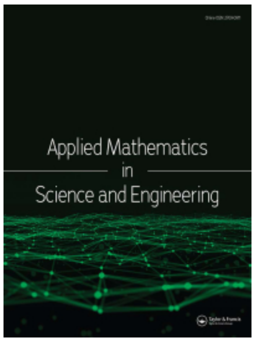

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Strongly $(g,h;\alpha - m)$-convex functions and the consequent Hermite–Hadamard-type inequalities

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ABSTRACT
In this paper, we define a new class of strongly $(g, h; \alpha - m)$-convex functions. Some important implications are listed and related with already known classes. Hermite–Hadamard-type inequalities are established for this new class of functions. Several particular cases are analysed. All the inequalities are established for Riemann–Liouville fractional integrals, and these are generalizations of ordinary integral inequalities.

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1. Introduction

First, we give some recently published results which were given by using exponentially $(\alpha, h\text{-}m)$-p-convexity defined in [1]. In [2], authors have proved inequalities for Riemann–Liouville (R-L) integrals which provide refinements of various fractional inequalities hold for different types of convexities. This paper gives a compact generalization of strongly convex functions (see Definition 2.1) by using a real-valued function, which is applied to construct R-L fractional integral inequalities of Hermite–Hadamard type. Convex functions, p-convex functions, exponentially convex functions, strongly convex functions of different kinds are unanimously included in the following definition.

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Definition 1.1: Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h: J \to \mathbb{R}$ be a non-negative function. Let $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R} \setminus \{0\}$. A function $f: I \to \mathbb{R}$ is said to be exponentially $(\alpha, h\text{-}m)$-p-convex, if

$$f\left(\left(ta^p + m(1-t)b^p\right)^{\frac{1}{p}}\right) \leq \frac{h(t^\alpha)f(a)}{e^{\zeta a}} + \frac{mh(1-t^\alpha)f(b)}{e^{\zeta b}} \tag{1}$$

holds provided $((ta^p + m(1-t)b^p)^{\frac{1}{p}}) \in I$ for $t \in (0,1)$, $\zeta \in \mathbb{R}$ and $(\alpha, m) \in [0,1]^2$.

The left and right Riemann–Liouville (R-L) fractional integrals are denoted by $I^\tau_{a^+}f$ and $I^\tau_{b^-}f$ respectively and represented in terms of equations by

$$I^\tau_{a^+}f(x) = \int_a^x \frac{f(t)}{\Gamma(\tau)(x-t)^{1-\tau}}\,\mathrm{d}t, \quad x > a, \tag{2}$$

$$I^\tau_{b^-}f(x) = \int_x^b \frac{f(t)}{\Gamma(\tau)(t-x)^{1-\tau}}\,\mathrm{d}\rho, \quad x < b, \tag{3}$$

where $\Gamma(.)$ is the gamma function and $f \in L_1[a,b]$.

By utilizing the above definitions of (R-L) fractional integrals and generalized convexity, the following results give two variants of the well-known Hermite–Hadamard inequality for (R-L) fractional integrals.

Theorem 1.1 ([2]). *Let $\psi : [a,b] \to \mathbb{R}$, $0 \leq a < bm$, $m \in (0,1]$ be strongly exponentially $(\alpha, h-m)$-convex function with modulus $C \geq 0$. If $h(x+y) \leq h(x)h(y)$, then we have the following inequality for (R-L) fractional integrals:*

$$\begin{aligned}
&\frac{1}{g(\eta)}\psi\left(\frac{bm+a}{2}\right) + \frac{mC\beta h(1)}{e^{2\eta b}(\beta+2)}\left\{(b-a)^2 + \frac{2(b-a)\left(\frac{a}{m}-mb\right)}{(\beta+1)}\right.\\
&\left.+\frac{2\left(\frac{a}{m}-mb\right)^2}{\beta(\beta+1)}\right\} \leq \frac{\Gamma(\beta+1)}{(bm-a)^\beta}\left[H\left(\frac{1}{2}\right)m^{\beta+1}I^\beta_{b^-}\psi\left(\frac{a}{m}\right)\right.\\
&\left.+h\left(\frac{1}{2^\alpha}\right)I^\beta_{a^+}\psi(mb)\right] \leq \beta\left\{\left[H\left(\frac{1}{2}\right)\frac{m\psi(b)}{e^{\eta b}} + h\left(\frac{1}{2^\alpha}\right)\frac{\psi(a)}{e^{\eta a}}\right]\right.\\
&\times\int_0^1 h(z^\alpha)z^{\beta-1}\,\mathrm{d}z + \left[H\left(\frac{1}{2}\right)\frac{m^2\psi\left(\frac{a}{m^2}\right)}{e^{\frac{\eta a}{m^2}}} + h\left(\frac{1}{2^\alpha}\right)\frac{m\psi(b)}{e^{\eta b}}\right]\\
&\left.\times\int_0^1 H(z)z^{\beta-1}\,\mathrm{d}z - \frac{Cmh(1)\left[(b-a)^2 + m\left(b-\frac{a}{m^2}\right)^2\right]}{\beta\left[e^{\eta(a+b)} + e^{\eta\left(b+\frac{a}{m^2}\right)}\right]}\right\},
\end{aligned} \tag{4}$$

with $\beta > 0$, where $H(z) = h(1-z^\alpha)$, $g(\eta) = \frac{1}{e^{\eta b}}$ for $\eta < 0$ and $g(\eta) = \frac{1}{e^{\eta a}}$ for $\eta \geq 0$.

Other variant of above inequality is stated in the following theorem.

Theorem 1.2. *Under the suppositions of Theorem* 1.1, *we have*

$$\frac{1}{g(\eta)}\psi\left(\frac{bm+a}{2}\right)+\frac{mC\beta h(1)}{2(\beta+2)\,\mathrm{e}^{2\eta b}}\left\{\frac{(b-a)^2}{2}+\frac{(b-a)\left(\frac{a}{m}-mb\right)(\beta+3)}{(\beta+1)}\right.$$
$$\left.+\frac{\left(\frac{a}{m}-mb\right)^2[\beta^2+5\beta+8]}{2\beta(\beta+1)}\right\}\leq\frac{2^\beta\Gamma(\beta+1)}{(bm-a)^\beta}\left[H\left(\frac{1}{2}\right)m^{\beta+1}I^\beta_{\left(\frac{a+bm}{2m}\right)^-}\psi\left(\frac{a}{m}\right)\right.$$
$$\left.+h\left(\frac{1}{2^\alpha}\right)I^\beta_{\left(\frac{a+bm}{2}\right)^+}\psi(mb)\right]\leq\beta\left\{\left[H\left(\frac{1}{2}\right)\frac{m^2\psi\left(\frac{a}{m^2}\right)}{\mathrm{e}^{\frac{\eta a}{m^2}}}+h\left(\frac{1}{2^\alpha}\right)\frac{m\psi(b)}{\mathrm{e}^{\eta b}}\right]\right.$$
$$\times\int_0^1 H\left(\frac{z}{2}\right)z^{\beta-1}\,\mathrm{d}z+\left[H\left(\frac{1}{2}\right)\frac{m\psi(b)}{\mathrm{e}^{\eta b}}+h\left(\frac{1}{2^\alpha}\right)\frac{\psi(a)}{\mathrm{e}^{\eta a}}\right]$$
$$\left.\times\int_0^1 h\left(\frac{z}{2}\right)^\alpha z^{\beta-1}\,\mathrm{d}z-\frac{mCh(1)\left((b-a)^2+m\left(b-\frac{a}{m^2}\right)^2\right)}{\beta\left[\mathrm{e}^{\eta(a+b)}+\mathrm{e}^{\eta\left(b+\frac{a}{m^2}\right)}\right]}\right\},\tag{5}$$

with $\beta>0$, *where* $g(\eta)=\frac{1}{\mathrm{e}^{\eta b}}$ *for* $\eta<0$ *and* $g(\eta)=\frac{1}{\mathrm{e}^{\eta a}}$ *for* $\eta\geq 0$.

In [1], the generalizations of above inequalities were also proved. Motivation in establishing the above inequalities is the following well-known Hermite–Hadamard inequality:

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\leq\frac{f(a)+f(b)}{2}\tag{6}$$

where f is convex function on $[a,b]$.

A convex function on $[a,b]$ satisfies the inequality: $f(tx+(1-t)y)\leq tf(x)+(1-t)f(y); t\in[0,1], x,y\in[a,b]$. The inequalities stated in above are actually generalizations/extensions of the Hermite–Hadamard inequality given in (6). Trend of exploring the Hermite–Hadamard inequality is very common, that is why a lot of articles and books have been published on it, see [1,3–6].

In the upcoming section, we define a new class of functions and its consequent definitions.

2. Auxiliary definitions

First, we are interested to give the definition of strongly $(g,h;\alpha-m)$-convex function and its consequences.

Definition 2.1: Let h be a non-negative function on $J\subset\mathbb{R}$, $(0,1)\subset J$, $h\neq 0$ and let g be a positive function on $I\subset\mathbb{R}$. A function $f:I\to\mathbb{R}$ is said to be strongly $(g,h;\alpha-m)$-convex if it is non-negative and satisfy the following inequality:

$$f\left(\lambda x+m(1-\lambda)y\right)\leq h(\lambda^\alpha)f.g(x)+mh(1-\lambda^\alpha)f.g(y)$$

$$- cmh(\lambda^{\alpha})h(1-\lambda^{\alpha})g(x)g(y)|y-x|^2, \tag{7}$$

where $\lambda \in [0,1]$, $x, y \in I$.

If we set $g(x) = \exp(-\eta x)$, $\eta \in \mathbb{R}$ in (7), we can the following inequality:

$$f\left(\lambda x + m(1-\lambda)y\right) \leq \frac{h(\lambda^{\alpha})f(x)}{e^{\eta x}} + \frac{mh(1-\lambda^{\alpha})f(y)}{e^{\eta y}} - \frac{cm}{e^{\eta(x+y)}}h(\lambda^{\alpha})h(1-\lambda^{\alpha})|y-x|^2. \tag{8}$$

A function satisfying the inequality (8) is called strongly exponentially $(\alpha, h-m)$-convex function, see [7, Definition 9]. By setting $\eta = 0$ in (8), the definition of strongly $(\alpha, h-m)$-convex function is obtained, while $c = 0$ in (8) gives the $(\alpha, h-m)$-convexity. Consequently, one can get several kinds of strongly convexities as well as exponential convexities. Next, we define a new class of functions for a strictly monotone continuous function F. We set $x = F(u)$ and $y = F(v)$ in (7) and obtain the following inequality:

$$\begin{aligned} f\left(\lambda F(u) + m(1-\lambda)F(v)\right) &\leq h(\lambda^{\alpha})f(F(u))g(F(u)) + mh(1-\lambda^{\alpha}) \\ & f(F(v))g(F(v)) - cmg(F(u))g(F(v))h(\lambda^{\alpha})h(1-\lambda^{\alpha})|F(v)-F(u)|^2. \end{aligned}$$

Now, replacing f with $f(F^{-1})$ and g with $g(F^{-1})$, in the above inequality we get

$$\begin{aligned} f\left(F^{-1}\left(\lambda F(u) + m(1-\lambda)F(v)\right)\right) &\leq h(\lambda^{\alpha})f(u)g(v) + mh(1-\lambda^{\alpha})f(v)g(v) \\ & - g(u)g(v)cmh(\lambda^{\alpha})h(1-\lambda^{\alpha})|v-u|^2. \end{aligned} \tag{9}$$

A function satisfying the inequality (9) will be called strongly quasi F-$(g, h; \alpha - m)$-convex function. By setting $F(u) = u^p$, $p \neq 0$ in (9), we get the following inequality:

$$\begin{aligned} f\left(\left(\lambda u^p + m(1-\lambda)v^p\right)^{\frac{1}{p}}\right) &\leq h(\lambda^{\alpha})f(u)g(v) + mh(1-\lambda^{\alpha})f(v)g(v) \\ & - g(u)g(v)cmh(\lambda^{\alpha})h(1-\lambda^{\alpha})|v-u|^2. \end{aligned} \tag{10}$$

A function satisfying the inequality (10) will be called strongly $(g, h; \alpha - m) - p$-convex function. If $g(x) = \exp(-\eta x)$, $\eta \in \mathbb{R}$ in (10), we can have the following inequality:

$$\begin{aligned} f\left(\left(\lambda u^p + m(1-\lambda)v^p\right)^{\frac{1}{p}}\right) &\leq \frac{h(\lambda^{\alpha})f(u)}{e^{\eta u}} + \frac{mh(1-\lambda^{\alpha})f(v)}{e^{\eta y}} \\ & - \frac{cm}{e^{\eta(u+v)}}h(\lambda^{\alpha})h(1-\lambda^{\alpha})|v-u|^2. \end{aligned} \tag{11}$$

A function satisfying the inequality (11) is called strongly exponentially $(\alpha, h-m) - p$-convex function [8]. In the forthcoming section by applying inequality (7) along with definitions of Riemann–Liouville fractional integrals, two variants of Hermite–Hadamard inequality are presented. Some new inequalities are deduced as applications of these variants.

3. Riemann–Liouville integral inequalities of Hermite–Hadamard type

First, we state and prove the following Hermite–Hadamard-type inequality for strongly $(g, h; \alpha - m)$-convex function.

Theorem 3.1. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be strongly $(g,h;\alpha-m)$-convex function as defined in Definition 2.1. Also, let g satisfies the condition $g(x) = g\left(\frac{mb+a-x}{m}\right)$. Then the following inequality holds:*

$$\begin{aligned}
&f\left(\frac{a+mb}{2}\right) + mch\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right)\left(mb-\frac{a}{m}\right)^2 \\
&\quad\times\left(\frac{\Gamma(\tau+1)I^\tau_{a^+}g^2(mb)}{(mb-a)^\tau} - \frac{2\Gamma(\tau+2)I^{\tau+1}_{a^+}g^2(mb)}{K(mb-a)^{\tau+1}} + \frac{\Gamma(\tau+3)I^{\tau+2}_{a^+}g^2(mb)}{K^2(mb-a)^{\tau+2}}\right) \\
&\le h\left(\frac{1}{2^\alpha}\right)\frac{\Gamma(\tau+1)I^\tau_{a^+}f.g(mb)}{(mb-a)^\tau} + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{\Gamma(\tau+1)I^\tau_{b^-}f.g\left(\frac{a}{m}\right)}{\left(b-\frac{a}{m}\right)^\tau} \\
&\le \tau M\left\{\left(h\left(\frac{1}{2^\alpha}\right)f(a) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(b)\right)\int_0^1 \rho^{\tau-1}h\left(\rho^\alpha\right)\mathrm{d}\rho\right. \\
&\quad\left.+m\left(h\left(\frac{1}{2^\alpha}\right)f(b) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{a}{m^2}\right)\right)\int_0^1 \rho^{\tau-1}h\left(1-\rho^\alpha\right)\mathrm{d}\rho\right\} \\
&\quad- mcg(b)\tau\int_0^1 \rho^{\tau-1}\left(h\left(\frac{1}{2^\alpha}\right)g(a)(b-a)^2 g\left(\rho a + m(1-\rho)b\right)\right. \\
&\quad\left.+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)g\left(\frac{a}{m^2}\right)\left(b-\frac{a}{m^2}\right)^2 g\left(\rho b + (1-\rho)\frac{a}{m}\right)\right)h(\rho^\alpha)h(1-\rho^\alpha)\,\mathrm{d}\rho,
\end{aligned} \tag{12}$$

where $M = \max\left(g(x)\right)$, $K = \frac{mb-\frac{a}{m}}{b-a+mb-\frac{a}{m}}$.

Proof: By using (7), one can have the following inequality:

$$\begin{aligned}
f\left(\frac{x+my}{2}\right) &\le h\left(\frac{1}{2^\alpha}\right)f.g(x) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f.g(y) \\
&\quad - mcg(x)g(y)h\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right)|x-y|^2,
\end{aligned} \tag{13}$$

where we have used $\lambda = \frac{1}{2}$, in (7). For $\rho \in [0,1]$, let $x = \rho a + m(1-\rho)b$ and $y = \rho b + (1-\rho)\frac{a}{m}$ in (13), we get

$$\begin{aligned}
f\left(\frac{a+mb}{2}\right) &\le h\left(\frac{1}{2^\alpha}\right)f.g\left(\rho a + m(1-\rho)b\right) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f.g\left(\rho b + (1-\rho)\frac{a}{m}\right) \\
&\quad - mch\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right)g(\rho a + m(1-\rho)b)g\left(\rho b + (1-\rho)\frac{a}{m}\right) \\
&\quad \times\left(\rho b + (1-\rho)\frac{a}{m} - (\rho a + m(1-\rho)b)\right)^2.
\end{aligned} \tag{14}$$

Multiplying the above inequality with $\rho^{\tau-1}$ on both sides and integrating over $[0,1]$, we have

$$f\left(\frac{a+mb}{2}\right)\int_0^1 \rho^{\tau-1}\,\mathrm{d}\rho \le h\left(\frac{1}{2^\alpha}\right)\int_0^1 \rho^{\tau-1}f.g\left(\rho a + m(1-\rho)b\right)\mathrm{d}\rho + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)$$

$$\times \int_0^1 \rho^{\tau-1} f.g\left(\rho b + (1-\rho)\frac{a}{m}\right) \mathrm{d}\rho - mch\left(\frac{1}{2^\alpha}\right) h$$
$$\times \left(1 - \frac{1}{2^\alpha}\right) \left(b - a + mb - \frac{a}{m}\right)^2$$
$$\times \int_0^1 \rho^{\tau-1} \left(\rho - \frac{mb - \frac{a}{m}}{b - a + mb - \frac{a}{m}}\right)^2$$
$$\times g(\rho a + m(1-\rho)b) g\left(\rho b + (1-\rho)\frac{a}{m}\right) \mathrm{d}\rho. \tag{15}$$

First, we evaluate the integral last integral in the above inequality as follows: Let $K = \frac{mb - \frac{a}{m}}{b - a + mb - \frac{a}{m}}$, $x = \rho a + m(1-\rho)b$. Then by using the condition $g(x) = g\left(\frac{mb + a - x}{m}\right)$, we have

$$\int_0^1 \rho^{\tau-1} \left(\rho - \frac{mb - \frac{a}{m}}{b - a + mb - \frac{a}{m}}\right)^2 g(\rho a + m(1-\rho)b) g\left(\rho b + (1-\rho)\frac{a}{m}\right) \mathrm{d}\rho$$
$$= \frac{K^2}{(mb-a)^\tau} \int_a^{mb} (mb - x)^{\tau-1} g^2(x) \,\mathrm{d}x - \frac{2K}{(mb-a)^{\tau+1}} \int_a^{mb} (mb - x)^\tau g^2(x) \,\mathrm{d}x$$
$$+ \frac{1}{(mb-a)^{\tau+2}} \int_a^{mb} (mb - x)^{\tau+1} g^2(x) \,\mathrm{d}x$$
$$= \frac{K^2 \Gamma(\tau)}{(mb-a)^\tau} I_{a+}^\tau g^2(mb) - \frac{2K\Gamma(\tau+1)}{(mb-a)^{\tau+1}} I_{a+}^{\tau+1} g^2(mb) + \frac{\Gamma(\tau+2)}{(mb-a)^{\tau+2}} I_{a+}^{\tau+2} g^2(mb).$$

With same substitution, we also have

$$\int_0^1 \rho^{\tau-1} f.g\left(\rho a + m(1-\rho)b\right) \mathrm{d}\rho = \frac{1}{(mb-a)^\tau} \int_a^{mb} (mb - x)^{\tau-1} f.g(x) \,\mathrm{d}x$$
$$= \frac{\Gamma(\tau)}{(mb-a)^\tau} I_{a+}^\tau f.g(mb).$$

Let $y = \rho b + (1-\rho)\frac{a}{m}$. Then we have

$$\int_0^1 \rho^{\tau-1} f.g\left(\rho b + (1-\rho)\frac{a}{m}\right) \mathrm{d}\rho = \frac{1}{\left(b - \frac{a}{m}\right)^\tau} \int_{\frac{a}{m}}^b \left(y - \frac{a}{m}\right)^{\tau-1} f.g(y) \,\mathrm{d}x$$
$$= \frac{\Gamma(\tau)}{\left(b - \frac{a}{m}\right)^\tau} I_{b-}^\tau f.g\left(\frac{a}{m}\right).$$

By using the above calculated integrals in (25), the following inequality can be obtained as

$$\frac{1}{\tau} f\left(\frac{a + mb}{2}\right) \le h\left(\frac{1}{2^\alpha}\right) \frac{\Gamma(\tau)}{(mb-a)^\tau} I_{a+}^\tau f.g(mb) + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right)$$
$$\times \frac{\Gamma(\tau)}{\left(b - \frac{a}{m}\right)^\tau} I_{b-}^\tau f.g\left(\frac{a}{m}\right) - mch\left(\frac{1}{2^\alpha}\right) h\left(1 - \frac{1}{2^\alpha}\right) \left(b - a + mb - \frac{a}{m}\right)^2$$

$$\times\left(\frac{K^2\Gamma(\tau)}{(mb-a)^\tau}I^\tau_{a^+}g^2(mb)-\frac{2K\Gamma(\tau+1)}{(mb-a)^{\tau+1}}I^{\tau+1}_{a^+}g^2(mb)+\frac{\Gamma(\tau+2)}{(mb-a)^{\tau+2}}I^{\tau+2}_{a^+}g^2(mb)\right). \tag{16}$$

From the above inequality, one can obtain the first inequality of (23). On the other hand by using Definition 2.1, one can obtain the inequality:

$$\begin{aligned}
&h\left(\frac{1}{2^\alpha}\right)f.g\left(\rho a+m(1-\rho)b\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f.g\left(\rho b+(1-\rho)\frac{a}{m}\right)\\
&\le h\left(\frac{1}{2^\alpha}\right)g\left(\rho a+m(1-\rho)b\right)\left(h(\rho^\alpha)f.g(a)+mh(1-\rho^\alpha)f.g(b)\right)\\
&\quad+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)g\left(\rho b+(1-\rho)\frac{a}{m}\right)\left(h(\rho^\alpha)f.g(b)+mh(1-\rho^\alpha)f.g\left(\frac{a}{m^2}\right)\right)\\
&\quad-mcg(b)\left(h\left(\frac{1}{2^\alpha}\right)g(a)(b-a)^2g\left(\rho a+m(1-\rho)b\right)\right.\\
&\quad\left.+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)g\left(\frac{a}{m^2}\right)\left(b-\frac{a}{m^2}\right)^2g\left(\rho b+(1-\rho)\frac{a}{m}\right)\right)h(\rho^\alpha)h(1-\rho^\alpha). \qquad (17)
\end{aligned}$$

Multiplying with $\rho^{\tau-1}$, and then integrating over $[0,1]$, one can get

$$\begin{aligned}
&h\left(\frac{1}{2^\alpha}\right)\int_0^1\rho^{\tau-1}f.g\left(\rho a+m(1-\rho)b\right)\mathrm{d}\rho\\
&\quad+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1\rho^{\tau-1}f.g\left(\rho b+(1-\rho)\frac{a}{m}\right)\mathrm{d}\rho\\
&\le h\left(\frac{1}{2^\alpha}\right)\left(f.g(a)\int_0^1\rho^{\tau-1}g\left(\rho a+m(1-\rho)b\right)h(\rho^\alpha)\,\mathrm{d}\rho\right.\\
&\quad\left.+mf.g(b)\int_0^1\rho^{\tau-1}g\left(\rho a+m(1-\rho)b\right)h(1-\rho^\alpha)\,\mathrm{d}\rho\right)\\
&\quad+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(f.g(b)\int_0^1\rho^{\tau-1}g\left(\rho b+(1-\rho)\frac{a}{m}\right)h(\rho^\alpha)\,\mathrm{d}\rho+mf.g\left(\frac{a}{m^2}\right)\right.\\
&\quad\left.\times\int_0^1\rho^{\tau-1}g\left(\rho b+(1-\rho)\frac{a}{m}\right)h(1-\rho^\alpha)\,\mathrm{d}\rho\right)\\
&\quad-mcg(b)\int_0^1\rho^{\tau-1}\left(h\left(\frac{1}{2^\alpha}\right)g(a)(b-a)^2\right.\\
&\quad\left.\times g\left(\rho a+m(1-\rho)b\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)g\left(\frac{a}{m^2}\right)\left(b-\frac{a}{m^2}\right)^2g\left(\rho b+(1-\rho)\frac{a}{m}\right)\right)\\
&\quad\times h(\rho^\alpha)h(1-\rho^\alpha)\,\mathrm{d}\rho. \qquad (18)
\end{aligned}$$

By substituting $\rho a + m(1-\rho)b = x$ and $\rho b + (1-\rho)\frac{a}{m} = y$ on the left-hand side of the above inequality (18), one can obtain

$$\begin{aligned}
&\frac{\Gamma(\tau+1)}{(mb-a)^{\tau}}\left(h\left(\frac{1}{2^{\alpha}}\right) I^{\tau}_{a+}(f.g)\,(mb) + m^{\tau+1} h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) I^{\tau}_{b-}(f.g)\left(\frac{a}{m}\right)\right)\\
&\le \tau h\left(\frac{1}{2^{\alpha}}\right)\left(f.g(a)\int_0^1 \rho^{\tau-1} g\left(\rho a + m(1-\rho)b\right) h(\rho^{\alpha})\,\mathrm{d}\rho\right.\\
&\left.\quad + mf.g(b)\int_0^1 \rho^{\tau-1} g\left(\rho a + m(1-\rho)b\right) h(1-\rho^{\alpha})\,\mathrm{d}\rho\right)\\
&\quad + m\tau h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\left(f.g(b)\int_0^1 \rho^{\tau-1} g\left(\rho b + (1-\rho)\frac{a}{m}\right) h(\rho^{\alpha})\,\mathrm{d}\rho + mf.g\left(\frac{a}{m^2}\right)\right.\\
&\quad \left.\times \int_0^1 \rho^{\tau-1} g\left(\rho b + (1-\rho)\frac{a}{m}\right) h(1-\rho^{\alpha})\,\mathrm{d}\rho\right)\\
&\quad - mcg(b)\tau \int_0^1 \rho^{\tau-1}\left(h\left(\frac{1}{2^{\alpha}}\right) g(a)(b-a)^2 g\left(\rho a + m(1-\rho)b\right)\right.\\
&\quad \left.+ mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) g\left(\frac{a}{m^2}\right)\left(b - \frac{a}{m^2}\right)^2 g\left(\rho b + (1-\rho)\frac{a}{m}\right)\right) h(\rho^{\alpha}) h(1-\rho^{\alpha})\,\mathrm{d}\rho.
\end{aligned} \tag{19}$$

From the above inequality, one can get the second inequality of (23). ■

Some implications of Theorem 3.1 are given in the forthcoming results.

Theorem 3.2. *The following inequality holds for strongly quasi F-$(g, h; \alpha - m)$-convex function defined in* (9):

$$\begin{aligned}
&f\left(\frac{F(u)+mF(v)}{2}\right) + mch\left(\frac{1}{2^{\alpha}}\right) h\left(1-\frac{1}{2^{\alpha}}\right)\left(mF(v) - \frac{F(u)}{m}\right)^2\\
&\quad\times\left(\frac{\Gamma(\tau+1) I^{\tau}_{F(u)+} g(F^{-1})^2(mF(v))}{(mF(v)-F(u))^{\tau}} - \frac{2\Gamma(\tau+2) I^{\tau+1}_{F(u)+} g(F^{-1})^2(mF(v))}{K(mF(v)-F(u))^{\tau+1}}\right.\\
&\quad\left. + \Gamma(\tau+3)\frac{I^{\tau+2}_{F(u)+} g(F^{-1})^2(mF(v))}{K^2(mF(v)-F(u))^{\tau+2}}\right)\\
&\le h\left(\frac{1}{2^{\alpha}}\right)\frac{\Gamma(\tau+1) I^{\tau}_{F(u)+} P(mF(v))}{(mF(v)-F(u))^{\tau}} + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{\Gamma(\tau+1) I^{\tau}_{F(v)-} P\left(\frac{F(u)}{m}\right)}{\left(F(v) - \frac{F(u)}{m}\right)^{\tau}}\\
&\le \tau M\left\{\left(h\left(\frac{1}{2^{\alpha}}\right) f(u) + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) f(v)\right)\int_0^1 \rho^{\tau-1} h\left(\rho^{\alpha}\right)\mathrm{d}\rho\right.\\
&\quad \left. + m\left(h\left(\frac{1}{2^{\alpha}}\right) f(v) + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) f\left(\frac{F(u)}{m^2}\right)\right)\int_0^1 \rho^{\tau-1} h\left(1-\rho^{\alpha}\right)\mathrm{d}\rho\right\}
\end{aligned}$$

$$
\begin{aligned}
&- mcg(v)\tau \int_0^1 \rho^{\tau-1} \left(h\left(\frac{1}{2^\alpha}\right) g(u)(F(v) - F(u))^2 g(F^{-1})\left(tF(u) + m(1-t)F(v)\right)\right. \\
&+ mh\left(\frac{2^\alpha - 1}{2^\alpha}\right) g\left(\frac{F(u)}{m^2}\right)\left(F(v) - \frac{F(u)}{m^2}\right)^2 g(F^{-1})\left.\left(tF(v) + (1-t)\frac{F(u)}{m}\right)\right) \\
&\times h(\rho^\alpha) h(1-\rho^\alpha)\,\mathrm{d}t, \qquad (20)
\end{aligned}
$$

where M and K are same as in above Theorem 3.1 *and* $P = f.g(F^{-1})$.

Proof: The inequality (20) can be obtained, first by setting $a = F(u)$, $b = F(v)$, where F is the strictly monotone function, in (23) and then replacing f with $f(F^{-1})$ and g with $g(F^{-1})$ in the resulting inequality. ■

In the continuation of above theorem, we have the following result.

Theorem 3.3. *The following inequalities hold for strongly* $(g, h; \alpha - m) - p$*-convex function:*

(i) *If* $p > 0$, *then we have*

$$
\begin{aligned}
&f\left(\frac{u^p + mv^p}{2}\right) + mch\left(\frac{1}{2^\alpha}\right) h\left(1 - \frac{1}{2^\alpha}\right)\left(mv^p - \frac{u^p}{m}\right)^2 \\
&\times \left(\Gamma(\tau+1)\frac{I^\tau_{u^p+} g^2 \circ w(mv^p)}{(mv^p - u^p)^\tau} - \frac{2\Gamma(\tau+2) I^{\tau+1}_{u^p+} g^2 \circ w(mv^p)}{K(mv^p - u^p)^{\tau+1}}\right. \\
&\left. + \frac{\Gamma(\tau+3) I^{\tau+2}_{u^p+} g^2 \circ w(mv^p)}{K^2(mv^p - u^p)^{\tau+2}}\right) \\
&\le h\left(\frac{1}{2^\alpha}\right) \frac{\Gamma(\tau+1) I^\tau_{u^p+}(f.g) \circ w(mv^p)}{(mv^p - u^p)^\tau} \\
&+ mh\left(\frac{2^\alpha - 1}{2^\alpha}\right) \frac{\Gamma(\tau+1) I^\tau_{v^p-}(f.g) \circ w\left(\frac{u^p}{m}\right)}{\left(v^p - \frac{u^p}{m}\right)^\tau} \\
&\le \tau M\left\{\left(h\left(\frac{1}{2^\alpha}\right) f(u) + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right) f(v)\right) \int_0^1 \rho^{\tau-1} h(\rho^\alpha)\,\mathrm{d}\rho\right. \\
&\left. + m\left(h\left(\frac{1}{2^\alpha}\right) f(v) + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right) f \circ w\left(\frac{u^p}{m^2}\right)\right) \int_0^1 \rho^{\tau-1} h(1-\rho^\alpha)\,\mathrm{d}\rho\right\} \\
&- mcg(v)\tau \int_0^1 \rho^{\tau-1}\left(h\left(\frac{1}{2^\alpha}\right) g(u)(v^p - u^p)^2 g \circ w(tu^p + m(1-t)v^p)\right. \\
&+ g \circ w\left(\frac{u^p}{m^2}\right)\left(v^p - \frac{u^p}{m^2}\right)^2
\end{aligned}
$$

$$\times\, g\circ w\left(tv^p+m(1-t)\frac{u^p}{m}\right)\Big)\, h(\rho^\alpha)h(1-\rho^\alpha)\,\mathrm{d}\rho, \tag{21}$$

where $w(x)=x^{\frac{1}{p}}; x\in[u^p, mv^p]$.

(ii) *If* $p<0$, *then we have*

$$\begin{aligned}
&f\left(\frac{u^p+mv^p}{2}\right)+mch\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right)\left(mv^p-\frac{u^p}{m}\right)^2\Big(\Gamma(\tau+1)\\
&\quad\times\frac{I^\tau_{u^p+}g^2\circ w(mv^p)}{(u^p-mv^p)^\tau}-\frac{2\Gamma(\tau+2)I^{\tau+1}_{u^p+}g^2\circ w(mv^p)}{K(u^p-mv^p)^{\tau+1}}\\
&\quad+\frac{\Gamma(\tau+3)I^{\tau+2}_{u^p+}g^2\circ w(mv^p)}{K^2(u^p-mv^p)^{\tau+2}}\Big)\\
&\le h\left(\frac{1}{2^\alpha}\right)\frac{\Gamma(\tau+1)I^\tau_{u^p-}(f.g)\circ w(mv^p)}{(u^p-mv^p)^\tau}+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\\
&\quad\times\frac{\Gamma(\tau+1)I^\tau_{v^p+}(f.g)\circ w\left(\frac{u^p}{m}\right)}{\left(\frac{u^p}{m}-v^p\right)^\tau}\\
&\le \tau M\left\{\left(h\left(\frac{1}{2^\alpha}\right)f(u)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(v)\right)\int_0^1\rho^{\tau-1}h\left(\rho^\alpha\right)\mathrm{d}\rho\right.\\
&\quad\left.+\,m\left(h\left(\frac{1}{2^\alpha}\right)f(v)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\circ w\left(\frac{u^p}{m^2}\right)\right)\int_0^1\rho^{\tau-1}h\left(1-\rho^\alpha\right)\mathrm{d}\rho\right\}\\
&\quad-mcg(v)\tau\int_0^1\rho^{\tau-1}\Big(h\left(\frac{1}{2^\alpha}\right)g(u)(v^p-u^p)^2g\circ w(tu^p+m(1-t)v^p)\\
&\quad+g\circ w\left(\frac{u^p}{m^2}\right)\left(v^p-\frac{u^p}{m^2}\right)^2g\circ w\left(tv^p+m(1-t)\frac{u^p}{m}\right)\Big)h(\rho^\alpha)h(1-\rho^\alpha)\,\mathrm{d}\rho,
\end{aligned} \tag{22}$$

where $w(x)=x^{\frac{1}{p}}; x\in[mv^p, u^p]$.

Proof: By setting $F(t)=t^p$ in (20), one can easily obtain the required inequalities. ■

Some connections of above theorems are summarized in the following remark.

Remark 3.1: (1) By choosing $g(x)=\exp(-\eta x)$ in Theorem 3.1 and using the condition $h(x+y)\le h(x)h(y)$ and after some computation, one can obtain [2, Theorem 3.1].

(2) By setting $c=0$, results for $(g,h;\alpha-m)$-convex, quasi F-$(g,h;\alpha-m)$-convex and $(g,h;\alpha-m)-p$-convex functions can be obtained. Also, results for almost all convexities via Riemann–Liouville functions can be recovered from above.

In the next theorem, we state and prove another Hermite–Hadamard-type inequality for Riemann–Liouville fractional integrals.

Theorem 3.4. *Under the assumptions of Theorem* 3.1, *we have*

$$
\begin{aligned}
& f\left(\frac{a+mb}{2}\right)+mch\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right)\left(mb-\frac{a}{m}\right)^2 \\
& \quad\times\left(\frac{\Gamma(\tau+1)I^{\tau}_{a+}g^2(mb)}{(mb-a)^\tau}-\frac{\Gamma(\tau+2)I^{\tau+1}_{a^+}g^2(mb)}{K(mb-a)^{\tau+1}}+\frac{\Gamma(\tau+3)I^{\tau+2}_{a^+}g^2(mb)}{4K^2(mb-a)^{\tau+2}}\right) \\
& \quad\leq h\left(\frac{1}{2^\alpha}\right)\frac{\Gamma(\tau+1)I^{\tau}_{\left\{\frac{a+mb}{2}\right\}^+}f.g(mb)}{(mb-a)^\tau}+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{\Gamma(\tau+1)I^{\tau}_{\left\{\frac{a+mb}{2m}\right\}^-}f.g\left(\frac{a}{m}\right)}{\left(b-\frac{a}{m}\right)^\tau} \\
& \quad\leq \tau M\left\{\left(h\left(\frac{1}{2^\alpha}\right)f(a)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(b)\right)\int_0^1\rho^{\tau-1}h\left(\left(\frac{\rho}{2}\right)^\alpha\right)\mathrm{d}\rho\right. \\
& \quad\quad\left.+m\left(h\left(\frac{1}{2^\alpha}\right)f(b)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{a}{m^2}\right)\right)\int_0^1\rho^{\tau-1}h\left(1-\left(\frac{\rho}{2}\right)^\alpha\right)\mathrm{d}\rho\right\} \\
& \quad\quad-mcg(b)\tau\int_0^1\left(g(a)(b-a)^2h\left(\frac{1}{2^\alpha}\right)g\left(\frac{\rho}{2}a+m\left(1-\frac{\rho}{2}\right)b\right)+mg\left(\frac{a}{m^2}\right)\right. \\
& \quad\quad\left.\times\left(b-\frac{a}{m^2}\right)^2h\left(\frac{2^\alpha-1}{2^\alpha}\right)g\left(\frac{\rho}{2}b+\left(1-\frac{\rho}{2}\right)\frac{a}{m}\right)\right)h\left(\left(\frac{\rho}{2}\right)^\alpha\right)h\left(1-\left(\frac{\rho}{2}\right)^\alpha\right)\mathrm{d}\rho,
\end{aligned}
\tag{23}
$$

where $M=\max\left(g(x)\right)$.

Proof: By setting $x=\frac{\rho}{2}a+m(1-\frac{\rho}{2})b$ and $y=\frac{\rho}{2}b+(1-\frac{\rho}{2})\frac{a}{m}$ in (13), we get

$$
\begin{aligned}
f\left(\frac{a+mb}{2}\right) &\leq h\left(\frac{1}{2^\alpha}\right)f.g\left(\frac{\rho}{2}a+m\left(1-\frac{\rho}{2}\right)b\right) \\
&\quad+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f.g\left(\frac{\rho}{2}b+\left(1-\frac{\rho}{2}\right)\frac{a}{m}\right) \\
&\quad-mch\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right)g\left(\frac{\rho}{2}a+m\left(1-\frac{\rho}{2}\right)b\right) \\
&\quad\times g\left(\frac{\rho}{2}b+\left(1-\frac{\rho}{2}\right)\frac{a}{m}\right)\left(\frac{\rho}{2}b+\left(1-\frac{\rho}{2}\right)\frac{a}{m}-\frac{\rho}{2}a-m\left(1-\frac{\rho}{2}\right)b\right)^2.
\end{aligned}
\tag{24}
$$

Multiplying the above inequality with $\rho^{\tau-1}$ on both sides and integrating over $[0,1]$, we have

$$
\begin{aligned}
& f\left(\frac{a+mb}{2}\right)\int_0^1\rho^{\tau-1}\,\mathrm{d}\rho \\
& \quad\leq h\left(\frac{1}{2^\alpha}\right)\int_0^1\rho^{\tau-1}f.g\left(\frac{\rho}{2}a+m\left(1-\frac{\rho}{2}\right)b\right)\mathrm{d}\rho \\
& \quad\quad+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1\rho^{\tau-1}f.g\left(\frac{\rho}{2}b+\left(1-\frac{\rho}{2}\right)\frac{a}{m}\right)\mathrm{d}\rho-mch\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right)
\end{aligned}
$$

$$\times \frac{\left(b-a+mb-\frac{a}{m}\right)^2}{4}\int_0^1 \rho^{\tau-1}(\rho-2K)^2 g\left(\frac{\rho}{2}a+m\left(1-\frac{\rho}{2}\right)b\right)$$

$$\times g\left(\frac{\rho}{2}b+\left(1-\frac{\rho}{2}\right)\frac{a}{m}\right)\mathrm{d}\rho. \tag{25}$$

The first inequality in (23) can be achieved by using the condition $g(x)=g(\frac{mb+a-x}{m})$ and substitutions $x=\frac{\rho}{2}a+m(1-\frac{\rho}{2})b, y=\frac{\rho}{2}b+(1-\frac{\rho}{2})\frac{a}{m}$ in the integrals appearing on right-hand side of the above inequality. By using Definition 2.1, one can have

$$h\left(\frac{1}{2^\alpha}\right)f.g\left(\frac{\rho}{2}a+m\left(1-\frac{\rho}{2}\right)b\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f.g\left(\frac{\rho}{2}b+\left(1-\frac{\rho}{2}\right)\frac{a}{m}\right)$$

$$\leq h\left(\frac{1}{2^\alpha}\right)g\left(\frac{\rho}{2}a+m\left(1-\frac{\rho}{2}\right)b\right)\left(h\left(\left(\frac{\rho}{2}\right)^\alpha\right)f.g(a)+mh\left(1-\left(\frac{\rho}{2}\right)^\alpha\right)f.g(b)\right)$$

$$+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)g\left(\frac{\rho}{2}b+\left(1-\frac{\rho}{2}\right)\frac{a}{m}\right)$$

$$\times\left(h\left(\left(\frac{\rho}{2}\right)^\alpha\right)f.g(b)+mh\left(1-\left(\frac{\rho}{2}\right)^\alpha\right)f.g\left(\frac{a}{m^2}\right)\right)$$

$$-mcg(b)\left\{g(a)(b-a)^2h\left(\frac{1}{2^\alpha}\right)g\left(\frac{\rho}{2}a+m\left(1-\frac{\rho}{2}\right)b\right)\right.$$

$$+mg\left(\frac{a}{m^2}\right)\left(b-\frac{a}{m^2}\right)^2h\left(\frac{2^\alpha-1}{2^\alpha}\right)$$

$$\left.g\left(\frac{\rho}{2}b+\left(1-\frac{\rho}{2}\right)\frac{a}{m}\right)\right\}h\left(\left(\frac{\rho}{2}\right)^\alpha\right)h\left(1-\left(\frac{\rho}{2}\right)^\alpha\right). \tag{26}$$

Multiplying with $\rho^{\tau-1}$, and then integrating over $[0,1]$, one can get

$$h\left(\frac{1}{2^\alpha}\right)\int_0^1\rho^{\tau-1}f.g\left(\frac{\rho}{2}a+m\left(1-\frac{\rho}{2}\right)b\right)\mathrm{d}\rho$$

$$+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1\rho^{\tau-1}f.g\left(\frac{\rho}{2}b+\left(1-\frac{\rho}{2}\right)\frac{a}{m}\right)\mathrm{d}\rho$$

$$\leq h\left(\frac{1}{2^\alpha}\right)\left(f.g(a)\int_0^1\rho^{\tau-1}g\left(\frac{\rho}{2}a+m\left(1-\frac{\rho}{2}\right)b\right)h\left(\left(\frac{\rho}{2}\right)^\alpha\right)\mathrm{d}\rho\right.$$

$$\left.+mf.g(b)\int_0^1\rho^{\tau-1}g\left(\frac{\rho}{2}a+m\left(1-\frac{\rho}{2}\right)b\right)h\left(1-\left(\frac{\rho}{2}\right)^\alpha\right)\mathrm{d}\rho\right)$$

$$+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(f.g(b)\int_0^1\rho^{\tau-1}g\left(\frac{\rho}{2}b+\left(1-\frac{\rho}{2}\right)\frac{a}{m}\right)\right.$$

$$\left.\times h\left(\left(\frac{\rho}{2}\right)^\alpha\right)\mathrm{d}\rho+mf.g\left(\frac{a}{m^2}\right)\int_0^1\rho^{\tau-1}g\left(\frac{\rho}{2}b+\left(1-\frac{\rho}{2}\right)\frac{a}{m}\right)h\left(1-\left(\frac{\rho}{2}\right)^\alpha\right)\mathrm{d}\rho\right)$$

$$-mcg(b)\int_0^1\left(g(a)(b-a)^2h\left(\frac{1}{2^\alpha}\right)g\left(\frac{\rho}{2}a+m\left(1-\frac{\rho}{2}\right)b\right)\right.$$

$$+mg\left(\frac{a}{m^2}\right)\left(b-\frac{a}{m^2}\right)^2h\left(\frac{2^\alpha-1}{2^\alpha}\right)$$

$$\times\, g\left(\frac{\rho}{2}b+\left(1-\frac{\rho}{2}\right)\frac{a}{m}\right)\Big)\, h\left(\left(\frac{\rho}{2}\right)^{\alpha}\right) h\left(1-\left(\frac{\rho}{2}\right)^{\alpha}\right) \mathrm{d}\rho. \tag{27}$$

Setting $\frac{\rho}{2}a+m(1-\frac{\rho}{2})b=x$, and $\frac{\rho}{2}b+(1-\frac{\rho}{2})\frac{a}{m}=y$, in integrals on the left hand side of the above inequality (18), we get

$$\begin{aligned}
&\frac{2^{\tau}\Gamma(\tau+1)}{(mb-a)^{\tau}}\left(h\left(\frac{1}{2^{\alpha}}\right) I^{\tau}_{\left\{\frac{a+mb}{2}\right\}^{+}}(f.g)\,(mb)+m^{\tau+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) I^{\tau}_{\left\{\frac{a+mb}{2m}\right\}^{-}}(f.g)\left(\frac{a}{m}\right)\right)\\
&\quad\le \tau h\left(\frac{1}{2^{\alpha}}\right)\left(f.g(a)\int_0^1 \rho^{\tau-1} g\left(\frac{\rho}{2}a+m\left(1-\frac{\rho}{2}\right)b\right) h\left(\left(\frac{\rho}{2}\right)^{\alpha}\right)\mathrm{d}\rho\right.\\
&\quad\quad\left.+\, mf.g(b)\int_0^1 \rho^{\tau-1} g\left(\frac{\rho}{2}a+m\left(1-\frac{\rho}{2}\right)b\right) h\left(1-\left(\frac{\rho}{2}\right)^{\alpha}\right)\mathrm{d}\rho\right)\\
&\quad\quad+\tau mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\left(f.g(b)\int_0^1 \rho^{\tau-1} g\left(\frac{\rho}{2}b+\left(1-\frac{\rho}{2}\right)\frac{a}{m}\right)\right.\\
&\quad\quad\left.\times\, h\left(\left(\frac{\rho}{2}\right)^{\alpha}\right)\mathrm{d}\rho+mf.g\left(\frac{a}{m^2}\right)\int_0^1 \rho^{\tau-1} g\left(\frac{\rho}{2}b+\left(1-\frac{\rho}{2}\right)\frac{a}{m}\right) h\left(1-\left(\frac{\rho}{2}\right)^{\alpha}\right)\mathrm{d}\rho\right)\\
&\quad\quad-\, mc\tau g(b)\int_0^1\left(g(a)(b-a)^2 h\left(\frac{1}{2^{\alpha}}\right) g\left(\frac{\rho}{2}a+m\left(1-\frac{\rho}{2}\right)b\right)\right.\\
&\quad\quad+\, mg\left(\frac{a}{m^2}\right)\left(b-\frac{a}{m^2}\right)^2 h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\\
&\quad\quad\times\, g\left(\frac{\rho}{2}b+\left(1-\frac{\rho}{2}\right)\frac{a}{m}\right)\Big)\, h\left(\left(\frac{\rho}{2}\right)^{\alpha}\right) h\left(1-\left(\frac{\rho}{2}\right)^{\alpha}\right)\mathrm{d}\rho.
\end{aligned}\tag{28}$$

By using M instead of g in first two terms of the right-hand side of above inequality, one can get the second required inequality. ■

Some implications of Theorem 3.2 are given in the forthcoming results.

Theorem 3.5. *The following inequality holds for strongly quasi F-$(g,h;\alpha-m)$-convex function defined in* (9):

$$\begin{aligned}
&f\left(F^{-1}\left(\frac{F(u)+mF(v)}{2}\right)\right)+mch\left(\frac{1}{2^{\alpha}}\right)h\left(1-\frac{1}{2^{\alpha}}\right)\left(mF(v)-\frac{F(u)}{m}\right)^2\Gamma(\tau+1)\\
&\quad\times\left(\frac{I^{\tau}_{F(u)^{+}}g(F^{-1})^2(mF(v))}{(mF(v)-F(u))^{\tau}}-\frac{\Gamma(\tau+2)I^{\tau+1}_{F(u)^{+}}g(F^{-1})^2(mF(v))}{K(mF(v)-F(u))^{\tau+1}}+\Gamma(\tau+3)\right.\\
&\quad\left.\times\,\frac{I^{\tau+2}_{F(u)^{+}}g(F^{-1})^2(mF(v))}{4K^2(mF(v)-F(u))^{\tau+2}}\right)\\
&\le h\left(\frac{1}{2^{\alpha}}\right)\frac{\Gamma(\tau+1)I^{\tau}_{\left\{\frac{F(u)+mF(v)}{2}\right\}^{+}}P\,(mF(v))}{(mF(v)-F(u))^{\tau}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \frac{\Gamma(\tau+1) I^{\tau}_{\left\{\frac{F(u)+mF(v)}{2m}\right\}^{-}} P\left(\frac{F(u)}{m}\right)}{\left(F(v)-\frac{F(u)}{m}\right)^{\tau}} \\
&\leq \tau M\left\{\left(h\left(\frac{1}{2^{\alpha}}\right) f(u)+m h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) f(v)\right)\right. \\
&\times \int_{0}^{1} \rho^{\tau-1} h\left(\left(\frac{\rho}{2}\right)^{\alpha}\right) \mathrm{d}\rho+m\left(h\left(\frac{1}{2^{\alpha}}\right) f(v)+m h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) f\left(\frac{F(u)}{m^{2}}\right)\right) \\
&\left.\times \int_{0}^{1} \rho^{\tau-1} h\left(1-\left(\frac{\rho}{2}\right)^{\alpha}\right) \mathrm{d}\rho\right\} \\
&-m c g(b) \tau \int_{0}^{1}\left(g(a)(F(v)-F(u))^{2} h\left(\frac{1}{2^{\alpha}}\right) g(F^{-1})\left(\frac{\rho}{2} F(u)+m\left(1-\frac{\rho}{2}\right) F(v)\right)\right. \\
&\left.+m g\left(\frac{F(u)}{m^{2}}\right)\left(F(v)-\frac{F(u)}{m^{2}}\right)^{2} h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right) g(F^{-1})\left(\frac{\rho}{2} F(v)+\left(1-\frac{\rho}{2}\right) \frac{F(u)}{m}\right)\right) \\
&\times h\left(\left(\frac{\rho}{2}\right)^{\alpha}\right) h\left(1-\left(\frac{\rho}{2}\right)^{\alpha}\right) \mathrm{d}\rho, \qquad (29)
\end{aligned}
$$

where $M = \max(g(x))$.

Proof: The inequality (29) can be obtained, first by setting $a = F(u)$, $b = F(v)$, where F is strictly monotone function, in (23) and then replacing f with $f(F^{-1})$ and g with $g(F^{-1})$ in the resulting inequality. ■

In the continuation of above theorem, we have the following result.

Theorem 3.6. *The following inequalities hold for strongly* $(g, h; \alpha - m) - p$*-convex function:*

(i) *If* $p > 0$, *then we have*

$$
\begin{aligned}
&f\left(\left(\frac{u^{p}+m v^{p}}{2}\right)^{\frac{1}{p}}\right)+m c h\left(\frac{1}{2^{\alpha}}\right) h\left(1-\frac{1}{2^{\alpha}}\right)\left(m v^{p}-\frac{u^{p}}{m}\right)^{2} \\
&\quad \times\left(\frac{\Gamma(\tau+1) I^{\tau}_{u^{p}+}(g \circ w)^{2}(m v^{p})}{(m v^{p}-u^{p})^{\tau}}-\frac{\Gamma(\tau+2) I^{\tau+1}_{u^{p}+}(g \circ w)^{2}(m v^{p})}{K(m v^{p}-u^{p})^{\tau+1}}+\Gamma(\tau+3)\right. \\
&\quad \left.\times \frac{I^{\tau+2}_{u^{p}+}(g \circ w)^{2}(m v^{p})}{4K^{2}(m v^{p}-u^{p})^{\tau+2}}\right) \\
&\quad \leq h\left(\frac{1}{2^{\alpha}}\right) \frac{\Gamma(\tau+1) I^{\tau}_{\left\{\frac{u^{p}+m v^{p}}{2}\right\}^{+}}(f.g) \circ w(m v^{p})}{(m v^{p}-u^{p})^{\tau}}+m h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)
\end{aligned}
$$

$$\times \frac{\Gamma(\tau+1)I^{\tau}_{\left\{\frac{u^p+mv^p}{2m}\right\}^-}(f.g)\circ w\left(\frac{u^p}{m}\right)}{\left(v^p-\frac{u^p}{m}\right)^{\tau}}$$

$$\leq \tau M\left\{\left(h\left(\frac{1}{2^\alpha}\right)f(u)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(v)\right)\right.$$

$$\times\int_0^1\rho^{\tau-1}h\left(\left(\frac{\rho}{2}\right)^\alpha\right)\mathrm{d}\rho+m\left(h\left(\frac{1}{2^\alpha}\right)f(v)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{u^p}{m^2}\right)\right)$$

$$\left.\times\int_0^1\int_0^1\rho^{\tau-1}h\left(1-\left(\frac{\rho}{2}\right)^\alpha\right)\mathrm{d}\rho\right\}-mcg(b)\tau$$

$$\times\left(g(a)(v^p-u^p)^2h\left(\frac{1}{2^\alpha}\right)g\circ w\left(\frac{\rho}{2}u^p+m\left(1-\frac{\rho}{2}\right)v^p\right)+mg\left(\frac{u^p}{m^2}\right)\right.$$

$$\left.\times\left(v^p-\frac{u^p}{m^2}\right)^2h\left(\frac{2^\alpha-1}{2^\alpha}\right)g\circ w\left(\frac{\rho}{2}v^p+\left(1-\frac{\rho}{2}\right)\frac{u^p}{m}\right)\right)$$

$$\times h\left(\left(\frac{\rho}{2}\right)^\alpha\right)h\left(1-\left(\frac{\rho}{2}\right)^\alpha\right)\mathrm{d}\rho, \tag{30}$$

where $w(x)=x^{\frac{1}{p}}; x\in[u^p,mv^p]$.

(ii) *If* $p<0$, *then we have*

$$f\left(\left(\frac{u^p+mv^p}{2}\right)^{\frac{1}{p}}\right)+mch\left(\frac{1}{2^\alpha}\right)h\left(1-\frac{1}{2^\alpha}\right)\left(mv^p-\frac{u^p}{m}\right)^2$$

$$\times\left(\frac{\Gamma(\tau+1)I^{\tau}_{u^p+}(g\circ w)^2(mv^p)}{(u^p-mv^p)^\tau}-\frac{\Gamma(\tau+2)I^{\tau+1}_{u^p+}(g\circ w)^2(mv^p)}{K(u^p-mv^p)^{\tau+1}}+\Gamma(\tau+3)\right.$$

$$\left.\times\frac{I^{\tau+2}_{u^p+}(g\circ w)^2(mv^p)}{4K^2(u^p-mv^p)^{\tau+2}}\right)$$

$$\leq h\left(\frac{1}{2^\alpha}\right)\frac{\Gamma(\tau+1)I^{\tau}_{\left\{\frac{u^p+mv^p}{2}\right\}^-}(f.g)\circ w(mv^p)}{(u^p-mv^p)^\tau}+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)$$

$$\times\frac{\Gamma(\tau+1)I^{\tau}_{\left\{\frac{u^p+mv^p}{2m}\right\}^+}(f.g)\circ w\left(\frac{u^p}{m}\right)}{\left(\frac{u^p}{m}-v^p\right)^\tau}$$

$$\leq \tau M\left\{\left(h\left(\frac{1}{2^\alpha}\right)f(u)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f(v)\right)\right.$$

$$\times\int_0^1\rho^{\tau-1}h\left(\left(\frac{\rho}{2}\right)^\alpha\right)\mathrm{d}\rho+m\left(h\left(\frac{1}{2^\alpha}\right)f(v)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{u^p}{m^2}\right)\right)$$

$$\left.\times\int_0^1\rho^{\tau-1}h\left(1-\left(\frac{\rho}{2}\right)^\alpha\right)\mathrm{d}\rho\right\}-mcg(b)\tau$$

$$\times \int_0^1 \left(g(a)(v^p - u^p)^2 h\left(\frac{1}{2^\alpha}\right) g \circ w\left(\frac{\rho}{2}u^p + m\left(1-\frac{\rho}{2}\right)v^p\right) + mg\left(\frac{u^p}{m^2}\right)\right.$$
$$\times \left(v^p - \frac{u^p}{m^2}\right)^2 h\left(\frac{2^\alpha - 1}{2^\alpha}\right) g \circ w\left(\frac{\rho}{2}v^p + \left(1-\frac{\rho}{2}\right)\frac{u^p}{m}\right)\Bigg) h\left(\left(\frac{\rho}{2}\right)^\alpha\right)$$
$$\times h\left(1 - \left(\frac{\rho}{2}\right)^\alpha\right) \mathrm{d}\rho, \tag{31}$$

where $w(x) = x^{\frac{1}{p}}; x \in [mv^p, u^p]$.

Proof: By setting $F(t) = t^p$ in (29), one can easily obtain the required inequalities. ■

Remark 3.2: (1) By choosing $g(x) = \exp(-\eta x)$ in Theorem 3.4 and using the condition $h(x+y) \leq h(x)h(y)$ and after some computation, one can obtain [2, Theorem 3.2].

(2) By setting $c = 0$, results for $(g, h; \alpha - m)$-convex, quasi F-$(g, h; \alpha - m)$-convex and $(g, h; \alpha - m) - p$-convex functions can be obtained. Also, results for almost all convexities via Riemann–Liouville functions can be recovered from the above.

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