,q)}. \end{aligned}$$

Furthermore, we get

$$L(\lambda,\sigma) = \frac{2q}{\Theta_3(\tau,\theta,q)} - \frac{q}{\Theta_3(\tau,\theta,q)}(\lambda^2+\sigma^2), \ (\lambda,\sigma) \in [0,1]^2.$$

So, we have

$$L(\lambda,\sigma) \le \max\{L(\lambda,\sigma) : (\lambda,\sigma) \in X\} = L(0,0) = \frac{2q}{\Theta_3(\tau,\theta,q)}.$$

Let $c = 2$. Then, since $y_2(2) = y_3(2) = 0$ and

$$y_1(2) = \frac{4q^3}{3\Theta(\tau, \theta, q)}.$$

The following function $L(\lambda, \sigma)$ is a constant.

$$L(\lambda, \sigma) = y_1(2) = \frac{4q^3}{3\Theta(\tau, \theta, q)}.$$

It is simple to demonstrate that the function $L(\lambda, \sigma)$ cannot reach its maximum value on the given set X when c belongs to the interval $(0, 2)$. As a result, we obtain the following.

$$|a_4| \leq \frac{2q}{\Theta_3(\tau, \theta, q)}.$$

□

From Theorem 1, we get the following findings for specific parameter values.

Corollary 1. *If $f(z)$ belongs to the class of functions denoted as $\mathcal{OB}^{\theta}_{\Sigma}(q)$, then*

$$|a_2| \leq \frac{2q}{[2]_q(e^{i\theta} + 3)}, \ |a_3| \leq \frac{2q}{[3]_q(2 + (e^{i\theta} + 1)[2]_q)} \text{ and } |a_4| \leq \frac{2q}{[4]_q(2 + (1 + e^{i\theta})[3]_q)}.$$

The outcomes achieved here are precise.

3. The Fekete–Szegö Inequality and the Second Hankel Determinant

Theorem 2. *Let $f \in \mathcal{OB}^{\theta}_{\tau,\Sigma}(q)$. Then,*

$$|a_2 a_4 - a_3^2| \leq \frac{4q^2}{\Theta_2^2(\tau, \theta, q)},$$

where

$$\Theta_2(\tau, \theta, q) = 2 + 2\tau([3]_q - 1) + (1 + e^{i\theta})[2]_q[3]_q.$$

The results obtained here are sharp.

Proof. Let $f \in \mathcal{OB}^{\theta}_{\tau,\Sigma}(q)$. Then, the equality $a_2 a_4 - a_3^2$ can be written in this following form using (34)–(36):

$$\begin{aligned} a_2 a_4 - a_3^2 = &\frac{q^2(3 - 2q^2)\Theta_1^3(\tau, \theta, q) - 12q^3\Theta_3(\tau, \theta, q)}{12\Theta_1^4(\tau, \theta, q)\Theta_3(\tau, \theta, q)} s_1^4 + \frac{q^3(s_2 - r_2)}{4\Theta_1^2(\tau, \theta, q)\Theta_2(\tau, \theta, q)} s_1^2 \\ &- \frac{q^2(s_2 + r_2)}{2\Theta_1(\tau, \theta, q)\Theta_3(\tau, \theta, q)} s_1^2 + \frac{q^2(s_3 - r_3)}{2\Theta_1(\tau, \theta, q)\Theta_3(\tau, \theta, q)} - \frac{q^2(s_2 - r_2)^2}{4\Theta_2^2(\tau, \theta, q)}. \end{aligned}$$

With the aid of equalities (37) and (38), the triangle inequality, and considering the assumptions where the absolute value of s_1 is denoted as c, the absolute value of e is denoted as λ, and the absolute value of μ is denoted as σ, we can make an approximation for

$$|a_2 a_4 - a_3^2| \leq Y_1(c) + Y_2(c)(\lambda + \sigma) + Y_3(c)(\lambda^2 + \sigma^2) + Y_4(c)(\lambda + \sigma)^2, \tag{39}$$

where

$$\begin{aligned}
\Upsilon_1(c) &= \frac{q^3[q\Theta_1^3(\tau,\theta,q)+6\Theta_3(\tau,\theta,q)]}{6\Theta_1^4(\tau,\theta,q)\Theta_3(\tau,\theta,q)}c^4+\frac{q^2(4-c^2)}{2\Theta_1(\tau,\theta,q)\Theta_3(\tau,\theta,q)}c \geq 0,\\
\Upsilon_2(c) &= \frac{q^3(4-c^2)}{8\Theta_1^2(\tau,\theta,q)\Theta_3(\tau,\theta,q)}c^2 \geq 0,\\
\Upsilon_3(c) &= \frac{q^2(4-c^2)(c-2)c}{8\Theta_1(\tau,\theta,q)\Theta_2(\tau,\theta,q)} \leq 0,\\
\Upsilon_4(c) &= \frac{q^2(4-c^2)^2}{16\Theta_2^2(\tau,\theta,q)} \geq 0.
\end{aligned}$$

We now define the function $D:\mathbb{R}^2 \longrightarrow \mathbb{R}$ as follows:

$$D(\lambda,\sigma)=\Upsilon_1(c)+\Upsilon_2(c)(\lambda+\sigma)+\Upsilon_3(c)(\lambda^2+\sigma^2)+\Upsilon_4(c)(\lambda+\sigma)^2,$$

for every pair of values (λ,σ) that belong to the interval $[0,1]$ and every value of c within the range of $(0,2)$, it is possible to find the maximum of the function $D(\lambda,\sigma)$ over the domain X.

It is necessary to examine the highest value for different parameter values of c because the coefficients $\Upsilon_1(c)$, $\Upsilon_2(c)$, $\Upsilon_3(c)$ and $\Upsilon_4(c)$ of the function $D(\lambda,\sigma)$ depend on the parameter c.

1. Let $c=0$. Since $\Upsilon_1(0)=\Upsilon_2(0)=\Upsilon_3(0)=0$ and

$$\Upsilon_4(0)=\frac{q^2}{\Theta_2^2(\tau,\theta,q)}$$

the function $D(\lambda,\sigma)$ written as follows

$$D(\lambda,\sigma)=\frac{q^2}{\Theta_2^2(\tau,\theta,q)}(\lambda+\sigma)^2,\ \ (\lambda,\sigma)\in X.$$

The maximum value of the function $D(\lambda,\sigma)$ is achieved at the edges of the enclosed square X, which is clearly observable.
Now, by using some techniques of differentiation on the function $D(\lambda,\sigma)$ with respect to λ, we get

$$D_\lambda(\lambda,\sigma)=\frac{2q^2}{\Theta_2^2(\tau,\theta,q)}(\lambda+\sigma),\ \ \sigma\in[0,1].$$

The function $D(\lambda,\sigma)$ is a monotonically increasing function concerning λ and reaches its peak value when λ equals 1, as indicated by $D_\lambda(\lambda,\sigma)\geq 0$. Therefore, the following relationship holds:

$$\max\{D(\lambda,\sigma):\sigma\in[0,1]\}=D(1,\sigma)=\frac{q^2}{\Theta_2^2(\tau,\theta,q)}(1+\sigma)^2,\ \sigma\in[0,1].$$

After employing the methods of differentiation to the function $D(1,\sigma)$, the result is as follows:

$$D'(1,\sigma)=\frac{2q^2}{\Theta_2^2(\tau,\theta,q)}(1+\sigma),\ \sigma\in[0,1].$$

Because the derivative of D with respect to σ at $\sigma = 1$ is positive ($D'(1, \sigma) > 0$), the function $D(1, \sigma)$ is monotonically increasing, and its maximum value is attained when $\sigma = 1$. Consequently,

$$\max\{D(1, \sigma) : \sigma \in [0, 1]\} = D(1, 1) = \frac{4q^2}{\Theta_2^2(\tau, \theta, q)}.$$

Thus, for $c = 0$, we have

$$D(\lambda, \sigma) \leq \max\{D(\lambda, \sigma) : (\lambda, \sigma) \in [0, 1]^2\} = D(1, 1) = \frac{4q^2}{\Theta_2^2(\tau, \theta, q)}.$$

Since $|a_2 a_4 - a_3^2| \leq D(\lambda, \sigma)$, we have

$$|a_2 a_4 - a_3^2| \leq \frac{4q^2}{\Theta_2^2(\tau, \theta, q)}.$$

2. Now, setting $c = 2$, for $Y_2(2) = Y_3(2) = Y_4(2) = 0$ and

$$Y_1(2) = \frac{8q^3[q\Theta_1^3(\tau, \theta, q) + 6\Theta_3(\tau, \theta, q)]}{3\Theta_1^4(\tau, \theta, q)\Theta_3(\tau, \theta, q)}$$

the function $D(\lambda, \sigma)$ is a constant, as follows

$$D(\lambda, \sigma) = Y_1(2) = \frac{8q^3[q\Theta_1^3(\tau, \theta, q) + 6\Theta_3(\tau, \theta, q)]}{3\Theta_1^4(\tau, \theta, q)\Theta_3(\tau, \theta, q)}.$$

Hence, we have

$$|a_2 a_4 - a_3^2| \leq \frac{8q^3[q\Theta_1^3(\tau, \theta, q) + 6\Theta_3(\tau, \theta, q)]}{3\Theta_1^4(\tau, \theta, q)\Theta_3(\tau, \theta, q)}$$

in the case of $c = 2$.

3. Given that c lies in the open interval between 0 and 2, our objective is to analyze the maximum value of the function $D(\lambda, \sigma)$. This analysis will consider the sign of some variables.

$$\chi(D(\lambda, \sigma)) = D_{\lambda\lambda}(\lambda, \mu) D_{\sigma\sigma}(\lambda, \sigma) - (D_{\lambda\sigma}(\lambda, \sigma))^2.$$

The equation

$$\chi(D(\lambda, \sigma)) = 4Y_3(c)[Y_3(c) + 2Y_4(c)]$$

is visible to us. We now consider two examples of the sign $\chi(D(\lambda, \sigma))$.

(a) Let $Y_3(c) + 2Y_4(c) \leq 0$ for the interval $c \in (0, 2)$. For this instance, since $D_{\lambda\sigma}(\lambda, \sigma) = D_{\sigma\lambda}(\lambda, \sigma) = 2Y_4(c) \geq 0$ and $\chi(D(\lambda, \sigma)) \geq 0$, basic calculus dictates that the function $D(\lambda, \sigma)$ cannot achieve a maximum within the boundaries of the square X.

(b) Additionally, suppose there exists a value c in the interval (0, 2) such that $Y_3(c) + 2Y_4(c) \geq 0$. Under this condition, where $\chi(D) \leq 0$, the function $D(\lambda, \sigma)$ cannot attain a maximum within the square region $X = [(\lambda, \sigma) : (\lambda, \sigma) \in [0, 1]^2]$.

As a result of these three instances, we write

$$|a_2 a_4 - a_3^2| \leq \frac{4q^2}{\Theta_2^2(\tau, \theta, q)}.$$

Therefore, the proof of Theorem 2 is now finished. □

Based on the specific parameter values, the following discoveries are obtained from Theorem 2.

Corollary 2. *Let* $f \in \mathcal{OB}_{\Sigma}^{\theta}(q)$*. Then,*

$$|a_2a_4 - a_3^2| \leq \frac{4q^2}{[3]_q(2 + (1 + e^{i\theta})[2]_q)}.$$

The results are sharp.

Corollary 3. *Let* $f \in \mathcal{OB}_{\Sigma}(q)$*. Then,*

$$|a_2a_4 - a_3^2| \leq \frac{2q^2}{[3]_q}.$$

The results are sharp.

Now, we will present the theorem related to the Fekete–Szegö inequality.

Theorem 3. *Let* $f \in \mathcal{OB}_{\tau,\Sigma}^{\theta}(q)$*,* $\xi \in \mathcal{C}$*. Then,*

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} \frac{4G(q)}{\Theta_1^2(\tau,\theta,q)} & |1-\xi| \leq G(q) \\ \\ \frac{4|1-\xi|}{\Theta_1^2(\tau,\theta,q)} & |1-\xi| \geq G(q). \end{cases}$$

where

$$G(q) = \frac{q \cdot \Theta_1^2(\tau,\theta,q)}{2\Theta_2(\tau,\theta,q)}.$$

The results obtained here are sharp.

Proof. Let $f \in \mathcal{OB}_{\tau,\Sigma}^{\theta}(q)$ and $\xi \in \mathcal{C}$. Then, from (34)–(38), we solve for the expression $a_3 - \xi a_2^2$ to be:

$$a_3 - \xi a_2^2 = \frac{q^2 s_1^2}{\Theta_1^2(\tau,\theta,q)}(1-\xi) + \frac{q(4 - s_1^2)}{4\Theta_2(\tau,\theta,q)}(e - \mu) \tag{40}$$

for some e, μ with $|e| \leq 1$ and $|\mu| \leq 1$.

If $|e| = \lambda$, $|\mu| = \sigma$, $|s_1| = c$ and applying triangle inequality to (40), we can then solve for the upper bound of $|a_3 - \xi a_2^2|$, as follows:

$$|a_3 - \xi a_2^2| \leq \frac{|1-\xi|q^2}{\Theta_1^2(\tau,\theta,q)}c^2 + \frac{q(4-c^2)(\lambda+\sigma)}{4\Theta_2(\tau,\theta,q)}, \quad (\lambda,\sigma) \in X, \tag{41}$$

for every value of c belonging to the interval $[0,2]$.

The function ϖ can be defined now as $\varpi : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

$$\varpi(\lambda,\sigma) = \frac{|1-\xi|q^2}{\Theta_1^2(\tau,\theta,q)}c^2 + \frac{q(4-c^2)(\lambda+\sigma)}{4\Theta_2(\tau,\theta,q)}, \quad (\lambda,\sigma) \in X,$$

for each $c \in (0,2)$. We have to check now that the function $\varpi(\lambda,\sigma)$ on X for each $c \in [0,2]$ is maximized.

The highest value of the function $\varpi(\lambda,\sigma)$ is clearly achieved at the boundaries of the enclosed square X.

Therefore, through straightforward differentiation of the function $\varpi(\lambda,\sigma)$ with respect to λ, we obtain

$$\varpi_\lambda(\lambda,\sigma) = \frac{q(4-c^2)}{4\Theta_2(\tau,\theta,q)}, \quad c \in [0,2]. \tag{42}$$

As $\varpi_\lambda(\lambda,\sigma)$ is greater than zero, the function $\varpi(\lambda,\sigma)$ shows a positive correlation with λ, leading to an increase as λ increases. The maximum value of this function is attained when $\sigma = 1$,

$$\max\{\varpi(\lambda,\sigma) : \sigma \in [0,1]\} = \varpi(1,\sigma) = \frac{|1-\xi|q^2}{\Theta_1^2(\tau,\theta,q)}c^2 + \frac{q(4-c^2)}{4\Theta_2(\tau,\theta,q)}(1+\sigma)$$

for each $\sigma \in [0,1]$ and $a \in [0,2]$.

Furthermore, by applying differentiation on $\varpi(1,\sigma)$, we have

$$\varpi'(1,\sigma) = \frac{q(4-c^2)}{4\Theta_2(\tau,\theta,q)}$$

for each $c \in [0,2]$.

If the condition $\varpi'(1,\sigma) > 0$ is met, the function $\varpi(1,\sigma)$ will exhibit a rising trend, and its maximum value will be attained at $\sigma = 1$. Therefore,

$$\max\{\varpi(1,\sigma) : \sigma \in [0,1]\} = \varpi(1,1) = \frac{|1-\xi|q^2}{\Theta_1^2(\tau,\theta,q)}c^2 + \frac{q(4-c^2)}{2\Theta_2(\tau,\theta,q)}.$$

Thus, we get

$$\varpi(\lambda,\mu) \le \max\{\varpi(\lambda,\sigma) : (\lambda,\sigma) \in [0,1]\} = \varpi(1,1) = \frac{|1-\xi|q^2}{\Theta_1^2(\tau,\theta,q)}c^2 + \frac{q(4-c^2)}{2\Theta_2(\tau,\theta,q)}.$$

Since $|a_3 - \xi a_2^2| \le \varpi(\lambda,\sigma)$, we get

$$|a_3 - \xi a_2^2| \le \frac{|1-\xi| - G(q)}{\Theta_1^2(\tau,\theta,q)}c^2 + \frac{4G(q)}{\Theta_1^2(\tau,\theta,q)}$$

where

$$G(q) = \frac{q \cdot \Theta_1^2(\tau,\theta,q)}{2\Theta_2(\tau,\theta,q)}.$$

Now, it is the right time to determine the maximum value of the function Ξ over the interval $[0,2]$ in the real number set.

$$\Xi(c) = \frac{|1-\xi| - G(q)}{\Theta_1^2(\tau,\theta,q)}c^2 + \frac{4G(q)}{\Theta_1^2(\tau,\theta,q)}.$$

By applying the principle of differentiation on the function $\Xi(c)$, we get

$$\Xi'(c) = \frac{2(|1-\xi| - G(q))}{\Theta_1^2(\tau,\theta,q)}c, \quad c \in [0,2].$$

Supposing $|1-\xi| \le G(q)$ and the maximum occurs at $c = 0$, then the function $\Xi(c)$ is a decreasing function since $\Xi'(c) \le 0$

$$\max\{\Xi(c) : c \in [0,2]\} = \Xi(0) = \frac{4G(q)}{\Theta_1^2(\tau,\theta,q)}.$$

Given that $\Xi'(c) \geq 0$, the function $\Xi(c)$ is monotonically increasing. When $|1-\xi| \geq G(q)$ and the maximum of the function is at $c = 2$, then

$$\max\{\Xi(c) : c \in [0,2]\} = \Xi(2) = \frac{4|1-\xi|}{\Theta_1^2(\tau,\theta,q)}.$$

We consequently arrive at

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} \frac{4G(q)}{\Theta_1^2(\tau,\theta,q)} & |1-\xi| \leq G(q) \\ \\ \frac{4|1-\xi|}{\Theta_1^2(\tau,\theta,q)} & |1-\xi| \geq G(q). \end{cases}$$

The outcome obtained in this case is sharp for $|1-\xi| \geq G(q)$. □

From Theorem 3, we get the following findings for specific parameter values.

Corollary 4. *Let $f \in \mathcal{OB}_{\Sigma}^{\theta}(q)$, $\xi \in \mathcal{C}$. Then,*

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} \frac{4G(q)}{[2]_q^2(3+e^{i\theta})^2} & |1-\xi| \leq G(q) \\ \\ \frac{4|1-\xi|}{[2]_q^2(3+e^{i\theta})^2} & |1-\xi| \geq G(q). \end{cases}$$

where

$$G(q) = \frac{q[2]_q^2(3+e^{i\theta})^2}{2[3]_q[2+(1+e^{i\theta})[2_q]]}.$$

The results obtained here are sharp.

Corollary 5. *Let $f \in \mathcal{OB}_{\Sigma}(q)$, $\xi \in \mathcal{C}$. Then,*

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} \frac{4G(q)}{[2]_q^2} & |1-\xi| \leq \frac{q[2]_q^2}{4[3]_q} \\ \\ \frac{4|1-\xi|}{[2]_q^2} & |1-\xi| \geq \frac{q[2]_q^2}{4[3]_q}. \end{cases}$$

The results obtained here are sharp.

Theorem 3 is stated as follows for the condition $\xi \in \mathbb{R}$.

Theorem 4. *Let $f \in \mathcal{OB}_{\tau,\Sigma}^{\theta}(q)$, $\xi \in \mathbb{R}$. Then,*

$$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{4(1-\xi)}{\Theta_1^2(\tau,\theta,q)} & \text{if } \xi \leq 1 - G(q) \\ \\ \frac{4G(q)}{\Theta_1^2(\tau,\theta,q)} & \text{if } 1 - G(q) \leq \xi \leq 1 + G(q) \\ \\ \frac{4(\xi-1)}{\Theta_1^2(\tau,\theta,q)} & \text{if } 1 + G(q) \leq \xi, \end{cases} \tag{43}$$

where

$$G(q) = \frac{q \cdot \Theta_1^2(\tau,\theta,q)}{2\Theta_2(\tau,\theta,q)}.$$

Proof. Let f belong to the class of functions denoted by $\mathcal{OB}^{\theta}_{\tau,\Sigma}(q)$, and let ξ be a real number. When ξ is a real number, the inequalities $|1-\xi| \geq G(q)$ and $|1-\xi| \leq G(q)$ are equivalent to the following conditions:

$$\xi \leq 1 - G(q) \text{ or } \xi \geq 1 + G(q),$$

$$1 - G(q) \leq \xi \leq 1 + G(q).$$

The theorem's conclusion is obtained from Theorem 3. □

Using the parameter $\xi = 1$, we have the following corollary:

Corollary 6. *Let $f \in \mathcal{OB}^{\theta}_{\tau,\Sigma}(q)$. Then,*

$$|a_3 - a_2^2| \leq \frac{2q}{2 + 2\tau([3]_q - 1) + (1 + e^{i\theta})[2]_q[3]_q}.$$

When ξ is set to zero, the subsequent corollary can be stated as follows:

Corollary 7. *Let $f \in \mathcal{OB}^{\theta}_{\tau,\Sigma}(q)$. Then,*

$$|s_3| \leq \begin{cases} \frac{4(1-\xi)}{\Theta_1^2(\tau,\theta,q)} & \text{if } G(q) \leq 1, \\ \\ \frac{4G(q)}{\Theta_1^2(\tau,\theta,q)} & \text{if } G(q) \geq 1 \end{cases} \tag{44}$$

where

$$G(q) = \frac{q \cdot \Theta_1^2(\tau,\theta,q)}{2\Theta_2(\tau,\theta,q)}.$$

From Theorem 4, we get the following findings for specific parameter values.

Corollary 8. *Let $f \in \mathcal{OB}_{\Sigma}(q)$, $\xi \in \mathbb{R}$. Then,*

$$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{1-\xi}{[2]_q^2} & \text{if } \xi \leq 1 - G(q) \\ \\ \frac{q}{[3]_q} & \text{if } 1 - G(q) \leq \xi \leq 1 + G(q) \\ \\ \frac{\xi-1}{[2]_q^2} & \text{if } 1 + G(q) \leq \xi, \end{cases} \tag{45}$$

where

$$G(q) = \frac{q[2]_q^2}{4[3]_q}.$$

4. Conclusions

To summarize, this study presents a fresh type of analytic functions called the q-calculus operator, defined through the utilization of the q-derivative operator and the q-version of the hyperbolic tangent function. The primary goals of this research involve computing coefficients, second Hankel determinants, and Fekete–Szegö estimates for this newly defined group of functions. Furthermore, the study seeks to investigate the upper bounds that determine the inclusion of functions $f(\vartheta)$ in this newly established category.

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