Coefficient Inequalities for *q*-Convex Functions with Respect to *q*-Analogue of the Exponential Function

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Abstract: In mathematical analysis, the *q*-analogue of a function refers to a modified version of the function that is derived from *q*-series expansions. This paper is focused on the *q*-analogue of the exponential function and investigates a class of convex functions associated with it. The main objective is to derive precise inequalities that bound the coefficients of these convex functions. In this research, the initial coefficient bounds, Fekete–Szegő problem, second and third Hankel determinant have been determined. These coefficient bounds provide valuable information about the behavior and properties of the functions within the considered class.

1. Introduction and Preliminaries

In the area of mathematical analysis and functional inequalities, the study of *q*-analogues has emerged as a fascinating and powerful area of research. These *q*-analogues, inspired by the theory of special functions and their properties, extend the classical notions of various mathematical functions. Among the many *q*-analogues, the *q*-analogue of the exponential function holds a special place, primarily due to its widespread applications in areas such as physics, combinatorics, and number theory.

The *q*-analogue of the exponential function is a significant generalization of the traditional exponential function and has been intensively investigated in recent years. One of the essential aspects of this study involves exploring the properties of a class of convex functions associated with the *q*-exponential function. These functions not only showcase intriguing behavior but also find relevance in various mathematical contexts, making their analysis of paramount importance. The motivation behind establishing coefficient bounds for the class of *q*-convex functions associated with the *q*-analog of the exponential function lies in understanding the properties and behavior of these functions in the context of *q*-calculus. Some important mathematical terms related to this research article will be discussed.

Suppose that $\mathcal{A}$ denotes the set of functions f represented by the power series

$$f(\tau) = \tau + \sum_{j=2}^{\infty} a_j \tau^j \qquad (1)$$

and be defined in open unit disk

$$U = \{\tau : \tau \in \mathbb{C} \text{ and } |\tau| < 1\}.$$

Clearly, f is univalent if

$$\tau_1 - \tau_2 \neq 0 \Rightarrow f(\tau_1) - f(\tau_2) \neq 0, \text{ where } \tau_1, \tau_2 \in U.$$

The set of all such functions is denoted by S.

The term "starlike function" refers to any function that maps a domain U with a starshaped domain, and the set of all such functions is represented by $\mathcal{S}^*$, that is

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : Re\left(\frac{\tau f'(\tau)}{f(\tau)} \right) > 0, \tau \in U \right\}.$$

Similarly, a function f is said to be a convex function if it maps a domain U with a convex-shaped domain, and all functions of this kind belong to the set C, that is

$$C = \left\{ f \in \mathcal{A} : Re\left(1 + \frac{\tau f''(\tau)}{f'(\tau)} \right) > 0, \tau \in U \right\}. \tag{2}$$

A function p, analytic in U, is said to be in the class $\mathcal{P}$ if it has the form

$$p(\tau) = 1 + \sum_{n=1}^{\infty} d_n \tau^n \tag{3}$$

and satisfies the conditions $p(0) = 1$ and $Re\{p(\tau)\} > 0$, for details see [1].

A function $w(\tau)$ defined in an open unit disk is said to be a Schwarz function if it satisfies the following conditions:

$$w(0) = 0 \text{ and } |w(\tau)| < 1 \text{ for } \tau \in U.$$

Subordination is an important tool to investigate the behavior of different subclasses of univalent functions. The concept of subordination was introduced by Lindelof [2]. Further, Rogosinski [3,4] and Littlewood [5] studied it in detail. Two functions, f_1 and f_2, which are analytic in U, where it is known that f_1 is subordinate to f_2 (denoted by $f_1 \prec f_2$), if there exists a Schwarz function w such that

$$f_2(\tau) = f_1(w(\tau)). \tag{4}$$

In particular, $f_1 \prec f_2 \Leftrightarrow f_1(0) = f_2(0)$ and $f_1(U) \subset f_2(U)$.

In the year 1992, Ma and Minda [6] used the technique of subordination and defined the general form of the family of univalent functions as follows:

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{A} : \frac{\tau f'(\tau)}{f(\tau)} \prec \varphi(\tau) \right\} \tag{5}$$

and

$$\mathcal{C}(\varphi) = \left\{ f \in \mathcal{A} : \frac{\left(\tau f'(\tau)\right)'}{f'(\tau)} \prec \varphi(\tau) \right\}. \tag{6}$$

where, φ is an analytic function along with the conditions $\varphi(0) > 0$ and $Re(\varphi(\tau)) > 0$ in U. If $\varphi(\tau) = \frac{1+\tau}{1-\tau}$, in (5) and (6), then well-known classes of starlike and convex functions are obtained. In recent years, a number of sub-families of the normalized analytic functions

have been studied as a special case of $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$. Recently, Cho et al. [7] chose $\varphi(\tau) = 1 + \sin \tau$ and defined a class $(\mathcal{S}^*_{\sin})$ of starlike functions:

$$\mathcal{S}^*_{\sin} = \left\{ f \in \mathcal{A} : \frac{\tau f'(\tau)}{f(\tau)} \prec 1 + \sin \tau \right\}.$$

Mediratta et al. [8] introduced the class of starlike functions associated with exponential function as

$$S^*_e = \left\{ f \in \mathcal{A} : \frac{\tau f'(\tau)}{f(\tau)} \prec e^{\tau}, \ (\tau \in U) \right\}.$$

They investigated the defined class and derived the relationship between the newly defined class and different existing subclasses of the class S. Furthermore, in [8], by using the Alexandar-type relation, the class of convex functions associated with exponential function was defined as

$$C_e = \left\{ f \in \mathcal{A} : \frac{(\tau(f'(\tau))'}{f'(\tau)} \prec e^{\tau}, \ \tau \in U \right\} \tag{7}$$

Pommerenke [9,10] defined the Hankel determinant for $f \in S$ of the form (1). The tth Hankel determinant is defined as

$$\mathcal{H}_{t,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+t-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+t} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+t-1} & a_{n+t-2} & \cdots & a_{n+2t-2} \end{vmatrix}, \tag{8}$$

where, n, t belongs to the set of natural numbers and $a_1 = 1$.

The following are three special cases of the Hankel determinant.

(i) A special case of Fekete–Szegő functional:

$$\mathcal{H}_{2,1}(f) = \left| a_3 - a_2^2 \right|,$$

where the famous Fekete–Szegő functional is defined by

$$\mathcal{H}_{2,1}(f) = \left| a_3 - \mu a_2^2 \right|,$$

where, μ is real or complex number.

(ii) Second Hankel determinant:

$$\mathcal{H}_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = \left| a_2 a_4 - a_3^2 \right|. \tag{9}$$

(iii) The third Hankel determinant is defined as

$$\mathcal{H}_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3 \left(a_2 a_4 - a_3^2 \right) - a_4 (a_4 - a_2 a_3) + a_5 \left(a_3 - a_2^2 \right). \tag{10}$$

For the details of the first two cases (see [11,12]), while Babalola [13] was the poinear who found $\mathcal{H}_{3,1}(f)$ for the family of classes of convex and starlike functions. Later, many researchers found $|\mathcal{H}_{3,1}(f)|$ for different subclasses of S, for this see [14–19].

Quantum calculus, also known as q-calculus, is an extension of traditional calculus that has gained significant interest in various fields of mathematics and physics, including geometric function theory. The introduction of quantum calculus in geometric function theory offers new insights into the behavior of complex functions and opens up exciting avenues for exploring the geometry of complex domains. In this context, quantum calculus allows us to investigate how q-analogues of derivative and integral operators affect the

properties of complex functions, their zeros, critical points, and singularities, among other aspects. For the detail (see [20–22]).

Jackson [23,24] defined the q-analogue of ordinary derivative, and further this operator was utilized by Ismail et al. [25] to define a q-analogue of starlike functions. The extension of Ismail et al.'s work associated with Janowski functions were given in [20]. Several new subclasses of starlike and convex functions, as well as sharp bounds for second- and third-order Hankal determinants, have been investigated recently by researchers using the q-derivative. Srivastava, for example, achieved the same work for close-to-convex functions in [26] and Mahmood et al. [27] did the same for the family of q-starlike functions and investigated the third-order Hankel determinant. Cotîrla et al. [28] investigated certain coefficient bounds for q-starlike functions based on a Ruscheweyh q-differential operator. A unique subclass of starlike functions connected to the sine function was constructed by Arif et al. [29] using the subordination technique, and the third Hankel determinant for this class was eventually discovered. Concurrently, Srivastava et al. [30] explored the Hankel and the Toeplitz determinants connected to the generalized conic domain and identified a new subclass of q-starlike functions. Zhang and Tang [31] worked on the sine function while Güney and Korfeci [32] created a new subclass of analytic functions and discovered the fourth-order Hankel determinant. By deriving bounds on the coefficients, researchers aim to gain insights into the growth and convergence properties of these functions, which can have applications in various mathematical and scientific fields. These bounds help elucidate the behavior of q-convex functions and their q-exponential analogs, contributing to a deeper understanding of their mathematical properties and potential applications. This article focuses on the q-analogue of the exponential function and investigate a class of convex functions associated with it. The main objective is to derive precise inequalities that bound the coefficients of these convex functions. This article contributes to the broader mathematical understanding of q-convex functions and their association with the q-analogue of the exponential function.

If $f \in \mathcal{A}$, then the q-derivative or q-difference operator is defined as [23]:

$$
D_q f(\tau) \quad = \quad \frac{f(\tau) - f(q\tau)}{(1-q)\tau}, \quad \tau \neq 0, q \in (0,1) \tag{11}
$$

$$
= \quad 1 + \sum_{n=2}^{\infty} [n]_q a_n \tau^{n-1},
$$

where

$$
[n]_q = \frac{1-q^n}{1-q},
$$

or

$$
[n]_q = \sum_{j=0}^{n-1} q^j.
$$

This study continues the numerous investigations of various authors on the estimation of Taylor–Maclaurin coefficients for a new subclass of class $\mathcal{A}$ of analytic functions on the unit disc U with coefficients $a_0 = 0$ and $a_1 = 1$. Among many more, just a small handful of investigations was highlighted in the aforementioned works. Using the notions of the q-difference operator D_q and function subordination $\prec$, the new class $\mathcal{C}(q,e)$ of q-convex functions connected to the q-analogue of the exponential function is defined; see the definition below. The bounds for coefficients a_2, a_3, a_4 and a_5 of the function $f \in \mathcal{C}(q,e)$ in terms of q, give inequalities for $a_2a_3 - a_4$, $a_3 - a_2^2$, $a_2a_4 - a_3^2$ for the Hankel determinant. This covers the analogically known results for the class $\mathcal{C}_e$ (as $q \to 1^-$). All the results obtained are new and quite interesting. Since q-calculus (quantum calculus) has wide modern applications in various fields of function theory and physics, the considered problems and results in the paper can be estimated as useful and significant, in particular, with respect to using of q-exponential function.

Definition 1. *A function $f \in \mathcal{A}$ belongs to the class $\mathcal{C}(q,e)$, if it satisfies the following subordination condition:*

$$\frac{D_q\left(\tau D_q f(\tau)\right)}{D_q f(\tau)} \prec e^{q\tau}, \quad \tau \in U. \tag{12}$$

Remark 1. *When $q \to 1^-$, then $\mathcal{C}(q,e) = \mathcal{C}_e$, defined in [33].*

Remark 2. *The image domain e^{qU} is shown in the Figure 1. A function f belongs to the class $\mathcal{C}(q,e)$ if and only if the function $\frac{D_q\left(\tau D_q f(\tau)\right)}{D_q f(\tau)}$ takes all values from the domain e^{qU}.*

(**a**) For $q = 0.2$ (**b**) For $q = 0.5$

(**c**) For $q = 0.7$ (**d**) For $q = 0.99$
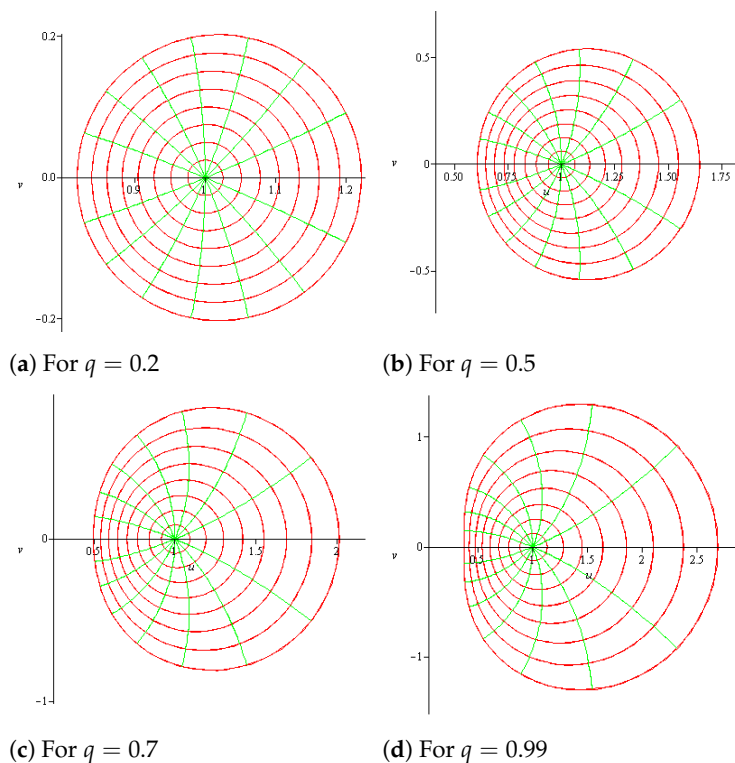
Figure 1. The images of e^{qU} for different values of q.

Remark 3. *A function f belongs to the class $\mathcal{C}(q,e)$ if and only if there exists an analytic function h, $h \prec e^{q\tau}$ such that*

$$f(z) = \int_0^z e^{J(t)} d_q t \tag{13}$$

where

$$J(t) = \int_0^t \frac{h(x) - 1}{x} d_q x.$$

In particular for $h = e^{qz}$, we have

$$J(t) = \int_0^t \frac{e^{qx} - 1}{x} d_q x.$$

The function f defined in (13) plays the role of extremal function for many problems over the class $\mathcal{C}(q,e)$.

The following set of lemmas will be used to investigate the coefficient problems for the class $\mathcal{C}(q,e)$.

Lemma 1 ([34]). *Let the function $p(\tau)$ be of the form (3), then*

$$|d_n| \leq 2, \ n \in \mathbb{N}. \tag{14}$$

Also,

$$|d_n - \mu d_i d_{n-i}| \leq 2, \ n > i, \ \mu \in [0,1]. \tag{15}$$

The equality holds for

$$f(\tau) = \frac{1+\tau}{1-\tau}.$$

Lemma 2 ([29]). *Let the function $p \in \mathcal{P}$, be given by (3), then*

$$|d_{n+t} - \nu d_n d_t| \leq 2, \ \text{for } 0 \leq \nu \leq 1 \tag{16}$$

$$\left| d_{n+2t} - \nu d_n d_t^2 \right| \leq 2(1 + 2\nu), \ \text{for } \nu \in \mathbb{R} \tag{17}$$

$$\left| d_2 - \nu d_1^2 \right| \leq \max\{2, 2|\nu - 1|\} \tag{18}$$

where ν is any complex or real number.

2. Main Results

This article extends the class of functions defined in (7) to q-analogue of exponential function with the help of q-differential operator (11) given in (12). The coefficient bounds, Fekete–Szegő functional and second Hankel determinant are determined. Furthermore, the third Hankel determinants will be found with the help of these results. Some well-known corollaries have been proved as well. The following are some main results of this research.

Theorem 1. *If $f \in \mathcal{C}(q, e)$, is of the form (12), then*

$$
\begin{aligned}
|a_2| &\leq \frac{1}{q[2]_q} \\
|a_3| &\leq \frac{(2q+1)}{q^2[3]_q[2]_q^2}, \\
|a_4| &\leq \frac{(q^4 + 5q^3 + 6q^2 + 4q + 1)}{q^3[4]_q[3]_q^2[2]_q^2}, \\
|a_5| &\leq \frac{(3q^6 + 9q^5 + 18q^4 + 21q^3 + 15q^2 + 6q + 1)}{q^4[2]_q^3[3]_q^2[4]_q[5]_q}.
\end{aligned}
$$

Proof. Let $f \in \mathcal{C}(q, e)$, then by definition of $\mathcal{C}(q, e)$,

$$\frac{D_q(\tau D_q f(\tau))}{D_q f(\tau)} \prec e^{q\tau}. \tag{19}$$

By the definition of subordination

$$\frac{D_q(\tau D_q f(\tau))}{D_q f(\tau)} = e^{q(\omega(\tau))}, \tag{20}$$

where

$$
\begin{aligned}
\omega(\tau) &= \frac{p(\tau)-1}{p(\tau)+1} \\
&= \frac{d_1}{2}\tau + \left(\frac{d_2}{2} - \frac{d_1^2}{4}\right)\tau^2 + \left(\frac{d_1^3}{8} + \frac{d_3}{2} - \frac{d_1 d_2}{2}\right)\tau^3 + \\
&\quad \left(\frac{d_4}{2} - \frac{d_1 d_3}{2} - \frac{d_2^2}{4} + \frac{3d_1^2 d_2}{8} - \frac{d_1^4}{16}\right)\tau^4 \\
&= \frac{1}{2}d_1\tau + \frac{1}{2}\left(d_2 - \frac{1}{2}d_1^2\right)\tau^2 + \frac{1}{2}\left(d_3 - d_{12} + \frac{1}{4}d_1^3\right)\tau^3 + \cdots.
\end{aligned}
\tag{21}
$$

In the light of (20) and (21), the following series is obtained:

$$
\begin{aligned}
e^{q(\omega(\tau))} &= 1 + \sum_{i=1}^{\infty} \frac{(\omega(\tau))^i}{[i]_q!} \\
&= 1 + \frac{d_1}{2}\tau + \left\{\frac{d_2}{2} - \frac{qd_1^2}{4[2]_q}\right\}\tau^2 + \left\{\frac{d_3}{2} - \frac{qd_1 d_2}{2[2]_q} + \frac{qd_1^3}{8[3]_q[2]_q}\right\}\tau^3 + \\
&\quad + \frac{1}{16[3]_q[2]_q}\left\{
\begin{array}{c}
(-q^3 + q^2 + q - 1)d_1^4 + 6q^3 d_1^2 d_2 - 8q[3]_q d_1 d_3 \\
-4q[3]_q d_2^2 + 8(q^3 + 2q^2 + 2q + 1)d_4
\end{array}
\right\}\tau^4 + \dots.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\frac{D_q(\tau D_q f(\tau))}{D_q f(\tau)} &= 1 + q[2]_q a_2\tau + \left\{q[2]_q[3]_q a_3 - q[2]_q^2 a_2^2\right\}\tau^2 + \\
&\quad \left\{q[3]_q[4]_q a_4 - [2]_q[3]_q\left([2]_q + [3]_q - 2\right)a_2 a_3 + q[2]_q^3 a_2^3\right\}\tau^3 + \\
&\quad \left\{
\begin{array}{c}
q[4]_q[5]_q a_5 - q[2]_q[3]_q^2 a_3^2 - [2]_q[4]_q\left([2]_q + [4]_q - 2\right)a_2 a_4 + \\
[2]_q^2[3]_q\left(2[2]_q + [3]_q - 3\right)a_2^2 a_3 - q[2]_q^4 a_2^4
\end{array}
\right\}\tau^4\dots
\end{aligned}
\tag{22}
$$

By comparing the corresponding coefficients of (21) and (22), obtain

$$
a_2 = \frac{d_1}{2q[2]_q},
\tag{23}
$$

$$
a_3 = \frac{1}{2q[2]_q[3]_q}\left\{d_2 + \frac{1 + q - q^2}{2q[2]_q}d_1^2\right\}
\tag{24}
$$

$$
a_4 = \frac{1}{2q[3]_q[4]_q}\left\{d_3 - \frac{2q^2 - q - 2}{2q[2]_q}d_1 d_2 + \frac{q^6 - 3q^4 - 2q^3 + 2q + 1}{4q^2[2]_q^2[3]_q}d_1^3\right\}
\tag{25}
$$

$$a_5 = \frac{1}{q[4]_q[5]_q} \left[\begin{array}{c} \left\{ \begin{array}{c} -\frac{(q-1)^2}{16[3]_q} + \frac{(q^2-q-1)^2}{16q^3[2]_q^3} + \frac{(q^2+q+2)(q^6-3q^4-2q^3+2q+1)}{16q^3[2]_q^2[3]_q^2} \\[3mm] +\frac{(q+3)(q^2-q-1)}{16q^3[2]_q^2} + \frac{1}{16q^3} \end{array} \right\} d_1^4 \\[8mm] +\left\{ \frac{q^2+q+2}{4q[3]_q} - \frac{q}{2[2]_q} \right\} d_1 d_3 + \left\{ \frac{1}{4q[2]_q} - \frac{q}{4[2]_q} \right\} d_2^2 \\[8mm] +\left\{ \begin{array}{c} \frac{3q^3}{8[2]_q[3]_q} - \frac{(2q^2-q-2)(q^2+q+2)}{8q^2[2]_q[3]_q} \\[3mm] -\frac{(q^2-q-1)}{4q^2[2]_q^2} - \frac{q+3}{8q^2[2]_q} \end{array} \right\} d_1^2 d_2 + \frac{1}{2} d_4 \end{array} \right]. \tag{26}$$

Using the Lemma 1 on (23) and (24), to obtain

$$|a_2| \le \frac{1}{q[2]_q},$$

$$|a_3| \le \frac{2q+1}{2q^2[2]_q[3]_q}.$$

Now taking the modulus of (25),

$$|a_4| = \frac{1}{2q[3]_q[4]_q} \left| d_3 - \frac{2q^2-q-2}{2q[2]_q} d_1 d_2 + \frac{q^6-3q^4-2q^3+2q+1}{4q^2[2]_q^2[3]_q} d_1^3 \right|.$$

By applying the triangle inequality, we reached the following result:

$$|a_4| \le \frac{1}{2q[3]_q[4]_q} \left\{ |d_3| + \frac{2+q-2q^2}{2q[2]_q} |d_1 d_2| + \frac{q^6-3q^4-2q^3+2q+1}{4q^2[2]_q^2[3]_q} |d_1|^3 \right\}.$$

With the help of Lemma 1, arrived at

$$|a_4| \le \frac{q^4+5q^3+6q^2+4q+1}{q^3[4]_q[3]_q^2[2]_q^2}.$$

Now, from (26),

$$|a_5| = \frac{1}{q[4]_q[5]_q} \left| \lambda d_1^4 + \alpha d_1 d_3 + \beta d_1^2 d_2 + \kappa d_2^2 + \frac{1}{2} d_4 \right|, \tag{27}$$

where

$$\lambda = -\frac{(q-1)^2}{16[3]_q} + \frac{(q^2-q-1)^2}{16q^3[2]_q^3} + \frac{(q^2+q+2)(q^6-3q^4-2q^3+2q+1)}{16q^3[2]_q^2[3]_q^2} +$$

$$\frac{(q+3)(q^2-q-1)}{16q^3[2]_q^2} + \frac{1}{16q^3},$$

$$\alpha = \frac{q^2+q+2}{4q[3]_q} - \frac{q}{2[2]_q},$$

$$\beta = \frac{3q^3}{8[2]_q[3]_q} - \frac{(q^2-q-1)}{4q^2[2]_q^2} - \frac{(2q^2-q-2)(q^2+q+2)}{8q^2[2]_q[3]_q} - \frac{q+3}{8q^2[2]_q},$$

$$\kappa = \frac{1}{4q[2]_q} - \frac{q}{4[2]_q}.$$

Applying triangle inequality on (27),

$$|a_5| \le \frac{1}{q[4]_q[5]_q}\left\{\left|\lambda d_1^4\right| + |\alpha d_1 d_3| + \left|\kappa d_1^2\right| + \frac{1}{2}|d_4|\right\}$$

Using values of λ, α, β, κ and applying Lemma 1, we obtain

$$|a_5| \le \frac{(3q^6 + 9q^5 + 18q^4 + 21q^3 + 15q^2 + 6q + 1)}{q^4[2]_q^3[3]_q^2[4]_q[5]_q}.$$

□

In Theorem 1, assume that $q \to 1^-$, then the following known consequence is.

Corollary 1 ([33]). *If $f \in \mathcal{C}(e)$ has the series as given in (1), then*

$$|a_2| \le \frac{1}{2}, \tag{28}$$

$$|a_3| \le \frac{1}{4},$$

$$|a_4| \le \frac{17}{144},$$

$$|a_5| \le \frac{73}{1440}. \tag{29}$$

Theorem 2. *If $f \in \mathcal{C}(q,e)$ is of the form (1), then for a complex number γ*

$$\left|a_3 - \gamma a_2^2\right| \le \frac{1}{q[2]_q[3]_q}\max\left\{1, \left|\frac{\gamma(q^2+q+1)-2q-1}{q[2]_q}\right|\right\}. \tag{30}$$

Proof. From (23) and (24), consider

$$\left|a_3 - \gamma a_2^2\right| = \frac{1}{2q[2]_q[3]_q}\left|d_2 - \frac{q^2-q-1}{2q[2]_q}d_1^2 - \frac{\gamma}{4q^2[2]_q^2}d_1^2\right|$$

$$= \frac{1}{2q[2]_q[3]_q}\left|d_2 - vd_1^2\right|,$$

where

$$v = \frac{1}{2q[2]_q}\left(\frac{\gamma(1+q+q^2)+q^2-q-1}{2q[2]_q}\right).$$

Now, using Lemma 2, arrived at

$$\left|a_3 - \gamma a_2^2\right| \leq \frac{1}{q[2]_q[3]_q} \max\left\{1, \left|\frac{\gamma(1+q+q^2)+q^2-q-1}{q[2]_q} - 1\right|\right\},$$

After some simplification, result obtained is

$$\left|a_3 - \gamma a_2^2\right| \leq \frac{1}{q[2]_q[3]_q} \max\left\{1, \left|\frac{\gamma(q^2+q+1)-2q-1}{q[2]_q}\right|\right\}. \tag{31}$$

□

In Theorem 2, assume that $q \to 1^-$, then the following known result is.

Corollary 2 ([33]). *If $f \in \mathcal{C}(e)$ is of the form (1), then for a complex number γ*

$$\left|a_3 - \gamma a_2^2\right| \leq \frac{1}{6} \max\left\{1, \frac{3}{2}|\gamma - 1|\right\}.$$

In Theorem 2, suppose $\gamma = 1$ and $q \to 1^-$, then the following known consequence is.

Corollary 3 ([33]). *If $f \in \mathcal{C}(e)$ is of the form (1), then*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{6}. \tag{32}$$

Theorem 3. *If $f \in C(q, e)$ is of the form (1), then*

$$|a_2 a_3 - a_4| \leq \frac{6q^3 + 11q^2 + 10q + 4}{2q[2]_q^2[3]_q^2[4]_q}.$$

Proof. By using (23)–(26),

$$|a_2 a_3 - a_4| = \left|\frac{3q^2 - q - 1}{4q^2[2]_q[3]_q[4]_q} d_1 d_2 - \frac{2q^4 - 3q^2 - 4q - 2}{8q[2]_q^2[3]_q^2[4]_q} d_1^3 - \frac{1}{2q[3]_q[4]_q} d_3\right|.$$

With the help of triangle inequality, we obtain

$$|a_2 a_3 - a_4| \leq \frac{1}{4q[3]_q[4]_q}\left\{\left|d_3 - \frac{3q^2 - q - 1}{q[2]_q} d_1 d_3\right| + \left|d_3 - \frac{(2+4q+3q^2-2q^4)}{2[2]_q^2[3]_q} d_1^3\right|\right\}$$

$$\leq \frac{1}{4q[3]_q[4]_q}\left\{\Psi_1(q) + \Psi_2(q)\right\}. \tag{33}$$

where

$$\Psi_1(q) = \left|d_3 - \frac{3q^2 - q - 1}{q[2]_q} d_1 d_3\right| \tag{34}$$

and

$$\Psi_2(q) = \left|d_3 - \frac{(2+4q+3q^2-2q^4)}{2[2]_q^2[3]_q} d_1^3\right|. \tag{35}$$

Using Lemma 2 on (34) and (35), we obtain

$$\Psi_1(q) \leq 2 \quad \text{and} \quad \Psi_2(q) \leq \frac{3 + 7q + 7q^2 + 3q^3 - q^4}{[2]_q^2[3]_q}.$$

Now, by applying these values on (33), we arrived at

$$|a_2a_3 - a_4| \leq \frac{6q^3 + 11q^2 + 10q + 4}{2q[2]_q^2[3]_q^2[4]_q}.\tag{36}$$

□

In Theorem 3, let $q \to 1^-$, then following is the well-known result.

Corollary 4. *If $f \in \mathcal{C}(e)$ is of the form (1), then*

$$|a_2a_3 - a_4| \leq \frac{31}{288}.$$

Theorem 4. *If $f \in \mathcal{C}(q, e)$ is of the form (12), then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{3q^4 + 5q^3 + 11q^2 + 10q + 4}{2q^2[2]_q^3[3]_q^2[4]_q}.$$

Proof. Using (23)–(25), for

$$\left|a_2a_4 - a_3^2\right| = \left| \begin{array}{c} \frac{1}{4q^2[2]_q[3]_q[4]_q}d_1d_3 - \frac{(2q+1)(2-q)}{16q[2]_q^3[3]_q^2[4]_q}d_1^4 - \\ \frac{(3q^2-q-1)}{8q^2[2]_q^2[3]_q^2[4]_q}d_1^2d_2 - \frac{1}{4q^2[2]_q^2[3]_q^2}d_2^2 \end{array} \right|$$

$$= \left| \begin{array}{c} \frac{1}{16q^2[2]_q[3]_q[4]_q}d_1\left\{d_3 - \frac{2(3q^2-q-1)}{[2]_q[3]_q}d_1d_2\right\} + \frac{1}{16q^2[2]_q[3]_q[4]_q} \\ d_1\left\{d_3 - \frac{q(2q+1)(2-q)}{[2]_q^2[3]_q}d_1^3\right\} + \frac{1}{8q^2[2]_q[3]_q[4]_q}\left\{d_1d_3 - \frac{2(q^3+q^2+q+1)}{[2]_q[3]_q}d_2^2\right\} \end{array} \right|.$$

Applying the triangle inequality to obtain

$$|a_2a_3 - a_4| \leq \frac{|d_1|}{16q^2[2]_q[3]_q[4]_q}\{\Phi_1(q) + \Phi_2(q)\} + \frac{1}{8q^2[2]_q^3[3]_q^2[4]_q}\Phi_3(q),\tag{37}$$

where

$$\Phi_1(q) = \left|d_3 - \frac{2(3q^2 - q - 1)}{[2]_q[3]_q}d_1d_2\right|,\tag{38}$$

$$\Phi_2(q) = \left|d_3 - \frac{q(2q + 1)(2 - q)}{[2]_q^2[3]_q}d_1^3\right|,\tag{39}$$

and

$$\Phi_3(q) = \left|d_1d_3 - \frac{2(q^3 + q^2 + q + 1)}{[2]_q[3]_q}d_2^2\right|.\tag{40}$$

Now, using Lemma 2, we obtain

$$\Phi_1(q) = |d_3 - \lambda_1 d_1 d_2| \leq 2, \text{ where } \lambda_1 = \frac{2(3q^2 - q - 1)}{[2]_q[3]_q} \in (0, 1) \text{ when } q \in \left(\frac{1 + \sqrt{13}}{6}, 1\right)\tag{41}$$

$$\Phi_2(q) = \left|d_3 - \lambda_2 d_1^3\right| \leq 2(1 + 2\lambda_2), \text{ where } \lambda_2 = \frac{q(2q + 1)(2 - q)}{[2]_q^2[3]_q} \in \mathbb{R} \text{ for } q \in (0, 1).$$

After some simplifications, this yields

$$\Phi_2(q) \leq \frac{2(q^4 - q^3 + 10q^2 + 7q + 1)}{[2]_q^2 [3]_q}. \tag{42}$$

Applying the triangle inequality on (40), we obtain

$$\Phi_3(q) \leq \frac{4 + 8(q^3 + q^2 + q + 1)}{[2]_q [3]_q}. \tag{43}$$

Using (41)–(43), the result is as follows:

$$\left| a_2 a_4 - a_3^2 \right| \leq \frac{3q^4 + 5q^3 + 11q^2 + 10q + 4}{2q^2 [2]_q^3 [3]_q^2 [4]_q}.$$

□

Considering $q \to 1^-$, in Theorem 4, to obtain the corollary, which is an improvement of the result proved in [33].

Corollary 5. *If $f \in C(e)$ has the series of the form as given in (1), then*

$$|a_4 - a_2 a_3| \leq \frac{11}{192}.$$

Theorem 5. *If $f \in C(q, e)$ is the form (12), then*

$$|H_{3,1}(f)| \leq \frac{6\{\mathcal{T}_1(q) + \mathcal{T}_2(q)\}}{4q^6 (q^2 + 1)^2 [2]_q^6 [3]_q^4 [5]_q},$$

where

$$\mathcal{T}_1(q) \;=\; 6q^{14} + 37q^{13} + 171q^{12} + 447q^{11} + 876q^{10} + 1412q^9 + 1945q^8$$

$$\mathcal{T}_2(q) \;=\; 2253q^7 + 2129q^6 + 1611q^5 + 958q^4 + 440q^3 + 149q^2 + 34q + 4$$

Proof. Since by (10)

$$|H_{3,1}(f)| \leq |a_5|\left|a_3 - a_2^2\right| + |a_4||a_4 - a_2 a_3| + |a_3|\left|a_2 a_4 - a_3^2\right|.$$

Now, using Theorems 1–4, the result given in the statement is proved. □

Considering $q \to 1^-$, in Theorem 5, we obtain the corollary, which is an improvement of the result proved in [33].

Corollary 6. *If $f \in C(e)$ is the form (1), then*

$$|H_{3,1}(f)| \leq \frac{1559}{8640}.$$

3. Discussion

This section acts as an introductory segment leading to the conclusion and aims to present a comparative analysis between the existing categories and the newly introduced class. Mathematician Lei Shi collaborated with H.M. Srivastava [33] to establish a category of convex functions that incorporate the exponential function. Additionally, Srivastava et al. [35] introduced a class of q-starlike functions connected to the q-exponential

function. Since the coefficient inequalities for q-convex functions with respect to q-analogues of functions, like the q-exponential, have applications in various areas of mathematics, particularly in q-analysis, combinatorics, and special functions theory. They provide insights into the behavior of q-convex functions and help establish relationships between these functions and q-analogues of classical functions. Consequently, in this research article, an essential category of convex functions linked to the q-analogue of the exponential function is introduced, and the study explores its bounds and associated properties. Also, certain specific instances of the category of convex functions linked to the exponential function have been demonstrated and validated.

4. Conclusions

This article provides a rigorous analysis of a class of convex functions associated with the q-analogue of the exponential function. The obtained coefficient bounds deepen our understanding of these functions and their properties, making significant contributions to the field of q-analogue theory and its applications. The Fekete–Szegő fractional and the third-order Hankel determinant constraints are defined for this class. Some known consequences of the main results were also highlighted.

It is worth mentioning that the q-analogues that have been considered in this article as well as in a remarkably large number of other earlier q-investigations on the subject for $0 < q < 1$ can easily (and possibly trivially) be translated into the corresponding (p, q)-analogues (with $0 < q < p \leqq 1$) by applying some straightforward parametric and argument variations, the additional parameter p being redundant (see, for example, Srivastava [36]).

This study examined a new family that may stimulate additional investigation into various topics, such as certain special families of univalent functions using the integro-differential operator [37] and the fractional q-difference operator [38].

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