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Article

On Certain Inequalities for Several Kinds of Strongly Convex Functions for *q*-*h*-Integrals

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Abstract: This article investigates inequalities for certain types of strongly convex functions by applying q-h-integrals. These inequalities provide the refinements of some well-known results that hold for (α, m)- and $(\hbar\text{-}m)$-convex and related functions. Inequalities for q-integrals are deducible by vanishing the parameter h. Some particular cases are discussed after proving the main results.

Keywords: q-derivative/integral; q-h-derivative/integral; convex function; strongly convex function; Hermite–Hadamard inequality

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1. Introduction

Naturally, questions of why and how occur in the mind when defining new concepts in different fields of science and engineering. These same questions were raised in the sixteenth century about derivatives/integrals of fractional orders. Ultimately, these questions led to a new field of fractional calculus based on integral and derivative operators of different kinds. Now, in this era, almost all subjects of science and engineering are studied with the help of fractional calculus. This field of fractional calculus has not only generalized the classical concepts of ordinary calculus but has also solved long-lasting unsolved real-world problems. For a detailed study, we refer the readers to [1–4].

Besides the fractional derivatives and fractional integrals, in the last two decades, q-derivatives and q-integrals have been studied very extensively. Subjects of diverse fields of sciences have been investigated in the sense of q-derivatives and q-integrals. For instance, q-difference equations are studied in [5,6], and q-analogues of special functions and Taylor series are studied in [7–9].

Fractional integral inequalities for convex functions published in the last decade are directly related to classical integral inequalities. Similarly, q-integral inequalities are generalizations of classical inequalities that hold for Riemann integrals. For a detailed study of q-integral inequalities, we refer the readers to [10–13].

The aim of this article is to study the integral inequalities of the Hermite–Hadamard type for various types of strongly convex functions via q-h-integrals. The results hold at the same time for q- and h-integrals in implicit form. Inequalities for q-integrals can be formulated explicitly and can also be deduced from q-h-integral inequalities.

Strongly convex functions give a generalization and refinement of convex functions. The established results present refinements of many published inequalities that hold for q-h-integrals and convex functions of several kinds. In the following, we give definitions of convex and strongly convex functions.

Definition 1. *A function $\xi : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if*

$$\xi(tu + (1-t)v) \leq t\xi(u) + (1-t)\xi(v), \tag{1}$$

is satisfied for all $u, v \in I$ and $t \in [0,1]$, where I is an interval in $\mathbb{R}$. If the inequality (1) holds in reverse order, then ξ is said to be concave on interval I.

The above inequality (1) can be preserved in different ways, which makes room to define new notions. In the recent literature of mathematical inequalities, such notions are named as modified convex, strongly convex, exponentially convex, refined convex, etc. Strong convexity is an extension of the inequality (1), which actually provides a refinement as well as a generalization of the notion of convexity.

In [14], the definition of strongly convex functions and in [15] the definition of strongly $\hbar$-convex functions are given.

Definition 2 ([14])**.** *Let D be a convex subset of X, and $(X, \|.\|)$ be a normed space. A function $\xi : D \longrightarrow \mathbb{R}$ is called strongly convex with modulus $C \geq 0$ if the following inequality*

$$\xi(tu + (1-t)v) \leq t\xi(u) + (1-t)\xi(v) - Ct(1-t)\|v-u\|^2, \tag{2}$$

is satisfied for all $u, v \in D$, $t \in [0,1]$.

Definition 3 ([15])**.** *Let I be an interval in $\mathbb{R}$, $\hbar : (0,1) \to (0,\infty)$ be a given function, and C be a positive constant. A function $\xi : D \to \mathbb{R}$ is strongly $\hbar$-convex with modulus C if*

$$\xi(tu + (1-t)v) \leq \hbar(t)\xi(u) + \hbar(1-t)\xi(v) - Ct(1-t)\|u-v\|^2, \tag{3}$$

is satisfied for all $u, v \in D$ and $t \in (0,1)$.

Two extensions of strongly convex functions are termed strongly $(\hbar$-$m)$- and strongly $(\alpha$-$m)$-convex functions; these are given in forthcoming definitions.

Definition 4 ([16])**.** *Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $\hbar : J \to \mathbb{R}$ be a non-negative function. A non-negative function $\xi : [0,b] \to \mathbb{R}$ is called strongly $(\hbar$-$m)$-convex with modulus $C \geq 0$ if*

$$\xi(tu + m(1-t)v) \leq \hbar(t)\xi(u) + m\hbar(1-t)\xi(v) - mC\hbar(t)\hbar(1-t)(u-v)^2, \tag{4}$$

is satisfied for all $u, v \in [0,b]$, $m \in [0,1]$ and $t \in (0,1)$.

Definition 5 ([17])**.** *A function $\xi : [0,b] \to \mathbb{R}$ is said to be strongly $(\alpha$-$m)$-convex with modulus $C \geq 0$, for $(\alpha, m) \in [0,1]^2$, if*

$$\xi(tu + m(1-t)v) \leq t^\alpha\xi(u) + m(1-t^\alpha)\xi(v) - Cmt^\alpha(1-t^\alpha)|v-u|^2, \tag{5}$$

is satisfied for all $u, v \in [0,b]$ and $t \in [0,1]$.

One can obtain the definition of a strongly convex function either by setting $h(t) = t$, $m = 1$ in (4) or $\alpha = 1$, $m = 1$ in (5). The above two definitions bring out the definition of strong m-convexity defined in [18]. The objective of this article is to develop q-h-integral inequalities by utilizing the above definitions of strongly convex functions. We give the definitions of the q-derivative, q-integral, q-h-derivative and q-h-integral in upcoming definitions. In this section, we let $I := [a,b]$, where $a,b \in \mathbb{R}$, $a < b$, and $C[a,b]$ denotes the class of continuous functions defined on $[a,b]$.

Definition 6 ([19])**.** *Let $\xi \in C[a,b]$ and $q \in (0,1)$. Then, $D_q\xi(u)$, defined by the quotient*

$$D_q\xi(u) = \frac{d_q\xi(u)}{d_q u} = \frac{\xi(qu) - \xi(u)}{(q-1)u},\ u \neq 0 \tag{6}$$

is known as the q-derivative of ξ at point $u \in [a,b]$, provided that $qu \in [a,b]$.

Definition 7 ([19])**.** *Let $\xi \in C[a,b]$ and $q \in (0,1)$. Then, the q-definite integral on I is defined by*

$$\int_a^u \xi(t)\ {}_a d_q t = (1-q)(u-a)\sum_{n=0}^{\infty} q^n \xi(q^n u + (1-q^n)a), \tag{7}$$

for $u \in I$.

Definition 8 ([20])**.** *Let $\xi \in C[a,b]$, $q \in (0,1)$ and $h \in \mathbb{R}$. Then, $C_h D_q \xi(u)$, defined by the quotient*

$$C_h D_q\xi(u) = \frac{{}_h d_q \xi(u)}{{}_h d_q u} = \frac{\xi(qu+h) - \xi(u)}{(q-1)u + qh},\ u \neq \frac{q}{1-q}h \tag{8}$$

is known as the q-h-derivative of ξ at point $u \in [a,b]$, provided that $qu + h \in [a,b]$.

By setting $h = 0$ in (8), one can obtain $C_0 D_q\xi(u) = D_q\xi(u)$.

Definition 9 ([20])**.** *Let $\xi \in C[a,b]$, $q \in (0,1)$ and $h \in \mathbb{R}$. Then, the following formulas*

$$\begin{aligned} I^{a+}_{q-h}\xi(u) &:= \int_a^u \xi(t)\ {}_h d_q t \\ &= ((1-q)(u-a) + qh)\sum_{n=0}^{\infty} q^n \xi(q^n a + (1-q^n)u + nq^n h),\ u > a, \end{aligned} \tag{9}$$

$$\begin{aligned} I^{b-}_{q-h}\xi(u) &:= \int_u^b \xi(t)\ {}_h d_q t \\ &= ((1-q)(b-u) + qh)\sum_{n=0}^{\infty} q^n \xi(q^n u + (1-q^n)b + nq^n h),\ u < b, \end{aligned} \tag{10}$$

are known as left and right q-h integrals.

Example 1. *Let $\xi(t) = (w+t)^2$, where w is a real constant. Then, $I^{a+}_{q-h}\xi(b)$ is calculated as follows:*

$$\begin{aligned} &\int_a^b (t+w)^2\ {}_h d_q t = ((1-q)(b-a) + qh)\sum_{n=0}^{\infty} q^n (q^n a + (1-q^n)b + nq^n h + w)^2 \\ &= \frac{((1-q)(b-a) + qh)}{1-q}\left(\frac{(b-a)^2}{1+q+q^2} + (b+w)^2 - \frac{2(b-a)(b+w)}{1+q}\right. \\ &\left. + \Big(h^2 U + 2h(b+w)S - 2(b-a)hT\Big)(1-q)\right), \end{aligned}$$

where

$$S = \sum_{n=0}^{\infty} q^{2n} n,\ T = \sum_{n=0}^{\infty} q^{3n} n,\ U = \sum_{n=0}^{\infty} q^{3n} n^2.$$

The rest of the paper is organized as follows: In the upcoming section, we derive q-h-integral inequalities for strongly convex functions of various types. These inequalities are formulated for symmetric and symmetric-like functions. Consequences of the main results are discussed after completing their proofs. The later results are generalizations of previous results.

2. Inequalities for q-h-Integrals via Strongly Convex Functions

First, we prove q-h-integral inequalities for strongly $\hbar$- and m-convex functions. Second, we prove the compact forms of the presented results by using strongly ($\hbar$-m)- and (α, m)-convex functions. All these results are proved by stating the symmetry and symmetry-like conditions.

Theorem 1. *Let ξ be strongly $\hbar$-convex and $\hbar\left(\frac{1}{2}\right) \neq 0$. Then, the forthcoming inequality for the left q-h-integral holds when ξ is symmetric about $\frac{a+u}{2}$:*

$$\begin{aligned}
&\frac{1}{2\hbar\left(\frac{1}{2}\right)}\xi\left(\frac{a+u}{2}\right) \leq \frac{1-q}{(1-q)(u-a)+qh_1}\int_a^u \xi(t)\ {}_{h_1}d_qt \\
&-\frac{C(u-a)^2}{2\hbar\left(\frac{1}{2}\right)}\left(\frac{1+q^3+2q(1-q)}{4(1+q)(1+q+q^2)}+\left(h^2U+hS-2hT\right)(1-q)\right) \\
&\leq \frac{1-q}{1-q+qh}\left(\xi(u)\int_0^1 \hbar(t)\ {}_hd_qt+\xi(a)\int_0^1 \hbar(1-t)\ {}_hd_qt\right)-C(u-a)^2 \\
&\times\left(\frac{q^2}{(1+q)(1+q+q^2)}+(1-q)\left(2hT-hS-h^2U\right)\right),
\end{aligned} \tag{11}$$

where $0 \leq a < b$, $u \in (a,b]$, $h_1 = (u-a)h$, $S = \sum_{n=0}^{\infty} q^{2n}n, T = \sum_{n=0}^{\infty} q^{3n}n$ and $U = \sum_{n=0}^{\infty} q^{3n}n^2$.

Proof. A strongly $\hbar$-convex function satisfies the following inequality:

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\xi\left(\frac{a+u}{2}\right) \leq \xi(ta+(1-t)u)+\xi(tu+(1-t)a)-\frac{C(u-a)^2}{\hbar\left(\frac{1}{2}\right)}\left(\frac{1}{2}-t\right)^2. \tag{12}$$

As a result, for q-h-integrals, one can obtain the upcoming inequality (13):

$$\begin{aligned}
&\frac{1}{\hbar\left(\frac{1}{2}\right)}\xi\left(\frac{a+u}{2}\right) \leq \frac{1-q}{(1-q)+qh}\left(\int_0^1 \xi(ta+(1-t)u)\ {}_hd_qt\right. \\
&\left.+\int_0^1 \xi(tu+(1-t)a)\ {}_hd_qt-\frac{C(u-a)^2}{\hbar\left(\frac{1}{2}\right)}\int_0^1\left(t-\frac{1}{2}\right)^2\ {}_hd_qt\right).
\end{aligned} \tag{13}$$

The function ξ is symmetric about $\frac{a+u}{2}$. By using this fact, we obtain the resulting inequality from (13) as follows:

$$\begin{aligned}
&\frac{1}{2\hbar\left(\frac{1}{2}\right)}\xi\left(\frac{a+u}{2}\right) \\
&\leq \frac{1-q}{(1-q)+qh}\left(\int_0^1 \xi(a+t(u-a))\ {}_hd_qt-\frac{C(u-a)^2}{2\hbar\left(\frac{1}{2}\right)}\int_0^1\left(t-\frac{1}{2}\right)^2\ {}_hd_qt\right).
\end{aligned} \tag{14}$$

From Example 1, taking $a = 0$, $b = 1$, and $w = -\frac{1}{2}$, we have

$$\int_0^1 \left(t - \frac{1}{2}\right)^2 {}_h d_q t = \frac{(1-q)+qh}{1-q}\left(\frac{q^3+2q(1-q)+1}{4(1+q)(1+q+q^2)} + \left(h^2U + hS - 2hT\right)(1-q)\right). \tag{15}$$

By using Formula (15) in the inequality (14), we obtain

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\xi\left(\frac{a+u}{2}\right) \leq \frac{1-q}{(1-q)+qh}\int_0^1 \xi(a+t(u-a))\ {}_h d_q t - \frac{C(u-a)^2}{2\hbar\left(\frac{1}{2}\right)}\left(\frac{1+q^3+2q(1-q)}{4(1+q)(1+q+q^2)} + \left(h^2U + hS - 2hT\right)(1-q)\right). \tag{16}$$

The upcoming identity for the left q-h-integrals holds:

$$\frac{(1-q)+qh}{(1-q)(u-a)+qh_1}\int_a^u \xi(t)\ {}_{h_1} d_q t = ((1-q)+qh)\sum_{n=0}^{\infty} q^n \xi(q^n a + (1-q^n)u + nq^n h_1) = \int_0^1 \xi(a+(u-a)t)\ {}_h d_q t. \tag{17}$$

The last integral appearing in (17) can be estimated for strongly $\hbar$-convex function ξ as follows:

$$\int_0^1 \xi(a+(u-a)t)\ {}_h d_q t \leq \xi(u)\int_0^1 \hbar(t)\ {}_h d_q t + \xi(a)\int_0^1 \hbar(1-t)\ {}_h d_q t - C(u-a)^2\int_0^1 t(1-t)\ {}_h d_q t. \tag{18}$$

Therefore, the following inequality can be derived from (17) and (18):

$$\frac{(1-q)+qh}{(1-q)(u-a)+qh_1}\int_a^u \xi(t)\ {}_{h_1} d_q t \leq \xi(u)\int_0^1 \hbar(t)\ {}_h d_q t + \xi(a)\int_0^1 \hbar(1-t)\ {}_h d_q t - C(u-a)^2\int_0^1 t(1-t)\ {}_h d_q t. \tag{19}$$

It is possible to evaluate the q-h-integral present in the above inequality as follows:

$$\int_0^1 t(1-t)\ {}_h d_q t = \frac{((1-q)+qh)}{1-q}\left(\frac{q^2}{(1+q)(1+q+q^2)} + (1-q)\left(2hT - hS - h^2U\right)\right). \tag{20}$$

The required inequality (11) can be achieved by using (17), (19) and (20) in (16). $\square$

A version of the above theorem for right q-h-integrals is stated in the next result. The proof is left for the reader.

Theorem 2. *Under assumptions of Theorem 1, assuming that ξ is symmetric about $\frac{b+u}{2}$, $u \in [a,b)$ instead of $\frac{a+u}{2}$, the forthcoming inequality holds for the right q-h-integral:*

$$\begin{aligned}
&\frac{1}{2\hbar\left(\frac{1}{2}\right)}\xi\left(\frac{b+u}{2}\right) \le \frac{1-q}{(1-q)(b-u)+qh_2}\int_u^b \xi(t)\ {}_{h_2}d_qt \\
&-\frac{C(b-u)^2}{2\hbar\left(\frac{1}{2}\right)}\left(\frac{1+q^3+2q(1-q)}{4(1+q)(1+q+q^2)}+\left(h^2U+hS-2hT\right)(1-q)\right) \\
&\le \frac{1-q}{1-q+qh}\left(\xi(b)\int_0^1 \hbar(t)\ {}_hd_qt+\xi(u)\int_0^1 \hbar(1-t)\ {}_hd_qt\right)-C(b-u)^2 \\
&\times\left(\frac{q^2}{(1+q)(1+q+q^2)}+(1-q)\left(2hT-hS-h^2U\right)\right),
\end{aligned} \tag{21}$$

where $h_2 = (b-u)h$.

Corollary 1. *The forthcoming inequality for the left q-h-integral holds for $\hbar$-convex functions by setting $C = 0$ in (11):*

$$\begin{aligned}
&\frac{1}{2\hbar\left(\frac{1}{2}\right)}\xi\left(\frac{a+u}{2}\right) \le \frac{1-q}{(1-q)(u-a)+qh_1}\int_a^u \xi(t)\ {}_{h_1}d_qt \\
&\le \frac{1-q}{1-q+qh}\left(\xi(u)\int_0^1 \hbar(t)\ {}_hd_qt+\xi(a)\int_0^1 \hbar(1-t)\ {}_hd_qt\right).
\end{aligned} \tag{22}$$

Remark 1. *It is noted that the inequality (22) was proved independently in [21]; the multiplier $\frac{1-q}{1-q+qh}$ appearing in the last term was missing.*

Corollary 2. *The forthcoming inequality for the left q-integral holds for $\hbar$-convex functions by setting $C = 0$ and $h = 0$ in (11):*

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\xi\left(\frac{a+u}{2}\right) \le \frac{1}{u-a}\int_a^u \xi(t)d_qt \le \xi(u)\int_0^1 \hbar(t)d_qt+\xi(a)\int_0^1 \hbar(1-t)d_qt$$

A version of the above Theorem 1 for strongly m-convex functions is stated and proved in the forthcoming result.

Theorem 3. *Let ξ be a strongly m-convex function. Then, the following left q-h-integral inequality holds when ξ satisfies the condition (i) $\xi\left(\frac{a+u-x}{m}\right) = \xi(x)$, $x \in (a,u)$*

$$\begin{aligned}
&\xi\left(\frac{a+u}{2}\right) \le \frac{(1-q)(1+m)}{2((1-q)(u-a)+qh_1)}\int_a^u \xi(t)\ {}_{h_1}d_qt \\
&-Cm(u-a)^2\left(\frac{1+q^3+2q(1-q)}{4(1+q)(1+q+q^2)}+\left(h^2U+hS-2hT\right)(1-q)\right) \\
&\le \frac{(1-q)+qh}{2(1-q)}\left(\xi(u)\left(\frac{q}{1+q}+(1-q)S\right)+m\xi\left(\frac{a}{m}\right)\left(\frac{1}{1+q}-(1-q)hS\right)\right) \\
&-Cm(u-a)^2\left(\frac{q^2}{(1+q)(1+q+q^2)}+(1-q)\left(2hT-hS-h^2U\right)\right).
\end{aligned} \tag{23}$$

Here, h_1, S, T and U are the same as those given in Theorem 1.

Proof. The forthcoming inequality holds for strongly m-convex function ξ:

$$\xi\left(\frac{a+u}{2}\right) \le \frac{1}{2}\left(\xi(a+t(u-a)) + m\xi\left(\frac{u-t(u-a)}{m}\right) - 2Cm(u-a)^2\left(t-\frac{1}{2}\right)^2\right). \tag{24}$$

The function ξ satisfies the condition (i). By using this fact, we obtain the resulting inequality from (24) as follows:

$$2\xi\left(\frac{a+u}{2}\right) \le \frac{(1-q)}{(1-q)+qh}\left((1+m)\int_0^1 \xi(a+t(u-a))\ {}_h d_q t \right. \\ \left. - 2Cm(u-a)^2 \int_0^1 \left(t-\frac{1}{2}\right)^2 {}_h d_q t\right). \tag{25}$$

The first integral of the right-hand side of the above inequality (25) is estimated by using the definition of the strongly m-convex function for ξ as follows:

$$\int_0^1 \xi(a+t(u-a))\ {}_h d_q t \le \xi(u)\int_0^1 t\ {}_h d_q t + m\xi\left(\frac{a}{m}\right)\int_0^1 (1-t)\ {}_h d_q t \\ - Cm(u-a)^2 \int_0^1 t(1-t)\ {}_h d_q t. \tag{26}$$

From (17), (25) and (26), we constitute the following inequality:

$$2\xi\left(\frac{a+u}{2}\right) \le \frac{(1-q)(1+m)}{(1-q)(u-a)+qh_1}\int_a^u \xi(t)\ {}_{h_1} d_q t \\ - \frac{2Cm(1-q)(u-a)^2}{(1-q)+qh}\int_0^1 \left(t-\frac{1}{2}\right)^2 {}_h d_q t \le \frac{1-q}{1-q+qh}\left(\xi(u)\int_0^1 t\ {}_h d_q t \right. \\ \left. + m\xi\left(\frac{a}{m}\right)\int_0^1 (1-t)\ {}_h d_q t - Cm(u-a)^2 \int_0^1 t(1-t)\ {}_h d_q t\right). \tag{27}$$

One can easily have

$$\int_0^1 t\ {}_h d_q t = \frac{(1-q)+qh}{1-q}\left(\frac{q}{1+q} + (1-q)hS\right), \tag{28}$$

and

$$\int_0^1 (1-t)\ {}_h d_q t = \frac{(1-q)+qh}{1-q}\left(\frac{q}{1+q} - (1-q)hS\right). \tag{29}$$

By using (20), (28), (29) and (15) in (27), one can obtain the required inequality. □

A version of the above theorem for right q-h-integrals is stated in the next result. The proof is left for the readers.

Theorem 4. *Let ξ be strongly m-convex. Assuming that ξ satisfies the condition (ii) $\xi\left(\frac{b+u-x}{m}\right) = \xi(x)$, $x \in (u,b)$, then the forthcoming inequality holds for the right q-h-integral:*

$$\xi\left(\frac{b+u}{2}\right) \leq \frac{(1-q)(1+m)}{2(1-q)(b-u)+qh_2} \int_u^b \xi(t)\ {}_{h_2}d_qt \tag{30}$$
$$- Cm(b-u)^2\left(\frac{1+q^3+2q(1-q)}{4(1+q)(1+q+q^2)} + \left(h^2U + hS - 2hT\right)(1-q)\right)$$
$$\leq \frac{(1-q)+qh}{2(1-q)}\left(\xi(b)\left(\frac{q}{1+q} + (1-q)S\right) + m\xi\left(\frac{u}{m}\right)\left(\frac{1}{1+q} - (1-q)hS\right)\right)$$
$$- Cm(b-u)^2\left(\frac{q^2}{(1+q)(1+q+q^2)} + (1-q)\left(2hT - hS - h^2U\right)\right),$$

where $h_2 = (b-u)h$.

Corollary 3. *The forthcoming left q-h-integral inequality for m-convex functions is obtained by setting* $C = 0$ *in* (23)*:*

$$\xi\left(\frac{a+u}{2}\right) \leq \frac{(1-q)(1+m)}{2((1-q)(u-a)+qh_1)} \int_a^u \xi(t)\ {}_{h_1}d_qt$$
$$\leq \frac{(1-q)+qh}{2(1-q)}\left(\xi(u)\left(\frac{q}{1+q} + (1-q)S\right) + m\xi\left(\frac{a}{m}\right)\left(\frac{1}{1+q} - ((1-q)hS)\right)\right).$$

Corollary 4. *The forthcoming left q-integral inequality for m-convex functions is obtained by setting* $C = 0$ *and* $h = 0$ *in* (23)*:*

$$\xi\left(\frac{a+u}{2}\right) \leq \frac{1+m}{2(u-a)} \int_a^u \xi(t)d_qt \leq \frac{1}{2(1+q)}\left(q\xi(u) + m\xi\left(\frac{a}{m}\right)\right).$$

A compact version of Theorems 1 and 3 is given by using the definition of strongly $(\hbar - m)$-convex function. This is stated and proved in the forthcoming theorem.

Theorem 5. *Let* ξ *be a strongly* $(h-m)$*-convex function. Then, the following left q-h-integral inequality holds when* ξ *satisfies the condition* (i) *of Theorem 3:*

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\xi\left(\frac{a+u}{2}\right) \leq \frac{(1-q)(1+m)}{(1-q)(u-a)+qh_1} \int_a^u \xi(t)\ {}_{h_1}d_qt \tag{31}$$
$$- Cm\hbar\left(\frac{1}{2}\right)(1+m)^2(a-u)^2\left(\frac{1}{1+q+q^2} + (am-u)^2\right.$$
$$\left. - \frac{2(am-u)}{1+q} + \left(h^2U + 2hS(am-u)^2 - 2hT\right)(1-q)\right)$$
$$\leq \frac{1-q}{(1-q)+qh}\left(\xi(u)\int_0^1 \hbar(t)\ {}_hd_qt + m\xi\left(\frac{a}{m}\right)\int_0^1 \hbar(1-t)\ {}_hd_qt\right.$$
$$\left. - Cm(mu-v)^2\int_0^1 \hbar(t)\hbar(1-t)\ {}_hd_qt\right).$$

Here, h_1*,* S*,* T *and* U *are the same as those given in Theorem 1.*

Proof. The forthcoming inequality holds for the strongly $(\hbar\text{-}m)$-convex function:

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\xi\left(\frac{a+u}{2}\right) \leq \xi(ta+(1-t)u) + m\xi\left(\frac{tu+(1-t)a}{m}\right) \tag{32}$$
$$- Cm\hbar\left(\frac{1}{2}\right)(1+m)^2(a-u)^2\left(t + \frac{mu-a}{(1+m)(a-u)}\right)^2, t \in [0,1].$$

The above inequality holds for the q-h-integral as follows:

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\xi\left(\frac{a+u}{2}\right) \leq \frac{1-q}{(1-q)+qh}\left(\int_0^1 \xi(ta+(1-t)u)\ {}_hd_qt\right. \tag{33}$$
$$+\int_0^1 m\xi\left(\frac{tu+(1-t)a}{m}\right)\ {}_hd_qt - Cm\hbar\left(\frac{1}{2}\right)(1+m)^2(a-u)^2$$
$$\int_0^1\left(t+\frac{mu-a}{(1+m)(a-u)}\right)^2\ {}_hd_qt, t\in[0,1].$$

The function ξ satisfies the condition (i). By using this fact, we obtain the resulting inequality from (33) as follows:

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\xi\left(\frac{a+u}{2}\right) \leq \frac{(1-q)}{(1-q)+qh}\left((1+m)\int_0^1 \xi(a+(u-a)t)\ {}_hd_qt\right. \tag{34}$$
$$\left. - Cm\hbar\left(\frac{1}{2}\right)(1+m)^2(a-u)^2\int_0^1\left(t+\frac{mu-a}{(1+m)(a-u)}\right)^2\ {}_hd_qt\right), t\in[0,1].$$

In Example 1, taking $a=0$, $b=1$, and $w=\frac{mu-a}{(1+m)(a-u)}$, we obtain

$$\int_0^1\left(t+\frac{mu-a}{(1+m)(a-u)}\right)^2\ {}_hd_qt = \frac{((1-q)+qh)}{1-q}\left(\frac{1}{1+q+q^2}+(am-u)^2\right. \tag{35}$$
$$\left.-\frac{2(am-u)}{1+q}+\left(h^2U+2hS(am-u)^2-2hT\right)(1-q)\right).$$

The first integral of the right-hand side of the above inequality (34) is estimated by using the definition of strongly $(h-m)$-convex function for ξ as follows:

$$\int_0^1 \xi(a+(u-a)t)\ {}_hd_qt \leq \xi(u)\int_0^1 \hbar(t)\ {}_hd_qt + m\xi\left(\frac{a}{m}\right)\int_0^1 \hbar(1-t)\ {}_hd_qt \tag{36}$$
$$- Cm(mu-a)^2\int_0^1 \hbar(t)\hbar(1-t)\ {}_hd_qt.$$

From (17), (34), (35) and (36), one can constitute the required inequality. □

Theorem 6. *Let ξ be a strongly $(h-m)$-convex function. Assuming that ξ satisfies the condition (ii) of Theorem 4, then the forthcoming inequality holds for the right q-h-integral:*

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\xi\left(\frac{u+b}{2}\right) \leq \frac{(1-q)(1+m)}{(1-q)(b-u)+qh_1}\int_u^b \xi(t)\ {}_{h_2}d_qt \tag{37}$$
$$- Cm\hbar\left(\frac{1}{2}\right)(1+m)^2(u-b)^2\left(\frac{1}{1+q+q^2}+(b-mu)^2\right.$$
$$\left.-\frac{2(b-mu)}{1+q}+\left(h^2U+2hS(b-mu)^2-2hT\right)(1-q)\right)$$
$$\leq \frac{1-q}{(1-q)+qh}\left(\xi(b)\int_0^1 \hbar(t)\ {}_hd_qt + m\xi\left(\frac{u}{m}\right)\int_0^1 \hbar(1-t)\ {}_hd_qt\right.$$
$$\left.- Cm(b-mu)^2\int_0^1 \hbar(t)\hbar(1-t)\ {}_hd_qt\right).$$

Corollary 5. *The forthcoming right q-h-integral inequality holds for $\hbar - m$-convex functions by setting $C = 0$ in (31):*

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\xi\left(\frac{a+u}{2}\right) \leq \frac{(1-q)+qh}{(1-q)(x-a)+qh_1}\int_a^u \xi(t)\ {}_{h_1}d_qt$$
$$\leq \frac{1-q}{(1-q)+qh}\left(\xi(u)\int_0^1 \hbar(t)\ {}_hd_qt + m\xi\left(\frac{a}{m}\right)\int_0^1 \hbar(1-t)\ {}_hd_qt\right).$$

In the upcoming theorem, we give the q-h-integral inequality for another well-known class of functions called the strongly (α, m)-convex function.

Theorem 7. *Let ξ be a strongly (α, m)-convex function. Then, the following left q-h-integral inequality holds when ξ satisfies the condition (i) of Theorem 3:*

$$2^\alpha\xi\left(\frac{a+u}{2}\right) \leq \frac{(1-q)(1+m(2^\alpha-1))}{(1-q)(u-a)+qh_1}\int_a^u \xi(t)\ {}_{h_1}d_qt \tag{38}$$
$$-2Cm(2^\alpha-1)(u-a)^2\left(\frac{1+q^3+2q(1-q)}{4(1+q)(1+q+q^2)} + \left(h^2U+hS-2hT\right)(1-q)\right)$$
$$\leq (1+m(2^\alpha-1))\left(\xi(u)\left(\frac{1+q-\alpha}{1+q} + h^\alpha P(1-q)\right) + m\xi\left(\frac{a}{m}\right)\right)$$
$$-\frac{Cm(u-a)^2(1+m(2^\alpha-1))(1-q)}{(1-q)+qh}\int_0^1 t^\alpha(1-t^\alpha)\ {}_hd_qt,$$

where h_1 is the same as in Theorem 1, and $P = \sum_{n=0}^{\infty}(q^n)^{1+\alpha}n^\alpha$.

Proof. The forthcoming inequality holds for strongly (α, m)-convex function ξ:

$$2^\alpha\xi\left(\frac{a+u}{2}\right) \leq \xi(ta+(1-t)u) + (2^\alpha-1)m\xi\left(\frac{tu+(1-t)a}{m}\right) \tag{39}$$
$$-Cm(2^\alpha-1)(u-a)^2(1-2t)^2.$$

The above inequality holds for the q-h-integrals as follows:

$$2^\alpha\xi\left(\frac{a+u}{2}\right) \leq \frac{1-q}{(1-q)+qh}\left(\int_0^1 \xi(ta+(1-t)u)\ {}_hd_qt\right. \tag{40}$$
$$+(2^\alpha-1)m\int_0^1 \xi\left(\frac{tu+(1-t)a}{m}\right)\ {}_hd_qt - 2Cm(2^\alpha-1)(u-a)^2$$
$$\left.\times\int_0^1\left(t-\frac{1}{2}\right)^2\ {}_hd_qt\right), t\in[0,1].$$

The function ξ satisfies the condition (i) of Theorem 3. By using this fact, we obtain the resulting inequality from (40) as follows:

$$2^\alpha\xi\left(\frac{a+u}{2}\right) \leq \frac{1-q}{(1-q)+qh}\left((1+m(2^\alpha-1))\int_0^1 \xi(a+(u-a)t)\ {}_hd_qt\right. \tag{41}$$
$$\left.-2Cm(2^\alpha-1)(u-a)^2\int_0^1\left(t-\frac{1}{2}\right)^2\ {}_hd_qt\right).$$

The first integral of the right-hand side of the above inequality (34) is estimated by using the definition of the strongly (α, m)-convex function for ξ as follows:

$$\int_0^1 \xi(a+(u-a)t)\ {}_h d_q t \le \xi(u) \int_0^1 t^\alpha\ {}_h d_q t \tag{42}$$
$$+ m\xi\left(\frac{a}{m}\right) \int_0^1 (1-t^\alpha)\ {}_h d_q t - Cm(u-a)^2 \int_0^1 t^\alpha(1-t^\alpha)\ {}_h d_q t.$$

Hence, from (17), (31), (41) and (42), we obtain the following inequality:

$$2\xi\left(\frac{a+u}{2}\right) \le \frac{(1+m(2^\alpha-1))(1-q)}{(1-q)(u-a)+qh_1} \int_a^u \xi(t)\ {}_{h_1} d_q t \tag{43}$$
$$-2Cm(2^\alpha-1)(u-a)^2 \int_0^1 \left(t-\frac{1}{2}\right)^2 {}_h d_q t\Big)$$
$$\le \frac{(1+m(2^\alpha-1))(1-q)}{(1-q)+qh}\left(\xi(u)\int_0^1 t^\alpha\ {}_h d_q t + m\xi\left(\frac{a}{m}\right)\int_0^1 (1-t^\alpha)\ {}_h d_q t\right.$$
$$\left.-\frac{Cm(u-a)^2(1+m(2^\alpha-1))(1-q)}{(1-q)+qh}\int_0^1 t^\alpha(1-t^\alpha)\ {}_h d_q t\right).$$

By the definition of the q-h-integral, we have

$$\int_0^1 t^\alpha\ {}_h d_q t = ((1-q)+qh)\sum_{n=0}^{\infty} q^n((1-q^n)+nq^n h)^\alpha,$$

and by using $(a+b)^\alpha \le a^\alpha + b^\alpha$ where $0<\alpha<1$, we have

$$\int_0^1 t^\alpha\ {}_h d_q t = ((1-q)+qh)\sum_{n=0}^{\infty} q^n((1-q^n)+nq^n h)^\alpha$$
$$\le ((1-q)+qh)\left(\sum_{n=0}^{\infty} q^n(1-q^n)^\alpha + h^\alpha \sum_{n=0}^{\infty}(q^n)^{1+\alpha} n^\alpha\right).$$

By using Bernoulli's inequality, we obtain

$$\int_0^1 t^\alpha\ {}_h d_q t \le ((1-q)+qh)\left(\sum_{n=0}^{\infty} q^n(1-\alpha q^n) + h^\alpha \sum_{n=0}^{\infty}(q^n)^{1+\alpha} n^\alpha\right)$$
$$= \frac{1-q+qh}{1-q}\left(\frac{1+q-\alpha}{1+q} + h^\alpha P(1-q)\right). \tag{44}$$

By using (44) and (15) in the inequalities (43), the required inequalities in (38) are obtained. □

Theorem 8. *Let ξ be a strongly (α, m)-convex function. Assuming that ξ satisfies the condition (ii) of Theorem 4, then the following inequality holds for right q-h-integral:*

$$2\xi\left(\frac{b+u}{2}\right) \le \frac{(1-q)(1+m(2^\alpha-1))}{(1-q)(b-u)+qh_2}\int_u^b \xi(t)\ {}_{h_2} d_q t \tag{45}$$
$$-2Cm(2^\alpha-1)(b-u)^2\left(\frac{1+q^3+2q(1-q)}{4(1+q)(1+q+q^2)} + \left(h^2 U + hS - 2hT\right)(1-q)\right)$$
$$\le (1+m(2^\alpha-1))\left(\xi(b)\left(\frac{1+q-\alpha}{1+q} + h^\alpha P(1-q)\right) + m\xi\left(\frac{u}{m}\right)\right)$$
$$-\frac{Cm(b-u)^2(1+m(2^\alpha-1))(1-q)}{1-q+qh}\int_0^1 t^\alpha(1-t^\alpha)\ {}_h d_q t.$$

Corollary 6. *The forthcoming inequalities for left q-h-integrals for (α, m)-convex functions are obtained by setting $C = 0$ in (38):*

$$2\xi\left(\frac{a+u}{2}\right) \leq \frac{(1-q)(1+m(2^\alpha-1))}{(1-q)(u-a)+qh_1}\int_a^u \xi(t)\ _{h_1}d_qt$$
$$\leq (1+m(2^\alpha-1))\left(\xi(u)\left(\frac{1+q-\alpha}{1+q}+h^\alpha P(1-q)\right)+m\xi\left(\frac{a}{m}\right)\right).$$

Corollary 7. *The forthcoming inequalities for the left q-integrals for (α, m)-convex functions are obtained by setting $C = 0$ and $h = 0$ in (38):*

$$2\xi\left(\frac{a+u}{2}\right) \leq \frac{1+m(2^\alpha-1)}{u-a}\int_a^u \xi(t)d_qt$$
$$\leq (1+m(2^\alpha-1))\left(\xi(u)\left(\frac{1+q-\alpha}{1+q}\right)+m\xi\left(\frac{a}{m}\right)\right).$$

3. Conclusions

The results proved in this article include q-h-integral inequalities for different types of strongly convex functions. Inequalities for q-integrals were deduced from implicit forms, which hold simultaneously for q- and h-integrals. All the results were proved for strongly convex and symmetric functions. In establishing the results for strongly (α, m)- and (h-m)-convex functions, some symmetric-like conditions were applied. Moreover, a strongly convex function provides the refinement of convex function provided the modulus $C \neq 0$, while for $C = 0$, the definition of convexity is recaptured. Hence, the provided results give refinements of the inequalities that exist in the literature for convex functions.

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