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Applications of Fractional Differential Operator to Subclasses of Uniformly q-Starlike Functions

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Abstract: In this paper, we use the concept of quantum (or q-) calculus and define a q-analogous of a fractional differential operator and discuss some of its applications. We consider this operator to define new subclasses of uniformly q-starlike and q-convex functions associated with a new generalized conic domain, $\Lambda_{\beta,\mathfrak{q},\gamma}$. To begin establishing our key conclusions, we explore several novel lemmas. Furthermore, we employ these lemmas to explore some important features of these two classes, for example, inclusion relations, coefficient bounds, Fekete–Szego problem, and subordination results. We also highlight many known and brand-new specific corollaries of our findings.

Keywords: q-calculus; q-difference operator; analytic functions; q-convex functions; conic domains; q-starlike functions; subordination

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1. Introduction and Motivation

Perhaps the most intriguing part of the complex function theory is the interaction between geometry and analysis. Such connections between analytic structure and geometric behavior are at the heart of the theory of univalent functions. The domains of these functions characterize a wide variety of appealing geometric structures and canonical types. As an example, the image of an open unit disk under a normalized analytic and univalent function, ξ, contains a disk, $|w| < \delta$. Moreover, the images of a few of the ranges of some of them specify starlike, convex, close-to-convex, some in certain directions, some uniformly convex (starlike), and so on. The ranges of these geometric functions are characteristic of certain geometries. Additionally, this area of study is also known as geometric function theory (GFT).

In GFT, researchers have shown particular interest in linear operators. What makes this research so important is that we are looking at the characteristics of many classes of functions under a certain linear operator at the same time. In 1915, Alexander [1] developed the first integral operator, which he effectively used in his study of analytical functions. This subfield of the analytic function theory of complex analysis, which includes derivative and fractional derivative operators, is the subject of active investigation. Recent works such as [2] demonstrate the relevance of differential and integral fractional operators to the scientific community. Intriguing new results have emerged from studies of differential and integral operators from a number of perspectives, including quantum (or q-) calculus, that have implications for other areas of physics and mathematics. Further investigation may reveal that such operators have a role in providing solutions to partial differential equations, given their significance in the study of differential equations via functional analysis and operator theory. In his survey-cum-expository review work, Srivastava [3] highlights the exciting operator applications that are emerging from such a methodology.

Many of these applications of the basic (or $\mathfrak{q}$-) calculus and the fractional basic (or $\mathfrak{q}$-) calculus in GFT have inspired the present work, in which we introduce and analyze new subclasses of uniformly $\mathfrak{q}$-starlike and $\mathfrak{q}$-convex functions associated with a new generalized conic domain.

Let the set of all functions ξ be denoted by A and of the form

$$\xi(\tau) = \tau + \sum_{n=2}^{\infty} a_n \tau^n. \tag{1}$$

These are normalized analytic functions in the unit disk $E = \{\tau : |\tau| < 1\}$. Let us denote by S the collection of all functions in A that are univalent in E (see [4]) .

For any two analytic functions, ξ and g, in E. We state as $\xi(\tau)$ is subordinate to $g(\tau)$, denoted by $\xi(\tau) \prec g(\tau)$ if there exists a Schwartz function $w(\tau)$ with $w(0) = 0$ and $|w(\tau)| < 1$, such that $\xi(\tau) = g(w(\tau))$. More importantly, if $g(\tau)$ is univalent in E, then ([4])

$$\xi(\tau) \prec g(\tau) \Leftrightarrow \xi(0) = g(0) \text{ and } \xi(E) \subset g(E).$$

For the analytic functions ξ and g, where

$$\xi(\tau) = \sum_{n=0}^{\infty} a_n \tau^n \text{ and } g(\tau) = \sum_{n=0}^{\infty} b_n \tau^n, \quad (\tau \in E),$$

then $(\xi * g)(\tau)$ convolution is defined as

$$(\xi * g)(\tau) = \sum_{n=0}^{\infty} a_n b_n \tau^n.$$

Let P stand for the famous class of functions p that are analytic in E, and have the series form:

$$p(\tau) = 1 + \sum_{n=1}^{\infty} c_n \tau^n,$$

such that

$$\text{Re}(p(\tau)) > 0.$$

In relation to a point a in E, a domain E is starlike if and only if every line segment connecting the point a to every other point in E entirely encircles the domain. Simply said, starlike refers to a domain that is starlike in relation to its origin. A domain E is convex if and only if it is starlike with regard to each and every point in E or if every line segment connecting between two points in E lies fully inside E. If a function ξ maps E onto a star-shaped (convex) domain, it is said to be a starlike (convex) function. All starlike and convex function classes are abbreviated as S^* and C, respectively. These classes are distinguished analytically by the inequalities:

$$\xi \in S^* \Leftrightarrow \text{Re}\left(\frac{\tau\xi'(\tau)}{\xi(\tau)}\right) > 0$$

and

$$\xi \in C \Leftrightarrow \text{Re}\left(1 + \frac{\tau\xi''(\tau)}{\xi(\tau)}\right) > 0.$$

For $0 \leq \gamma < 1$, let $C(\gamma)$ and $S^*(\gamma)$ be the subclasses of S composed of convex and starlike functions of order γ, respectively. Analytically we can write

$$\mathrm{Re}\left(1+\frac{\tau\xi''(\tau)}{\xi(\tau)}\right) > \gamma, \ \ \tau \in E,$$

and

$$\mathrm{Re}\left(\frac{\tau\xi'(\tau)}{\xi(\tau)}\right) > \gamma, \ \tau \in E.$$

It has been widely accepted that for a convex (starlike) function ξ, its image under E and any circles within E centered at the origin are convex (starlike) arcs. Nevertheless, argumentation is needed to determine whether the characteristic generally applies to circles with the center at any other point, let us say ζ. Goodman [5] provided the answer to this issue by defining uniformly convex and starlike functions. After much deliberation, Ronning [6] and Ma and Minda [7] proposed the one variable characterization of these functions, defining them as follows:

The set of uniformly starlike functions, denoted by ST, includes the function $\xi \in \mathcal{A}$ if and only if

$$\mathrm{Re}\left(\frac{\tau\xi'(\tau)}{\xi(\tau)}\right) > \left|\frac{\tau\xi'(\tau)}{\xi(\tau)} - 1\right|.$$

The class of uniformly convex functions, denoted by UCV, includes the function $\xi \in \mathcal{A}$ if and only if

$$\mathrm{Re}\left(1+\frac{\tau\xi''(\tau)}{\xi'(\tau)}\right) > \left|\frac{\tau\xi''(\tau)}{\xi'(\tau)}\right|.$$

Later in [8], Kanas and Wisniowska investigated the classes $\beta - ST$ and $\beta - UCV$ defined as

$$\xi(\tau) \in \beta - ST \Leftrightarrow \xi(\tau) \in \mathcal{A} \text{ and } \mathrm{Re}\left(\frac{\tau\xi'(\tau)}{\xi(\tau)}\right) > \beta\left|\frac{\tau\xi'(\tau)}{\xi(\tau)} - 1\right|, \quad \tau \in E$$

and

$$\xi(\tau) \in \beta - UCV \Leftrightarrow \xi(\tau) \in \mathcal{A} \text{ and } 1 + \mathrm{Re}\left(\frac{\tau\xi''(\tau)}{\xi'(\tau)}\right) > \beta\left|\frac{\tau\xi''(\tau)}{\xi'(\tau)}\right|, \quad \tau \in E.$$

Note that $\xi(\tau) \in \beta - UCV \Leftrightarrow \tau\xi'(\tau) \in \beta - ST$.

To map the intersection of E and any disk center ζ, $|\zeta| \leq \beta$ onto a convex domain, it was proven mathematically in [8] that the class $\beta - UCV$, for $\beta \geq 0$, is a subclass of univalent functions. Thus, the concept of β-uniform convexity extends the definition of convexity and ζ is the origin and $\beta = 0$; then $\beta - UCV = C$ (see [4]), and for $\beta = 1$, then $\beta - UCV = UCV$. This class was first described by Goodman [5] and has been extensively investigated by Ronning [9] and Ma and Minda [7]. It should be pointed out that the $\beta - UCV$ class really began much earlier in [10] with some extra criteria but without the geometric meaning.

In the previous section, we said that Kanas and Wisniowska (see [8]) proposed and analyzed the class β-UCV and subsequently the corresponding class β-ST. These classes in the conic domain Λ_β, $(\beta \geq 0)$ were then defined by Kanas and Wisniowska (see [8]) as follows:

$$\Lambda_\beta = \left\{s + it : s > \beta\sqrt{(s-1)^2 + t^2}\right\},$$

or

$$\Lambda_\beta = \{w : \mathrm{Re}\, w > |w - 1|\}.$$

Remark 1. *For $\beta = 0$, this domain is the right half plane; for $0 < \beta < 1$, it is a hyperbola; for $\beta = 1$, it is a parabola; and for $\beta > 1$, it is an ellipse.*

For these conic regions, the functions $p_\beta(\tau)$ (see [8]) play the role of extremal functions.

In [11], Al-Oboudi et al. used the idea of conic domain $\Lambda_{\beta,\gamma}$ and defined new subclasses of starlike and convex functions where

$$\Lambda_{\beta,\gamma} = \left\{ s + it : s > \beta\sqrt{(s-1)^2 + t^2} + \gamma \right\},$$

or

$$\Lambda_{\beta,\gamma} = \{ w : \operatorname{Re} w > |w - 1| + \gamma \}.$$

From elementary computations, $\partial\Lambda_{\beta,\gamma}$ represents the conic sections symmetric about the real axis.

The following functions serve as extremal functions in various conic domains:

$$p_{\beta,\gamma}(\tau) = \left\{ \begin{array}{ll} \frac{1+(1-2\gamma)\tau}{1-\tau}, & \beta = 0, \\ 1 + \frac{2(1-\gamma)}{\pi^2}\left(\log\frac{1+\sqrt{\tau}}{1-\sqrt{\tau}}\right)^2, & \beta = 1, \\ \frac{1-\gamma}{1-\beta^2}\cos\left\{\left(\frac{2}{\pi}\arccos\beta\right)i\log\frac{1+\sqrt{\tau}}{1-\sqrt{\tau}}\right\} - \frac{\beta^2-\gamma}{1-\beta^2}, & 0 < \beta < 1, \\ \frac{(1-\gamma)}{\beta^2-1}\sin\left(\frac{\pi}{2K(t)}\right)\int\limits_0^{\frac{u(\tau)}{\sqrt{t}}} \frac{1}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}.dx + \frac{\beta^2-\gamma}{\beta^2-1}, & \beta > 1. \end{array} \right\} \tag{2}$$

where

$$u(\tau) = \frac{\tau - \sqrt{t}}{1 - \sqrt{t\tau}}, \; t \in (0,1),$$

t is chosen such that $K(t)$ is Legendre's complete elliptic integral of the first kind, while $K'(t)$ is a complementary integral of $K(t)$, $\beta = \cosh\frac{\pi K'(t)}{4K(t)}$.

For $\beta = 0$, we have

$$p_{0,\gamma}(\tau) = 1 + 2(1-\gamma)\tau + 2(1-\gamma)\tau + \cdots.$$

For $\beta = 1$, ([9,12]), we obtain

$$p_{1,\gamma}(\tau) = 1 + (1-\gamma)\tau + \frac{16}{3\pi^2}(1-\gamma)\tau + \cdots.$$

For $0 < \beta < 1$, (see [7]), we obtain

$$p_{\beta,\gamma}(\tau) = 1 + \frac{(1-\gamma)}{1-\beta^2}\sum_{n=1}^{\infty}\left[\sum_{l=1}^{2n} 2^l \binom{B}{l}\binom{2n-1}{2n-l}\right]\tau,$$

where $B = \frac{2}{\pi}\arccos\beta$. Finally, for $\beta > 1$, we have

$$\begin{aligned} p_{\beta,\gamma}(\tau) &= 1 + \frac{\pi^2(1-\gamma)}{4\sqrt[2]{t}(\beta^2-1)K^2(t)(1+t)} \\ &\quad \times \left\{ \tau + \frac{4K^2(t)(t^2+6t+1) - \pi^2}{24\sqrt[2]{t}K^2(t)(1+t)} + \ldots \right\}. \end{aligned} \tag{3}$$

For details, see [7,9,12]. From (3), we have

$$P_{\beta,\gamma}(\tau) = 1 + Q_1\tau + Q_2\tau + \cdots, \tag{4}$$

where

$$Q_1 = \frac{\pi^2(1-\gamma)}{4\sqrt[2]{t}(\beta^2-1)K^2(t)(1+t)}, \tag{5}$$

$$Q_2 = \frac{\pi^2(1-\gamma)}{4\sqrt[2]{t}(\beta^2-1)K^2(t)(1+t)} \times \frac{4K^2(t)(t^2+6t+1)-\pi^2}{24\sqrt[2]{t}K^2(t)(1+t)}. \tag{6}$$

The motivation and use of the $\mathfrak{q}$ calculus may be seen in the fact that it is used to study many families of analytic functions with wide-ranging applications in mathematics and related subjects. The quantum (or $\mathfrak{q}$-) calculus is also extensively employed in the context of approximation theory, especially for a number of operators, such as the convergence of operators to functions in the real and complex domains. Jackson (see [13]) was the first scholar to define the $\mathfrak{q}$-analogue of the classical derivative and integral and explain some of its applications. The $\mathfrak{q}$-beta function was subsequently used by Aral and Gupta to create the $\mathfrak{q}$-Baskakov–Durrmeyer operator (see [14]), and the $\mathfrak{q}$-Picard and $\mathfrak{q}$-Gauss–Weierstrass singular integral operators were investigated in [15]. In addition, a Ruscheweyh $\mathfrak{q}$-differential operator was initially presented by Kanas and Raducanu (see [16]), and its applications for multivalent functions were studied by Arif et al. (see [17]). In the meanwhile, [18] explored $\mathfrak{q}$-calculus via the lens of convolution. In recent years, several researchers have defined and explored several $\mathfrak{q}$-analogous differential operators for analytic, multivalent, and meromorphic functions, and discussed applications of these operators in various contexts; for more information, see [3,19,20].

Now, for your convenience, we provided the most basic definitions of quantum (or $\mathfrak{q}$-) calculus.

Definition 1. *For $\gamma, \mathfrak{q} \in \mathbb{C}$, the $\mathfrak{q}$-shifted factorial $(\gamma,\mathfrak{q})_j$ is defined by*

$$(\gamma,\mathfrak{q})_j = \prod_{j=0}^{j-1}\left(1-\gamma\mathfrak{q}^j\right), \quad (j \in \mathbb{N} = \{1,2,3\ldots\}). \tag{7}$$

If $\gamma \neq \mathfrak{q}^{-m}$, $(m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\})$, then it can be written as

$$(\gamma,\mathfrak{q})_\infty = \prod_{j=0}^{\infty}\left(1-\gamma\mathfrak{q}^j\right), \quad (\gamma \in \mathbb{C} \text{ and } |\mathfrak{q}| < 1), \tag{8}$$

when $\gamma \neq 0$ and $\mathfrak{q} \geq 1$, $(\gamma,\mathfrak{q})_\infty$ diverges. Therefore, whenever we use $(\gamma,\mathfrak{q})_\infty$, then $|\mathfrak{q}| < 1$ will be assumed.

Definition 2. *Below is a precise expression for $(\gamma,\mathfrak{q})_j$ in (7) in terms of the $\mathfrak{q}$-gamma function:*

$$\Gamma_{\mathfrak{q}}(\gamma) = \frac{(1-\mathfrak{q})^{1-\alpha}(\mathfrak{q},\mathfrak{q})_\infty}{(\mathfrak{q}^a,\mathfrak{q})_\infty}, \quad (0 < \mathfrak{q} < 1),$$

or

$$(\mathfrak{q}^\gamma,\mathfrak{q})_j = \frac{(1-\mathfrak{q}^j)\digamma_{\mathfrak{q}}(\gamma+j)}{\digamma_{\mathfrak{q}}(\gamma)}, \quad (j \in \mathbb{N}),$$

and $\mathfrak{q}$-factorial $[j]_{\mathfrak{q}}!$ defined by

$$[j]_{\mathfrak{q}}! = \prod_{j=1}^{i}[i], \quad (i \in \mathbb{N}). \tag{9}$$

Definition 3 ([21]). *For* $\mathfrak{q} \in (0,1)$*, we have the following definition of the* $\mathfrak{q}$*-number:*

$$[x]_{\mathfrak{q}} = \left\{ \begin{array}{ll} \frac{1-\mathfrak{q}^x}{1-\mathfrak{q}}, & (t \in \mathbb{C}), \\ \sum\limits_{n=0}^{i-1} \mathfrak{q}^i = 1 + \mathfrak{q} + \mathfrak{q}^2 + \cdots + \mathfrak{q}^{i-1} & (x = i \in \mathbb{N}) \end{array} \right\}. \tag{10}$$

Definition 4. *For* $\mathfrak{q} \in (0,1)$*, we have the following definition of* $[n]_{\mathfrak{q}}!$*:*

$$[n]_{\mathfrak{q}}! = \left\{ \begin{array}{ll} 1 & (n = 0), \\ \prod\limits_{n=1}^{i} [n]_{\mathfrak{q}} & (i \in \mathbb{N}) \end{array} \right\}. \tag{11}$$

Definition 5. *The notation* $[x]_{n,\mathfrak{q}}$*,* $x \in \mathbb{C}$ *for the* $\mathfrak{q}$*-generalized Pochhammer is given by*

$$[x]_{n,\mathfrak{q}} = \frac{(\mathfrak{q},\mathfrak{q})_n}{(1-\mathfrak{q})n} = \left\{ \begin{array}{ll} 1, & (n = 0), \\ [x]_{\mathfrak{q}}[x+1]_{\mathfrak{q}}[x+2]_{\mathfrak{q}} \cdots [x+n-1]_{\mathfrak{q}}, & (n \in \mathbb{N}) \end{array} \right\}.$$

Additionally, the $\mathfrak{q}$-gamma function can be characterized as

$$\Gamma_{\mathfrak{q}}(x+1) = [x]_{\mathfrak{q}}\Gamma_{\mathfrak{q}}(x) \quad \text{and} \quad \Gamma_{\mathfrak{q}}(1) = 1.$$

Definition 6 ([13]). *For* $\xi \in \mathcal{A}$*, the* $\mathfrak{q}$*-derivative operator (*$\mathfrak{q}$*-difference operator) can be written as*

$$\partial_{\mathfrak{q}}\xi(\tau) = \frac{\xi(\mathfrak{q}\tau) - \xi(\tau)}{(\mathfrak{q}-1)\tau}, \quad \tau \in E. \tag{12}$$

From (1) and (12), we have

$$\partial_{\mathfrak{q}}\xi(\tau) = 1 + \sum_{n=2}^{\infty} [n]_{\mathfrak{q}} a_n \tau^{n-1}.$$

For $n \in \mathbb{N}$ *and* $\tau \in E$*, we have*

$$\partial_{\mathfrak{q}}\tau = [n]_{\mathfrak{q}}\tau^{n-1}, \quad \partial_{\mathfrak{q}}\left\{ \sum_{n=1}^{\infty} a_n \tau \right\} = \sum_{n=1}^{\infty} [n]_{\mathfrak{q}} a_n \tau^{n-1}.$$

We can observe that

$$\lim_{\mathfrak{q}\to 1^-} \partial_{\mathfrak{q}}\xi(\tau) = \xi'(\tau).$$

Let

$$\varphi(a,c;\tau) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\tau^n, \quad \tau \in E;\ c \neq 0, -1, -2, \cdots,$$

Using the definition of the Pochhammer symbol in terms of Gamma functions by

$$(\mu)_{\mathfrak{q}} = \frac{\Gamma_{\mathfrak{q}}(\mu+n)}{\Gamma_{\mathfrak{q}}(\mu)} = \left\{ \begin{array}{ll} 1 & n = 0, \\ (\mu)_{\mathfrak{q}}(\mu+1)_{\mathfrak{q}} \cdots (\mu+n-1)_{\mathfrak{q}} & n \in \mathbb{N} = \{1, 2, \cdots\} \end{array} \right\}.$$

Note that for the derivative of negative order, it is the integral defined below:

Definition 7. *For* $\alpha > 0$*, the fractional* $\mathfrak{q}$*-integral operator (see [22]) defined by*

$$I^{\alpha}_{\mathfrak{q},\tau}\xi(\tau) = I^{-\alpha}_{\mathfrak{q},\tau}\xi(\tau) = \frac{1}{\Gamma_{\mathfrak{q}}(\alpha)} \int_0^{\tau} (\tau - t\mathfrak{q})_{\alpha-1}\xi(t) d_{\mathfrak{q}}(t), \tag{13}$$

where the $\mathfrak{q}$*-binomial function* $(\tau - t\mathfrak{q})_{\alpha-1}$ *is defined by*

$$(\tau - t\mathfrak{q})_{\alpha-1} = \tau^{\alpha-1}{}_1\Phi_0\left(\mathfrak{q}^{-v+1}, -, \mathfrak{q}, t\mathfrak{q}^{\alpha}/\tau\right).$$

The representation of series ${}_1\Phi_0$ *is given by*

$${}_1\Phi_0(a, -, \mathfrak{q}, \tau) = 1 + \sum_{j=1}^{\infty} \frac{(a,\mathfrak{q})_j}{(\mathfrak{q},\mathfrak{q})_j}\tau, \quad (|\mathfrak{q}| < 1, |\tau| < 1).$$

The last equality is called $\mathfrak{q}$*-binomial theorem (see [23]). For further details, see [3].*

Note that the integral defined above is the derivative of negative order.

Definition 8. *Let the smallest possible integer be m.* $Ð_{\mathfrak{q}}^{\alpha}$ *is the extended fractional* $\mathfrak{q}$*-derivative of order* α*, and it can be defined as*

$$Ð_{\mathfrak{q}}^{\alpha}\xi(\tau) = Ð_{\mathfrak{q}}^{m}\left(Ð_{\mathfrak{q},\tau}^{m-\alpha}\xi(\tau)\right). \tag{14}$$

We find from (14) that

$$Ð_{\mathfrak{q}}^{\alpha}\tau^{j} = \frac{\Gamma_{\mathfrak{q}}(j+1)}{\Gamma_{\mathfrak{q}}(j+1-\alpha)}\tau, \quad (0 \le \alpha,\ j > -1).$$

Remark 2. *The case of* $-\infty < \alpha < 0$*,* $Ð_{\mathfrak{q}}^{\alpha}$ *denotes a fractional* $\mathfrak{q}$*-integral of order* α*.*

Remark 3. *The case of* $0 \le \alpha < 2$*,* $Ð_{\mathfrak{q}}^{\alpha}$ *denotes a fractional* $\mathfrak{q}$*-derivative of order* α*.*

Definition 9. *([24]). Selvakumaran et al. defined the* $(\alpha, \mathfrak{q})$*-differintegral operator* $\Lambda_{\mathfrak{q}}^{\alpha} : \mathcal{A} \to \mathcal{A}$ *as*

$$\begin{aligned} \Lambda_{\mathfrak{q}}^{\alpha}\xi(\tau) &= \frac{\Gamma_{\mathfrak{q}}(2-\alpha)}{\Gamma_{\mathfrak{q}}(2)}\tau Ð_{\mathfrak{q}}^{\alpha}\xi(\tau) \\ &= \tau + \sum_{j=2}^{\infty} \frac{\Gamma_{\mathfrak{q}}(2-\alpha)\Gamma_{\mathfrak{q}}(j+1)}{\Gamma_{\mathfrak{q}}(2)\Gamma_{\mathfrak{q}}(j+1-\alpha)} a_j\tau^j, \quad \tau \in E, \qquad (15) \\ &= \varphi(2, 2-\alpha; \tau) * \xi(\tau). \qquad (16) \end{aligned}$$

where

$$\alpha < 2,\ 0 < \mathfrak{q} < 1.$$

Remark 4. *When* $\mathfrak{q} \to 1^{-}$*, then we have the Owa and Srivastava operator defined in [25].*

In this article, using the $(\alpha, \mathfrak{q})$-differintegral operator $\Lambda_{\mathfrak{q}}^{\alpha}\xi$ and $\mathfrak{q}$-difference operator, we now define the $\mathfrak{q}$-analogous of the linear multiplier fractional differential operator $(Ð_{\lambda,\mathfrak{q}}^{n,\alpha})$ as follows:

Definition 10. *The linear multiplier* $\mathfrak{q}$*-fractional differential operator* $Ð_{\lambda,\mathfrak{q}}^{n,\alpha}$ *is defined as follows:*

$$\begin{aligned} Ð_{\mathfrak{q}}^{0}\xi(\tau) &= \xi(\tau), \\ Ð_{\lambda,\mathfrak{q}}^{1,\alpha}\xi(\tau) &= (1-\lambda)\Lambda^{\alpha}\xi(\tau) + \lambda\tau\partial_{\mathfrak{q}}(\Lambda^{\alpha}\xi(\tau) = Ð_{\lambda,\mathfrak{q}}^{\alpha}(\xi(\tau)), \quad \lambda \ge 0, 0 \le \alpha < 1, \qquad (17) \end{aligned}$$

$$Ð_{\lambda,\mathfrak{q}}^{2,\alpha}\xi(\tau) = Ð_{\lambda,\mathfrak{q}}^{\alpha}(Ð_{\lambda,\mathfrak{q}}^{1,\alpha}\xi(\tau)),$$

$$\vdots$$

$$Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau) = Đ^{\alpha}_{\lambda,\mathfrak{q}}(Đ^{n-1,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau)), \quad n \in N. \tag{18}$$

From (1), (15), (17), and (18), we see that

$$Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau) = \tau + \sum_{n=2}^{\infty} \Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)a_n\tau^n, \qquad n \in N_0 = N \cup \{0\}, \tag{19}$$

where

$$\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda) = \left[\frac{\Gamma_{\mathfrak{q}}(n+1)\Gamma_{\mathfrak{q}}(2-\alpha)}{\Gamma_{\mathfrak{q}}(2)\Gamma_{\mathfrak{q}}(n+1-\alpha)}(1+\lambda([n]_{\mathfrak{q}}-1))\right]^k. \tag{20}$$

From (16) *and* (20), $Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau)$ *can be written in terms of convolution as*

$$Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau) = \underbrace{\left[(\varphi(2,2-\alpha;\tau) * g_{\lambda,\mathfrak{q}}(\tau)) * \cdots * (\varphi(2,2-\alpha;\tau) * g_{\lambda,\mathfrak{q}}(\tau))\right]} * \xi(\tau), \tag{21}$$

where

$$g_{\lambda,\mathfrak{q}}(\tau) = \tau + \sum_{n=2}^{\infty}[1 + ([n]_{\mathfrak{q}} - 1)\lambda]\tau^n.$$

Remark 5. *When $\mathfrak{q} \to 1^-$, in Definition 10, then we obtain the operator defined in [11].*

We now propose a definition based on quantum (or $\mathfrak{q}$-) calculus and the concept of subordination, which is as follows:

Definition 11. *Let $\beta \in [0,\infty)$, $\mathfrak{q} \in (0,1)$, and $\gamma \in \mathbb{C}\backslash\{0\}$. It is claimed that a function $p(\tau)$ belongs to the class $\beta - P_{\mathfrak{q},\gamma}$ if and only if*

$$p(\tau) \prec P_{\beta,\gamma,\mathfrak{q}}(\tau), \tag{22}$$

where

$$P_{\beta,\gamma,\mathfrak{q}}(\tau) = \frac{2P_{\beta,\gamma}(\tau)}{(\mathfrak{q}+1) + (1-\mathfrak{q})P_{\beta,\gamma}(\tau)}, \tag{23}$$

and $P_{\beta,\gamma}(\tau)$ is given in (2).

Geometrically, all values of the function $p(\tau) \in \beta - P_{\mathfrak{q},\gamma}$ take from the domain $\Lambda_{\beta,\mathfrak{q},\gamma}$ which we will describe as

$$\Lambda_{\beta,\mathfrak{q},\gamma} = \Lambda_{\beta,\mathfrak{q}} + \gamma \tag{24}$$

where

$$\Lambda_{\beta,\mathfrak{q}} = \left\{w : \mathrm{Re}\left(\frac{(\mathfrak{q}+1)w}{(\mathfrak{q}-1)w+2}\right) > \beta\left|\frac{(\mathfrak{q}+1)w}{(\mathfrak{q}-1)w+2} - 1\right|\right\}.$$

The domain $\Lambda_{\beta,\mathfrak{q},\gamma}$ is denoted by the generalized conic domain.

Remark 6. *When $\mathfrak{q} \to 1^-$, then*

$$\begin{aligned}\Lambda_{\beta,\mathfrak{q},\gamma} &= \Lambda_{\beta,\gamma} = \{w : \mathrm{Re}(w) > \beta|w-1| + \gamma\} \\ &= \left\{s + it : s > \beta\sqrt{(s-1)^2 + t^2} + \gamma\right\},\end{aligned}$$

where $\Lambda_{\beta,\mathfrak{q},\gamma}$ is studied in [11].

Remark 7. *When $\gamma = 0$ and $\mathfrak{q} \to 1-$, then $\Lambda_{\beta,\mathfrak{q},\gamma} = \Lambda_{\beta}$. This conic domain was determined by Kanas and Wisniowska [8,26].*

Remark 8. *When $\gamma = 0$ and $\mathfrak{q} \to 1-$, in Definition 11, then $\beta - P_{\mathfrak{q},\gamma} = P(p_\beta)$ introduced by Kanas and Wisniowska in [8,26].*

Remark 9. *When $\gamma = 0$, $\beta = 0$, and $\mathfrak{q} \to 1-$, in Definition 11, then $\beta - P_{\mathfrak{q},\gamma} = P$.*

Definition 12. *The class $SP^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma)$ is defined as the set of all functions $\xi \in \mathcal{A}$ satisfying the condition*

$$\mathrm{Re}\left(\frac{\tau\partial_{\mathfrak{q}}(Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))}{Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau)}\right) > \beta\left|\frac{\tau\partial_{\mathfrak{q}}(Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))}{Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau)} - 1\right| + \gamma. \tag{25}$$

Remark 10. *When $\mathfrak{q} \to 1^-$, then we have a known class of analytic functions investigated by Al-Oboudi and Al-Amoudi in [11].*

Definition 13. *The class $UCV^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma)$ is defined as the set of all functions $\xi \in \mathcal{A}$ satisfying the condition*

$$\mathrm{Re}\left(1 + \frac{\tau\partial_{\mathfrak{q}}(\partial_{\mathfrak{q}}Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))}{\partial_{\mathfrak{q}}(Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))}\right) > \beta\left|\frac{\tau\partial_{\mathfrak{q}}(\partial_{\mathfrak{q}}Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))}{\partial_{\mathfrak{q}}(Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))}\right| + \gamma. \tag{26}$$

Remark 11. *When $\mathfrak{q} \to 1^-$, then we have a known class of analytic functions defined by Al-Oboudi and Al-Amoudi in [11].*

It is clear that

$$\xi \in UCV^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma) \Leftrightarrow \tau\partial_{\mathfrak{q}}\xi \in SP^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma) \tag{27}$$

and that

$$UCV^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma) \subset SP^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma).$$

Geometric interpretation: From (25) and (26), $\xi \in UCV^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma)$ and $\xi \in SP^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma)$ if and only if $p(\tau) = 1 + \frac{\tau\partial_{\mathfrak{q}}(\partial_{\mathfrak{q}}Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))}{\partial_{\mathfrak{q}}(Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))}$ and $p(\tau) = \frac{\tau\partial_{\mathfrak{q}}(Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))}{Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau)}$ take all values in the conic domain $\Lambda_{\beta,\mathfrak{q},\gamma}$ given in (24). We can write the conditions (25) and (26) in the form

$$\frac{\tau\partial_{\mathfrak{q}}(Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))}{Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau)} \prec P_{\beta,\gamma,\mathfrak{q}}(\tau) \tag{28}$$

and

$$1 + \frac{\tau\partial_{\mathfrak{q}}(\partial_{\mathfrak{q}}Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))}{\partial_{\mathfrak{q}}(Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))} \prec P_{\beta,\gamma,\mathfrak{q}}(\tau). \tag{29}$$

where $P_{\beta,\gamma,\mathfrak{q}}(\tau)$ is given by (23) and $P_{\beta,\gamma}(\tau)$ is given by (2).

By virtue of (25) and (26) and using the characteristics of the domain $\Lambda_{\beta,\mathfrak{q},\gamma}$ given in (24), we have

$$\mathrm{Re}\left(\frac{\tau\partial_{\mathfrak{q}}(Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))}{Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau)}\right) > \frac{\beta(\mathfrak{q}+1) + (3-\mathfrak{q})\gamma}{(1+\beta)((1-\mathfrak{q})+2)} \tag{30}$$

and

$$\mathrm{Re}\left(1 + \frac{\tau\partial_{\mathfrak{q}}(\partial_{\mathfrak{q}}Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))}{\partial_{\mathfrak{q}}(Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))}\right) > \frac{\beta(\mathfrak{q}+1) + (3-\mathfrak{q})\gamma}{(1+\beta)((1-\mathfrak{q})+2)}, \tag{31}$$

which means that

$$\xi \in SP^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma) \Rightarrow Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi \in ST\left(\frac{\beta(\mathfrak{q}+1) + (3-\mathfrak{q})\gamma}{(1+\beta)((1-\mathfrak{q})+2)}\right) \subseteq ST \tag{32}$$

and

$$\xi \in UCV^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma) \Rightarrow \text{Đ}^{n,\alpha}_{\lambda,\mathfrak{q}}\xi \in CV\left(\frac{\beta(\mathfrak{q}+1)+(3-\mathfrak{q})\gamma}{(1+\beta)((1-\mathfrak{q})+2)}\right) \subseteq CV. \tag{33}$$

Remark 12. *When we take $\mathfrak{q} \to 1^-$, in (30)–(33), we have the following special cases studied in [11].*

$$\text{Re}\left(\frac{\tau(Đ^{n,\alpha}_{\lambda}\xi(\tau))'}{Đ^{n,\alpha}_{\lambda}\xi(\tau)}\right) > \frac{\beta+\gamma}{1+\beta}$$

and

$$\text{Re}\left(1+\frac{\tau\left(Đ^{n,\alpha}_{\lambda}\xi(\tau)\right)''}{\left(Đ^{n,\alpha}_{\lambda}\xi(\tau)\right)'}\right) > \frac{\beta+\gamma}{1+\beta},$$

which means that

$$\xi \in SP^{n}_{\alpha,\lambda}(\beta,\gamma) \Rightarrow Đ^{n,\alpha}_{\lambda}\xi \in ST\left(\frac{\beta+\gamma}{1+\beta}\right) \subseteq ST$$

and

$$\xi \in UCV^{n}_{\alpha,\lambda}(\beta,\gamma) \Rightarrow Đ^{n,\alpha}_{\lambda}\xi \in CV\left(\frac{\beta+\gamma}{1+\beta}\right) \subseteq CV.$$

2. A Set of Lemmas

Here, we provide several lemmas that may be used to further explore the paper's key outcomes.

Lemma 1 ([27])**.** *Let $p(\tau) = 1 + \sum_{n=1}^{\infty} p_n\tau^n$ be analytic in E and satisfy $\text{Re}\, p(\tau) > 0$ for τ in E, then*

$$\left|p_2 - \mu p_1^2\right| \leq 2\max\{1, |2\mu-1|\}, \;\; \mu \in \mathbb{C}.$$

Lemma 2 (see [28])**.** *Let $p(\tau) = 1 + \sum_{n=1}^{\infty} p_n\tau^n \prec \xi(\tau) = 1 + \sum_{n=1}^{\infty} C_n\tau^n$, and if $\xi(\tau)$is convex univalent in E, then*

$$|p_n| \leq |C_1|, \quad n \geq 1.$$

Lemma 3 ([29])**.** *Let ξ and g be starlike of order $\frac{1}{2}$. Then so is $\xi * g$.*

Lemma 4 ([30])**.** *Let ξ and g be univalent starlike functions of order $\frac{1}{2}$. Then, for every function $\xi \in \mathcal{A}$, we have*

$$\frac{\xi(\tau) * g(\tau)\xi(\tau)}{\xi(\tau) * g(\tau)} \in \overline{co}(\xi(E)), \quad \tau \in E,$$

where $\overline{co}$ denotes the closed convex hull.

New lemmas are explored here that will be useful in establishing the article's results.

Lemma 5. *Let $\beta \in [0,\infty)$ be fixed and*

$$p_{\beta,\gamma,\mathfrak{q}}(\tau) = \frac{2P_{\beta,\gamma}(\tau)}{(\mathfrak{q}+1)+(1-\mathfrak{q})P_{\beta,\gamma}(\tau)}. \tag{34}$$

Then

$$P_{\beta,\gamma,\mathfrak{q}}(\tau) = 1 + \frac{2}{\mathfrak{q}+1}Q_1\tau + \left\{\frac{2}{\mathfrak{q}+1}Q_2 - \frac{2(1-\mathfrak{q})}{\mathfrak{q}+1}Q_1^2\right\}\tau + \cdots,$$

where Q_1 and Q_2 are given by (5) and (6).

Proof. From (34), we have

$$\begin{aligned}
P_{\beta,\gamma,\mathfrak{q}}(\tau) &= \frac{2P_{\beta,\gamma,\mathfrak{q}}(\tau)}{(\mathfrak{q}+1)+(1-\mathfrak{q})P_{\beta,\gamma}(\tau)} \\
&= \frac{2}{\mathfrak{q}+1}\{P_{\beta,\gamma}(\tau)\} - \frac{2(1-\mathfrak{q})}{(\mathfrak{q}+1)^2}\left\{P^2_{\beta,\gamma}(\tau)\right\} + \frac{2(1-\mathfrak{q})^2}{(\mathfrak{q}+1)^3}\left\{P^3_{\beta,\gamma}(\tau)\right\} \\
&\quad - \frac{2(1-\mathfrak{q})^3}{(\mathfrak{q}+1)^4}\left\{P^4_{\beta,\gamma}(\tau)\right\} + \cdots . \qquad (35)
\end{aligned}$$

By using (4) in (35), we have

$$\begin{aligned}
P_{\beta,\gamma,\mathfrak{q}}(\tau) &= \left(\frac{2}{\mathfrak{q}+1} - \frac{2(\mathfrak{q}+1)}{(\mathfrak{q}+1)^2} + \frac{2(1-\mathfrak{q})^2}{(\mathfrak{q}+1)^3} + \cdots\right) \\
&\quad + \left(\frac{2}{\mathfrak{q}+1}Q_1 - \frac{4(1-\mathfrak{q})}{(\mathfrak{q}+1)^2}Q_1 + \frac{6(1-\mathfrak{q})^2}{(1-\mathfrak{q})^3}Q_1 + \ldots\right)\tau \\
&\quad + \left\{\left(\frac{2}{\mathfrak{q}+1} - \frac{4(1-\mathfrak{q})}{(\mathfrak{q}+1)^2} + \frac{6(1-\mathfrak{q})^2}{(\mathfrak{q}+1)^3} + \ldots\right)Q_2 \right. \\
&\quad \left. - \left(\frac{2(1-\mathfrak{q})}{(\mathfrak{q}+1)^2} + \frac{6(1-\mathfrak{q})^2}{(\mathfrak{q}+1)^3} + \ldots\right)Q_1^2\right\}\tau, \\
&= \sum_{n=1}^{\infty} \frac{2(-1)^{n-1}(1-\mathfrak{q})^{n-1}}{(\mathfrak{q}+1)^n} + \sum_{n=1}^{\infty} \frac{2n(-1)^{n-1}(1-\mathfrak{q})^{n-1}}{(\mathfrak{q}+1)^n}Q_1\tau \\
&\quad + \left\{\sum_{n=1}^{\infty} \frac{2n(-1)^{n-1}(1-\mathfrak{q})^{n-1}}{(\mathfrak{q}+1)^n}Q_2 \right. \\
&\quad \left. - \sum_{n=1}^{\infty} \frac{2(2n-1)(-1)^{n-1}(1-\mathfrak{q})^{n-1}}{(\mathfrak{q}+1)^{n+1}}Q_1^2\right\}\tau^2 + \cdots . \qquad (36)
\end{aligned}$$

The series $\sum\limits_{n=1}^{\infty} \frac{2(-1)^{n-1}(1-\mathfrak{q})^{n-1}}{(\mathfrak{q}+1)^n}$, $\sum\limits_{n=1}^{\infty} \frac{2n(-1)^{n-1}(1-\mathfrak{q})^{n-1}}{(\mathfrak{q}+1)^n}$, and $\sum\limits_{n=1}^{\infty} \frac{2(2n-1)(-1)^{n-1}(1-\mathfrak{q})^{n-1}}{(\mathfrak{q}+1)^{n+1}}$ are convergent and convergent to 1, $\frac{2}{\mathfrak{q}+1}$ and $\frac{2(1-\mathfrak{q})}{(\mathfrak{q}+1)}$.

Therefore, (36) becomes

$$P_{\beta,\gamma,\mathfrak{q}}(\tau) = 1 + \frac{2}{\mathfrak{q}+1}Q_1\tau + \left\{\frac{2}{\mathfrak{q}+1}Q_2 - \frac{2(1-\mathfrak{q})}{(\mathfrak{q}+1)}Q_1^2\right\}\tau^2 + \cdots . \qquad (37)$$

With this, the proof of Lemma 5 is finished. □

Lemma 6. *Let* $P(\tau) = 1 + \sum\limits_{n=1}^{\infty} p_n\tau^n \in \beta - P_{\mathfrak{q},\gamma}$, *then*

$$|p_n| \leq \frac{2}{\mathfrak{q}+1}|Q_1|, \qquad n \geq 1.$$

Proof. By Definition 11, $p(\tau) \in \beta - P_{\mathfrak{q},\gamma}$ if

$$p(\tau) \prec P_{\beta,\gamma,\mathfrak{q}}(\tau), \qquad (38)$$

where $\beta \in [0,\infty)$, and $P_{\beta,\gamma,\mathfrak{q}}(\tau)$ is given by (2).

By using (37) in (38), we have

$$P(\tau) \prec 1 + \frac{2}{\mathfrak{q}+1}Q_1\tau + \left\{\frac{2}{\mathfrak{q}+1}Q_2 - \frac{2(1-\mathfrak{q})}{(\mathfrak{q}+1)}Q_1^2\right\}\tau^2 + \cdots. \tag{39}$$

Now by using Lemma 5 on (39), we have

$$|p_n| \leq \frac{2}{\mathfrak{q}+1}|Q_1|.$$

With this, the proof of Lemma 6 is finished. □

Lemma 7. *Let $\Lambda_{\mathfrak{q}}^{\alpha}\xi$ be in the class $SP_{\alpha,\lambda}^{n,\mathfrak{q}}(\beta,\gamma)$. Then ξ is in the class $SP_{\alpha,\lambda}^{n,\mathfrak{q}}(\beta,\gamma)$.*

Proof. Let $\Lambda_{\mathfrak{q}}^{\alpha}\xi \in SP_{\alpha,\lambda}^{n,\mathfrak{q}}(\beta,\gamma)$; then from (32), $Đ_{\lambda,\mathfrak{q}}^{n,\alpha}\Lambda_{\mathfrak{q}}^{\alpha}\xi \in ST$. Using (16) and (21), we can write $Đ_{\lambda,\mathfrak{q}}^{n,\alpha}\xi$ in terms of $Đ_{\lambda,\mathfrak{q}}^{n,\alpha}\Lambda_{\mathfrak{q}}^{\alpha}\xi$ as follows:

$$Đ_{\lambda,\mathfrak{q}}^{n,\alpha}\xi(\tau) = \varphi(2-\alpha,2;\tau) * Đ_{\lambda,\mathfrak{q}}^{n,\alpha}\Lambda_{\mathfrak{q}}^{\alpha}\xi,$$

and by convolution properties, we obtain

$$\tau\partial_{\mathfrak{q}}\left(Đ_{\lambda,\mathfrak{q}}^{n,\alpha}\xi(\tau)\right) = \varphi(2-\alpha,2;\tau) * \tau\partial_{\mathfrak{q}}\left(Đ_{\lambda,\mathfrak{q}}^{n,\alpha}\Lambda_{\mathfrak{q}}^{\alpha}\xi\right).$$

Using Lemma 4 and (32), we obtain

$$\begin{aligned} \frac{\tau\partial_{\mathfrak{q}}\left(Đ_{\lambda,\mathfrak{q}}^{n,\alpha}\xi(\tau)\right)}{Đ_{\lambda,\mathfrak{q}}^{n,\alpha}\xi(\tau)} &= \frac{\varphi(2-\alpha,2;\tau) * \left[\frac{\tau\partial_{\mathfrak{q}}\left(Đ_{\lambda,\mathfrak{q}}^{n,\alpha}\Lambda_{\mathfrak{q}}^{\alpha}\xi(\tau)\right)}{Đ_{\lambda,\mathfrak{q}}^{n,\alpha}\Lambda_{\mathfrak{q}}^{\alpha}\xi(\tau)}\right]Đ_{\lambda,\mathfrak{q}}^{n,\alpha}\Lambda_{\mathfrak{q}}^{\alpha}\xi(\tau)}{\varphi(2-\alpha,2;\tau) * \tau\partial_{\mathfrak{q}}\left(Đ_{\lambda,\mathfrak{q}}^{n,\alpha}\Lambda_{\mathfrak{q}}^{\alpha}\xi\right)} \\ &\in \overline{co}\left(\frac{\tau\partial_{\mathfrak{q}}\left(Đ_{\lambda,\mathfrak{q}}^{n,\alpha}\Lambda_{\mathfrak{q}}^{\alpha}\xi(\tau)\right)}{Đ_{\lambda,\mathfrak{q}}^{n,\alpha}\Lambda_{\mathfrak{q}}^{\alpha}\xi(\tau)}(E)\right) \subseteq \Lambda_{\beta,\mathfrak{q},\gamma}. \end{aligned}$$

Therefore, $\xi \in SP_{\alpha,\lambda}^{n,\mathfrak{q}}(\beta,\gamma)$. □

Lemma 8. *Let $\Lambda_{\mathfrak{q}}^{\alpha}\xi$ be in the class $UCV_{\alpha,\lambda}^{n,\mathfrak{q}}(\beta,\gamma)$. Then ξ is in the class $UCV_{\alpha,\lambda}^{n,\mathfrak{q}}(\beta,\gamma)$.*

Proof. By virtue of (27) and Lemma 7, we have

$$\begin{aligned} \Lambda_{\mathfrak{q}}^{\alpha}\xi &\in UCV_{\alpha,\lambda}^{n,\mathfrak{q}}(\beta,\gamma) \Leftrightarrow \tau\partial_{\mathfrak{q}}\Lambda_{\mathfrak{q}}^{\alpha}\xi \in SP_{\alpha,\lambda}^{n,\mathfrak{q}}(\beta,\gamma) \\ &\Leftrightarrow \Lambda_{\mathfrak{q}}^{\alpha}\tau\partial_{\mathfrak{q}}\xi \in SP_{\alpha,\lambda}^{n,\mathfrak{q}}(\beta,\gamma) \\ &\Leftrightarrow \tau\partial_{\mathfrak{q}}\xi \in SP_{\alpha,\lambda}^{n,\mathfrak{q}}(\beta,\gamma) \\ &\Leftrightarrow \xi \in UCV_{\alpha,\lambda}^{n,\mathfrak{q}}(\beta,\gamma). \end{aligned}$$

Therefore, $\xi \in UCV_{\alpha,\lambda}^{n,\mathfrak{q}}(\beta,\gamma)$. □

Lemma 9. *Let $0 \leq \beta < 1$ and $\frac{1}{2} \leq \gamma < 1$ or $\beta \geq 1$ and $0 \leq \gamma < 1$. If $\xi \in SP_{\alpha,\lambda}^{n,\mathfrak{q}}(\beta,\gamma)$, then $Đ_{\lambda,\mathfrak{q}}^{n,\alpha}\xi \in ST(\frac{1}{2})$.*

Proof. From (30), the result is obvious, where $\frac{\beta(\mathfrak{q}+1)+(3-\mathfrak{q})\gamma}{(1+\beta)((1-\mathfrak{q})+2)} \geq \frac{1}{2}$ for $0 \leq \beta < 1$ and $\frac{1}{2} \leq \gamma < 1$ or $\beta \geq 1$ and $0 \leq \gamma < 1$. □

3. Main Results

Theorem 1. *Let* $0 \leq \beta < \infty$, $\mathfrak{q} \in (0,1)$, $\mu \in \mathbb{C}$, *and let* $\xi(\tau) \in SP^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma)$, *and* ξ *is of the form (1). Then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2|Q_1|}{\left([3]_{\mathfrak{q}} - 1\right)\Psi_{3,k,\mathfrak{q}}(\alpha,\lambda)(\mathfrak{q}+1)} \max\{1, |2v-1|\} \tag{40}$$

where v *is given by* (48) *below.*

Proof. If $\xi(\tau) \in SP^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma)$, then by definition,

$$SP^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma) \prec P_{\beta,\gamma,\mathfrak{q}}(\tau)),$$

there is a Schwartz function w with $w(0) = 0$ and $|w(\tau)| < 1$, in such a way that

$$\frac{\tau\partial_{\mathfrak{q}}(\text{Đ}^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))}{\text{Đ}^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau)} = P_{\beta,\gamma,\mathfrak{q}}(w(\tau)). \tag{41}$$

Let $h \in P$, defined as

$$h(\tau) = \frac{1+w(\tau)}{1-w(\tau)} = 1 + c_1\tau + c_2\tau + \cdots.$$

This can be written as

$$w(\tau) = \frac{c_1}{2}\tau + \frac{1}{2}(c_2 - \frac{c_1^2}{2})\tau^2 + \cdots.$$

Now

$$\begin{aligned}\frac{\tau\partial_{\mathfrak{q}}(\text{Đ}^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))}{\text{Đ}^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau)} &= 1 + ([2]_{\mathfrak{q}} - 1)\Psi_{2,k,\mathfrak{q}}(\alpha,\lambda)a_2\tau \\ &\quad + \Big\{([3]_{\mathfrak{q}} - 1)\Psi_{3,k,\mathfrak{q}}(\alpha,\lambda)a_3 - \\ &\quad ([2]_{\mathfrak{q}} - 1)\Psi^2_{2,k,\mathfrak{q}}(\alpha,\lambda)a_2^2\Big\}\tau^2 + \cdots \end{aligned} \tag{42}$$

Similarly,

$$\begin{aligned}& P_{\beta,\gamma,\mathfrak{q}}(w(\tau)) \\ &= 1 + \frac{Q_1c_1}{(\mathfrak{q}+1)}\tau + \frac{1}{(\mathfrak{q}+1)}\left\{\frac{Q_2c_1^2}{2} + (c_2 - \frac{c_1^2}{2})Q_1 - \frac{(1-\mathfrak{q})Q_1^2c_1^2}{2}\right\}\tau^2 + \cdots \end{aligned} \tag{43}$$

By using (42) and (43) in (41) and comparing both sides, we obtain

$$a_2 = \frac{Q_1c_1}{([2]_{\mathfrak{q}} - 1)(\mathfrak{q}+1)\Psi_{2,k,\mathfrak{q}}(\alpha,\lambda)} \tag{44}$$

and

$$\begin{aligned}& ([3]_{\mathfrak{q}} - 1)\Psi_{3,k,\mathfrak{q}}(\alpha,\lambda)a_3 - ([2]_{\mathfrak{q}} - 1)\Psi^2_{2,k,\mathfrak{q}}(\alpha,\lambda)a_2^2 \\ &= \frac{1}{(\mathfrak{q}+1)}\left\{\frac{Q_2c_1^2}{2} + (c_2 - \frac{c_1^2}{2})Q_1 - \frac{(1-\mathfrak{q})Q_1^2c_1^2}{2}\right\} \end{aligned} \tag{45}$$

After some simple calculation of (45), we obtain

$$a_3 = \frac{Q_1}{([3]_{\mathfrak{q}} - 1)\Psi_{3,k,\mathfrak{q}}(\alpha,\lambda)(\mathfrak{q}+1)}\left(c_2 - \left(\frac{1}{2} - \frac{Q_2}{2Q_1} + \frac{(1-\mathfrak{q})Q_1}{2} - \frac{Q_1}{([2]_{\mathfrak{q}} - 1)(\mathfrak{q}+1)}\right)c_1^2\right) \tag{46}$$

and

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|Q_1|}{([3]_{\mathfrak{q}} - 1)\Psi_{3,k,\mathfrak{q}}(\alpha,\lambda)(\mathfrak{q}+1)}\left\{c_2 - vc_1^2\right\}, \tag{47}$$

where

$$\begin{aligned} v &= \frac{1}{2} - \frac{Q_2}{2Q_1} + \frac{(1-\mathfrak{q})Q_1}{2} - \frac{Q_1}{([2]_{\mathfrak{q}} - 1)(\mathfrak{q}+1)} \\ &\quad + \frac{\mu Q_1([3]_{\mathfrak{q}} - 1)\Psi_{3,k,\mathfrak{q}}(\alpha,\lambda)}{([2]_{\mathfrak{q}} - 1)^2(\mathfrak{q}+1)\Psi_{2,k,\mathfrak{q}}^2(\alpha,\lambda)}. \end{aligned} \tag{48}$$

Now by using Lemma 1 on (47), we have

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2|Q_1|}{\left([3]_{\mathfrak{q}} - 1\right)\Psi_{3,k,\mathfrak{q}}(\alpha,\lambda)(\mathfrak{q}+1)} \max\{1, |2v-1|\}.$$

This concludes the proof of Theorem 1. □

Theorem 2. *A function ξ of the type (1) is in $SP_{\alpha,\lambda}^{n,\mathfrak{q}}(\beta,\gamma)$ if*

$$\sum_{n=2}^{\infty}\{[n]_{\mathfrak{q}}(1+\beta) - (\beta+\gamma)\}\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)|a_n| \leq 1-\gamma, \tag{49}$$

where $\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)$ is given by (20).

Proof. It suffices to show that

$$\beta\left|\frac{\tau\partial_{\mathfrak{q}}(Đ_{\lambda,\mathfrak{q}}^{n.\alpha}\xi(\tau))}{Đ_{\lambda,\mathfrak{q}}^{n.\alpha}\xi(\tau)} - 1\right| - \mathrm{Re}\left\{\frac{\tau\partial_{\mathfrak{q}}(Đ_{\lambda,\mathfrak{q}}^{n.\alpha}\xi(\tau))}{Đ_{\lambda,\mathfrak{q}}^{n.\alpha}\xi(\tau)} - 1\right\} < 1-\gamma.$$

We have

$$\begin{aligned} &\beta\left|\frac{\tau\partial_{\mathfrak{q}}(Đ_{\lambda,\mathfrak{q}}^{n.\alpha}\xi(\tau))}{Đ_{\lambda,\mathfrak{q}}^{n.\alpha}\xi(\tau)} - 1\right| - \mathrm{Re}\left\{\frac{\tau\partial_{\mathfrak{q}}(Đ_{\lambda,\mathfrak{q}}^{n.\alpha}\xi(\tau))}{Đ_{\lambda,\mathfrak{q}}^{n.\alpha}\xi(\tau)} - 1\right\} \\ \leq\ &(1+\beta)\left|\frac{\tau\partial_{\mathfrak{q}}(Đ_{\lambda,\mathfrak{q}}^{n.\alpha}\xi(\tau))}{Đ_{\lambda,\mathfrak{q}}^{n.\alpha}\xi(\tau)} - 1\right| \\ =\ &(1+\beta)\left|\frac{\tau + \sum_{n=2}^{\infty}[n]_{\mathfrak{q}}\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)a_n\tau^n}{\tau + \sum_{n=2}^{\infty}\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)a_n\tau^n} - 1\right| \end{aligned}$$

$$= (1+\beta)\left|\frac{\tau + \sum_{n=2}^{\infty}[n]_{\mathfrak{q}}\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)a_n\tau^n - \tau - \sum_{n=2}^{\infty}\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)a_n\tau^n}{\tau + \sum_{n=2}^{\infty}\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)a_n\tau^n}\right|$$

$$
\begin{aligned}
&= (1+\beta)\left|\frac{\tau\left(1+\sum\limits_{n=2}^{\infty}[n]_{\mathfrak{q}}\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)a_n\tau^n - 1 - \sum\limits_{n=2}^{\infty}\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)a_n\tau^n\right)}{\tau\left(1+\sum\limits_{n=2}^{\infty}\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)a_n\tau^n\right)}\right| \\
&= (1+\beta)\left|\frac{\sum\limits_{n=2}^{\infty}([n]_{\mathfrak{q}}-1)\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)a_n\tau^n}{1+\sum\limits_{n=2}^{\infty}\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)a_n\tau^n}\right| \\
&< (1+\beta)\frac{\sum\limits_{n=2}^{\infty}([n]_{\mathfrak{q}}-1)\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)|a_n|}{1-\sum\limits_{n=2}^{\infty}\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)|a_n|},
\end{aligned}
$$

Therefore,

$$
(1+\beta)\frac{\sum\limits_{n=2}^{\infty}([n]_{\mathfrak{q}}-1)\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)|a_n|}{1-\sum\limits_{n=2}^{\infty}\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)|a_n|} \le 1-\gamma.
$$

After some simple calculation, we have

$$
\sum_{n=2}^{\infty}([n]_{\mathfrak{q}}(1+\beta)-(\beta+\gamma))\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)|a_n| \le 1-\gamma.
$$

Hence, this completes our result. □

When we take $\mathfrak{q} \to 1-$, then Theorem 2 makes use of a well-established finding from [11].

Corollary 1. *The set $SP^{n}_{\alpha,\lambda}(\beta,\gamma)$ contains the function f of the type (1), if*

$$
\sum_{n=2}^{\infty} n\{n(1+\beta)-(\beta+\gamma)\}\Psi_n(\alpha,\lambda)|a_n| \le 1-\gamma,
$$

where

$$
\Psi_{n,k}(\alpha,\lambda) = \left\{\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}(1+\lambda(n-1))\right\}^k.
$$

Remark 13. *The outcomes of Theorem 2 and Corollary 1 in [31] are the same whether we consider $\mathfrak{q} \to 1-, n=1, \alpha=0, \lambda=0,$ and $\beta=1$.*

Remark 14. *We can obtain the result that is proved in [32], if we consider $\mathfrak{q} \to 1-, \alpha=0, \lambda=1,$ and $\gamma=0$ in Theorem 2.*

Theorem 3. *The class $UCV^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma)$ has the function ξ of the type (1) if*

$$
\sum_{n=2}^{\infty}[n]_{\mathfrak{q}}\{[n]_{\mathfrak{q}}(1+\beta)-(\beta+\gamma)\}\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)|a_n| \le 1-\gamma.
$$

Proof. Use the same technique of Theorem 2; we obtain the proof of Theorem 3. □

When we take $\mathfrak{q} \to 1-$, then from Theorem 3, we have a known result studied in [11].

Corollary 2. *The class $UCV^{n}_{\alpha,\lambda}(\beta,\gamma)$ has the function ξ of the type (1) if*

$$\sum_{n=2}^{\infty}\{n(1+\beta)-(\beta+\gamma)\}\Psi_{n}(\alpha,\lambda)|a_n|\leq 1-\gamma,$$

where

$$\Psi_{n,k}(\alpha,\lambda)=\left\{\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}(1+\lambda(n-1))\right\}^{k}.$$

Theorem 4. *The class $SP^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma)$ has the function ξ of the type (1). Then*

$$|a_2|\leq\frac{2|Q_1|}{(\mathfrak{q}+1)([2]_{\mathfrak{q}}-1)\Psi_{2,k,\mathfrak{q}}(\alpha,\lambda)}\tag{50}$$

and

$$|a_n|\leq\frac{2|Q_1|}{(\mathfrak{q}+1)([n]\mathfrak{q}-1)\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)}\prod_{j=2}^{n-1}\left(1+\frac{2|Q_1|}{(\mathfrak{q}+1)[j]_{\mathfrak{q}}-1}\right).\tag{51}$$

Proof. Let $\xi\in SP^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma)$; then

$$\frac{\tau\partial_{\mathfrak{q}}(\text{Đ}^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))}{\text{Đ}^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau)}\prec P_{\mathfrak{q},\beta,\gamma}(\tau).$$

Define

$$\mathfrak{q}(\tau)=\frac{\tau\partial_{\mathfrak{q}}(\text{Đ}^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))}{\text{Đ}^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau)}=1+\sum_{n=1}^{\infty}p_n\tau^n,$$

implying

$$\tau\partial_{\mathfrak{q}}(\text{Đ}^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))=\mathfrak{q}(\tau)\left(\text{Đ}^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau)\right)$$

$$\begin{aligned}\tau\left(1+\sum_{n=2}^{\infty}[n]_{\mathfrak{q}}\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)a_n\tau^n\right)&=\left(1+\sum_{n=2}^{\infty}p_n\tau^n\right)\left(\tau+\sum_{n=2}^{\infty}\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)a_n\tau^n\right),\\ \sum_{n=2}^{\infty}([n]_{\mathfrak{q}}-1)\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)a_n\tau^n&=\sum_{n=2}^{\infty}p_n\tau^n+\sum_{n=2}^{\infty}\left(\sum_{j=1}^{n-1}c_j\Psi_{n-j,n,\mathfrak{q}}(\alpha,\lambda)a_{n-j}\right)\tau^n\end{aligned}$$

When we evaluate the τ-coefficient of each side, we obtain

$$a_n=\frac{1}{([n]_{\mathfrak{q}}-1)\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)}\sum_{j=1}^{n-1}p_j\Psi_{n-j,k,\mathfrak{q}}(\alpha,\lambda)a_{n-j},\qquad a_1=1.$$

Taking the mod and applying Lemma 6, we have

$$|a_n|\leq\frac{2|Q_1|}{([n]_{\mathfrak{q}}-1)(\mathfrak{q}+1)\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)}\sum_{j=1}^{n-1}\Psi_{n-j,k,\mathfrak{q}}(\alpha,\lambda)a_{n-j},\qquad a_1=1,\tag{52}$$

For $n=2$ in (52), we obtain

$$|a_2|\leq\frac{2|Q_1|}{(\mathfrak{q}+1)\left([2]_{\mathfrak{q}}-1\right)\Psi_{2,k,\mathfrak{q}}(\alpha,\lambda)}.$$

Therefore, the result is true for $n = 2$. Let $n = 3$ in (52); we have

$$|a_3| \leq \frac{2|Q_1|}{(\mathfrak{q}+1)([3]_{\mathfrak{q}}-1)\Psi_{3,k,\mathfrak{q}}(\alpha,\lambda)}\left[\Psi_{2,k,\mathfrak{q}}(\alpha,\lambda)|a_2|+1\right]$$

$$|a_3| \leq \frac{2|Q_1|}{(\mathfrak{q}+1)([3]_{\mathfrak{q}}-1)\Psi_{3,k,\mathfrak{q}}(\alpha,\lambda)}\left[1+\frac{2|Q_1|}{(\mathfrak{q}+1)([2]_{\mathfrak{q}}-1)}\right].$$

Therefore, the result is true for $n = 3$. Let $n = 4$ in (52); we have

$$\begin{aligned}|a_4| &\leq \frac{2|Q_1|}{(\mathfrak{q}+1)([4]_{\mathfrak{q}}-1)\Psi_{4,k,\mathfrak{q}}(\alpha,\lambda)}\times \\ &\left\{\left(\frac{2|Q_1|}{(\mathfrak{q}+1)[3]_{\mathfrak{q}}-1}\right)\left(1+\frac{2|Q_1|}{(\mathfrak{q}+1)([2]_{\mathfrak{q}}-1)}\right)+\left(1+\frac{2|Q_1|}{(\mathfrak{q}+1)([2]_{\mathfrak{q}}-1)}\right)\right\}, \\ &= \frac{2|Q_1|}{(\mathfrak{q}+1)([4]_{\mathfrak{q}}-1)\Psi_{4,n,\mathfrak{q}}(\alpha,\lambda)}\times\left\{\left(1+\frac{2|Q_1|}{(\mathfrak{q}+1)([2]_{\mathfrak{q}}-1)}\right)\left(1+\frac{2|Q_1|}{(\mathfrak{q}+1)([3]_{\mathfrak{q}}-1)}\right)\right\}.\end{aligned}$$

If n is 4, then the conclusion holds. Mathematical induction allows us to derive

$$|a_n| \leq \frac{2|Q_1|}{(\mathfrak{q}+1)([n]_{\mathfrak{q}}-1)\Psi_{n,k,\mathfrak{q}}(\alpha,\lambda)}\prod_{j=2}^{n-1}\left(1+\frac{2|Q_1|}{(\mathfrak{q}+1)[j]_{\mathfrak{q}}-1}\right).$$

□

Theorem 5. *Let* $0 \leq \mu \leq \alpha < 1$, $\mathfrak{q} \in (0,1)$. *Then*

$$SP^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma) \subseteq SP^{n,\mathfrak{q}}_{\mu,\lambda}(\beta,\gamma)$$

where

$$(0 \leq \beta < 1 \text{ and } \frac{1}{2} \leq \gamma < 1) \text{ or } (\beta \geq 1 \text{ and } 0 \leq \gamma < 1).$$

Proof. Let $\xi \in SP^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma)$. Then by (21) and convolution properties, we have

$$Đ^{n,\mu}_{\lambda,\mathfrak{q}}\xi(\tau) = \underbrace{\left[(\varphi(2,2-\mu;\tau)*g_{\lambda,\mathfrak{q}}(\tau))*\cdots*(\varphi(2,2-\mu;\tau)*g_{\lambda,\mathfrak{q}}(\tau))\right]}_{\text{n-times}}*\xi(\tau)$$

$$Đ^{n,\mu}_{\lambda,\mathfrak{q}}\xi(\tau) = \underbrace{\left[\varphi(2-\alpha,2-\mu;\tau)*\cdots*\varphi(2-\alpha,2-\mu;\tau)\right]}_{\text{n-times}}*Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau)$$

and

$$\tau\partial_{\mathfrak{q}}(Đ^{n,\mu}_{\lambda,\mathfrak{q}}\xi(\tau)) = \underbrace{\left[\varphi(2-\alpha,2-\mu;\tau)*\cdots*\varphi(2-\alpha,2-\mu;\tau)\right]}_{\text{n-times}}*\tau\partial_{\mathfrak{q}}(Đ^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau)).$$

Additio ally, it is known that [33]

$$\varphi(2-\alpha,2-\mu;\tau) \in ST\left(\frac{1}{2}\right).$$

Therefore, we obtain by repeatedly using Lemma 3 n–times

$$\underbrace{\left[\varphi(2-\alpha,2-\mu;\tau)*\cdots*\varphi(2-\alpha,2-\mu;\tau)\right]}\in ST(\frac{1}{2}).$$

For ($0 \leq \beta < 1$ and $\frac{1}{2} \leq \gamma < 1$) or ($\beta \geq 1$ and $0 \leq \gamma < 1$), we have, by Lemma 9,

$$Ð^{n,\alpha}_{\lambda,\mathfrak{q}}\xi \in ST\left(\frac{1}{2}\right)$$

Using Lemma 4, we obtain

$$\frac{\tau\partial_{\mathfrak{q}}(Ð^{n,\mu}_{\lambda,\mathfrak{q}}\xi(\tau))}{Ð^{n,\mu}_{\lambda,\mathfrak{q}}\xi(\tau)} = \frac{\underbrace{[\varphi(2-\alpha,2-\mu;\tau)*\cdots*\varphi(2-\alpha,2-\mu;\tau)]}*\left(\frac{\tau\partial_{\mathfrak{q}}(Ð^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau)}{Ð^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau)}\left(Ð^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau)\right)\right)}{\underbrace{[\varphi(2-\alpha,2-\mu;\tau)*\cdots*\varphi(2-\alpha,2-\mu;\tau)]}*Ð^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau)}$$

$$\in \overline{co} \in \left(\frac{\tau\partial_{\mathfrak{q}}(Ð^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau))}{Ð^{n,\alpha}_{\lambda,\mathfrak{q}}\xi(\tau)}(E)\right) \subseteq \Lambda_{\beta,\mathfrak{q},\gamma}.$$

Thus, $\xi \in SP^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma)$. □

Remark 15. *When we take $\mathfrak{q} \to 1-$, then from Theorem 5, we have a known result studied in [11]. The result explored in [11] is obtained from Theorem 5 when $\mathfrak{q} \to 1-$.*

Remark 16. *If we consider $\mathfrak{q} \to 1-$, $n = 1$, $\lambda = 0$, and $\beta = 0$, in Theorem 5, we obtain the result given in [34].*

Remark 17. *If we consider $\mathfrak{q} \to 1-$, $n = 1$, $\lambda = 0$, $\gamma = 0$, and $\beta = 1$, in Theorem 5, we obtain the result given in [35].*

Theorem 6. *Let $0 \leq \mu \leq \alpha < 1$, $\mathfrak{q} \in (0,1)$. Then*

$$UCV^{n,\mathfrak{q}}_{\alpha,\lambda}(\beta,\gamma) \subseteq UCV^{n,\mathfrak{q}}_{\mu,\lambda}(\beta,\gamma)$$

where

$$(0 \leq \beta < 1 \text{ and } \frac{1}{2} \leq \gamma < 1) \text{ or } (\beta \geq 1 \text{ and } 0 \leq \gamma < 1).$$

Proof. The proof of Theorem 6 can be obtained by using the same method as that used to prove Theorem 5. □

Remark 18. *When we take $\mathfrak{q} \to 1-$, then from Theorem 6, we have a known result studied in [11].*

4. Conclusions

The operators of $\mathfrak{q}$-fractional calculus have been addressed and effectively implemented in a number of recent and continuing publications; see [36]. From the fractional calculus $\mathfrak{q}$-Pochhammer symbol, several writers have expanded the notions of fractional $\mathfrak{q}$-integral and fractional $\mathfrak{q}$-derivative by proposing many different lower limits of integration. Many of these applications of the basic (or $\mathfrak{q}$-) calculus and the fractional basic (or $\mathfrak{q}$-) calculus in the geometric function theory of complex analysis have inspired the present work.

We were motivated to conduct this research after reading Srivastava's survey-cum-expository review essay [3], in which he describes the application of both the fundamental (or $\mathfrak{q}$-) calculus and the fractional (or $\mathfrak{q}$-) calculus to the study of geometric functions. In Section 1, we discussed some background ideas that are presented in this article and also used the $\mathfrak{q}$-calculus operator theory and successfully defined the $\mathfrak{q}$-analogous of a fractional differential operator for analytic functions. Considering this operator, we defined subclasses of $\mathfrak{q}$-starlike and $\mathfrak{q}$-convex functions. In Section 2, we mentioned some known lemmas, and also we proved some new lemmas. In Section 3, by utilizing these lemmas, we examined several useful properties, such as inclusion relations, coefficient bounds, Fekete–Szego problem, and subordination results. We also highlight many known and brand-new particular cases of our findings.

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