ARTICLE IN PRESS

Ain Shams Engineering Journal xxx (xxxx) xxx

Contents lists available at ScienceDirect

Ain Shams Engineering Journal

journal homepage: https://www.sciencedirect.com

Full Length Article

A study of new quantum Montgomery identities and general Ostrowski like inequalities

Muhammad Uzair Awan [a], Muhammad Zakria Javed [a], Huseyin Budak [b], Y.S. Hamed [c], Jong-Suk Ro [d,e,*]

[a] Department of Mathematics, Government College University, Faisalabad, Pakistan
[b] Department of Mathematics, Faculty of Science and Arts, D'uzce University, D'uzce, Turkey
[c] Department of Mathematics and Statistics, College of Science, Taif University, P. O. Box 11099, Taif 21944, Saudi Arabia
[d] School of Electrical and Electronics Engineering, Chung-Ang University, Dongjak-gu, Seoul, 06974, Republic of Korea
[e] Department of Intelligent Energy and Industry, Chung-Ang University, Dongjak-gu, Seoul, 06974, Republic of Korea

ARTICLE INFO

MSC:
26A33
26A51
26D07
26D10
26D15
26D20

Keywords:
Montgomery
Identity
Mercer
Convex
Function
Hölder's inequality

ABSTRACT

The main objective of this paper is to analyze the Montgomery identities and Ostrowski like inequalities, within the framework of quantum calculus. The study utilizes ${}_{\varpi_3}q$ and ${}^{\varpi_4}q$ differentiable functions to establish two new Montgomery identities, which are essential for the development of our main results which are new quantum estimates of general Ostrowski type inequality. The study involves numerous techniques, including q-identities, Hölder-like inequalities, and the Jensen-Mercer inequality for convex mappings, to derive the main outcomes of the paper. Additionally, the study presents special cases, numerical validation, and graphical illustrations to support the main results.

1. Introduction

A set $\mathcal{C} \subset \mathbb{R}$ is said to be convex, if

$$(1-\delta)\varpi_3 + \delta\varpi_4 \in \mathcal{C}, \quad \forall \varpi_3, \varpi_4 \in \mathcal{C}, \delta \in [0,1].$$

Similarly, a function $\Xi : \mathcal{C} \to \mathbb{R}$ is said to be convex, if

$$\Xi((1-\delta)\varpi_3 + \delta\varpi_4) \le (1-\delta)\Xi(\varpi_3) + \delta\Xi(\varpi_4), \quad \forall \varpi_3, \varpi_4 \in \mathcal{C}, \delta \in [0,1].$$

Convex analysis is the branch of mathematics that revolves around the concept of convex sets and functions. It has a wide range of applications in various fields of applied sciences, such as Economics, Optimization theory, Engineering, Functional analysis, Fixed point theory, Differential equations, and Mathematical inequalities. Among these, Mathematical inequalities have gained significant attention from researchers in recent decades, as it is a vast and notable area of analysis. Integral inequalities are particularly important due to their numerous applications, especially in error analysis. The growth of this field owes much to the impact of convex mappings and their generalizations. Many well-known mathematical

* Corresponding author at: School of Electrical and Electronics Engineering, Chung-Ang University, Dongjak-gu, Seoul, 06974, Republic of Korea.
E-mail addresses: awan.uzair@gmail.com (M.U. Awan), zakriajaved071@gmail.com (M.Z. Javed), hsyn.budak@gmail.com (H. Budak), yasersalah@tu.edu.sa (Y.S. Hamed), jongsukro@gmail.com (J.-S. Ro).

https://doi.org/10.1016/j.asej.2024.102683
Received 24 August 2023; Received in revised form 9 January 2024; Accepted 1 February 2024

inequalities, including the Hemite-Hadamard inequality, Minkowski inequality, Hölder-type inequalities, Young's inequality, and others, are direct consequences of convex mappings.

One can consider Jensen's inequality as an alternative characterization of a convex function, which can be expressed as follows: Let Ξ be a convex function on $[\varpi_3, \varpi_4]$, then for all $x_i \in [\varpi_3, \varpi_4]$ and $\mu_i \in [0,1]$, $i = 1, 2, \ldots, n$, with $\sum_{i=1}^{n} \mu_i = 1$, we have

$$\Xi\left(\sum_{i=1}^{n} \mu_i x_i\right) \le \sum_{i=1}^{n} \mu_i \Xi(x_i).$$

For details, see [1]. In 2004, Mercer introduced another form of Jensen's inequality through a distinct approach, which can be stated as follows: Suppose $\Xi : [\varpi_3, \varpi_4] \to \mathbb{R}$ is a convex function, then

$$\Xi\left(\varpi_3 + \varpi_4 - \sum_{i=1}^{n} \mu_i x_i\right) \le \Xi(\varpi_3) + \Xi(\varpi_4) - \sum_{i=1}^{n} \mu_i \Xi(x_i).$$

For more details, see [2–4].

Presented below is the renowned Ostrowski's type inequality which applies to a differentiable function $\Xi : [\varpi_3, \varpi_4] \to \mathbb{R}$ over the interval (ϖ_3, ϖ_4)

$$\left|\Xi(x) - \frac{1}{\varpi_4 - \varpi_3}\int_{\varpi_3}^{\varpi_4} \Xi(x)\mathrm{d}x\right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{\varpi_3+\varpi_4}{2}\right)^2}{(\varpi_4 - \varpi_3)^2}\right](\varpi_4 - \varpi_3)||\Xi'||_\infty,$$

$\forall x \in [\varpi_3, \varpi_4]$, where $||\Xi'||_\infty = \sup_{s\in(\varpi_3,\varpi_4)} |\Xi'(x)| < \infty$.

Quantum calculus and analysis are extensively utilized in various domains, including special functions, combinatorics, number theory, geometric function theory, difference equations, and integral inequalities, to discover fresh extensions of established discoveries. Nonetheless, Tariboon and Ntouyas discovered that applying Jackson q derivatives to impulsive difference equations has certain limitations. To overcome these limitations, they proposed a q-derivative and integral operator defined over finite intervals, as demonstrated below:

Definition 1.1 *([5])*. Assuming that $\Xi : [\varpi_3, \varpi_4] \subseteq \mathbb{R} \to \mathbb{R}$ is a continuous function and $u \in J$, the following statement holds:

$${}_{\varpi_3}D_q\Xi(e) = \frac{\Xi(e) - \Xi(qe + (1-q)\varpi_3)}{(1-q)(e - \varpi_3)}, \; u \neq \varpi_3, 0 < q < 1.$$

If $\varpi_3 Dq\Xi(u)$ exists for all $u \in J$, then we classify Ξ as being ${}_{\varpi_3}q$-differentiable on J. Moreover, the widely recognized q-number is defined as:

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1}, \; q \in (0,1).$$

Moreover, we will revisit the concept of the quantum q_{ϖ_3}-integral, which was initially proposed by Tariboon and Ntouyas in [5], and examine it over a limited interval.

Definition 1.2 *([5])*. Assuming that $\Xi : [\varpi_3, \varpi_4] \to \mathbb{R}$ is a continuous function, the following statement holds:

$$\int_{\varpi_3}^{\varpi_4} \Xi(e)_{\varpi_3} d_q e = (1-q)(\varpi_4 - \varpi_3)\sum_{n=0}^{\infty} q^n \Xi(q^n\varpi_4 + (1-q^n)\varpi_3) = (\varpi_4 - \varpi_3)\int_0^1 \Xi((1-\delta)\varpi_3 + \delta\varpi_4)d_q\delta.$$

The following theorem describes some essential features of the $\varpi_3 q$ integral operator in connection with the $\varpi_3 q$-integral, which are significant for our analysis.

Theorem 1.1 *([5]). If $\Xi : [\varpi_3, \varpi_4] \to \mathbb{R}$ is a continuous function and $u \in [\varpi_3, \varpi_4]$, then*

$${}_{\varpi_3}D_q \int_{\varpi_3}^{z} \Xi(u)_{\varpi_3}\mathrm{d}_q u = \Xi(z).$$

$$\int_c^z {}_{\varpi_3}D_q\Xi(u)_{\varpi_3}\mathrm{d}_q u = \Xi(z) - \Xi(c).$$

Lemma 1.1 *([5]). For continuous functions $\Xi, g : [\varpi_3, \varpi_4] \to \mathbb{R}$, we have*

$$\int_0^c g(\delta)_{\varpi_3}D_q\Xi(\delta\varpi_4 + (1-\delta)\varpi_3)\mathrm{d}_q\delta$$

$$= \frac{g(\delta)\Xi(\delta\varpi_4 + (1-\delta)\varpi_3)}{\varpi_4 - \varpi_3}\Big|_0^c - \frac{1}{\varpi_4 - \varpi_3}\int_0^c D_q g(\delta)\Xi(q\delta\varpi_4 + (1-q\delta)\varpi_3)\mathrm{d}_q\delta.$$

Bermuda et al. [6] introduced the concept of right quantum derivatives and definite integrals in [6], which are presented below:

Definition 1.3 *([6]).* Let $\Xi : [\varpi_3, \varpi_4] \to \mathbb{R}$ be a continuous function and $e \in [\varpi_3, \varpi_4]$, then

$$^{\varpi_4}D_q\Xi(e) = \frac{\Xi(qe + (1-q)\varpi_4) - \Xi(e)}{(1-q)(\varpi_4 - e)}, \quad e < \varpi_4.$$

Definition 1.4 *([6]).* Let $\Xi : [\varpi_3, \varpi_4] \to \mathbb{R}$ be a continuous function. Then, the q^{ϖ_4}-definite integral on $[\varpi_3, \varpi_4]$ is defined as:

$$\int_{\varpi_3}^{\varpi_4} \Xi(\delta)\,^{\varpi_4}d_q\delta = (1-q)(\varpi_4 - \varpi_3)\sum_{n=0}^{\infty} q^n \Xi(q^n\varpi_3 + (1-q^n)\varpi_4) = (\varpi_4 - \varpi_3)\int_0^1 \Xi(\delta\varpi_3 + (1-\delta)\varpi_4)d_q\delta.$$

Now we rewrite some further results.

Theorem 1.2 *([6]). If $\Xi : [\varpi_3, \varpi_4] \to \mathbb{R}$ is a continuous function and $e \in [\varpi_3, \varpi_4]$, then*

$$^{\varpi_4}D_q \int_z^{\varpi_4} \Xi(e)\,^{\varpi_4}\mathrm{d}_q e = -\Xi(z).$$

$$\int_z^{\varpi_4} {}^{\varpi_4}D_q\Xi(e)\,^{\varpi_4}\mathrm{d}_q e = \Xi(\varpi_4) - \Xi(z).$$

Lemma 1.2 *([6]). For continuous functions $\Xi, g : [\varpi_3, \varpi_4] \to \mathbb{R}$, we have*

$$\int_0^c g(\delta)\,^{\varpi_4}D_q\Xi(\delta\varpi_3 + (1-\delta)\varpi_4)\mathrm{d}_q\delta$$

$$= \frac{1}{\varpi_4 - \varpi_3}\int_0^c D_q g(\delta)\Xi(q\delta\varpi_3 + (1-q\delta)\varpi_4)\mathrm{d}_q\delta - \frac{g(\delta)\Xi(\delta\varpi_3 + (1-\delta)\varpi_4)}{\varpi_4 - \varpi_3}\Big|_0^c.$$

In [7], Noor et al. demonstrated the quantum version of the Ostrowski inequality, which is expressed as follows:
Suppose that $\Xi : [\varpi_3, \varpi_4] \to \mathbb{R}$ is a q-differentiable function. If $|_{\varpi_3}D_q\Xi(x)| \le M$, then

$$\left|\Xi(x) - \frac{1}{\varpi_4 - \varpi_3}\int_{\varpi_3}^{\varpi_4} \Xi(u)_{\varpi_3}\mathrm{d}_q u\right| \le \frac{qM[(x-\varpi_3)^2 + (\varpi_4 - x)^2]}{[2]_q(\varpi_4 - \varpi_3)}.$$

Over the last few years, numerous researchers have contributed to the advancement and extension of Ostrowski and Hermite-Hadamard type inequalities in diverse directions. Alomari et al. [8] introduced generalized convexity, while Basci and Baleanu [9] proposed fractional Ostrowski-like inequalities employing Hilfer fractional operators. Yildiz and Cotrila [10] established Hermite-Hadamard type inequalities for Riemann-Liouville fractional operators and Green functions. Zaheer et al. [11] studied well-known inequalities using majorization and strong convexity. Qurashi et al. [12] derived Ostrowski type inequalities over fractal sets with applications, and Kashuri et al. [13] developed weighted Ostrowski's type inequalities. Wang et al. [14] established q-Fractional Ostrowski's type inequalities involving generalized convexity, while Ogulmus and Sarikaya [15] examined fractional Hermite-Hadamard type inequalities. In 2022 Du and Zhou [16] investigated the interval-valued fractional analogs of Hermite-Hadamard type inequalities involving exponential fractional operators for two variables and coordinated interval-valued convexity. Budak et al. [17] investigated the midpoint type inequalities involving generalized fractional operators to obtain the unified form of existing fraction variants. Zhang et al. [18] derived the quantum integral inequalities through generalized convexity. IN 2023 Du and Peng [19] established the trapezium-type inequalities by making use of the idea of fractional multiplicative calculus. İscan et al. [20] proposed weighted Hermite-Hadamard-Mercer type inequalities, and You et al. [21] investigated Hermite-Hadamard-Mercer type inequalities associated with harmonic convex functions. In [22], the generalized Jensen-Mercer inequality and fractional concepts were used to derive new Hermite-Hadamard type inequalities, and Faisal et al. [23] established new Hermite-Hadamard-Mercer type inequalities using majorization theory and generalized Mercer inequality. Bin-Mohsin et al. [24] used strongly harmonic convexity to obtain some Hadamard-Mercer-type inequalities within the framework of fractional concepts. Additionally, Butt et al. [25] presented Ostrowski-Mercer type inequalities involving generalized fractional operators with respect to a monotone function ψ, and Sial et al. [26] evaluated new Ostrowski-Mercer type inequalities for differentiable convex functions and their applications. For more information, please see [27–30].

The Montgomery identity is a well-established mathematical tool utilized to establish various classical integral inequalities. Over the last few decades, substantial research has been conducted on this identity, leading to significant advancements in the field.

Lemma 1.3. *If $\Xi : [\varpi_3, \varpi_4] \to \mathbb{R}$ be a differentiable mapping on (ϖ_3, ϖ_4), then*

$$\Xi(x) - \frac{1}{\varpi_4 - \varpi_3}\int_{\varpi_3}^{\varpi_4} \Xi(x)\mathrm{d}x = \int_{\varpi_3}^{x} \frac{x - \varpi_3}{\varpi_4 - \varpi_3}\Xi'(u)\mathrm{d}u + \int_x^{\varpi_4} \frac{x - \varpi_4}{\varpi_4 - \varpi_3}\Xi'(u)\mathrm{d}u.$$

Equivalently, it can be transformed as: If $\Xi : [\varpi_3, \varpi_4] \to \mathbb{R}$ be a differentiable mapping on (ϖ_3, ϖ_4), then:

$$\Xi(x) - \frac{1}{\varpi_4 - \varpi_3} \int_{\varpi_3}^{\varpi_4} \Xi(x) \mathrm{d}x$$
$$= (\varpi_4 - \varpi_3) \left[\int_0^{\frac{\varpi_4 - x}{\varpi_4 - \varpi_3}} \delta \Xi'(\delta\varpi_3 + (1-\delta)\varpi_4) \mathrm{d}\delta + \int_{\frac{\varpi_4 - x}{\varpi_4 - \varpi_3}}^1 (\delta - 1) \Xi'(\delta\varpi_3 + (1-\delta)\varpi_4) \mathrm{d}\delta \right].$$

Kunt et al. [31] proposed the quantum Montgomery identity for ${}_{\varpi_3}q$ differentiable function, which is given as:

Lemma 1.4. *If $\Xi : [\varpi_3, \varpi_4] \to \mathbb{R}$ be a continuous and ${}_{\varpi_3}q$ differentiable function on (ϖ_3, ϖ_4), then*

$$\Xi(x) - \frac{1}{\varpi_4 - \varpi_3} \int_{\varpi_3}^{\varpi_4} \Xi(x) {}_{\varpi_3}\mathrm{d}_q x$$
$$= (\varpi_4 - \varpi_3) \left[\int_0^{\frac{x - \varpi_3}{\varpi_4 - \varpi_3}} q\delta \, {}_{\varpi_3}D_q \Xi(\delta\varpi_4 + (1-\delta)\varpi_3) \mathrm{d}_q\delta + \int_{\frac{x - \varpi_3}{\varpi_4 - \varpi_3}}^1 (q\delta - 1) {}_{\varpi_3}D_q \Xi(\delta\varpi_4 + (1-\delta)\varpi_3) \mathrm{d}_q\delta \right].$$

In [32] authors have used the right quantum derivative as a tool to obtain a new Montgomery identity, which is followed as:

Lemma 1.5. *If $\Xi : [\varpi_3, \varpi_4] \to \mathbb{R}$ be a continuous and ${}_{\varpi_3}q$ differentiable function on (ϖ_3, ϖ_4), then*

$$\Xi(x) - \frac{1}{\varpi_4 - \varpi_3} \int_{\varpi_3}^{\varpi_4} \Xi(x) {}_{\varpi_3}\mathrm{d}_q x$$
$$= (\varpi_4 - \varpi_3) \left[\int_0^{\frac{\varpi_4 - x}{\varpi_4 - \varpi_3}} q\delta \, {}^{\varpi_4}D_q \Xi(\delta\varpi_3 + (1-\delta)\varpi_4) \mathrm{d}_q\delta + \int_{\frac{\varpi_4 - x}{\varpi_4 - \varpi_3}}^1 (q\delta - 1) \, {}^{\varpi_4}D_q \Xi(\delta\varpi_3 + (1-\delta)\varpi_4) \mathrm{d}_q\delta \right].$$

The Montgomery identity has gained recognition as a valuable tool for formulating classical integral inequalities, and it has been the subject of extensive research in recent years. Butt et al. [33] utilized the generalized Montgomery identity to derive a range of new identities and used these equalities to investigate Popoviciu and Ostrowski-like inequalities for higher-order convex mappings. Aglic [34] obtained fractional bounds for Ostrowski-type inequalities using the Montgomery identity. Vivas Cortez et al. [35] derived a new generalized Montgomery identity and established error bounds for the rectangular rule using preinvex functions. Ali et al. [36] explored the well-known Ostrowski inequality over rectangles in the context of q-calculus associated with Montgomery equality and coordinated convex functions. Kunt et al. [31] employed quantum concepts to develop Montgomery identity and related inequalities. Mehmood et al. [37] applied the Montgomery identity to obtain new extensions of Popoviciu-type inequalities using new Green's functions, while Kalsoom et al. [38,39] derived q and (p, q) estimates using the Montgomery identity.

Chu et al. [40] introduced right p, q derivatives, and integrals and established some Ostrowski-type inequalities, and Ali et al. [41] estimated new bounds for Ostrowski-type inequalities involving twice q-differentiable convex functions. In [42] authors obtained the rectified forms of Jensen inequality and their applications through Montgomery identity.

Recently, Jensen-Mercer and related inequalities have been extensively studied using different approaches. In [43] authors computed some Hadamard-Mercer-type inequalities via quantum calculus. Bin-Mohsin et al. [44] employed a combined approach of majorization theory and quantum calculus to establish some Jensen-Mercer and related inequalities.

The primary objective of this paper is to develop novel quantum results based on the Montgomery identities and associated general Ostrowski inequalities. The study is divided into three main sections. The first section provides a brief overview of the research background and essential terminologies required for further analysis. The second section presents the main findings of the research. Finally, the third section includes numerical verification and graphical analysis. The techniques and ideas presented in this article are expected to pique the interest of curious readers.

2. Main results

In the following section, we present our key findings, which are divided into two sub-sections. Firstly, we demonstrate two fundamental q-identities of the Montgomery type. Secondly, we establish new approximations of general Ostrowski type inequalities and various general Hermite-Hadamard type inequalities, which are regarded as particular cases of our primary theorems.

2.1. New quantum Montgomery identities

We now present our first q-Montgomery identity.

Lemma 2.1. *Let* $\Xi : [\varpi_3, \varpi_4] \to \mathbb{R}$ *be a* q*-differentiable function, then*

$$\Xi(u) - \frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x} \Xi(z)\,{}_{\varpi_3+\varpi_4-y}\mathrm{d}_q z$$
$$= (y-x)\left[\int_0^{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}} q\delta\ {}_{\varpi_3+\varpi_4-y}D_q\Xi(\delta(\varpi_3+\varpi_4-x)+(1-\delta)(\varpi_3+\varpi_4-y))\,\mathrm{d}_q\delta\right.$$
$$\left. + \int_{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}^{1} (q\delta-1)\ {}_{\varpi_3+\varpi_4-y}D_q\Xi(\delta(\varpi_3+\varpi_4-x)+(1-\delta)(\varpi_3+\varpi_4-y))\,\mathrm{d}_q\delta\right], \tag{2.1}$$

where $\varpi_3+\varpi_4-y < \varpi_3+\varpi_4-x, u \in (\varpi_3+\varpi_4-y, \varpi_3+\varpi_4-x)$ *and* $\delta \in [0,1]$.

Proof. Consider the right-hand side of (2.1),

$$I = (y-x)\left[\int_0^{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}} q\delta\ {}_{\varpi_3+\varpi_4-y}D_q\Xi(\delta(\varpi_3+\varpi_4-x)+(1-\delta)(\varpi_3+\varpi_4-y))\,\mathrm{d}_q\delta\right.$$
$$\left. + \int_{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}^{1} (q\delta-1)\ {}_{\varpi_3+\varpi_4-y}D_q\Xi(\delta(\varpi_3+\varpi_4-x)+(1-\delta)(\varpi_3+\varpi_4-y))\,\mathrm{d}_q\delta\right]$$
$$= (y-x)\left[\int_0^{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}} q\delta\ {}_{\varpi_3+\varpi_4-y}D_q\Xi(\delta(\varpi_3+\varpi_4-x)+(1-\delta)(\varpi_3+\varpi_4-y))\,\mathrm{d}_q\delta\right.$$
$$+ \int_0^1 (q\delta-1)\ {}_{\varpi_3+\varpi_4-y}D_q\Xi(\delta(\varpi_3+\varpi_4-x)+(1-\delta)(\varpi_3+\varpi_4-y))\,\mathrm{d}_q\delta$$
$$\left. - \int_0^{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}} (q\delta-1)\ {}_{\varpi_3+\varpi_4-y}D_q\Xi(\delta(\varpi_3+\varpi_4-x)+(1-\delta)(\varpi_3+\varpi_4-y))\,\mathrm{d}_q\delta\right]$$
$$= (y-x)\left[\int_0^1 (q\delta-1)\ {}_{\varpi_3+\varpi_4-y}D_q\Xi(\delta(\varpi_3+\varpi_4-x)+(1-\delta)(\varpi_3+\varpi_4-y))\,\mathrm{d}_q\delta\right.$$
$$\left. + \int_0^{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}} {}_{\varpi_3+\varpi_4-y}D_q\Xi(\delta(\varpi_3+\varpi_4-x)+(1-\delta)(\varpi_3+\varpi_4-y))\,\mathrm{d}_q\delta\right]$$
$$= (y-x)[I_1 + I_2]. \tag{2.2}$$

Now q-integrating by parts, we get

$$I_1 = \int_0^1 (q\delta-1)\ {}_{\varpi_3+\varpi_4-y}D_q\Xi(\delta(\varpi_3+\varpi_4-x)+(1-\delta)(\varpi_3+\varpi_4-y))\,\mathrm{d}_q\delta$$
$$= \frac{(q-1)\Xi(\varpi_3+\varpi_4-x)+\Xi(\varpi_3+\varpi_4-y)}{y-x}$$
$$- \frac{q}{y-x}\int_0^1 \Xi(q\delta(\varpi_3+\varpi_4-x)+(1-q\delta)(\varpi_3+\varpi_4-y))\mathrm{d}_q\delta$$
$$= \frac{(q-1)\Xi(\varpi_3+\varpi_4-x)+\Xi(\varpi_3+\varpi_4-y)}{y-x}$$

$$
\begin{aligned}
&-\frac{(1-q)}{y-x}\sum_{n=0}^{\infty}q^{n+1}\Xi(q^{n+1}(\varpi_3+\varpi_4-x)+(1-q^{n+1})(\varpi_3+\varpi_4-y))\\
&=\frac{\Xi(\varpi_3+\varpi_4-y)}{y-x}-\frac{(1-q)(y-x)}{(y-x)^2}\sum_{n=0}^{\infty}q^{n}\Xi(q^{n}(\varpi_3+\varpi_4-x)+(1-q^{n})(\varpi_3+\varpi_4-y))\\
&=\frac{\Xi(\varpi_3+\varpi_4-y)}{y-x}-\frac{1}{(y-x)^2}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x}\Xi(z)\,{}_{\varpi_3+\varpi_4-y}\mathrm{d}_q z.
\end{aligned}
$$

And similarly,

$$
\begin{aligned}
I_2&=\int_0^{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}{}_{\varpi_3+\varpi_4-y}D_q\Xi(\delta(\varpi_3+\varpi_4-x)+(1-\delta)(\varpi_3+\varpi_4-y))\,\mathrm{d}_q\delta\\
&=\frac{\Xi(u)-\Xi(\varpi_3+\varpi_4-y)}{y-x}.
\end{aligned}
\tag{2.3}
$$

Substituting the values of I_1 and I_2 in I, we conclude our required result. □

Now we give the Montgomery identity with respect to right ${}^{\varpi_4}q$-derivatives.

Lemma 2.2. *Let $\Xi:[\varpi_3,\varpi_4]\to\mathbb{R}$ be a ${}^{\varpi_4}q$-differentiable function, then*

$$
\begin{aligned}
&\frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x}\Xi(z)\,{}^{\varpi_3+\varpi_4-x}\mathrm{d}_q z-\Xi(u)\\
&=(y-x)\Bigg[\int_0^{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}q\delta\,{}^{\varpi_3+\varpi_4-x}D_q\Xi(\delta(\varpi_3+\varpi_4-y)+(1-\delta)(\varpi_3+\varpi_4-x))\,\mathrm{d}_q\delta\\
&+\int_{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}^{1}(q\delta-1)\,{}^{\varpi_3+\varpi_4-x}D_q\Xi(\delta(\varpi_3+\varpi_4-y)+(1-\delta)(\varpi_3+\varpi_4-x))\,\mathrm{d}_q\delta\Bigg],
\end{aligned}
\tag{2.4}
$$

where $\varpi_3+\varpi_4-y<\varpi_3+\varpi_4-x$, $u\in(\varpi_3+\varpi_4-y,\varpi_3+\varpi_4-x)$ and $\delta\in[0,1]$.

Proof. Consider the right-hand side of (2.4),

$$
\begin{aligned}
I&=\Bigg[\int_0^{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}q\delta\,{}^{\varpi_3+\varpi_4-x}D_q\Xi(\delta(\varpi_3+\varpi_4-y)+(1-\delta)(\varpi_3+\varpi_4-x))\,\mathrm{d}_q\delta\\
&\quad+\int_{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}^{1}(q\delta-1)\,{}^{\varpi_3+\varpi_4-x}D_q\Xi(\delta(\varpi_3+\varpi_4-y)+(1-\delta)(\varpi_3+\varpi_4-x))\,\mathrm{d}_q\delta\Bigg]\\
&=(y-x)\Bigg[\int_0^{1}(q\delta-1)\,{}^{\varpi_3+\varpi_4-x}D_q\Xi(\delta(\varpi_3+\varpi_4-y)+(1-\delta)(\varpi_3+\varpi_4-x))\,\mathrm{d}_q\delta\\
&\quad+\int_0^{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}{}^{\varpi_3+\varpi_4-x}D_q\Xi(\delta(\varpi_3+\varpi_4-y)+(1-\delta)(\varpi_3+\varpi_4-x))\,\mathrm{d}_q\delta\Bigg]\\
&=(y-x)[I_1+I_2].
\end{aligned}
\tag{2.5}
$$

Now q-integrating by parts, we get

$$
I_1=\int_0^1(q\delta-1)\,{}^{\varpi_3+\varpi_4-x}D_q\Xi(\delta(\varpi_3+\varpi_4-y)+(1-\delta)(\varpi_3+\varpi_4-x))\,\mathrm{d}_q\delta
$$

$$
\begin{aligned}
&= \frac{(1-q)\Xi(\varpi_3+\varpi_4-y)-\Xi(\varpi_3+\varpi_4-x)}{y-x}\\
&+ \frac{q}{y-x}\int_0^1 \Xi(q\delta(\varpi_3+\varpi_4-y)+(1-q\delta)(\varpi_3+\varpi_4-x))\mathrm{d}_q\delta\\
&= \frac{(1-q)\Xi(\varpi_3+\varpi_4-y)-\Xi(\varpi_3+\varpi_4-x)}{y-x}\\
&+ \frac{(1-q)}{y-x}\sum_{n=0}^{\infty} q^{n+1}\Xi(q^{n+1}(\varpi_3+\varpi_4-y)+(1-q^{n+1})(\varpi_3+\varpi_4-x))\\
&= -\frac{\Xi(\varpi_3+\varpi_4-x)}{y-x}+\frac{(1-q)(y-x)}{(y-x)^2}\sum_{n=0}^{\infty} q^{n}\Xi(q^{n}(\varpi_3+\varpi_4-y)+(1-q^{n})(\varpi_3+\varpi_4-x))\\
&= -\frac{\Xi(\varpi_3+\varpi_4-x)}{y-x}+\frac{1}{(y-x)^2}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x}\Xi(z)\,{}^{\varpi_3+\varpi_4-x}\mathrm{d}_q z.
\end{aligned}
$$

And similarly,

$$
\begin{aligned}
I_2 &= \int_0^{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}} {}^{\varpi_3+\varpi_4-x}D_q\Xi(\delta(\varpi_3+\varpi_4-y)+(1-\delta)(\varpi_3+\varpi_4-x))\,\mathrm{d}_q\delta\\
&= \frac{\Xi(\varpi_3+\varpi_4-x)-\Xi(u)}{y-x}.
\end{aligned}
\tag{2.6}
$$

Substituting the values of I_1 and I_2 in I, we conclude our required result. □

2.2. General Ostrowski type inequalities

In this section, we employ the auxiliary results derived in the previous sub-section and the Jensen-Mercer inequality for convex functions.

Theorem 2.1. *Let $\Xi : [\varpi_3,\varpi_4] \to \mathbb{R}$ be continuous q-differentiable function and if $|{}_{\varpi_3+\varpi_4-y}D_q\Xi|^r$ is a convex function, where $\varpi_3+\varpi_4-y < \varpi_3+\varpi_4-x$ and $r \geq 1$ then,*

$$
\begin{aligned}
&\left|\Xi(u)-\frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x}\Xi(z)\mathrm{d}z\right|\\
&\leq (y-x)\Bigg[w_1^{1-\frac{1}{r}}(\varpi_3,\varpi_4,x,y,q)\Big(w_1(\varpi_3,\varpi_4,x,y,q)(|{}_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_3)|^r+|{}_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_4)|^r)\\
&-w_2(\varpi_3,\varpi_4,x,y,q)|{}_{\varpi_3+\varpi_4-y}D_q\Xi(x)|^r-w_3(\varpi_3,\varpi_4,x,y,q)|{}_{\varpi_3+\varpi_4-y}D_q\Xi(y)|^r\Big)^{\frac{1}{r}}\\
&+w_4^{1-\frac{1}{r}}(\varpi_3,\varpi_4,x,y,q)\Big(w_4(\varpi_3,\varpi_4,x,y,q)(|{}_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_3)|^r+|{}_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_4))|^r\\
&-w_5(\varpi_3,\varpi_4,x,y,q)|{}_{\varpi_3+\varpi_4-y}D_q\Xi(x)|^r-w_6(\varpi_3,\varpi_4,x,y,q)|{}_{\varpi_3+\varpi_4-y}D_q\Xi(y)|^r\Big)^{\frac{1}{r}}\Bigg],
\end{aligned}
$$

where

$$
w_1(\varpi_3,\varpi_4,x,y,q)=\int_0^{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}} q\delta\,\mathrm{d}_q\delta=\frac{q}{[2]_q}\left(\frac{u-(\varpi_3+\varpi_4-y)}{y-x}\right)^2.
$$

$$
w_2(\varpi_3,\varpi_4,x,y,q)=\int_0^{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}} q\delta^2\,\mathrm{d}_q\delta=\frac{q}{[3]_q}\left(\frac{u-(\varpi_3+\varpi_4-y)}{y-x}\right)^3.
$$

$$
w_3(\varpi_3,\varpi_4,x,y,q)=\int_0^{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}} q\delta(1-\delta)\,\mathrm{d}_q\delta=w_1(\varpi_3,\varpi_4,x,y,q)-w_2(\varpi_3,\varpi_4,x,y,q).
$$

$$
\begin{aligned}
&w_4(\varpi_3,\varpi_4,x,y,q)\\
&=\int_{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}^{1}(1-q\delta)\,\mathrm{d}_q\delta=\frac{(1-q)}{[2]_q}\left(\frac{(\varpi_3+\varpi_4-x)-u}{y-x}\right)+\frac{q}{[2]_q}\left(\frac{(\varpi_3+\varpi_4-x)-u}{y-x}\right)^2.
\end{aligned}
$$

$$w_5(\varpi_3,\varpi_4,x,y,q) = \int_{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}^{1} (1-q\delta)\delta \, d_q\delta = \frac{1}{[2]_q[3]_q} - \frac{q}{[2]_q}\left(\frac{u-(\varpi_3+\varpi_4-y)}{y-x}\right)^2 + \frac{q}{[3]_q}\left(\frac{u-(\varpi_3+\varpi_4-y)}{y-x}\right)^3.$$

$$w_6(\varpi_3,\varpi_4,x,y,q) = \int_{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}^{1} (1-q\delta)(1-\delta) \, d_q\delta = w_4(\varpi_3,\varpi_4,x,y,q) - w_5(\varpi_3,\varpi_4,x,y,q).$$

Proof. By making use of Lemma 2.1, modulus property, power mean inequality and convexity property of $|_{\varpi_3+\varpi_4-y}Df|^r$ of the function, then

$$\begin{aligned}
&\left|\Xi(u) - \frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x} \Xi(z)_{\varpi_3+\varpi_4-y} d_q z\right| \\
&\le (y-x)\left[\int_0^{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}} q\delta \left|_{\varpi_3+\varpi_4-y}D_q\Xi(\delta(\varpi_3+\varpi_4-x)+(1-\delta)(\varpi_3+\varpi_4-y))\right| d_q\delta \right.\\
&\left. + \int_{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}^{1} (1-q\delta) \left|_{\varpi_3+\varpi_4-y}D_q\Xi(\delta(\varpi_3+\varpi_4-x)+(1-\delta)(\varpi_3+\varpi_4-y))\right| d_q\delta\right] \\
&\le (y-x)\left[\left(\int_0^{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}} q\delta \, d_q\delta\right)^{1-\frac{1}{r}} \left(\int_0^{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}} q\delta \left|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_3+\varpi_4-(\delta x+(1-\delta)y))\right|^r d_q\delta\right)^{\frac{1}{r}}\right.\\
&\left. + \left(\int_{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}^{1} (1-q\delta) \, d_q\delta\right)^{1-\frac{1}{r}} \left(\int_{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}^{1} (1-q\delta) \left|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_3+\varpi_4-(\delta x+(1-\delta)y))\right|^r d_q\delta\right)^{\frac{1}{r}}\right] \\
&\le (y-x)\left[\left(\int_0^{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}} q\delta \, d_q\delta\right)^{1-\frac{1}{r}} \left(\int_0^{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}} q\delta \left(|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_3)|^r + |_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_4)|^r\right.\right.\right.\\
&\left.\left. -\delta|_{\varpi_3+\varpi_4-y}D_q\Xi(x)|^r - (1-\delta)|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_3)|^r\right) d_q\delta\right)^{\frac{1}{r}} + \left(\int_{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}^{1} (1-q\delta) \, d_q\delta\right)^{1-\frac{1}{r}} \\
&\left(\int_{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}^{1} (1-q\delta)\left(|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_3)|^r + |_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_4)|^r\right.\right.\\
&\left.\left.\left. -\delta|_{\varpi_3+\varpi_4-y}D_q\Xi(x)|^r - (1-\delta)|_{\varpi_3+\varpi_4-y}D_q\Xi(y)|^r\right) d_q\delta\right)^{\frac{1}{r}}\right].
\end{aligned}$$

After simple computations, we obtain our required result. □

Now we discuss some special cases of Theorem 2.1.

- If $|_{\varpi_3+\varpi_4-y}D_q\Xi| \le M$ with $M \ge 0$ and $r=1$, then we obtain the following new Ostrowski-Mercer inequality:

$$\left|\Xi(u) - \frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x} \Xi(z) d_q z\right|$$

$$\leq \frac{M}{(y-x)[2]_q}\left[q((\varpi_3+\varpi_4-x)-u)^2+(1-q)(y-x)((\varpi_3+\varpi_4-x)-u)+q(u-(\varpi_3+\varpi_4-x))^2\right].$$

- If we take $r=1$ in Theorem 2.1, then following bounds for Ostrowski-Mercer inequality hold:

$$\left|\Xi(u)-\frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x}\Xi(z)\mathrm{d}_q z\right|$$
$$\leq (y-x)\Big[(w_1(\varpi_3,\varpi_4,x,y,q)+w_4(\varpi_3,\varpi_4,x,y,q))\left(|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_3)|+|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_4)|\right)$$
$$-(w_2(\varpi_3,\varpi_4,x,y,q)+w_5(\varpi_3,\varpi_4,x,y,q))|_{\varpi_3+\varpi_4-y}D_q\Xi(x)|$$
$$-(w_3(\varpi_3,\varpi_4,x,y,q)+w_6(\varpi_3,\varpi_4,x,y,q))|_{\varpi_3+\varpi_4-y}D_q\Xi(y)|\Big],$$

where $w_i(\varpi_3,\varpi_4,x,y,q), i=1,2,3,\ldots,6$ are defined in Theorem 2.1.

- If we take $u=\varpi_3+\varpi_4-\frac{x+y}{2}$ in Theorem 2.1, then following bound for general Hermite-Hadamard inequality holds:

$$\left|\Xi\left(\varpi_3+\varpi_4-\frac{x+y}{2}\right)-\frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x}\Xi(z)\mathrm{d}z\right|$$
$$\leq (y-x)\Big[{}_1w_1^{1-\frac{1}{r}}(q)\Big({}_1w_1(q)(|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_3)|^r+|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_4)|^r)$$
$$-{}_1w_2(q)|_{\varpi_3+\varpi_4-y}D_q\Xi(x)|^r-{}_1w_3(q)|_{\varpi_3+\varpi_4-y}D_q\Xi(y)|^r\Big)^{\frac{1}{r}}$$
$$+{}_1w_4^{1-\frac{1}{r}}(q)\Big({}_1w_4(q)(|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_3)|^r+|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_4))|^r$$
$$-{}_1w_5(q)|_{\varpi_3+\varpi_4-y}D_q\Xi(x)|^r-{}_1w_6(q)|_{\varpi_3+\varpi_4-y}D_q\Xi(y)|^r\Big)^{\frac{1}{r}}\Big],$$

where

$${}_1w_1(q)=\frac{q}{4[2]_q},\quad {}_1w_2(q)=\frac{q}{8[3]_q},\quad {}_1w_3(q)={}_1w_1(q)-{}_1w_2(q).$$
$${}_1w_4(q)=\frac{(1-q)}{2[2]_q}+\frac{q}{4[2]_q}.\quad {}_1w_5(q)=\frac{1}{[2]_q[3]_q}-\frac{q}{4[2]_q}+\frac{q}{8[3]_q}.$$
$${}_1w_6(q)={}_1w_4(q)-{}_1w_5(q).$$

Theorem 2.2. *Let $\Xi:[\varpi_3,\varpi_4]\to\mathbb{R}$ be continuous q-differentiable function and if $|_{\varpi_3+\varpi_4-y}D_q\Xi|^s$ is a convex function, where $\varpi_3+\varpi_4-y<\varpi_3+\varpi_4-x$ and $r,s>1$ with $\frac{1}{r}+\frac{1}{s}=1$, then*

$$\left|\Xi(u)-\frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x}\Xi(z)\mathrm{d}z\right|$$
$$\leq (y-x)\Bigg[\frac{q}{[r+1]_q^{\frac{1}{r}}}\left(\frac{u-(\varpi_3+\varpi_4-y)}{y-x}\right)^{1+\frac{1}{r}}$$
$$\Bigg(\frac{u-(\varpi_3+\varpi_4-y)}{y-x}(|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_3)|^s+|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_4)|^s)-\frac{1}{[2]_q}\left(\frac{u-(\varpi_3+\varpi_4-y)}{y-x}\right)^2$$
$$|_{\varpi_3+\varpi_4-y}D_q\Xi(x)|^s-|_{\varpi_3+\varpi_4-y}D_q\Xi(y)|^s\left(\frac{u-(\varpi_3+\varpi_4-y)}{y-x}-\frac{1}{[2]_q}\left(\frac{u-(\varpi_3+\varpi_4-y)}{y-x}\right)^2\right)\Bigg)^{\frac{1}{s}}$$
$$+\left(\int_{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}^{1}(1-q\delta)^r\mathrm{d}_q\delta\right)^{\frac{1}{r}}\Bigg(\frac{(\varpi_3+\varpi_4-x)-u}{y-x}(|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_3)|^s+|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_4))|^s$$
$$-\left(\frac{1}{[2]_q}-\frac{1}{[2]_q}\left(\frac{u-(\varpi_3+\varpi_4-y)}{y-x}\right)^2\right)|_{\varpi_3+\varpi_4-y}D_q\Xi(x)|^s-\Bigg(\frac{q}{[2]_q}-\frac{(u-(\varpi_3+\varpi_4-y))}{y-x}$$
$$+\frac{1}{[2]_q}\left(\frac{u-(\varpi_3+\varpi_4-y)}{y-x}\right)^2\Bigg)|_{\varpi_3+\varpi_4-y}D_q\Xi(y)|^s\Bigg)^{\frac{1}{s}}\Bigg].$$

Proof. By making use of Lemma 2.1, modulus property, Hölder's inequality and convexity property of $|_{\varpi_3+\varpi_4-y}D_q f|^s$ of the function, then

$$\left|\Xi(u)-\frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x}\Xi(z)\,_{\varpi_3+\varpi_4-y}\mathrm{d}_q z\right|$$

$$\leq (y-x)\left[\int_0^{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}} q\delta\left|_{\varpi_3+\varpi_4-y}D_q\Xi(\delta(\varpi_3+\varpi_4-x)+(1-\delta)(\varpi_3+\varpi_4-y))\right|\mathrm{d}_q\delta\right.$$

$$\left.+\int_{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}^{1}(1-q\delta)\left|_{\varpi_3+\varpi_4-y}D_q\Xi(\delta(\varpi_3+\varpi_4-x)+(1-\delta)(\varpi_3+\varpi_4-y))\right|\mathrm{d}_q\delta\right]$$

$$\leq (y-x)\left[\left(\int_0^{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}(q\delta)^r\mathrm{d}_q\delta\right)^{\frac{1}{r}}\left(\int_0^{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}\left|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_3+\varpi_4-(\delta x+(1-\delta)y))\right|^s\mathrm{d}_q\delta\right)^{\frac{1}{s}}\right.$$

$$\left.+\left(\int_{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}^{1}(1-q\delta)^r\mathrm{d}_q\delta\right)^{\frac{1}{r}}\left(\int_{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}^{1}\left|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_3+\varpi_4-(\delta x+(1-\delta)y))\right|^s\mathrm{d}_q\delta\right)^{\frac{1}{s}}\right]$$

$$\leq (y-x)\left[\left(\int_0^{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}(q\delta)^r\mathrm{d}_q\delta\right)^{\frac{1}{r}}\left(\int_0^{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}\left(|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_3)|^s+|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_4)|^s\right.\right.\right.$$

$$\left.\left.-\delta|_{\varpi_3+\varpi_4-y}D_q\Xi(x)|^s-(1-\delta)|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_3)|^s\right)\mathrm{d}_q\delta\right)^{\frac{1}{r}}+\left(\int_{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}^{1}(1-q\delta)^r\mathrm{d}_q\delta\right)^{\frac{1}{r}}$$

$$\left(\int_{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}^{1}(|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_3)|^s+|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_4)|^s\right.$$

$$\left.\left.-\delta|_{\varpi_3+\varpi_4-y}D_q\Xi(x)|^s-(1-\delta)|_{\varpi_3+\varpi_4-y}D_q\Xi(y)|^s)\mathrm{d}_q\delta\right)^{\frac{1}{s}}\right].$$

After simplifying, we acquire our required result. □

- If $|_{\varpi_3+\varpi_4-y}D_q\Xi|\leq M$ with $M\geq 0$, then we obtain the following new Ostrowski-Mercer inequality:

$$\left|\Xi(u)-\frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x}\Xi(z)\mathrm{d}z\right|$$

$$\leq M(y-x)\left[\frac{q}{[r+1]_q^{\frac{1}{r}}}\left(\frac{u-(\varpi_3+\varpi_4-y)}{y-x}\right)^2+\left(\int_{\frac{u-(\varpi_3+\varpi_4-y)}{y-x}}^{1}(1-q\delta)^r\mathrm{d}_q\delta\right)^{\frac{1}{r}}\left(\frac{(\varpi_3+\varpi_4-x)-u}{y-x}\right)^{\frac{1}{s}}\right].$$

- If we take $u=\varpi_3+\varpi_4-\frac{x+y}{2}$ in Theorem 2.2, then the following bound for general Hermite-Hadamard inequality holds:

$$\left|\Xi(\varpi_3+\varpi_4-\frac{x+y}{2})-\frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x}\Xi(z)\mathrm{d}z\right|$$

$$\le (y-x)\Bigg[\frac{q}{[r+1]_q^{\frac{1}{r}}}\left(\frac{1}{2}\right)^{1+\frac{1}{r}}\Bigg(\frac{1}{2}(|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_3)|^s+|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_4)|^s)$$

$$-\frac{1}{4[2]_q}|_{\varpi_3+\varpi_4-y}D_q\Xi(x)|^s-|_{\varpi_3+\varpi_4-y}D_q\Xi(y)|^s\left(\frac{1}{2}-\frac{1}{4[2]_q}\right)\Bigg)^{\frac{1}{s}}$$

$$+\left(\int_{\frac{1}{2}}^{1}(1-q\delta)^r\,\mathrm{d}_q\delta\right)^{\frac{1}{r}}\Bigg(\frac{1}{2}(|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_3)|^s+|_{\varpi_3+\varpi_4-y}D_q\Xi(\varpi_4))|^s$$

$$-\left(\frac{1}{[2]_q}-\frac{1}{4[2]_q}\right)|_{\varpi_3+\varpi_4-y}D_q\Xi(x)|^s-\left(\frac{q}{[2]_q}-\frac{1}{2}+\frac{1}{4[2]_q}\right)|_{\varpi_3+\varpi_4-y}D_q\Xi(y)|^s\Bigg)^{\frac{1}{s}}\Bigg].$$

Theorem 2.3. *Let $\Xi:[\varpi_3,\varpi_4]\to\mathbb{R}$ be continuous q-differentiable function and if $|^{\varpi_3+\varpi_4-x}D_q\Xi|^r$ is a convex function, where $\varpi_3+\varpi_4-y<\varpi_3+\varpi_4-x$ and $r\ge 1$ then,*

$$\left|\frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x}\Xi(z)\mathrm{d}z-\Xi(u)\right|$$

$$\le (y-x)\Big[w_7^{1-\frac{1}{r}}(\varpi_3,\varpi_4,x,y,q)\big(w_7(\varpi_3,\varpi_4,x,y,q)(|^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_3)|^r+|^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_4)|^r)$$

$$-w_9(\varpi_3,\varpi_4,x,y,q)|^{\varpi_3+\varpi_4-x}D_q\Xi(x)|^r-w_8(\varpi_3,\varpi_4,x,y,q)|^{\varpi_3+\varpi_4-x}D_q\Xi(y)|^r\big)^{\frac{1}{r}}$$

$$+w_{10}^{1-\frac{1}{r}}(\varpi_3,\varpi_4,x,y,q)\big(w_{10}(\varpi_3,\varpi_4,x,y,q)(|^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_3)|^r+|^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_4))|^r$$

$$-w_{12}(\varpi_3,\varpi_4,x,y,q)|^{\varpi_3+\varpi_4-x}D_q\Xi(x)|^r-w_{11}(\varpi_3,\varpi_4,x,y,q)|^{\varpi_3+\varpi_4-x}D_q\Xi(y)|^r\big)^{\frac{1}{r}}\Big],$$

where

$$w_7(\varpi_3,\varpi_4,x,y,q)=\int_0^{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}q\delta\,\mathrm{d}_q\delta=\frac{q}{[2]_q}\left(\frac{(\varpi_3+\varpi_4-x)-u}{y-x}\right)^2.$$

$$w_8(\varpi_3,\varpi_4,x,y,q)=\int_0^{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}q\delta^2\,\mathrm{d}_q\delta=\frac{q}{[3]_q}\left(\frac{(\varpi_3+\varpi_4-x)-u}{y-x}\right)^3.$$

$$w_9(\varpi_3,\varpi_4,x,y,q)=\int_0^{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}q\delta(1-\delta)\,\mathrm{d}_q\delta=w_7(\varpi_3,\varpi_4,x,y,q)-w_8(\varpi_3,\varpi_4,x,y,q).$$

$$w_{10}(\varpi_3,\varpi_4,x,y,q)=\int_{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}^{1}(1-q\delta)\,\mathrm{d}_q\delta$$

$$=\frac{(1-q)}{[2]_q}\left(\frac{u-(\varpi_3+\varpi_4-y)}{y-x}\right)+\frac{q}{[2]_q}\left(\frac{u-(\varpi_3+\varpi_4-y)}{y-x}\right)^2.$$

$$w_{11}(\varpi_3,\varpi_4,x,y,q)=\int_{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}^{1}(1-q\delta)\delta\,\mathrm{d}_q\delta$$

$$=\frac{1}{[2]_q[3]_q}-\frac{q}{[2]_q}\left(\frac{(\varpi_3+\varpi_4-x)-u}{y-x}\right)^2+\frac{q}{[3]_q}\left(\frac{(\varpi_3+\varpi_4-x)-u}{y-x}\right)^3.$$

$$w_{12}(\varpi_3,\varpi_4,x,y,q)=\int_{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}^{1}(1-q\delta)(1-\delta)\,\mathrm{d}_q\delta=w_{10}(\varpi_3,\varpi_4,x,y,q)-w_{11}(\varpi_3,\varpi_4,x,y,q).$$

Proof. By making use of Lemma 2.2, modulus property, power mean inequality and convexity property of $|_{\varpi_3+\varpi_4-y}D_q f|^r$ of the function, then

$$\left|\frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x}\Xi(z)_{\varpi_3+\varpi_4-y}\mathrm{d}_q z-\Xi(u)\right|$$

$$\leq (y-x)\left[\int_0^{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}} q\delta\left|{}^{\varpi_3+\varpi_4-x}D_q\Xi(\delta(\varpi_3+\varpi_4-y)+(1-\delta)(\varpi_3+\varpi_4-x))\right|\mathrm{d}_q\delta\right.$$

$$\left.+\int_{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}^{1}(1-q\delta)\left|{}^{\varpi_3+\varpi_4-x}D_q\Xi(\delta(\varpi_3+\varpi_4-y)+(1-\delta)(\varpi_3+\varpi_4-x))\right|\mathrm{d}_q\delta\right]$$

$$\leq (y-x)\left[\left(\int_0^{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}} q\delta\,\mathrm{d}_q\delta\right)^{1-\frac{1}{r}}\left(\int_0^{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}} q\delta\left|{}^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_3+\varpi_4-(\delta y+(1-\delta)x))\right|^r\mathrm{d}_q\delta\right)^{\frac{1}{r}}\right.$$

$$+\left(\int_{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}^{1}(1-q\delta)\,\mathrm{d}_q\delta\right)^{1-\frac{1}{r}}\times$$

$$\left.\left(\int_{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}^{1}(1-q\delta)\left|{}^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_3+\varpi_4-(\delta y+(1-\delta)x))\right|^r\mathrm{d}_q\delta\right)^{\frac{1}{r}}\right]$$

$$\leq (y-x)\left[\left(\int_0^{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}} q\delta\,\mathrm{d}_q\delta\right)^{1-\frac{1}{r}}\left(\int_0^{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}} q\delta\left(|{}^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_3)|^r+|{}^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_4)|^r\right.\right.\right.$$

$$\left.\left.-(1-\delta)|{}^{\varpi_3+\varpi_4-x}D_q\Xi(x)|^r-\delta|{}^{\varpi_3+\varpi_4-x}D_q\Xi(y)|^r\right)\mathrm{d}_q\delta\right)^{\frac{1}{r}}+\left(\int_{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}^{1}(1-q\delta)\,\mathrm{d}_q\delta\right)^{1-\frac{1}{r}}$$

$$\left(\int_{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}^{1}(1-q\delta)\left(|{}^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_3)|^r+|{}^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_4)|^r\right.\right.$$

$$\left.\left.\left.-\delta|{}^{\varpi_3+\varpi_4-x}D_q\Xi(y)|^r-(1-\delta)|{}^{\varpi_3+\varpi_4-x}D_q\Xi(x)|^r\right)\mathrm{d}_q\delta\right)^{\frac{1}{r}}\right]$$

After simple computations, we obtain our required result. □

Now we discuss some special cases of Theorem 2.3.

- If $|{}^{\varpi_3+\varpi_4-x}D_q\Xi|\leq M$ with $M\geq 0$ and $r=1$, then we obtain the following new Ostrowski-Mercer inequality:

$$\left|\frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}{}^{\varpi_3+\varpi_4-x}\Xi(z)^{\varpi_3+\varpi_4-x}\mathrm{d}_q z-\Xi(u)\right|$$

$$\leq \frac{M}{(y-x)[2]_q}\left[q((\varpi_3+\varpi_4-x)-u)^2+(1-q)(y-x)((\varpi_3+\varpi_4-x)-u)+q(u-(\varpi_3+\varpi_4-x))^2\right].$$

- If we take $r = 1$ in Theorem 2.3, then following bounds for Ostrowski-Mercer inequality hold:

$$\left|\Xi(u) - \frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x}\Xi(z)\mathrm{d}_q z\right|$$
$$\leq (y-x)\left[(w_7(\varpi_3,\varpi_4,x,y,q)+w_{10}(\varpi_3,\varpi_4,x,y,q))\left(|^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_3)|+|^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_4)|\right)\right.$$
$$-(w_9(\varpi_3,\varpi_4,x,y,q)+w_{12}(\varpi_3,\varpi_4,x,y,q))|^{\varpi_3+\varpi_4-x}D_q\Xi(x)|$$
$$\left.-(w_8(\varpi_3,\varpi_4,x,y,q)+w_{11}(\varpi_3,\varpi_4,x,y,q))|^{\varpi_3+\varpi_4-y}D_q\Xi(y)|\right],$$

where $w_i(\varpi_3,\varpi_4,x,y,q), i = 7,8,9,10,11,12$ are defined in Theorem 2.3.
- If we take $u = \varpi_3+\varpi_4-\frac{x+y}{2}$ in Theorem 2.3, then following bound for general Hermite-Hadamad inequality holds:

$$\left|\frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x}\Xi(z)\mathrm{d}_q z - \Xi\left(\varpi_3+\varpi_4-\frac{x+y}{2}\right)\right|$$
$$\leq (y-x)\left[{}_1w_7^{1-\frac{1}{r}}(q)\left({}_1w_7(q)(|^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_3)|^r+|^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_4)|^r)\right.\right.$$
$$\left.-{}_1w_9(q)|^{\varpi_3+\varpi_4-x}D_q\Xi(x)|^r - {}_1w_8(q)|^{\varpi_3+\varpi_4-x}D_q\Xi(y)|^r\right)^{\frac{1}{r}}$$
$$+{}_1w_{10}^{1-\frac{1}{r}}(q)\left({}_1w_{10}(q)(|^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_3)|^r+|^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_4))|^r\right.$$
$$\left.\left.-{}_1w_{12}(q)|^{\varpi_3+\varpi_4-x}D_q\Xi(x)|^r - {}_1w_{11}(q)|^{\varpi_3+\varpi_4-x}D_q\Xi(y)|^r\right)^{\frac{1}{r}}\right],$$

where

$${}_1w_7(q) = \frac{q}{4[2]_q},\quad {}_1w_8(q) = \frac{q}{8[3]_q},\quad {}_1w_9(q) = {}_1w_1(7) - {}_1w_2(8).$$
$${}_1w_{10}(q) = \frac{(1-q)}{2[2]_q}+\frac{q}{4[2]_q}.\quad {}_1w_{11}(q) = \frac{1}{[2]_q[3]_q}-\frac{q}{4[2]_q}+\frac{q}{8[3]_q}.$$
$${}_1w_{12}(q) = {}_1w_{10}(q) - {}_1w_{11}(q).$$

Theorem 2.4. *Let $\Xi : [\varpi_3,\varpi_4] \to \mathbb{R}$ be continuous q-differentiable function and if $|^{\varpi_3+\varpi_4-x}D_q\Xi|^s$ is a convex function, where $\varpi_3+\varpi_4-y < \varpi_3+\varpi_4-x$ and $r, s > 1$ with $\frac{1}{r}+\frac{1}{s} = 1$ then,*

$$\left|\frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x}\Xi(z)\mathrm{d}z - \Xi(u)\right|$$
$$\leq (y-x)\left[\left(\frac{q}{[r+1]_q}\right)^{\frac{1}{r}}\left(\frac{(\varpi_3+\varpi_4-x)-u}{y-x}\right)^{1+\frac{1}{r}}\left(\frac{(\varpi_3+\varpi_4-x)-u}{y-x}(|^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_3)|^s\right.\right.$$
$$+|^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_4)|^s) - \frac{1}{[2]_q}\left(\frac{(\varpi_3+\varpi_4-x)-u}{y-x}\right)^2|^{\varpi_3+\varpi_4-x}D_q\Xi(y)|^s$$
$$\left.-|^{\varpi_3+\varpi_4-x}D_q\Xi(x)|^s\left(\frac{(\varpi_3+\varpi_4-x)-u}{y-x}-\frac{1}{[2]_q}\left(\left(\frac{(\varpi_3+\varpi_4-x)-u}{y-x}\right)^2\right)\right)\right)^{\frac{1}{s}}$$
$$+\left(\int_{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}^{1}(1-q\delta)^r\mathrm{d}_q\delta\right)^{\frac{1}{r}}\left(\frac{(\varpi_3+\varpi_4-x)-u}{y-x}(|^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_3)|^s+|^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_4))|^s\right.$$
$$-\left(\frac{1}{[2]_q}-\frac{1}{[2]_q}\left(\frac{(\varpi_3+\varpi_4-x)-u}{y-x}\right)^2\right)|^{\varpi_3+\varpi_4-x}D_q\Xi(y)|^s$$
$$\left.\left.-\left(\frac{q}{[2]_q}-\frac{(\varpi_3+\varpi_4-x)-u}{y-x}+\frac{1}{[2]_q}\left(\frac{(\varpi_3+\varpi_4-x)-u}{y-x}\right)^2\right)|^{\varpi_3+\varpi_4-x}D_q\Xi(x)|^s\right)^{\frac{1}{s}}\right].$$

Proof. By making use of Lemma 2.2, modulus property, Hölder's inequality and convexity property of $|_{\varpi_3+\varpi_4-y}D_q f|^r$ of the function, then

$$\left|\frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x}\Xi(z)_{\varpi_3+\varpi_4-y}\mathrm{d}_q z - \Xi(u)\right|$$

$$\leq (y-x)\left[\int_0^{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}} q\delta \left|{}^{\varpi_3+\varpi_4-x}D_q\Xi(\delta(\varpi_3+\varpi_4-y)+(1-\delta)(\varpi_3+\varpi_4-x))\right| \mathrm{d}_q\delta\right.$$

$$\left.+\int_{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}^{1} (1-q\delta) \left|{}^{\varpi_3+\varpi_4-x}D_q\Xi(\delta(\varpi_3+\varpi_4-y)+(1-\delta)(\varpi_3+\varpi_4-x))\right| \mathrm{d}_q\delta\right]$$

$$\leq (y-x)\left[\left(\int_0^{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}} (q\delta)^r \mathrm{d}_q\delta\right)^{\frac{1}{r}} \times\right.$$

$$\left(\int_0^{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}} \left|{}^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_3+\varpi_4-((1-\delta)x+\delta y))\right|^s \mathrm{d}_q\delta\right)^{\frac{1}{s}}$$

$$+\left(\int_{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}^{1} (1-q\delta)^r \mathrm{d}_q\delta\right)^{\frac{1}{r}} \times$$

$$\left.\left(\int_{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}^{1} \left|{}^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_3+\varpi_4-((1-\delta)x+\delta y))\right|^s \mathrm{d}_q\delta\right)^{\frac{1}{s}}\right]$$

$$\leq (y-x)\left[\left(\int_0^{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}} (q\delta)^r \mathrm{d}_q\delta\right)^{\frac{1}{r}} \left(\int_0^{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}} \left(|{}^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_3)|^s+|{}^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_4)|^s\right.\right.\right.$$

$$\left.\left.-(1-\delta)|{}^{\varpi_3+\varpi_4-x}D_q\Xi(x)|^s-\delta|{}^{\varpi_3+\varpi_4-x}D_q\Xi(y)|^s\right) \mathrm{d}_q\delta\right)^{\frac{1}{r}}+\left(\int_{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}^{1} (1-q\delta)^r \mathrm{d}_q\delta\right)^{\frac{1}{r}}$$

$$\times\left(\int_{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}^{1} (|{}^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_3)|^s+|{}^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_4)|^s\right.$$

$$\left.\left.-(1-\delta)|{}^{\varpi_3+\varpi_4-x}D_q\Xi(x)|^s-\delta|{}^{\varpi_3+\varpi_4-x}D_q\Xi(y)|^s)\mathrm{d}_q\delta\right)^{\frac{1}{s}}\right].$$

After simplifying, we acquire our required result. □

Now we discuss some special cases of Theorem 2.4.

- If $|{}^{\varpi_3+\varpi_4-x}D_q\Xi| \leq M$ with $M>0$, then we obtain the following new Ostrowski-Mercer inequality:

$$\left|\frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x} \Xi(z)\,{}^{\varpi_3+\varpi_4-x}\mathrm{d}z-\Xi(u)\right|$$

$$\leq M(y-x)\left[\frac{q}{[r+1]_q^{\frac{1}{r}}}\left(\frac{(\varpi_3+\varpi_4-x)-u}{y-x}\right)^2+\left(\int_{\frac{(\varpi_3+\varpi_4-x)-u}{y-x}}^{1}(1-q\delta)^r\mathrm{d}_q\delta\right)^{\frac{1}{r}}\left(\frac{u-(\varpi_3+\varpi_4-y)}{y-x}\right)^{\frac{1}{s}}\right].$$

- If we take $u=\varpi_3+\varpi_4-\frac{x+y}{2}$ in Theorem 2.4, then the following bound for general Hermite-Hadamard inequality holds:

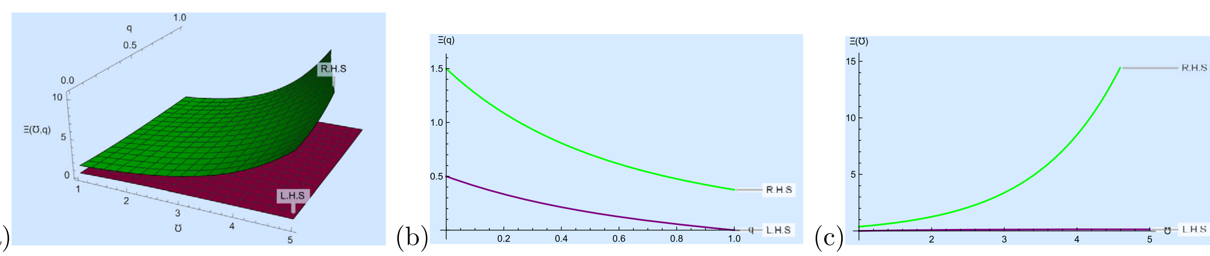
$$\left|\frac{1}{y-x}\int_{\varpi_3+\varpi_4-y}^{\varpi_3+\varpi_4-x}\Xi(z)\mathrm{d}z-\Xi(\varpi_3+\varpi_4-\frac{x+y}{2})\right|$$
$$\leq(y-x)\left[\frac{q}{[r+1]_q}^{\frac{1}{r}}\left(\frac{1}{2}\right)^{1+\frac{1}{r}}\left(\frac{1}{2}(|^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_3)|^s+|^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_4)|^s)\right.\right.$$
$$\left.-\frac{1}{4[2]_q}|^{\varpi_3+\varpi_4-x}D_q\Xi(y)|^s-|^{\varpi_3+\varpi_4-x}D_q\Xi(x)|^s\left(\frac{1}{2}-\frac{1}{4[2]_q}\right)\right)^{\frac{1}{s}}$$
$$+\left(\int_{\frac{1}{2}}^{1}(1-q\delta)^r\mathrm{d}_q\delta\right)^{\frac{1}{r}}\left(\frac{1}{2}(|^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_3)|^s+|^{\varpi_3+\varpi_4-x}D_q\Xi(\varpi_4))|^s\right.$$
$$\left.\left.-\left(\frac{1}{[2]_q}-\frac{1}{4[2]_q}\right)|^{\varpi_3+\varpi_4-x}D_q\Xi(y)|^s-\left(\frac{q}{[2]_q}-\frac{1}{2}+\frac{1}{4[2]_q}\right)|^{\varpi_3+\varpi_4-x}D_q\Xi(x)|^s\right)^{\frac{1}{s}}\right].$$

3. Numerical examples and graphical analysis

In this section, we present our primary results through numerical examples and graphical representations.

- Assuming Theorem 2.1, if we set $u=\varpi_3+\varpi_4-\frac{x+y}{2}$, $r=1$, $\Xi(m)=m^{\mho}$, and $\varpi_3=1$, $\varpi_4=4$, $x=2$, and $y=3$, then:

$$\left|\left(\frac{1}{2}\right)^{\mho}-\frac{1}{[\mho]_q}\right|\leq[\mho]_q\left(\frac{(|(-1)^{n-1}+2^{n-1}|)}{2}-\frac{1}{2(1+q)}+\frac{8-2q^3}{8(1+q)(1+q+q^2)}\right).$$

(a) 

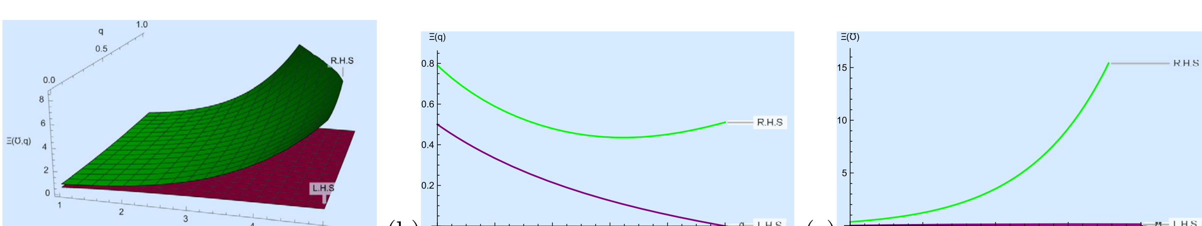
- For Fig. (a), we take $q\in(0,1)$ and $\mho\in[1,5]$ as variables to illustrate a graph between left and right hand side of Theorem 2.1.
- For Fig. (b), we take $q\in(0,1)$ as variables to illustrate a graph between left and right hand side of Theorem 2.1.
- For Fig. (c), we take $q\in(0,1)$ and $\mho\in[1,5]$ as variables to illustrate a graph between left and right hand side of Theorem 2.1.

If we take $q=\frac{1}{2}$, then we have

$$0.16667<0.702381.$$

- Under the assumption of Theorem 2.2, if we take $u=\varpi_3+\varpi_4-\frac{x+y}{2}, r=s=2$ and $\Xi(m)=m^2$ with $\varpi_3=1,\varpi_4=4,x=2$ and $y=3$, then

$$\left|\left(\frac{1}{2}\right)^{\mho}-\frac{1}{[\mho]_q}\right|\leq\frac{0.35q}{\sqrt{1+q+q^2}}\sqrt{\frac{1+2^{2n-2}}{2}-\frac{(1+2q)}{4(1+q)}}$$
$$+\sqrt{\frac{2-q+q^2}{4(1+q)}-\frac{8q+q^2+q^3}{8(1+q)(1+q+q^2)}}\times\sqrt{\frac{1+2^{2n-2}}{2}-\frac{(2q-1)}{4(1+q)}}.$$

(a) 

- For Fig. (a), we take $q\in(0,1)$ and $\mho\in[1,5]$ as variables to illustrate a graph between left and right hand side of Theorem 2.2.

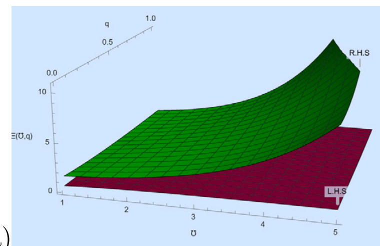
- For Fig. (b), we take $q \in (0,1)$ as variables to illustrate a graph between left and right hand side of Theorem 2.2.
- For Fig. (c), we take $q \in (0,1)$ and $\mho \in [1,5]$ as variables to illustrate a graph between left and right hand side of Theorem 2.2.

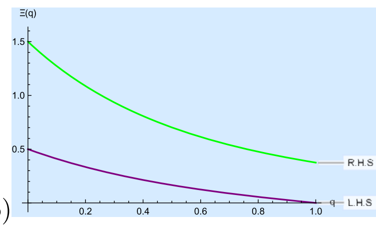
If we take $q = \frac{1}{2}$, then we have

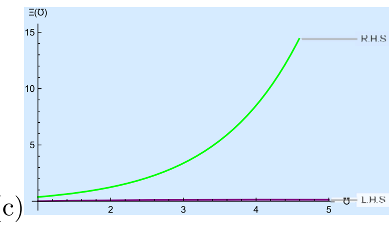
$$0.16667 < 0.450694.$$

- Under the assumption of Theorem 2.3, if we take $u = \varpi_3 + \varpi_4 - \frac{x+y}{2}, r = s = 2$ and $\Xi(m) = m^{\mho}$ with $\varpi_3 = 1, \varpi_4 = 4, x = 2$ and $y = 3$, then

$$\left|\frac{(-1)^{\mho+2}}{[\mho+1]_q} - \left(\frac{-1}{2}\right)^{\mho}\right| \leq [\mho]_q \left(\frac{1 + |(-2)^{2\mho-2}|}{2(1+q)} - |(-1)^{2\mho-2}| \left(\frac{1}{2(1+q)} - \frac{8-2q^3}{8(1+q)(1+q+q^2)}\right)\right).$$

(a) 

(b) 

(c) 

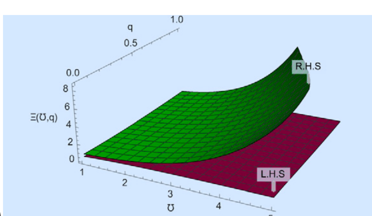
- For Fig. (a), we take $q \in (0,1)$ and $\mho \in [1,5]$ as variables to illustrate a graph between left and right hand side of Theorem 2.3.
- For Fig. (b), we take $q \in (0,1)$ as variables to illustrate a graph between left and right hand side of Theorem 2.3.
- For Fig. (c), we take $q \in (0,1)$ and $\mho \in [1,5]$ as variables to illustrate a graph between left and right hand side of Theorem 2.3.

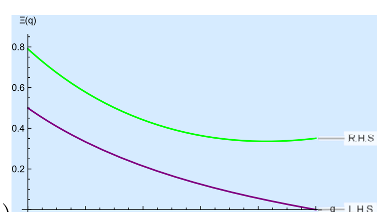
If we take $q = \frac{1}{2}$, then we have

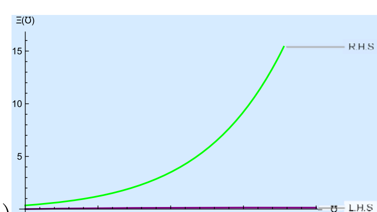
$$0.16667 < 0.702381.$$

- Under the assumption of Theorem 2.4, if we take $u = \varpi_3 + \varpi_4 - \frac{x+y}{2}, r = s = 2$ and $\Xi(m) = m^{\mho}$ with $\varpi_3 = 1, \varpi_4 = 4, x = 2$ and $y = 3$, then

$$\begin{aligned}
&\left|\frac{(-1)^{\mho+2}}{[\mho+1]_q} - \left(\frac{-1}{2}\right)^{\mho}\right| \\
&\leq \frac{0.35q[\mho]_q}{\sqrt{1+q+q^2}} \sqrt{\frac{1+|(-2)^{2n-2}|}{2} - |(-1)^{2n-2}| \frac{(1+2q)}{4(1+q)}} \\
&\quad + [\mho]_q \sqrt{\frac{2-q+q^2}{4(1+q)} - \frac{8q+q^2+q^3}{8(1+q)(1+q+q^2)}} \times \sqrt{\frac{1+|(-2)^{2n-2}|}{2} - |(-1)^{2n-2}| \frac{(2q-1)}{4(1+q)}}.
\end{aligned}$$

(a) 

(b) 

(c) 

- For Fig. (a), we take $q \in (0,1)$ and $\mho \in [1,5]$ as variables to illustrate a graph between left and right hand side of Theorem 2.4.
- For Fig. (b), we take $q \in (0,1)$ as variables to illustrate a graph between left and right hand side of Theorem 2.4.
- For Fig. (c), we take $q \in (0,1)$ and $\mho \in [1,5]$ as variables to illustrate a graph between left and right hand side of Theorem 2.4.

If we take $q = \frac{1}{2}$, then we have

$$0.16667 < 0.396687.$$

4. Conclusion

The analysis of these integral inequalities using quantum calculus is an essential topic of research today due to its wide range of applications. The q-analogues of various integral inequalities provide us with error estimates for well-known quantum quadrature rules such as Ostrowski, Simpson, Newton, Maclaurin, and Milne type rules. Many researchers have attempted to find precise bounds involving the Montgomery identity, Hermite-Hadamard inequality, Fink identity, Green functions, Taylor series, and other mathematical concepts. In this paper, we have obtained two new q-Montgomery identities and derived new upper bounds for the general Ostrowski and Hermite-Hadamard inequalities by using the convexity property of the functions and quantum calculus approach. These findings will aid in proving numerous mathematical inequalities involving different classes of convexity, bounding properties of functions, Lipschitz mappings, and bounded variation. The present study is the first attempt regarding the quantum Montgomery identities, and it is hoped that these identities will open new avenues for further research.

Declaration of competing interest

The authors declare that they have no competing interests.

Acknowledgement

The research work of Jong-Suk Ro is supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. NRF-2022R1A2C2004874) and by the Korea Institute of Energy Technology Evaluation and Planning (KETEP) and the Ministry of Trade, Industry and Energy (MOTIE) of the Republic of Korea (No. 20214000000280). The authors are also thankful to the editor and anonymous reviewers for their valuable comments and suggestions.

References

[1] Pavic Z. Geometric and analytic connections of the Jensen and Hermite-Hadamard inequality. Math Sci Appl E-Notes 2016;4(1):69–76.
[2] Mercer AM. A variant of Jensen's inequality. JIPAM J Inequal Pure Appl Math 2003;4(4):73.
[3] Khan MA, Khan GA, Jameel M, Khan KA, Kilicman A. New refinements of Jensen-Mercer's inequality. J Comput Theor Nanosci 2015;12(11):4442–9.
[4] Matkovic A, Pecaric J, Peric I. A variant of Jensen's inequality of Mercer's type for operators with applications. Linear Algebra Appl 2006;418(2–3):551–64.
[5] Tariboon J, Ntouyas SK. Quantum calculus on finite intervals and applications to impulsive difference equations. Adv Differ Equ 2013;2013:1–9.
[6] Bermudo S, Korus P, Napoles Valdes JN. On q-Hermite-Hadamard inequalities for general convex functions. Acta Math Hung 2020;162:364–74.
[7] Noor MA, Awan MU, Noor KI. Quantum Ostrowski inequalities for q-differentiable convex functions. J Math Inequal 2016;10(4):1013–8.
[8] Alomari M, Darus M, Dragomir SS, Cerone P. Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense. Appl Math Lett 2010;23(9):1071–6.
[9] Basci Y, Baleanu D. Ostrowski type inequalities involving ψ-Hilfer fractional integrals. Mathematics 2019;7(9):770.
[10] Yildiz C, Cotirla LI. Examining the Hermite-Hadamard inequalities for k-fractional operators using the Green function. Fractal Fract 2023;7(2):161.
[11] Zaheer Ullah S, Khan MA, Khan ZA, Chu YM. Integral majorization type inequalities for the functions in the sense of strong convexity. J Funct Spaces 2019;2019.
[12] Al Qurashi M, Rashid S, Khalid A, Karaca Y, Chu YM. New computations of Ostrowski-type inequality pertaining to fractal style with applications. Fractals 2021;29(05):2140026.
[13] Kashuri A, Meftah B, Mohammed PO, Lupas AA, Abdalla B, Hamed YS, Abdeljawad T. Fractional weighted Ostrowski-type inequalities and their applications. Symmetry 2021;13(6):968.
[14] Wang X, Khan KA, Ditta A, Nosheen A, Awan KM, Mabela RM. New developments on Ostrowski type inequalities via-fractional integrals involving-convex functions. J Funct Spaces 2022;2022.
[15] Ogulmus H, Sarikaya ZM. Hermite-Hadamard-Mercer type inequalities for fractional integrals. Filomat 2021;35(7):2425–36.
[16] Du T, Zhou T. On the fractional double integral inclusion relations having exponential kernels via interval-valued co-ordinated convex mappings. Chaos Solitons Fractals 2022;156:111846.
[17] Budak H, Kara H, Kapucu R. New midpoint type inequalities for generalized fractional integral. Comput Methods Differ Equ 2022;10(1):93–108.
[18] Zhang Y, Du TS, Wang H, Shen YJ. Different types of quantum integral inequalities via (α, m) convexity. J Inequal Appl 2018;2018(1):264.
[19] Du T, Peng Y. Hermite-Hadamard type inequalities for multiplicative Riemann-Liouville fractional integrals. J Comput Appl Math 2023:115582.
[20] İscan I. Weighted Hermite-Hadamard-Mercer type inequalities for convex functions. Numer Methods Partial Differ Equ 2021;37(1):118–30.
[21] You X, Ali MA, Budak H, Reunsumrit J, Sitthiwirattham T. Hermite-Hadamard-Mercer-type inequalities for harmonically convex mappings. Mathematics 2021;9(20):2556.
[22] Vivas-Cortez M, Awan MU, Javed MZ, Kashuri A, Noor MA, Noor KI. Some new generalized k-fractional Hermite-Hadamard-Mercer type integral inequalities and their applications. AIMS Math 2022;7:3203–20.
[23] Faisal S, Khan MA, Iqbal S. Generalized Hermite-Hadamard-Mercer type inequalities via majorization. Filomat 2022;36(2):469–83.
[24] Bin-Mohsin B, Javed MZ, Awan MU, Mihai MV, Budak H, Khan AG, et al. Type inequalities in the setting of fractional calculus with applications. Symmetry 2022;14(10):2187.
[25] Butt SI, Nosheen A, Nasir J, Khan KA, Matendo Mabela R. New fractional Mercer-Ostrowski type inequalities with respect to monotone function. Math Probl Eng 2022:2022.
[26] Sial IB, Patanarapeelert N, Ali MA, Budak H, Sitthiwirattham T. On some new Ostrowski-Mercer-type inequalities for differentiable functions. Axioms 2022;11(3):132.
[27] Abdeljawad T, Ali MA, Mohammed PO, Kashuri A. On inequalities of Hermite-Hadamard-Mercer type involving Riemann-Liouville fractional integrals. AIMS Math 2020;5:7316–31.
[28] Dragomir SS, Rassias TM, editors. Ostrowski type inequalities and applications in numerical integration. Dordrecht: Kluwer Academic; 2002.
[29] Akhtar N, Awan MU, Javed MZ, Rassias MT, Mihai MV, Noor MA, et al. Ostrowski type inequalities involving harmonically convex functions and applications. Symmetry 2021;13(2):201.
[30] Mohsen BB, Awan MU, Javed MZ, Noor MA, Noor KI. Some new Ostrowski-type inequalities involving σ-fractional integrals. J Math 2021;(2021):1–2.
[31] Kunt M, Kashuri A, Du T, Baidar AW. Quantum Montgomery identity and quantum estimates of Ostrowski type inequalities. AIMS Math 2020;5(6):5439–57.
[32] Sitthiwirattham T, Ali MA, Budak H, Abbas M, Chasreechai S. Montgomery identity and Ostrowski-type inequalities via quantum calculus. Open Math 2021;19(1):1098–109.
[33] Butt SI, Khan KA, Pecaric J. Popoviciu type inequalities via Green function and generalized Montgomery identity. Math Inequal Appl 2015;18(4):1519–38.
[34] Aglic Aljinovic A. Montgomery identity and Ostrowski type inequalities for Riemann-Liouville fractional integral. J Math 2014;2014.
[35] Vivas-Cortez M, Kashuri A, Liko R, Hernandez JE. Some new q-integral inequalities using generalized quantum Montgomery identity via preinvex functions. Symmetry 2020;12(4):553.
[36] Ali MA, Chu YM, Budak H, Akkurt A, Yildirim H, Zahid MA. Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables. Adv Differ Equ 2021;2021:1–26.
[37] Mehmood N, Agarwal RP, Butt SI, Pecaric J. New generalizations of Popoviciu-type inequalities via new Green's functions and Montgomery identity. J Inequal Appl 2017;2017:1–7.
[38] Kalsoom HU, Ali MA, Abbas MU, Budak HU, Murtaza GH. Generalized quantum Montgomery identity and Ostrowski type inequalities for preinvex functions. TWMS J Pure Appl Math 2022;13(1):72–90.
[39] Kalsoom H, Vivas-Cortez M, Abidin MZ, Marwan M, Khan ZA. Montgomery identity and Ostrowski-type inequalities for generalized quantum calculus through convexity and their applications. Symmetry 2022;14(7):1449.
[40] Chu YM, Awan MU, Talib S, Noor MA, Noor KI. New post quantum analogues of Ostrowski-type inequalities using new definitions of left-right (p, q)-derivatives and definite integrals. Adv Differ Equ 2020;2020(1):634.
[41] Ali MA, Budak H, Akkurt A, Chu YM. Quantum Ostrowski-type inequalities for twice quantum differentiable functions in quantum calculus. Open Math 2021;19(1):440–9.
[42] Khan KA, Niaz T, Pecaric D, Peccaric J. Refinement of Jensen's inequality and estimation of f- and Renyi divergence via Montgomery identity. J Inequal Appl 2018;2018:1–22.
[43] Bin-Mohsin B, Saba M, Javed MZ, Awan MU, Budak H, Nonlaopon K. A quantum calculus view of Hermite-Hadamard-Jensen-Mercer inequalities with applications. Symmetry 2022;14(6):1246.
[44] Bin-Mohsin B, Javed MZ, Awan MU, Budak H, Kara H, Noor MA. Quantum integral inequalities in the setting of majorization theory and applications. Symmetry 2022;14(9):1925.