*fractal and fractional*

*Article*

# On Inequalities and Filtration Associated with the Nonlinear Fractional Operator

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**Abstract:** In this paper, we study a new filtration class $\mathcal{MF}^{\mu}_{\alpha,\beta}$, associated with the filtration of infinitesimal generators, by using the nonlinear fractional differential operator and study certain properties, like sharp Fekete–Szegö inequalities and filtration problems.

**Keywords:** infinitesimal generator; Fekete–Szegö inequality; semi-group; differential subordination; fractional differential operator

**MSC:** 30C35; 30C50; 30D05; 37C25; 30C45



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## 1. Introduction and Preliminaries

Assume that $\mathcal{H}$ is a family of analytic functions. For a natural $j$ and $a \in \mathbb{C}$ and let

$$\mathcal{H}^j_a := \left\{ f \in \mathcal{H} : \; f(\xi) = a + \xi + a_{j+1}\xi^{j+1} + a_{j+2}\xi^{j+2} + \dots, \; \xi \in \mathbb{E} \right\} \tag{1}$$

be the subclass of $\mathcal{H}$. Furthermore, we deduce that

$$\mathcal{H}^j_0 := \left\{ f : f(\xi) = \xi + a_{j+1}\xi^{j+1} + \dots, \frac{f^{(j+1)}(0)}{(j+1)!} = a_{j+1} \;\; \xi \in \mathbb{E} \right\}. \tag{2}$$

For $j = 1$, we observe that

$$\mathcal{H}^1_0 = \mathcal{A} := \left\{ f : f(\xi) = \xi + a_2\xi^2 + a_3\xi^3 + \dots, \; \xi \in \mathbb{E} \right\} \tag{3}$$

and $\Omega = \{s \in \mathcal{A} : s(0) = 0\}$. The family $\mathcal{A}$ is given by (3). For $\lambda \in \mathbb{C}$, let $\phi(f,\lambda) = a_3 - \lambda a_2^2$ be a quadratic functional. The problem related to this function involves the derivation of sharp estimates for the functional absolute values $\phi(.,\lambda)$ for certain types of functions over the class $\mathcal{A}$. Keogh and Merks [1] showed that

$$|\phi(f,\lambda)| \le \begin{cases} \max\left(\frac{1}{3}, |1-\lambda|\right), & f \in \mathcal{C} \\ \max\left(\frac{1}{2}, |1-\lambda|\right), & f \in \mathcal{S}^*_{\frac{1}{2}} \end{cases}, \tag{4}$$

and these inequalities are best possible. In [2], it is shown that for the functions $f$ such that $\Re\text{e}\frac{f(\xi)}{\xi} > \frac{1}{2}$, $\xi \in \mathbb{E}$, the following sharp result holds

$$|\phi(f,\lambda)| \le \max(1, |1-\lambda|).$$

Consider the mapping $f \in \mathcal{A}$, defined by the limit given by

$$\lim_{t\to 0^+} \frac{\xi - u_t(\xi)}{t} = f(\xi), \quad t \ge 0.$$

This mapping is called an infinitesimal generator of one-parameter family of a continuous semi-group, if, for $\xi \in \mathbb{E}$, the Cauchy problem,

$$\begin{cases} \frac{\partial}{\partial t}(u(\xi,t)) + f(u(\xi,t)) = 0, \\ u(\xi,t) = \xi, t = 0 \end{cases} \tag{5}$$

has a unique solution $u = u(\xi,t) \in \mathbb{E}$, $t \ge 0$. In this case, the solution $u = u(\xi,t) \in \mathbb{E}$, $t \ge 0$ of (5) forms a semi-group of holomorphic or analytic self-mappings in $\mathbb{E}$ generated by $f$; for details, see [3–5]. The family of all generators, known as a semi-complete vector field on $\mathbb{E}$, is expressed by $\mathcal{G}$. Another important form of the family $\mathcal{G}$ is studied by Berkson and Porta [6] and can be restated below.

**Theorem 1.** *For $f \in \mathcal{A}$ with $f(\xi) \neq 0$, $f$ is a generator on $\mathbb{E}$, if and only if $\exists$ a point $\tau \in \overline{\mathbb{E}}$ and a function $h \in \mathcal{A}$ with $\Re\text{e}h(\xi) \ge 0$ :*

$$f(\xi) = h(\xi)(\tau - \xi)(\xi\overline{\tau} - 1),\ \xi \in \mathbb{E}.$$

In particular, $f \in \mathcal{A}$ is a generator if and only if $\Re\text{e}\left(\frac{f(\xi)}{\xi}\right) > 0$. We express the family of such a generator using $\mathcal{G}$. The condition $\Re\text{e}\left(\frac{f(\xi)}{\xi}\right) > 0$ seems simple, but often it is hard to verify. The condition $\Re\text{e}(f'(\xi)) > 0$, provided by Noshiro and Warschawski [4,7], independently implies that a function $f \in \mathcal{A}$ is univalent. It can be taken as sufficient for $f \in \mathcal{A}$ to consider it a generator. All infinitesimal generators may not be univalent. Then, $\Re\text{e}(f'(\xi)) > 0$ cannot ensure that $f \in \mathcal{G}$.

In recent years, many remarkable developments have been made in the study of the generation theory of one parametric semi-group of analytic functions. This study not only answers many diverse questions from different areas of mathematical analysis, but also deals with significantly new developments in the initial and boundary value partial differential equations, approximation theory and the theory of singular integrals. The generation theory of semi-groups has been used in Markov stochastic processes and in the theory of branching processes. This leads to its involvement in one-dimensional complex analysis and is the main motivation of this work.

In this paper, our main task is to establish a connection between the Fekete–Szegö quadratic functional and the class of infinitesimal generators $\mathcal{G}$. For this purpose, we define a class $\mathcal{MF} = \{f_s, s > 0\}$, of all $f \in \mathcal{A}$, such that $\phi(f,\lambda)$ satisfies the sharp estimates $\sup_{f\in\mathcal{MF}}|\phi(f,\lambda)| = \max(s, |1-\lambda|)$. Moreover, in the case of an invertible function $f \in \mathcal{A}$, we have

$$\sup_{f\in\mathcal{MF}}\left|\phi\left(f^{-1},\lambda\right)\right| = \sup_{f\in\mathcal{MF}}|\phi(f,\lambda)|.$$

**Definition 1.** *A filtration of $\mathcal{G}$ is a family $\mathcal{MF} = \{\wp_s : s \in [c,d], \wp_s \subseteq \mathcal{G}\}$, where $c$, $d \in [-\infty, +\infty]$ and $c < d$ : $\wp_s \subseteq \wp_t$, whenever $c \le s \le t \le d$. Furthermore, a filtration is strict if $\wp_s \subset \wp_t$, $s < t$.*

We are focused on building a relationship between the family $\mathcal{G}$ of infinitesimal generators and the Fekete–Szegö functional $\phi(f,\lambda)$. Here, we establish a more generalized mechanism for $f \in \mathcal{A}$ to be in the family $\mathcal{G}$. Our task is to establish a connection between the family of infinitesimal generators $\mathcal{G}$ and the Fekete–Szegö functional $\phi(f,\lambda)$ by determining a filtration $\{\wp_s,\ s>0\}$ such that $\sup_{f\in\mathcal{MF}}|\phi(f,\lambda)| = \max(s,|1-\lambda|)$. Here, we assume a more generalized condition for $f \in \mathcal{A}$ to be in $\mathcal{G}$ by:

**Definition 2.** *Assume that $h(\xi) = \frac{f(\xi)}{\xi}$ is strongly starlike of order $\alpha = \frac{1}{\mu}$, such that*

$$\left|\arg\left(\xi^{-1}f(\xi)\right)\right| \le \frac{\pi}{2\mu},\quad 0 < \frac{1}{\mu} \le 1,\ \xi \in \mathbb{E}.$$

*A function $f \in \mathcal{A}$ is in $\mathcal{G}$ if and only if $\Re e\left(\frac{f(\xi)}{\xi}\right)^{\mu} > 0$ and $-1 \le \mu \le 1$.*

By using Definition 2, we now develop some new filtration families by using a nonlinear differential operator

$$\wp^{\mu}_{\alpha,\beta}(f)(\xi) = \alpha\left(\frac{f(\xi)}{\xi}\right)^{\mu} + (\beta\mu+\Lambda)w(\xi) + \Lambda\frac{\xi\mu w'(\xi)}{1-\mu+\mu w(\xi)}, \tag{6}$$

where $w(\xi) = \frac{\xi f'(\xi)}{f(\xi)}$, $\alpha,\beta \in \mathbb{R}$, $\Lambda = 1-\alpha-\beta$, $-1 \le \mu \le 1$ and $\xi \in \mathbb{E}$, and we establish sharp bounds on the modulus of $\phi(f,\lambda)$ over these filtration classes.

For the sake of completeness, we are required to obtain the following important results from Geometric Function Theory.

## 2. Preliminaries

Let $\Psi : \mathbb{C}^3 \times \mathbb{E} \to \mathbb{C}$ and let $h \in \mathcal{S}$ in $\mathbb{E}$. If $p$ is analytic in $\mathbb{E}$ and satisfies the nonlinear second order differential subordination

$$\Psi\left(p(\xi), \xi p'(\xi), \xi^2 p''(\xi); \xi\right) \prec h(\xi), \tag{7}$$

then $p$ is called a solution of (7). The function $q \in \mathcal{S}$ is called a dominant of the solutions of (7), or simply a dominant, if $p(\xi) \prec q(\xi)$ for all $p$ satisfying (7). A dominant $q : q(\xi) \prec \tilde{q}(\xi)$ for all dominants $q$ of (7) is said to be the best dominant of (7).

**Lemma 1.** *Let $\beta,\tau \in \mathbb{C}$ and $\beta \ne 0$. Let $h_1 \in \mathcal{A}$ and $\Re e[\beta h_1(\xi)+\tau] > 0$. Then, the solution $h$ of*

$$h(\xi) + \frac{\xi h'(\xi)}{\beta h(\xi)+\tau} = h_1(\xi)$$

*satisfies $\Re e[\beta h(\xi)+\tau] > 0 : h(0) = c$.*

For the details of Lemma 1, see [8].

**Lemma 2.** *Let $\tau \in \mathbb{R}$, $f \in \mathcal{A}$ and $\Omega \subset \mathbb{C}$. Then, $\Re e f(\xi) > \tau$, $\xi \in \mathbb{E}$ if and only if the functional $s$ is defined by*

$$s(\xi) = \frac{f(0)-f(\xi)}{2\tau - f(0) - f(\xi)} \in \mathcal{A}. \tag{8}$$

**Lemma 3.** *Suppose $h : h(0) = 1$, $\Omega \subset \mathbb{C}$ and $\Psi : \mathbb{C}^3 \times \mathbb{E} \to \mathbb{C}$ satisfy*

$$\Psi(i\rho, \sigma, u+iv; \xi) \notin \Omega, \qquad (\xi \in \mathbb{E}),$$

*for $\rho,\sigma,u,\sigma \in \mathbb{R}$, $\sigma \le -\frac{1+\rho^2}{2}$ and $\sigma + u \le 0$. If, for the functional $\Psi$, we have*

$$\Psi\left(h, \xi h', \xi^2 h''\right)(\xi) \in \Omega \qquad (\xi \in \mathbb{E}),$$

*then* $\Re eh(\xi) > 0, \xi \in \mathbb{E}$.

For the proof of the both Lemmas 2 and 3, we refer to [9].

**Lemma 4.** *If* $w \in \mathcal{A}$ *and* $w(\xi) = \sum_{j=1}^{\infty} b_j \xi^j$, *then* $|b_2| \leq 1 - |b_1|^2$*and* $|b_2 - sb_1^2| \leq \max(1, |s|)$ *for* $s \in \mathbb{C}$.

A detailed proof of Lemma 4 can be found in [1].

**Lemma 5.** *Let* $f \in \mathcal{A} : |f(\xi)| < 1$. *If* $|f(\xi)|$ *attains its highest value at* $\xi_0$, *then*

$$\frac{\xi_0 f'(\xi_0)}{f(\xi_0)} = m, \; m \geq 1.$$

For detailed information of Lemma 5, see [10].

## 3. A Function as an Infinitesimal Generator

In the first Theorem, we drive some sufficient conditions on $\wp^{\mu}_{\alpha,\beta}(f)$ which ensure that $f$ is a generator.

**Theorem 2.** *For* $\alpha, \beta, \mu \in \mathbb{R}$ *and* $-1 \leq \mu \leq 1$, *consider that*

$$\Delta_1 = \Big\{ s = x + iy : \mu\beta - \alpha + 1 \geq x; \;\; (x - 1 - \beta\mu)^2 - \alpha^2 \geq y^2; \;\; \alpha \geq 0 \Big\},$$

*and*

$$\Delta_2 = \Big\{ s = x + iy : x > \mu\beta - \alpha + 1; \;\; y^2 \leq (x - 1 - \beta\mu)^2 - \alpha^2; \;\; \alpha < 0 \Big\}.$$

*For the image domain* $\Delta \subset \mathbb{C}$ *such that*

$$\Delta = \begin{cases} \mathbb{C} \setminus \Delta_1, & \alpha \geq 0 \\ \mathbb{C} \setminus \Delta_2, & \alpha < 0 \end{cases}, \tag{9}$$

*if, for* $f \in \mathcal{A}$ *and* $\wp^{\mu}_{\alpha,\beta}(f) \subseteq \Delta$, *then* $f \in \mathcal{G}$.

**Proof.** For the mapping $f \in \mathcal{A}$ and $h(\xi) = \left(\frac{f(\xi)}{\xi}\right)^{\mu}$, we see that

$$\mu \frac{\xi f'(\xi)}{f(\xi)} = \mu + \frac{\xi h'(\xi)}{h(\xi)}$$

and

$$\frac{\mu\xi\left(\frac{\xi f'(\xi)}{f(\xi)}\right)'}{1 + \mu\left(\frac{\xi f'(\xi)}{f(\xi)} - 1\right)} + \frac{\xi f'(\xi)}{f(\xi)} = \frac{\left(\xi^2 h'(\xi)\right)'}{(\xi h(\xi))'} + 1.$$

Thus, for $\Lambda = 1 - \alpha - \beta$, we find that

$$\begin{aligned} \wp^{\mu}_{\alpha,\beta}(f)(\xi) &= \mu\beta + \Lambda + \alpha h(\xi) + \frac{\beta\xi h'(\xi)}{h(\xi)} + \Lambda\frac{\left(\xi^2 h'(\xi)\right)'}{(\xi h(\xi))'} \\ &= 1 - \alpha + \beta(\mu - 1) + \alpha h(\xi) + \frac{\beta\xi h'(\xi)}{h(\xi)} + \Lambda\frac{\left(\xi^2 h'(\xi)\right)'}{(\xi h(\xi))'}. \end{aligned}$$

By choosing $r = h(\xi)$, $s = \xi h'(\xi)$ and $t = \xi^2 h''(\xi)$, we study the admissibility conditions as:

For $\Lambda = 1 - \beta - \alpha$, consider that

$$\Psi(r, s, t; \xi) = 1 - \alpha + \beta(\mu - 1) + \alpha r + \beta \frac{s}{r} + \Lambda \frac{2s + t}{s + r},$$

and

$$\Psi\left(h, \xi h', \xi^2 h''; \xi\right) = 1 - \alpha + \beta(\mu - 1) + \alpha h(\xi) + \beta \frac{\xi h'(\xi)}{h(\xi)} + \Lambda \frac{\left(\xi^2 h'(\xi)\right)'}{\left(\xi h(\xi)\right)'}.$$

In view of Lemma 3, we prove that $f \in \mathcal{G}$. To establish this inclusion, it is enough to prove that $h$ maps $\mathbb{E}$ onto the right half plane. For this, we need to show that

$$\Psi(i\rho, \sigma, u + i\sigma; \xi) \notin \Delta, \text{ when } \rho,\ \sigma,\ u,\ \sigma \in \mathbb{R},\ \sigma \leq -\frac{1}{2}\left(1 + \rho^2\right),\ \sigma + u \leq 0. \tag{10}$$

For $\Lambda = 1 - \alpha - \beta$, we see that

$$X = \Re e \Psi(i\rho, \sigma, u + i\sigma; \xi) = 1 - \alpha + \beta(\mu - 1) + \frac{\Lambda}{\rho^2 + \sigma^2}[(u + 2\sigma)\sigma + v\rho],$$

and

$$Y = \operatorname{Im} \Psi(i\rho, \sigma, u + i\sigma; \xi) = \Lambda \frac{v\sigma - (u + 2\sigma)\rho}{\rho^2 + \sigma^2} - \frac{\beta\sigma}{\rho} + \alpha\rho.$$

This implies that

$$\frac{Y - \alpha\rho + \frac{\beta\sigma}{\rho}}{X - 1 + \alpha - \beta(\mu - 1)} = \frac{v\sigma - (u + 2\sigma)\rho}{(u + 2\sigma)\sigma + v\rho},$$

or we can write that

$$\frac{Y - \alpha\rho + \frac{\beta\sigma}{\rho}}{(X - 1 + \alpha - \beta\mu) + \beta} = \frac{\sigma}{\rho} - \frac{(u + 2\sigma)\left(\rho^2 + \alpha\sigma^2\right)}{\rho(v\rho + (u + 2\sigma)\sigma)}. \tag{11}$$

Moreover, if we let $\kappa = X - 1 + \alpha - \beta\mu$ and

$$Y_{\sigma,\rho,u} = \frac{(u + 2\sigma)\left(\rho^2 + \sigma^2\right)}{(v\rho + (u + 2\sigma)\sigma)},$$

then (11) becomes

$$Y - \alpha\rho + \frac{\beta\sigma}{\rho} = (\kappa + \beta)\frac{\sigma}{\rho} - \frac{\kappa + \beta}{\rho} Y_{\sigma,\rho,u},$$

or

$$Y = \rho\alpha + \kappa\frac{\sigma}{\rho} - \frac{\kappa + \beta}{\rho} Y_{\sigma,\rho,u}.$$

Therefore, we have

$$Y \neq \rho\left[\alpha + \kappa\frac{\sigma}{\rho^2}\right] = Y_{\sigma,\rho,u}(\kappa), \text{ where } \rho \in \mathbb{R} \text{ and } \sigma \leq -\frac{1}{2}\left(1 + \rho^2\right).$$

Condition (10) holds if every point of $\Omega = \{s \in \mathcal{A} : s(0) = 0\}$ is found on the graph of $Y_{\sigma,\rho,u}$, for some $\sigma$ and $\rho$. Next, we analyze the range of $Y_{\sigma,\rho,u}(\kappa)$.
Case I: For $\alpha \geq 0$, if we take $\kappa > 0$, then by letting

$$y = \rho\left[\alpha + \kappa\frac{\sigma}{\rho^2}\right], \tag{12}$$

we see that $\alpha\rho^2 - \rho y + \sigma\kappa = 0$ implies that

$$\rho = \frac{y \pm \sqrt{y^2 - 4\alpha\sigma\kappa}}{2\alpha}. \tag{13}$$

Furthermore, by using (12) for $\sigma \leq \frac{-(1+\rho^2)}{2}$, we have

$$y = \alpha\rho - \kappa\frac{(1+\rho^2)}{2\rho}, \text{ implies that } \rho = \frac{-y \pm \sqrt{y^2 - (\kappa - 2\alpha)\kappa}}{2(\kappa - 2\alpha)}.$$

Taking $y^2 - (\kappa - 2\alpha)\kappa \geq 0$, we have $y \geq \sqrt{(\kappa - 2\alpha)\kappa}$, where $(\kappa - 2\alpha)\kappa$ is taken to be positive. Also,

$$\begin{aligned} \left|Y_{\sigma,\rho,u}(\kappa)\right| &= \left|\alpha - \kappa\frac{1}{2\rho^2}\left(1+\rho^2\right)\right||\rho| \geq \left[\alpha - \kappa\frac{(1+\rho^2)}{2\rho^2}\right]|\rho| \\ &\geq \sqrt{(\kappa - 2\alpha)\kappa} = \sqrt{(X - 1 - \beta\mu)^2 - \alpha^2}. \end{aligned}$$

Hence $Y_{\sigma,\rho,u}(X) = \sqrt{(X - 1 - \beta\mu)^2 - \alpha^2}$ holds for all reals. For $y \in \mathbb{R}$, we can select a $\rho$ given by (13), so that (12) holds.
Case II: For $\kappa \leq 0$, we have

$$\left|Y_{\sigma,\rho,u}(\kappa)\right| \geq \sqrt{(\kappa - 2\alpha)\kappa} = \sqrt{(\kappa - \alpha)^2 - \alpha^2} = \sqrt{(X - 1 - \beta\mu)^2 - \alpha^2},$$

where we minimize $\left(\alpha - \kappa\frac{(1+\rho^2)}{2\rho^2}\right)|\rho|$ in respect to $\rho$. Thus, for $\alpha \geq 0$ and $\kappa \leq 0$, $\left|Y_{\sigma,\rho,u}(X)\right|^2$ assumes all values greater than or equal to $(\kappa - 2\alpha)\kappa = (-1 + X - \beta\mu)^2 - \alpha^2$. Therefore, if $\kappa \leq 0$, then $X \leq 1 - \alpha + \beta\mu$, and in this case the range of $\wp^{\mu}_{\alpha,\beta}$ is

$$\mathbb{C}\backslash\Delta_1 = \left\{s = x + iy : x \leq 1 - \alpha + \beta\mu;\ \ y^2 \geq (x - 1 - \beta\mu)^2 - \alpha^2;\ \ \alpha \geq 0\right\}.$$

Case III: Again, for $\kappa \leq 0$, we have

$$\begin{aligned} \left|Y_{\sigma,\rho,u}(\kappa)\right| &= -\left[\alpha - \kappa\frac{1+\rho^2}{2\rho^2}\right]|\rho| \geq |\rho|\left(\kappa\frac{1+\rho^2}{2\rho^2} - \alpha\right) \geq \sqrt{(\kappa - 2\alpha)\kappa} \\ &= (-1 + X - \beta\mu)^2 - \alpha^2. \end{aligned}$$

For $\alpha < 0$, $\kappa \geq 0$, we observe that $X > 1 - \alpha + \beta\mu$ and $y^2 \geq (x - 1 - \beta\mu)^2 - \alpha^2$, and, in this case, the range for $\wp^{\mu}_{\alpha,\beta}$ is

$$\mathbb{C}\backslash\Delta_2 = \left\{s = x + iy : x > 1 - \alpha + \beta\mu;\ \ y^2 \geq (x - 1 - \beta\mu)^2 - \alpha^2;\ \ \alpha < 0\right\}.$$

Thus, for $\alpha < 0$, the union of graphs of $Y_{\sigma,\rho,u}(\kappa)$ lies in the set $\mathbb{C}\backslash\Delta_2$; $\forall\ \rho \in \mathbb{R}$ and $\sigma \leq -\frac{1}{2}(1+\rho^2)$.

Combining the above cases, as well as applying Lemma 3, we complete the proof of the above Theorem. □

**Remark 1.** *From the geometry of the regions $\Delta_1$ and $\Delta_2$ along with $f \in \mathcal{A}$, $-1 \leq \mu \leq 1$, $\alpha, \beta \in \mathbb{R}$ and $\xi \in \mathbb{E}$, if $\alpha \geq 0$; $\frac{\alpha+\beta}{2} \geq 1 - \alpha + \beta\mu : \beta \geq \frac{2-3\alpha}{1-2\mu}$, then we obtain the results listed below.*

**Corollary 1.** *Let $\alpha, \beta \in \mathbb{R}$, $f \in \mathcal{A}$, $\alpha, \beta \in \mathbb{R}$, $-1 \leq \mu \leq 1$ and $\xi \in \mathbb{E}$. If either $\alpha \geq 0$; $\beta < \alpha$ and*

$$\Re e\wp^{\mu}_{\alpha,\beta}(f)(\xi) > \beta\mu - \alpha + 1,$$

*or* $\alpha < 0; \beta > \alpha$ *and*

$$\Re e \wp^{\mu}_{\alpha,\beta}(f)(\xi) < \beta\mu - \alpha + 1,$$

*then* $f \in \mathcal{G}$.

## 4. Maximization of the Fekete-Szegö Functional

We define a new class $\mathcal{MF}^{\mu}_{\alpha,\beta}$ in connection with the nonlinear operator $\wp^{\mu}_{\alpha,\beta}(f)$.

**Definition 3.** *For* $f \in \mathcal{A}$, *we have* $f \in \mathcal{MF}^{\mu}_{\alpha,\beta}$, *that is*

$$\Re e \wp^{\mu}_{\alpha,\beta}(f)(\xi) > \frac{\alpha+\beta}{2},$$

*where* $\beta \geq \frac{2-3\alpha}{1-2\mu}$ *or*

$$\Re e \wp^{\mu}_{\alpha,\beta}(f)(\xi) > \frac{1-\alpha(1+\mu)}{(1-2\mu)}.$$

Thus, we note that
For $\mu = 1$ and $\alpha + \beta \geq 2$, we have $\mathcal{MF}^{\mu}_{\alpha,\beta} = \emptyset$.
For $\alpha = 1$, $\beta = 0$ and $-1 \leq \mu \leq 1$, we have $\mathcal{MF}^{\mu}_{1,0} = \mathcal{G}$.
For $\alpha = 0$ and $-1 \leq \mu \leq 1$, we have $\mathcal{MF}^{\mu}_{\alpha,\beta} = \mathcal{M}_{1-\beta}$ of order $\frac{\beta}{2}$, where $\beta \geq \frac{2}{1-2\mu}$.

**Remark 2.** *For* $\alpha \geq 0$, $-1 \leq \mu \leq 1$ *and* $\beta \leq \frac{2-3\alpha}{1-2\mu}$, *we have* $\frac{\alpha+\beta}{2} \geq 1 - \alpha + \beta\mu$ *and* $\mathcal{MF}^{\mu}_{\alpha,\beta} \subset \mathcal{G}$.

In the next Theorem, we work out the conditions of $\alpha$, $\beta$ and $\mu$, so that for $\phi(.,\lambda)$, we obtain $|\phi(f,\lambda)| \leq \max(s, |1-\lambda|)$.

**Theorem 3.** *Let* $f \in \mathcal{A}$, $\alpha, \beta \in \mathbb{R}$, $-1 \leq \mu \leq 1$ *and* $\xi \in \mathbb{E}$ *satisfy*

$$\alpha - (2\mu - 3)\beta < 2, \quad (-3\mu - 2)\alpha - (2\mu + 1)\beta < 2(2\mu + 1),$$

*and*

$$\lambda_0 = (\alpha + 2\beta)\mu + 2(1 - \alpha - \beta)(2\mu + 1),$$

*such that* $\beta = \frac{2(\mu+1)}{4+3\mu-6\mu^2}$. *If we denote the level set of function*

$$\varphi\binom{\mu}{\alpha,\beta} = \frac{2 + (2\mu - 3)\beta - \alpha}{(\alpha + 2\beta)\mu + 2(1 - \alpha - \beta)(2\mu + 1)} > 0,$$

*then* $|\phi(f,\lambda)| \leq \max\Big(\varphi\binom{\mu}{\alpha,\beta}|1-\lambda|\Big)$ *for* $\lambda \in \mathbb{C}$, *over the family* $\mathcal{MF}^{\mu}_{\alpha,\beta}$.

**Proof.** Let $f \in \mathcal{A}$ be in the family $\mathcal{MF}^{\mu}_{\alpha,\beta}$. Then, $\wp^{\mu}_{\alpha,\beta}(f)(\xi)$ is obtained by using

$$\alpha\left(\frac{f(\xi)}{\xi}\right)^{\mu} = \alpha + \alpha\mu a_2\xi + \alpha\left[\mu a_3 + \frac{\mu(\mu-1)}{2}a_2^2\right]\xi^2 + ...,$$

$$\beta\mu\frac{\xi f'(\xi)}{f(\xi)} = \beta\mu + \beta\mu a_2\xi + \beta\mu\Big(2a_3 - a_2^2\Big)\xi^2 + ...,$$

and

$$\frac{\mu\xi\left(\frac{\xi f'(\xi)}{f(\xi)}\right)'}{\mu\frac{\xi f'(\xi)}{f(\xi)} - \mu + 1} + \frac{\xi f'(\xi)}{f(\xi)} = 1 + (1+\mu)a_2\xi + \Big[(1+2\mu)2a_3 - (\mu+1)^2a_2^2\Big]\xi^2 + ...$$

in (6), as seen below

$$\wp^{\mu}_{\alpha,\beta}(f)(\xi) = 1-(1-\mu)\beta+(\mu+\Lambda)a_2\xi+\left[\lambda_0 a_3-\lambda_1 a_2^2\right]\xi^2+..., \quad (14)$$

where

$$\Lambda = 1-\alpha-\beta, \lambda_0 = (-3\mu-2)\alpha-(2\mu+1)(\beta+2)$$

and

$$\lambda_1 = \left(1-\frac{3\alpha}{2}-\beta\right)\mu^2-\left(\frac{3}{2}\alpha+\beta-2\right)\mu+\Lambda.$$

Now, consider that

$$s(\xi) = \frac{\wp^{\mu}_{\alpha,\beta}(f)(\xi)-\wp^{\mu}_{\alpha,\beta}(f)(0)}{-2\backsim+\wp^{\mu}_{\alpha,\beta}(f)(0)+\wp^{\mu}_{\alpha,\beta}(f)(\xi)}.$$

By using (14), we see that

$$s(\xi) = \frac{(\Lambda+\mu)a_2\xi}{2+\mu_1\beta-\alpha}+\left[\frac{(\lambda_0+\lambda_1 a_2^2)a_3}{2+\mu_1\beta-\alpha}+\frac{(\Lambda+\mu)^2a_2^2}{[2+\mu_1\beta-\alpha]^2}\right]\xi^2+...,$$

where $\mu_1 = 2\mu-3$. If we take $s(\xi) = \sum_{j=1}^{\infty} b_j\xi^j$ and $\mu_1 = 2\mu-3$, then we note that

$$s(\xi) = \frac{(\Lambda+\mu)a_2\xi}{2+\mu_1\beta-\alpha}+\left[\frac{(\lambda_0+\lambda_1 a_2^2)a_3}{2+\mu_1\beta-\alpha}+\frac{(\Lambda+\mu)^2a_2^2}{[2+\mu_1\beta-\alpha]^2}\right]\xi^2+\sum_{j=3}^{\infty} b_j\xi^j.$$

On comparison, we have

$$b_1 = \frac{(\Lambda+\mu)}{2+\mu_1\beta-\alpha}a_2 = \tau_0 a_2,$$

$$b_2 = \frac{\lambda_0}{2+\mu_1\beta-\alpha}\left[a_3-\frac{\lambda_1[\alpha-2-\mu_1\beta]-(\Lambda+\mu)^2}{[2+\mu_1\beta-\alpha]\lambda_0}a_2^2\right] = \tau_1\left[a_3-\tau_2 a_2^2\right],$$

where $\mu_1 = 2\mu-3$, $\tau_0 = \frac{(1-\alpha-\beta+\mu)}{2+\mu_1\beta-\alpha}$, $\tau_1 = \frac{\lambda_0}{2+\mu_1\beta-\alpha}$, $\tau_2 = \frac{\lambda_1(\alpha-2-\mu_1\beta)-(1-\alpha-\beta+\mu)^2}{(2+\mu_1\beta-\alpha)\lambda_0}$ and $\tau_3 = \tau_1\tau_2$. Thus, with the help of Lemma 4, we obtain the following relation

$$\left|b_2-sb_1^2\right| = \left|\tau_1\left[a_3-\tau_2 a_2^2\right]-s\tau_0^2a_2^2\right| = \left|\tau_1 a_3-\left(\tau_3+s\tau_0^2\right)a_2^2\right|,$$

or we see that

$$\left|b_2-sb_1^2\right| = |\tau_1|\left|a_3-\frac{(\tau_3+s\tau_0^2)}{\tau_1}a_2^2\right|.$$

If we denote $\lambda = \frac{1}{\tau_1}\left(\tau_3+s\tau_0^2\right)$, then we write

$$\left|a_3-\lambda a_2^2\right| = \frac{1}{|\tau_1|}\left|b_2-sb_1^2\right|,$$

or

$$\left|a_3-\lambda a_2^2\right| = |\tau|\left|b_2-sb_1^2\right|, \text{ where } |\tau| = \frac{1}{|\tau_1|}$$

and the level set of functions is obtained from

$$\varphi\binom{\mu}{\alpha,\beta} = \left[\frac{\lambda_0}{(2+(2\mu-3)\beta-\alpha)}\right]^{-1} = \frac{(2+(2\mu-3)\beta-\alpha)}{(-3\mu-2)\alpha-(2\mu+1)(\beta+2)}.$$

On setting

$$\alpha = 2 + (2\mu - 3)\beta \quad \text{and} \quad (-3\mu - 2)\alpha - (2\mu + 1)(\beta + 2) = 0,$$

we obtain

$$\beta = \frac{2(\mu+1)}{4 + 3\mu - 6\mu^2}.$$

Hence, the level sets are rays starting from the points $\left(2 + \frac{2(\mu+1)(2\mu-3)}{4+3\mu-6\mu^2}, \frac{2(\mu+1)}{4+3\mu-6\mu^2}\right)$ and lying under the lines $\alpha - (2\mu - 3)\beta = 2$ and $(-3\mu - 2)\alpha - (2\mu + 1)\beta = 2(2\mu + 1)$. □

## 5. Filtration Problems for Some Related Classes

In this section, we take $\alpha = 0$ and consider the class $\mathcal{MF}^{\mu}_{0,\beta}$, consisting of functions $f \in \mathcal{A}$, that satisfy the inequality

$$\Re e\left\{(1 - \beta + \beta\mu)w(\xi) + \frac{(1-\beta)\mu\xi w'(\xi)}{\mu(w(\xi) - 1) + 1}\right\} > \frac{\beta}{2},$$

or we see that

$$\Re e\left\{\{1 + \beta(\mu - 1)\}w(\xi) + \frac{(1-\beta)\mu\xi w'(\xi)}{\mu(w(\xi) - 1) + 1}\right\} > \frac{\beta}{2}, \tag{15}$$

where $w(\xi) = \frac{\xi f'(\xi)}{f(\xi)}$. This can also be rewritten as:

$$\mathcal{MF}^{\mu}_{0,\beta} = \left\{f \in \mathcal{A} : 2\Re e\wp^{\mu}_{0,\beta}(f)(\xi) > \beta \text{ for } \beta \geq \frac{2}{1 - 2\mu}\right\}.$$

**Theorem 4.** *For $\beta \neq 1$, we have*

$$f \in \mathcal{MF}^{\mu}_{0,\beta} \Longleftrightarrow \left[1 - \mu + \mu\frac{\xi f'(\xi)}{f(\xi)}\right]^{1-\beta}[f(\xi)]^{1+\beta\mu-\beta} \in \mathcal{S}^*.$$

**Proof.** From (15), we note that

$$g(\xi) = [1 - \mu + \mu w(\xi)]^{1-\beta}[f(\xi)]^{1+\beta\mu-\beta},$$

or we see that

$$\frac{\xi g'(\xi)}{g(\xi)} = \{1 - \beta + \beta\mu\}w(\xi) + (1 - \beta)\mu\frac{\xi w'(\xi)}{1 - \mu + \mu w(\xi)}.$$

where $w(\xi) = \frac{\xi f'(\xi)}{f(\xi)}$. Thus, we obtain the desired conclusion. □

We also assume $\alpha = 0$ and $\mu = 1$ to consider the class $\mathcal{MF}^{1}_{0,\beta}$, consisting of functions $f \in \mathcal{A}$, satisfying

$$\mathcal{MF}^{1}_{0,\beta} = \left\{f \in \mathcal{A} : \Re e\left\{(1 - \beta)\frac{(\xi f'(\xi))'}{f'(\xi)} + \beta\frac{\xi f'(\xi)}{f(\xi)}\right\} > \frac{\beta}{2}\right\}.$$

The function in this class is equivalent to the $(1 - \beta)$-convex function of order $\frac{\beta}{2}$, first seen in [11] and then investigated by others. For more details, see [12–15]. Moreover, we obtain the following result, as shown by Elin, see [9].

**Corollary 2.** *For $\beta < 2$ and $\beta \neq 1$, we have*

$$f \in \mathcal{MF}^{1}_{0,\beta} \Longleftrightarrow g(\xi) = \xi\left[f'(\xi)\right]^{\frac{2-2\beta}{2-\beta}}\left[\frac{f(\xi)}{\xi}\right]^{\frac{2\beta}{2-\beta}} \in \mathcal{S}^{*},$$

*where*

$$f(\xi) = \left[\frac{1}{1-\beta}\int_0^{\xi} s^{\frac{\beta}{1-\beta}}\left(\frac{g(s)}{s}\right)^{\frac{2-\beta}{2-2\beta}} ds\right]^{1-\beta}.$$

**Theorem 5.** *Let $0 \leq \beta \leq 1$, $-1 \leq \mu \leq 1$ and $f \in \mathcal{MF}^{\mu}_{0,\beta}$, such that,*

$$\Re e \wp^{\mu}_{0,\beta}(f)(\xi) = \Re e\left\{\{1+\beta(\mu-1)\}w(\xi) + \frac{\mu(1-\beta)\xi w'(\xi)}{\mu(w(\xi)-1)+1}\right\} > \frac{\beta}{2},$$

*where $w(\xi) = \frac{\xi f'(\xi)}{f(\xi)}$. Then, $\Re e\frac{\xi f'(\xi)}{f(\xi)} > \frac{1}{2}$, $\xi \in \mathbb{E}$, that is, $f \in \mathcal{S}^{*}_{\frac{1}{2}}$.*

**Proof.** Here, we use Lemma 2, which implies that $\Re e\frac{\xi f'(\xi)}{f(\xi)} > \frac{1}{2}$, if and only if the function $s$ is defined by (8), such that

$$s(\xi) = 1 - \frac{1}{w(\xi)}, w(\xi) = \frac{\xi f'(\xi)}{f(\xi)},$$

which, on differentiation, leads to the following.

$$\frac{1}{1-s(\xi)} = w(\xi), \text{ and } 1 - w(\xi) + \frac{\xi f''(\xi)}{f'(\xi)} = \frac{\xi s'(\xi)}{1-s(\xi)}. \tag{16}$$

Here, we make use of Jack's Lemma 5 to achieve our task. We assume, on the contrary, that $s$ is not an analytic self-mapping of $\mathbb{E}$. Then, for $\xi_0 \in \mathbb{E} : |s(\xi)| < 1$ for all $|\xi| < |\xi_0|$ and $|s(\xi_0)| = 1$. By Lemma 5, we have

$$\frac{\xi_0 s'(\xi_0)}{s(\xi_0)} = m \geq 1.$$

Using notation $s(\xi_0) = a + ib$, for some $a, b \in \mathbb{R} : a^2 + b^2 = 1$, we have

$$\Re e\left\{\frac{1}{1-s(\xi_0)}\right\} = \frac{1-a}{1-2a+(a^2+b^2)} = \frac{1}{2}.$$

Hence, for $0 \leq \beta \leq 1$ and $-1 \leq \mu \leq 1$, (16) yields

$$\begin{aligned} &\Re e \wp^{\mu}_{0,\beta}(f)(\xi_0) \\ &= \Re e\left\{\frac{1}{2}\{1+\beta(\mu-1)\} + \frac{m(1-\beta)\mu s(\xi_0)}{\left(1-\frac{1}{2}\mu\right)(1-s(\xi_0))^2} - \frac{\beta}{2}\right\} \\ &= \frac{1+\beta(\mu-1)}{2} - \frac{(1-\beta)\mu}{2-\mu} - \frac{\beta}{2} \leq 0, \end{aligned}$$

which contradicts our assumption. This completes the desired proof of the Theorem. □

For $\alpha = 0$ and $\mu = 1$, we obtain the following corollary, as seen in [9,13].

**Corollary 3.** *If* $0 \leq \beta \leq 1$, $\mu = 1$ *and* $f \in \mathcal{MF}^{1}_{0,\beta}$, *that is,*

$$\Re e \wp^{1}_{0,\beta}(\xi_0) = \Re e \left\{ \beta \frac{\xi f'(\xi)}{f(\xi)} + (1-\beta) \frac{(\xi f'(\xi))'}{f'(\xi)} \right\} > \frac{\beta}{2},$$

*then we have the assertion* $\Re e \frac{\xi f'(\xi)}{f(\xi)} > \frac{1}{2}$, *that is,* $f \in \mathcal{S}^*\left(\frac{1}{2}\right)$.

**Theorem 6.** *For* $\beta < \beta_1 \leq 1$, *we have* $\mathcal{MF}^{\mu}_{0,\beta} \subseteq \mathcal{MF}^{\mu}_{0,\beta_1}$.

**Proof.** Let $f \in \mathcal{MF}^{\mu}_{0,\beta}$. Then,

$$\Re e \wp^{\mu}_{0,\beta}(f)(\xi) > \frac{\beta}{2} : \beta \geq \frac{2}{1-2\mu}, \ -1 \leq \mu \leq 1$$

implies that there exists $s(\xi) \in \Omega = \{s \in \mathcal{A} : s(0) = 0\}$, such that

$$\begin{aligned} \wp^{\mu}_{0,\beta}(f)(\xi) &= \frac{(1-\beta)\mu\xi w'(\xi)}{1-\mu+\mu w(\xi)} + (1-\beta+\beta\mu)w(\xi); \qquad \left(w(\xi) = \frac{\xi f'(\xi)}{f(\xi)}\right) \\ &= \frac{1-(1-\mu)\beta}{1-s(\xi)} + \frac{(1-\beta)\mu s(\xi)}{[1-s(\xi)][(1-\mu)(1-s(\xi))+\mu]} \\ &= \frac{1}{1-s(\xi)} \frac{[1-\beta(1-\mu)][(1-\mu)(1-s(\xi))+\mu] + (1-\beta)\mu s(\xi)}{\mu+(1-\mu)(1-s(\xi))}. \end{aligned}$$

Furthermore, by letting, $A = 1 + \beta\mu - \beta$, $B = 1 - \mu$ and $C = \mu - \beta\mu$, we note that

$$\wp^{\mu}_{0,\beta}(f)(\xi) - \frac{1}{2}\beta = \frac{1}{1-s(\xi)} \frac{A[B(1-s(\xi))+\mu] + Cs(\xi)}{\mu + B(1-s(\xi))} - \frac{1}{2}\beta$$

or we can write

$$\wp^{\mu}_{0,\beta}(f)(\xi) = \frac{2A(B+\mu) - \beta(B+\mu) + [\beta(\mu+2B) - 2(AB-C)]s(\xi) - \beta B s^2(\xi)}{2(B+\mu) + 2Bs^2(\xi) - 2(2B+\mu)s(\xi)} + \frac{1}{2}\beta.$$

By using (8), we note that

$$s_\beta(\xi) = -\frac{\wp^{\mu}_{0,\beta}(f)(0) - \wp^{\mu}_{0,\beta}(f)(\xi)}{\wp^{\mu}_{0,\beta}(f)(\xi) - 2\backsim + \wp^{\mu}_{0,\beta}(f)(0)}.$$

That is,

$$s_\beta(\xi) = \frac{1 - \wp^{\mu}_{0,\beta}(f)(\xi) - (1-\mu)\beta}{-\wp^{\mu}_{0,\beta}(f)(\xi) - 1 + \beta + (1-\mu)\beta}. \tag{17}$$

Similarly for $\beta_1$, we consider that

$$s_{\beta_1}(\xi) = \frac{\wp^{\mu}_{0,\beta_1}(f)(\xi) + (1-\mu)\beta_1 - 1}{1 - \beta_1 - (1-\mu)\beta_1 + \wp^{\mu}_{0,\beta_1}(f)(\xi)},$$

which is analytic in the disk or neighbourhood $\mathbb{E}$ and zero at origin. We rewrite (17) as

$$[s_\beta(\xi)]\left[1 - \beta - (1-\mu)\beta + \wp^{\mu}_{0,\beta_1}(f)(\xi)\right] = \wp^{\mu}_{0,\beta_1}(f)(\xi) - 1 + \beta(1-\mu). \tag{18}$$

Similarly, we see that

$$[s_{\beta_1}(\xi)]\left[1 - \beta_1 + (\mu-1)\beta_1 + \wp^{\mu}_{0,\beta_1}(f)(\xi)\right] = \wp^{\mu}_{0,\beta_1}(f)(\xi) - 1 - (\mu-1)\beta_1. \tag{19}$$

Furthermore, we define the function

$$\wp^{\mu}_{0,\frac{1}{2}}(f)(\xi) = \frac{\frac{1}{2}\mu\xi\left(\frac{\xi f'(\xi)}{f(\xi)}\right)'}{1-\mu+\mu\frac{\xi f'(\xi)}{f(\xi)}} - 1 + \beta(1-\mu),$$

or

$$h(\xi) = \mu\frac{\xi\left(\frac{\xi f'(\xi)}{f(\xi)}\right)'}{1-\mu+\mu\frac{\xi f'(\xi)}{f(\xi)}} + (1+\mu)\frac{\xi f'(\xi)}{f(\xi)},$$

where $h(\xi) = 2\wp^{\mu}_{0,\frac{1}{2}}(f)(\xi)$. Then, (18) implies that

$$\begin{aligned}&(\beta-1)\left[1-s_{\beta}(\xi)\right]h(\xi)\\&=\{(\mu-1)\beta-1\}+(1-2\beta)\mu\left[1-s_{\beta}(\xi)\right]\frac{\xi f'(\xi)}{f(\xi)}-\{(\mu-2)\beta+1\}s_{\beta}(\xi).\end{aligned} \tag{20}$$

We solve (20) to obtain the value of $h(\xi)$ as

$$h(\xi) = \frac{1}{(1-\beta)\left[1-s_{\beta}(\xi)\right]} + \frac{1-2\beta}{1-\beta}\mu\frac{\xi f'(\xi)}{f(\xi)} + \frac{s_{\beta}(\xi)}{1-s_{\beta}(\xi)} + \frac{\beta}{1-\beta}(1-\mu). \tag{21}$$

Furthermore, Equation (19) leads to

$$(1-\beta_1)h(\xi) = (1-2\beta_1)\mu\frac{\xi f'(\xi)}{f(\xi)} + \frac{1}{1-s_{\beta_1}(\xi)} + \frac{(1-\beta_1)s_{\beta_1}(\xi)}{1-s_{\beta_1}(\xi)} + (1-\mu)\beta_1. \tag{22}$$

From (21) and (22), we see that

$$\begin{aligned}&\frac{1}{(1-\beta)\left[1-s_{\beta}(\xi)\right]} + \left\{\frac{\beta_1}{1-\beta_1} - \frac{\beta}{1-\beta}\right\}\mu\frac{\xi f'(\xi)}{f(\xi)} + \frac{s_{\beta}(\xi)}{1-s_{\beta}(\xi)} - \beta\left(\frac{\mu-1}{1-\beta}\right)\\&= \frac{1-\mu}{1-\beta_1}\beta_1 + \frac{s_{\beta_1}(\xi)}{1-s_{\beta_1}(\xi)} + \frac{1}{(1-\beta_1)\left[1-s_{\beta_1}(\xi)\right]}.\end{aligned} \tag{23}$$

On the contrary, we may assume that $s_{\beta_1}(\xi)$ will not be a self-mapping of $\mathbb{E}$. Therefore, $\exists$ $\xi_0 \in \mathbb{E} : \left|s_{\beta_1}(\xi)\right| < 1$ for $|\xi| < |\xi_0|$ and $\left|s_{\beta_1}(\xi_0)\right| = 1$. Substitute $\xi = \xi_0$ in (23) to have

$$\frac{s_{\beta}(\xi)}{1-s_{\beta}(\xi)} = \frac{-1}{2}, \quad \frac{1}{1-s_{\beta}(\xi)} = \frac{1}{2},$$

and

$$\begin{aligned}&\left\{\frac{\beta_1}{1-\beta_1} - \frac{\beta}{1-\beta}\right\}\mu\Re e\frac{\xi_0 f'(\xi_0)}{f(\xi_0)} + \Re e\frac{1}{(1-\beta)\left[1-s_{\beta}(\xi_0)\right]} + \Re e\frac{s_{\beta}(\xi_0)}{1-s_{\beta}(\xi_0)} - \beta\frac{\mu-1}{1-\beta}\\&= \Re e\frac{1}{(1-\beta_1)\left[1-s_{\beta_1}(\xi_0)\right]} + \Re e\frac{s_{\beta_1}(\xi_0)}{1-s_{\beta_1}(\xi_0)} - \beta_1\left(\frac{\mu-1}{1-\beta_1}\right).\end{aligned}$$

Or, we see that

$$\left\{\frac{\beta_1}{1-\beta_1} - \frac{\beta}{1-\beta}\right\}\mu\Re e\frac{\xi_0 f'(\xi_0)}{f(\xi_0)} + \frac{1}{2-2\beta} - \frac{1}{2} - \beta\frac{\mu-1}{1-\beta} > \frac{1}{2-2\beta_1} - \frac{1}{2} + \frac{1-\mu}{1-\beta_1}\beta_1.$$

By using Theorem 5, we obtain

$$\left\{\frac{\beta_1}{1-\beta_1}-\frac{\beta}{1-\beta}\right\}\mu\Re e\frac{\xi_0 f'(\xi_0)}{f(\xi_0)}+\frac{1}{2-2\beta}-\frac{\mu-1}{1-\beta}\beta$$
$$>\frac{1}{2}\left\{\frac{\beta_1}{1-\beta_1}-\frac{\beta}{1-\beta}\right\}\mu+\frac{1}{2-2\beta}-\frac{\mu-1}{1-\beta}\beta=\frac{1}{2(1-\beta_1)}-\frac{\mu-1}{1-\beta_1}\beta_1.$$

This leads to a contradiction to our assumption. Therefore, it is obvious that the family $\left\{\mathcal{MF}^{\mu}_{0,\beta}, 1\le\mu\le 1\right\}$ is a filtration. □

**Corollary 4.** *If $\mu=1$, then for $\beta<\beta_1\le 1$, we have $\mathcal{MF}^1_{0,\beta}\subseteq\mathcal{MF}^1_{0,\beta_1}$, where*

$$\wp^{\mu}_{0,\beta}(f)(\xi)=\frac{1}{2}\beta-\left(\frac{1}{2}\beta-1\right)\frac{1+s(\xi)}{1-s(\xi)}.$$

For the reference of the above inequality, see [9].

**6. Interpolation to Fekete-Szegö Functional**

In the next Theorem, we work out the interpolation result to estimate the Fekete–Szegö functional $\phi(f,\lambda)$, given by (4).

**Theorem 7.** *If the family $\left\{\mathcal{MF}^{\mu}_{0,\beta},\ -1\le\mu\le 1\right\}$ is a filtration of $\mathcal{G}$ such that $\mathcal{MF}^{\mu}_{0,\beta}\subset\mathcal{G}$ is satisfied, then*

$$\sup_{f\in\mathcal{MF}^{\mu}_{0,\beta}}|\phi(f,\lambda)|\le\max\left(\mu_{0,\beta},|1-\lambda|\right),\ for\ \lambda\in\mathbb{C},$$

*over the family $\mathcal{MF}^{\mu}_{0,\beta}$, where $\mu_{0,\beta}=\frac{2+(2\mu-3)\beta}{2\beta\mu+2(1-\beta)(2\mu+1)}$.*

**Proof.** In Theorem 6, we already proved that the family $\left\{\mathcal{MF}^{\mu}_{0,\beta}, -1\le\mu\le 1\right\}$ is a filtration of $\mathcal{G}$. Moreover, applying Theorem 3 for $\alpha=0$ gives the desired result. □

As a special case, if we take $\alpha=0$, $\mu=1$ and $\beta\in\mathbb{R}$, we obtain the following corollary, as seen in [9].

**Corollary 5.** *The family $\left\{\mathcal{MF}^1_{0,\beta}\right\}$ is a filtration of $\mathcal{G}$ and satisfies $\mathcal{MF}^1_{0,\beta}\subset S^*\left(\frac{1}{2}\right)$. Also,*

$$\sup_{f\in\mathcal{MF}^1_{0,\beta}}|\phi(f,\lambda)|\le\max\left(\mu_{0,\beta},|1-\lambda|\right),$$

*where $\lambda\in\mathbb{C}$ and $\mu_{0,\beta}=\frac{2-\beta}{6-4\beta}$. In view of (4), the supremum is obtained; whenever $\beta=0$ and $\beta=1$, this estimate is sharp for $\beta\in(0,1)$ such that there exist two functions, $f_1$ and $f_2\in\mathcal{MF}^1_{0,\beta}$, so that*

$$\wp^1_{0,\beta}(f_1)(\xi)=\left(1-\frac{\beta}{2}\right)\frac{1+\xi}{1-\xi}+\frac{\beta}{2},$$

*and*

$$\wp^1_{0,\beta}(f_2)(\xi)=\left(1-\frac{\beta}{2}\right)\frac{1+\xi^2}{1-\xi^2}+\frac{\beta}{2}$$

*are constructed from (6) and satisfy the Briot–Bouquet differential equation. Hence, $|\phi(f_1,\lambda)|\le|1-\lambda|$, $\lambda\in\mathbb{C}$ and $|\phi(f_2,\lambda)|\le\frac{2-\beta}{6-4\beta}$.*

## 7. Some Filtration Classes

Here, we investigate the case where $\beta = -\alpha + 1$ for $\alpha < 2$. The class $\mathcal{MF}^{\mu}_{\alpha,1-\alpha}$ contains functions given by

$$\wp^{\mu}_{\alpha,1-\alpha}(f)(\xi) = \mu(1-\alpha)\frac{\xi f'(\xi)}{f(\xi)} + \alpha\left(\frac{f(\xi)}{\xi}\right)^{\mu},$$

and

$$\mathcal{MF}^{\mu}_{\alpha,1-\alpha} = \left\{f \in \mathcal{A} : \Re e\wp^{\mu}_{\alpha,1-\alpha}(f)(\xi) > \frac{1}{2}\right\}.$$

For $\mu = 1$, we obtain the known result of Marx-Strohhäcker given by

$$\Re e\left(\frac{\xi f'(\xi)}{f(\xi)}\right) > \frac{1}{2} \iff \Re e\left(\frac{f(\xi)}{\xi}\right) > \frac{1}{2},$$

and, hence, $\mathcal{S}^*\left(\frac{1}{2}\right) \subset \mathcal{MF}^{\mu}_{\alpha,1-\alpha}$ for any $\alpha < 2$.

**Theorem 8.** *For $f \in \mathcal{S}^*\left(\frac{1}{2}\right)$, $\Re e\left(\frac{f(\xi)}{\xi}\right)^{\mu} > \frac{1}{2}$ and $\mathcal{S}^*\left(\frac{1}{2}\right) \subset \mathcal{MF}^{\mu}_{\alpha,1-\alpha}$, $-1 \leq \mu \leq 1$. The sharpness occurs for the function $\xi(1-\xi^n)^{-\frac{1}{\mu}}$.*

**Proof.** Assume that

$$h(\xi) = 2\,\frac{[f(\xi)]^{\mu}}{\xi^{\mu}} - 1$$

and

$$\frac{1}{\mu}\frac{\xi h'(\xi)}{h(\xi)+1} + 1 = \frac{\xi f'(\xi)}{f(\xi)}.$$

For $s = \xi h'(\xi)$, $r = h(\xi)$, $\Psi(r,s) = \frac{s}{(r+1)\mu} + 1 : \Psi(i\rho,\sigma) = \frac{\sigma}{\mu(\rho^2+1)} + 1$, if $\rho \in \mathbb{R}$ and $\sigma \leq -\mu\left(\frac{\rho^2+1}{2}\right)$, then $\Re e[\Psi(i\rho,\sigma)] = \Re e\left(\frac{\sigma}{(\rho^2+1)\mu} + 1\right) \leq 0$. Since $f \in \mathcal{S}^*\left(\frac{1}{2}\right)$, we write

$$\Re e\left[\Psi\left(h, \xi h'\right)(\xi)\right] > 0.$$

Therefore, $\Re e h(\xi) > \frac{1}{2}$, and this implies that $\Re e\left(\frac{f(\xi)}{\xi}\right)^{\mu} > \frac{1}{2}$. □

**Theorem 9.** *Let $\alpha < \alpha_1 \leq 1$. Then, we have $\mathcal{MF}^{\mu}_{\alpha,1-\alpha} \subseteq \mathcal{MF}^{\mu}_{\alpha_1,1-\alpha_1}$.*

**Proof.** Suppose that $f \in \mathcal{MF}^{\mu}_{\alpha,1-\alpha}$ for $-1 \leq \mu \leq 1$, $\xi \in \mathbb{E}$. Then, $\Re e\wp^{\mu}_{\alpha,1-\alpha}(f)(\xi) > \frac{1}{2}$ proves the existence of a function $s_{\beta}(\xi)$ so that we can write that

$$s_{\beta}(\xi) = \frac{\wp^{\mu}_{\alpha,1-\alpha}(f)(\xi) + 2 - \mu + \alpha\mu - \alpha}{-1 + \mu - \alpha\mu + \alpha + \wp^{\mu}_{\alpha,1-\alpha}(f)(\xi)}.$$

Then, from the above functional equation, we see that

$$\begin{aligned} s_{\alpha}(\xi)&\left[(1-\alpha)\mu\frac{\xi f'(\xi)}{f(\xi)} - 1 + \mu - \alpha\mu + \alpha + \alpha\left(\frac{f(\xi)}{\xi}\right)^{\mu}\right] \\ &= \alpha\left(\frac{f(\xi)}{\xi}\right)^{\mu} - 1 - (1-\mu)(1-\alpha) + (1-\alpha)\mu\frac{\xi f'(\xi)}{f(\xi)}. \end{aligned} \tag{24}$$

Furthermore, we see that

$$-2\wp^{\mu}_{-\frac{1}{2},\frac{1}{2}}(f)(\xi) = -h_1(\xi) = -\mu\frac{\xi f'(\xi)}{f(\xi)} + \left(\frac{f(\xi)}{\xi}\right)^{\mu}. \tag{25}$$

By using (25) in (24), we observe that

$$\{\alpha+\mu(1-\alpha)\}\frac{(f(\xi))^{\mu}}{\xi^{\mu}}-(1-\alpha)\mu\left\{\frac{(f(\xi))^{\mu}}{\xi^{\mu}}-\frac{\xi f'(\xi)}{f(\xi)}\right\}=\frac{1-(1-\alpha)\mu_1[1+s_\alpha(\xi)]}{1-s_\alpha(\xi)},$$

or

$$h_1(\xi)=\frac{1-(1-\alpha)\mu_1[1+s_\alpha(\xi)]}{[1-s_\alpha(\xi)](1-\alpha)}-\frac{\{\mu-\alpha\mu_1\}\left(\frac{f(\xi)}{\xi}\right)^{\mu}}{1-\alpha}, \tag{26}$$

where $\mu_1=1-\mu$. Similarly, in the case of $\alpha_1$, we have

$$\begin{aligned}&s_{\alpha_1}(\xi)\left[\alpha_1\left(\frac{f(\xi)}{\xi}\right)^{\mu}+(1-\alpha_1)\mu\frac{\xi f'(\xi)}{f(\xi)}-\mu_1(1-\alpha_1)\right]\\&=\alpha_1\left(\frac{f(\xi)}{\xi}\right)^{\mu}+(1-\alpha_1)\mu\frac{\xi f'(\xi)}{f(\xi)}+\mu_1(1-\alpha_1)-1\end{aligned}$$

and

$$h_1(\xi)=\frac{1-(1-\alpha_1)\mu_1[1+s_{\alpha_1}(\xi)]}{(1-\alpha_1)[1-s_{\alpha_1}(\xi)]}-\frac{\{\mu+\alpha_1\mu_1\}\left(\frac{f(\xi)}{\xi}\right)^{\mu}}{1-\alpha}, \tag{27}$$

where $\mu_1=1-\mu$. Equating (26) and (27), we observe that

$$\begin{aligned}&\frac{1-(1-\alpha)\mu_1[1+s_\alpha(\xi)]}{(1-\alpha)[1-s_\alpha(\xi)]}-\frac{\{\mu+\alpha\mu_1\}\left(\frac{f(\xi)}{\xi}\right)^{\mu}}{1-\alpha}\\&=\frac{1-(1-\alpha_1)\mu_1[1+s_{\alpha_1}(\xi)]}{(1-\alpha_1)[1-s_{\alpha_1}(\xi)]}-\frac{\{\mu+\alpha_1\mu_1\}\left(\frac{f(\xi)}{\xi}\right)^{\mu}}{1-\alpha_1}\end{aligned}$$

or we write

$$\begin{aligned}&\left\{\frac{\alpha_1}{1-\alpha_1}-\frac{\alpha}{1-\alpha}\right\}\left(\frac{f(\xi)}{\xi}\right)^{\mu}+\frac{1}{(\alpha-1)[s_\alpha(\xi)-1]}-\mu_1\frac{1+s_\alpha(\xi)}{1-s_\alpha(\xi)}\\&=\left[\frac{1}{[1-s_{\alpha_1}(\xi)](1-\alpha_1)}+\mu_1\frac{1+s_{\alpha_1}(\xi)}{s_{\alpha_1}(\xi)-1}\right]\end{aligned}$$

or, equivalently, we have

$$\begin{aligned}&\left\{\frac{\alpha_1}{1-\alpha_1}-\frac{\alpha}{1-\alpha}\right\}\left(\frac{f(\xi)}{\xi}\right)^{\mu}+\frac{1}{(1-\alpha)[1-s_\alpha(\xi)]}-\frac{\mu_1}{1-s_\alpha(\xi)}-\frac{\mu_1 s_\alpha(\xi)}{1-s_\alpha(\xi)}\\&=-\mu_1\left[\frac{1}{1-s_{\alpha_1}(\xi)}+\frac{s_{\alpha_1}(\xi)}{1-s_{\alpha_1}(\xi)}\right]+\frac{1}{(1-\alpha_1)[1-s_{\alpha_1}(\xi)]},\end{aligned} \tag{28}$$

where $\mu_1=1-\mu$. It is clear that $s_{\alpha_1}(0)=0$ and we assume, on the contrary, that $s_{\alpha_1}(\xi)$ is not a self-mapping of $\mathbb{E}$. Then, there exists $\xi_0\in\mathbb{E}:|s_{\alpha_1}(\xi)|<1\ \forall\ |\xi|<|\xi_0|$ while $|s_{\alpha_1}(\xi_0)|=1$. We substitute $\xi=\xi_0$ in (28) to have

$$\begin{aligned}&\left\{\frac{\alpha_1}{1-\alpha_1}-\frac{\alpha}{1-\alpha}\right\}\Re e\left(\frac{f(\xi)}{\xi}\right)^{\mu}-(1-\mu)\Re e\frac{1+s_\alpha(\xi)}{1-s_\alpha(\xi)}+\Re e\frac{1}{(1-\alpha)[1-s_\alpha(\xi)]}\\&>-\frac{1}{2}\left[\frac{1+\alpha}{1-\alpha}-\frac{\alpha_1}{1-\alpha_1}\right].\end{aligned}$$

Now, applying Theorem 5, we see that

$$-\Re e\frac{1}{(1-\alpha_1)[1-s_{\alpha_1}(\xi)]}-\Re e\left[\frac{(1-\mu)s_{\alpha_1}(\xi)}{1-s_{\alpha_1}(\xi)}+\frac{1-\mu}{1-s_{\alpha_1}(\xi)}\right]>\frac{1}{2(1-\alpha_1)}.$$

Hence, we find that

$$-\frac{1}{2}\left[\frac{1+\alpha}{1-\alpha} - \frac{\alpha_1}{1-\alpha_1}\right] > \frac{1}{2(1-\alpha_1)}$$

or

$$-\left\{\frac{\alpha}{1-\alpha} - \frac{\alpha_1}{1-\alpha_1}\right\}\Re e\left(\frac{f(\xi)}{\xi}\right)^{\mu} + \frac{1}{2(\alpha-1)} > \frac{1}{2(1-\alpha_1)}$$

or

$$-\left\{\frac{\alpha(1-\alpha_1)-\alpha_1(1-\alpha)}{(1-\alpha)(1-\alpha_1)}\right\}\Re e\left(\frac{f(\xi)}{\xi}\right)^{\mu} > \frac{1}{2}\left[\frac{1-\alpha+1-\alpha_1}{(1-\alpha_1)(1-\alpha)}\right].$$

This gives that

$$\Re e\left(\frac{f(\xi)}{\xi}\right)^{\mu} < \frac{1}{2}\left[\frac{\alpha+\alpha_1-2}{\alpha-\alpha_1}\right].$$

This is clearly contradictory to our assumption. Thus, the proof of our Theorem is completed. □

These Theorems show that the class $\mathcal{MF}^{\mu}_{\alpha,1-\alpha}$, with $\frac{1}{2} \leq \alpha < 2$, constructs a filtration for generators along with sharp estimates over the bounds of the Fekete–Szegö quadratic functional.

**Theorem 10.** *Let $f \in \mathcal{MF}^{\mu}_{\alpha,1-\alpha}$ and $\frac{1}{2} \leq \alpha < 2$. Then, $\mathcal{MF}^{\mu}_{\alpha,1-\alpha} \subset \mathcal{G}$. Furthermore, for $\frac{1}{2} \leq \alpha \leq 1$, each $f \in \mathcal{MF}^{\mu}_{\alpha,1-\alpha}$ leads a semi-group $\{\wp_t,\ t \geq 0\} \subset \mathcal{A}$, which satisfies*

$$|\wp_t(\xi)| \leq e^{\frac{1-2\alpha}{2\alpha}t}|\xi|,\ \ \alpha = 1, \frac{1}{2} \text{ and } \xi \in \mathbb{E}.$$

*Furthermore, the family $\left\{\mathcal{MF}^{\mu}_{\alpha,1-\alpha}\right\}$ is a filtration of $\mathcal{G}$ such that*

$$\sup_{f \in \mathcal{MF}^{\mu}_{\alpha,1-\alpha}} |\phi(f,\lambda)| \leq \max\{\mu_{\alpha,1-\alpha}, |1-\lambda|\} \text{ for } \lambda \in \mathbb{C}.$$

**Proof.** Suppose that $f \notin \mathcal{G} : \Re\mathrm{e}\left(\frac{f(\xi)}{\xi}\right)^{\mu} > 0$ and consider that $s(\xi)$ is as defined by (8) with $\tau = 0$, such that

$$s(\xi) = \frac{(w(\xi))^{\mu}-1}{(w(\xi))^{\mu}+1},$$

and

$$\left(\frac{f(\xi)}{\xi}\right)^{\mu} = \frac{1+s(\xi)}{1-s(\xi)},\quad w(\xi) = \frac{f(\xi)}{\xi}.$$

Moreover,

$$\mu\frac{\xi f'(\xi)}{f(\xi)} = \frac{\xi s'(\xi)}{1-s(\xi)} + \frac{\xi s'(\xi)}{1+s(\xi)} + \mu.$$

For some fixed $\xi \in \mathbb{E}$, $s(\xi) \in \mathbb{E} \Longleftrightarrow \Re\mathrm{e}\left(\frac{f(\xi)}{\xi}\right)^{\mu} > 0$. In view of our supposition, there exists a $\xi_0 \in \mathbb{E}$ such that $\Re\mathrm{e}\left(\frac{f(\xi)}{\xi}\right)^{\mu} < 0 : |\xi| < |\xi_0|$ while $\Re\mathrm{e}\left(\frac{f(\xi_0)}{\xi_0}\right)^{\mu} < 0$ and, hence, $|s(\xi_0)| = 1$. Therefore, by Lemma 5, there is $m \geq 1$, such that $\xi_0 s'(\xi_0) = ms(\xi_0)$. A straightforward calculation leads to

$$\begin{aligned}
\wp^{\mu}_{\alpha,1-\alpha}(f)(\xi_0) &= (1-\alpha)\mu\frac{\xi_0 f'(\xi_0)}{f(\xi_0)} + \alpha\left(\frac{f(\xi_0)}{\xi_0}\right)^{\mu} \\
&= \alpha\frac{1+s(\xi_0)}{1-s(\xi_0)} + \left(\frac{\xi_0 s'(\xi_0)}{1+s(\xi_0)} + \frac{\xi_0 s'(\xi_0)}{1-s(\xi_0)} + \mu\right)(1-\alpha) \\
&= \alpha\frac{1+s(\xi_0)}{1-s(\xi_0)} + \left(\frac{ms(\xi_0)}{1+s(\xi_0)} + \frac{ms(\xi_0)}{1-s(\xi_0)} + \mu\right)(1-\alpha). \qquad (29)
\end{aligned}$$

Using $s(\xi_0) = a + ib,\, a, b \in \mathbb{R} : a^2 + b^2 = 1$, we have

$$\Re e\left\{\frac{1}{1 - s(\xi_0)}\right\} = \frac{1 - a}{1 - 2a + a^2 + b^2} = \frac{1}{2},$$

$$\Re e\frac{s(\xi_0)}{1 - s(\xi_0)} = \Re e\frac{(a + ib)((1 - a) + ib)}{1 - 2a + b^2 + a^2} = \frac{a - (a^2 + b^2)}{2(1 - a)} = -\frac{1}{2},$$

$$\Re e\left\{\frac{1}{1 + s(\xi_0)}\right\} = \Re e\left\{\frac{(1 + a) - ib}{1 + 2a + b^2 + a^2}\right\} = \frac{(1 + a)}{2(1 + a)} = \frac{1}{2},$$

and, finally, we see that

$$\Re e\frac{s(\xi_0)}{1 + s(\xi_0)} = \Re e\frac{(a + ib)((1 + a) - ib)}{1 + 2a + b^2 + a^2} = \Re e\frac{a + 1}{2(1 + a)} = \frac{1}{2}.$$

From (29), we note that

$$\begin{aligned}\Re e\wp^{\mu}_{\alpha,1-\alpha}(\xi_0) &= \Re e\left[\alpha\frac{1 + s(\xi_0)}{1 - s(\xi_0)} + \left(\frac{ms(\xi_0)}{1 + s(\xi_0)} + \frac{ms(\xi_0)}{1 - s(\xi_0)} + \mu\right)(1 - \alpha)\right] \\ &= (1 - \alpha)m\mu,\ m \geq 1.\end{aligned}$$

Since $f \in \mathcal{MF}^{\mu}_{\alpha,1-\alpha}$ and $\Re e\wp^{\mu}_{\alpha,1-\alpha}(f)(\xi_0) = (1 - \alpha)\mu$, we conclude that $\alpha < \frac{1}{2}$. Thus, $\mathcal{MF}^{\mu}_{\alpha,1-\alpha} \subset \mathcal{G}$ whenever $\alpha \geq \frac{1}{2}$ (by (9)). For $\alpha = 1$, we observe that

$$\wp^{\mu}_{1,0}(f)(\xi) = \left(\frac{f(\xi)}{\xi}\right)^{\mu},\ -1 \leq \mu \leq 1$$

and, when $\alpha = \frac{1}{2}$, we observe that

$$\wp^{\mu}_{\frac{1}{2},\frac{1}{2}}(f)(\xi) = \frac{1}{2}\left[\left(\frac{f(\xi)}{\xi}\right)^{\mu} + \mu\frac{\xi f'(\xi)}{f(\xi)}\right].$$

In both of these cases, $f \in \mathcal{MF}^{\mu}_{\alpha,(1-\alpha)}$ leads to a semi-group $\{\wp_t : t \geq 0\} \subset \mathcal{A}$, which satisfies the Cauchy problem given by (5). The mapping $f \in \mathcal{A}$, defined by

$$\left[\lim_{t\to 0^+}\frac{\xi - \wp_t(\xi)}{t}\right]^{\mu} = f_1(\xi) = (f(\xi))^{\mu},\quad t \geq 0$$

with $\wp_t(\xi) = \exp(-at)\xi$, $a = \frac{1-2\alpha}{2\alpha} \in \mathbb{C}$, such that

$$\left[\lim_{t\to 0^+}\frac{\xi - \exp(-at)\xi}{t}\right]^{\mu} = \xi^{\mu}\left[\lim_{t\to 0^+}\frac{1 - \exp(-at)}{t}\right]^{\mu} = a^{\mu}\xi^{\mu} = (f(\xi))^{\mu},\quad t \geq 0,$$

is obviously an infinitesimal generator for some one-parameter family of semi-group, and for every $\xi \in \mathbb{E}$, the problem from (5) clearly possess a unique solution $u = u_t(\xi) \in \mathbb{E}$, $t \geq 0$ such that $|\wp_t(\xi)| \leq e^{\frac{1-2\alpha}{2\alpha}t}|\xi|$, $\alpha = 1, \frac{1}{2}$ and $\xi \in \mathbb{E}$. Therefore, we take $\alpha \in \left(\frac{1}{2}, 1\right)$. Suppose that

$$w(\xi) = \left(\frac{f(\xi)}{\xi}\right)^{\mu} - \left(\frac{1 - 2\alpha}{2\alpha}\right)^{\mu}.$$

Then,

$$\frac{\xi w'(\xi)}{w(\xi) + \left(\frac{1-2\alpha}{2\alpha}\right)^{\mu}} + \mu = \mu\frac{\xi f'(\xi)}{f(\xi)}.$$

$$\begin{aligned}\wp^{\mu}_{\alpha,\delta}(f)(\xi) &= \alpha\left(\frac{f(\xi)}{\xi}\right)^{\mu} + \delta\mu\frac{\xi f'(\xi)}{f(\xi)}; \qquad (\delta = 1-\alpha)\\ &= \alpha w(\xi) + \alpha\left(\frac{\delta-\alpha}{2\alpha}\right)^{\mu} + \frac{\xi w'(\xi)}{\frac{w(\xi)}{\delta\mu} + \frac{1}{\delta\mu}\left(\frac{\delta-\alpha}{2\alpha}\right)^{\mu}} + \delta\mu^2\\ &= \alpha w(\xi) + \frac{w'(\xi)}{\frac{h(\xi)}{\delta\mu} + \frac{1}{\delta\mu}\left(\frac{\delta-\alpha}{2\alpha}\right)^{\mu}} + \delta\mu^2 + \alpha\left(\frac{\delta-\alpha}{2\alpha}\right)^{\mu}\\ &= \alpha w(\xi) + \frac{\xi w'(\xi)}{\frac{1}{\delta\mu}w(\xi) + \frac{1}{\delta\mu}\left(\frac{\delta-\alpha}{2\alpha}\right)^{\mu}} + \delta\mu^2 + \left(\frac{\delta-\alpha}{2\alpha}\right)^{\mu}.\end{aligned}$$

From the above calculations and for $\delta = 1-\alpha$, we see that

$$\wp^{\mu}_{\alpha,\delta}(f)(\xi) - \delta\mu^2 - \left(\frac{\delta-\alpha}{2\alpha}\right)^{\mu} = \alpha w(\xi) + \frac{\xi w'(\xi)}{\frac{1}{\delta\mu}w(\xi) + \frac{1}{\delta\mu}\left(\frac{\delta-\alpha}{2\alpha}\right)^{\mu}}.$$

Hence, from Lemma 1, we see that the solution $w(\xi) = h(\xi)$ of the above equation is analytic in $\mathbb{E}$ and $\Re e h(\xi) = \Re e\left(\frac{f(\xi)}{\xi}\right)^{\mu} > \frac{1}{2} > 0$, $\xi \in \mathbb{E}$. Therefore, in this case $\wp^{\mu}_{\alpha,1-\alpha}(f)(\xi)$ also generates a semi-group and the family $\left\{\mathcal{MF}^{\mu}_{\alpha,1-\alpha}\right\}$ is a filtration of $\mathcal{G}$. Moreover, we let $\alpha \in \mathbb{R}$, $f \in \mathcal{A}$, $-1 \le \mu \le 1$ and $\xi \in \mathbb{E}$ such that

$$\mu_{\alpha,1-\alpha} = \frac{2 + (2\mu-3)(1-\alpha) - \alpha}{(2-\alpha)\mu} > 0.$$

Then, $\sup_{f\in\mathcal{MF}^{\mu}_{\alpha,1-\alpha}}|\phi(f,\lambda)| \le \max(\mu_{\alpha,1-\alpha}, |1-\lambda|)$ for $\lambda \in \mathbb{C}$, over the family $\mathcal{MF}^{\mu}_{\alpha,1-\alpha}$. □

**8. Open Problems**

Recall that, by Theorem 3, $|\phi(f,\lambda)| \le \max\left(\varphi\left({}^{\mu}_{\alpha,\beta}\right)|1-\lambda|\right)$ for $\lambda \in \mathbb{C}$, over the family $\mathcal{MF}^{\mu}_{\alpha,\beta}$, where the level set of function denoted by

$$\varphi\left({}^{\mu}_{\alpha,\beta}\right) = \frac{2 + (2\mu-3)\beta - \alpha}{(\alpha+2\beta)\mu + 2(1-\alpha-\beta)(2\mu+1)} > 0,$$

where $\alpha, \beta \in \mathbb{R}$, $-1 \le \mu \le 1$ and $\xi \in \mathbb{E}$, satisfies

$$\alpha - (2\mu-3)\beta < 2, \quad (-3\mu-2)\alpha - (2\mu+1)\beta < 2(2\mu+1)$$

and

$$\lambda_0 = (\alpha+2\beta)\mu + 2(1-\alpha-\beta)(2\mu+1)$$

such that $\beta = \frac{2(\mu+1)}{4+3\mu-6\mu^2}$. Therefore, the following open problems are raised.

**Q1:** Is this estimate sharp for all $\alpha, \beta \in \mathbb{R}$ and $-1 \le \mu \le 1$, which satisfy

$$\alpha - (2\mu-3)\beta < 2, \quad (-3\mu-2)\alpha - (2\mu+1)\beta < 2(2\mu+1)$$

and

$$\lambda_0 = (\alpha+2\beta)\mu + 2(1-\alpha-\beta)(2\mu+1)$$

such that $\beta = \frac{2(\mu+1)}{4+3\mu-6\mu^2}$.

**Q2:** What values of $\alpha, \beta$ and $\mu$ provide the class $\mathcal{MF}^{\mu}_{\alpha,\beta}$ in connection with the nonlinear operator $\wp^{\mu}_{\alpha,\beta}(f)$, consisting of univalent functions?

(The only cases we know the affirmative answer for are $\alpha = 0, \mu = 1$ and $\beta < 2$.)

## 9. Conjecture

The filtrations constructed in Theorems 7 and 10 are strict by definition.

## 10. Conclusions

This research is avid to a systematic and comprehensive analysis of linearization models for one-parameter continuous semi-groups, functional equations, different classes of univalent functions and their applications to various problems of complex dynamics, in order to establish a connection between the Fekete–Szegö functional and the class of infinitesimal generators.

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