Journal of Inequalities and Applications
a SpringerOpen Journal

Check for updates

Best approximation of $(\mathcal{G}_1, \mathcal{G}_2)$-random operator inequality in matrix Menger Banach algebras with application of stochastic Mittag-Leffler and $\mathbb{H}$-Fox control functions

Safoura Rezaei Aderyani[1], Reza Saadati[1*] , Themistocles M. Rassias[2] and Choonkil Park[3]

*Correspondence:
rsaadati@iust.ac.ir; rsaadati@eml.cc
[1]School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran
Full list of author information is available at the end of the article

Abstract

We stabilize pseudostochastic $(\mathcal{G}_1, \mathcal{G}_2)$-random operator inequality using a class of stochastic matrix control functions in matrix Menger Banach algebras. We get an approximation for stochastic $(\mathcal{G}_1, \mathcal{G}_2)$-random operator inequality by means of both direct and fixed point methods. As an application, we apply both stochastic Mittag-Leffler and $\mathbb{H}$-fox control functions to get a better approximation in a random operator inequality.

MSC: 54H20; 46L05; 39B62; 43A22

Keywords: Stochastic matrix control function; Mittag-Leffler; $(\mathcal{G}_1, \mathcal{G}_2)$-random operator inequality; Menger Banach algebra; $\mathbb{H}$-Fox function

1 Introduction and preliminaries

The theory of special functions, such as Mittag-Leffler function, hypergeometric function, Wright function, $\mathbb{H}$-Fox function, and so on, encircles a significant segment of mathematics. In recent centuries, the necessity of solving problems taking place in various fields of science motivated the advancement of the theory of special functions. These functions have extensive applications in a variety of different fields, together with material science and engineering science, biology, chemistry, mathematical physics, and both applied and pure mathematics. The interested readers can review the literature [1–4].

In 1903, the Swedish mathematician Gosta Mittag-Leffler presented a generalization of the exponential function and introduced some properties of this function. In 1905, Wiman introduced its general form. Mittag-Leffler function naturally appears as the solution of fractional order integro-differential equations and particularly in the investigations of electric networks, random walks, fluid flow, superdiffusive transport, the fractional generalization of the kinetic equation, diffusive transport akin to diffusion, and in the study of complex systems. During the last 20 years, the interest in Mittag-Leffler function has remarkably increased among scientists and engineers, owing to it wide potential in applications. We suggest the readers to consult the literature [5, 6].

Springer

In 1961, Charles Fox presented a generalization of the Meijer G-function and the Fox–Wright function. $\mathbb{H}$-Fox function is defined by a Mellin–Barnes integral in the context of symmetrical Fourier kernels. It has a number of great applications, most notably in fractional calculus and statistics. Also, it plays a significant role in a wide range of response-related topics, like reaction–diffusion, theoretical physics, and mathematical probability theory. For more details, see [7, 8].

In [9], the authors introduced and applied the following additive (ρ_1, ρ_2)-functional inequalities in complex Banach spaces:

$$\begin{aligned}
&\big\| f(x + y + z) - f(x) - f(y) - f(z) \big\| \\
&\quad \leq \big\| \rho_1 \big[f(x + z) - f(x) - f(z) \big] \big\| + \big\| \rho_2 \big[f(y + z) - f(y) - f(z) \big] \big\|,
\end{aligned}$$

in which ρ_1, ρ_2 are fixed nonzero complex numbers with $|\rho_1| + |\rho_2| < 2$.

Here, we introduce a class of stochastic matrix control functions and apply them to approximate the following pseudostochastic additive $(\mathcal{G}_1, \mathcal{G}_2)$-random operator inequalities in matrix Menger Banach algebras:

$$\begin{aligned}
&\Phi_\Theta^{\mathcal{Q}(j,\mathsf{S}+\mathsf{R}+\mathsf{A})-\mathcal{Q}(j,\mathsf{S})-\mathcal{Q}(j,\mathsf{R})-\mathcal{Q}(j,\mathsf{A})} \\
&\quad \succeq \Phi_\Theta^{\mathcal{G}_1[\mathcal{Q}(j,\mathsf{S}+\mathsf{A})-\mathcal{Q}(j,\mathsf{S})-\mathcal{Q}(j,\mathsf{A})]} \circledast_M \Phi_\Theta^{\mathcal{G}_2[\mathcal{Q}(j,\mathsf{R}+\mathsf{A})-\mathcal{Q}(j,\mathsf{R})-\mathcal{Q}(j,\mathsf{A})]},
\end{aligned} \tag{1.1}$$

where $0 \neq \mathcal{G}_1, \mathcal{G}_2 \in \mathbb{C}$ are fixed and $\max\{|\mathcal{G}_1|, |\mathcal{G}_2|\} < 2$.

Now, let $\mho = [0, 1]$ and

$$\operatorname{diag} \mathsf{N}_n(\mho) = \left\{ \begin{bmatrix} \mathfrak{M}_1 & & \\ & \ddots & \\ & & \mathfrak{M}_n \end{bmatrix} = \operatorname{diag}[\mathfrak{M}_1, \dots, \mathfrak{M}_n], \mathfrak{M}_1, \dots, \mathfrak{M}_n \in \mho \right\}.$$

We denote $\mathfrak{M} := \operatorname{diag}[\mathfrak{M}_1, \dots, \mathfrak{M}_n] \preceq \mathfrak{N} := \operatorname{diag}[\mathfrak{N}_1, \dots, \mathfrak{N}_n]$ if $\mathfrak{M}_i \leq \mathfrak{N}_i$ for any $1 \leq i \leq n$, and note that $\mathbf{0} = \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix}$ and $\mathbf{1} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$.

Next, we define a generalized t-norm on $\operatorname{diag} \mathsf{N}_n(\mho)$.

Definition 1.1 ([10]) A generalized t-norm on $\operatorname{diag} \mathsf{N}_n(\mho)$ is an operation $\odot : \operatorname{diag} \mathsf{N}_n(\mho) \times \operatorname{diag} \mathsf{N}_n(\mho) \to \operatorname{diag} \mathsf{N}_n(\mho)$ satisfying the following conditions:

(1) $(\forall \mathfrak{M} \in \operatorname{diag} \mathsf{N}_n(\mho))(\mathfrak{M} \odot \mathbf{1}) = \mathfrak{M})$ (boundary condition);

(2) $(\forall (\mathfrak{M}, \mathfrak{N}) \in (\operatorname{diag} \mathsf{N}_n(\mho))^2)(\mathfrak{M} \odot \mathfrak{N} = \mathfrak{N} \odot \mathfrak{M})$ (commutativity);

(3) $(\forall (\mathfrak{M}, \mathfrak{N}, \mathfrak{I}) \in (\operatorname{diag} \mathsf{N}_n(\mho)^3)(\mathfrak{M} \odot (\mathfrak{N} \odot \mathfrak{I}) = (\mathfrak{M} \odot \mathfrak{N}) \odot \mathfrak{I})$ (associativity);

(4) $(\forall (\mathfrak{M}, \mathfrak{M}', \mathfrak{N}, \mathfrak{N}') \in (\operatorname{diag} \mathsf{N}_n(\mho^4)(\mathfrak{M} \preceq \mathfrak{M}'$ and $\mathfrak{N} \preceq \mathfrak{N}' \Longrightarrow \mathfrak{M} \odot \mathfrak{N} \preceq \mathfrak{M}' \odot \mathfrak{N}'$ (monotonicity).

For every $\mathfrak{M}, \mathfrak{N} \in \operatorname{diag} \mathsf{N}_n(\mho)$ and all sequences $\{\mathfrak{M}_k\}$ and $\{\mathfrak{N}_k\}$ converging to $\mathfrak{M}$ and $\mathfrak{N}$, respectively, if we have

$$\lim_k (\mathfrak{M}_k \odot \mathfrak{N}_k) = \mathfrak{M} \odot \mathfrak{N},$$

then $\odot$ on $\operatorname{diag} \mathsf{N}_n(\mho)$ is continuous, see [11, 12]. Consider the following examples of a continuous generalized t-norm:

(1) Let $\odot_P : \operatorname{diag} \mathsf{N}_n(\mho) \times \operatorname{diag} \mathsf{N}_n(\mho) \to \operatorname{diag} \mathsf{N}_n(\mho)$ such that

$$\mathfrak{M} \odot_P \mathfrak{N} = \operatorname{diag}[\mathfrak{M}_1,\ldots,\mathfrak{M}_n] \odot_P \operatorname{diag}[\mathfrak{N}_1,\ldots,\mathfrak{N}_n] = \operatorname{diag}[\mathfrak{M}_1.\mathfrak{N}_1,\ldots,\mathfrak{M}_n.\mathfrak{N}_n].$$

Then $\odot_P$ is a continuous generalized t-norm.

(2) Let $\odot_M : \operatorname{diag} \mathsf{N}_n(\mho) \times \operatorname{diag} \mathsf{N}_n(\mho) \to \operatorname{diag} \mathsf{N}_n(\mho)$ such that

$$\mathfrak{M} \odot_M \mathfrak{N} = \operatorname{diag}[\mathfrak{M}_1,\ldots,\mathfrak{M}_n] \odot_M \operatorname{diag}[\mathfrak{N}_1,\ldots,\mathfrak{N}_n]$$
$$= \operatorname{diag}\big[\min\{\mathfrak{M}_1,\mathfrak{N}_1\},\ldots,\min\{\mathfrak{M}_n,\mathfrak{N}_n\}\big].$$

Then $\odot_M$ is a continuous generalized t-norm.

(3) Let $\odot_L : \operatorname{diag} \mathsf{N}_n(\mho) \times \operatorname{diag} \mathsf{N}_n(\mho) \to \operatorname{diag} \mathsf{N}_n(\mho)$ such that

$$\mathfrak{M} \odot_L \mathfrak{N} = \operatorname{diag}[\mathfrak{M}_1,\ldots,\mathfrak{M}_n] \odot_L \operatorname{diag}[\mathfrak{N}_1,\ldots,\mathfrak{N}_n]$$
$$= \operatorname{diag}\big[\max\{\mathfrak{M}_1 + \mathfrak{N}_1 - 1, 0\},\ldots,\max\{\mathfrak{M}_n + \mathfrak{N}_n - 1, 0\}\big].$$

Then $\odot_L$ is a continuous generalized t-norm.

Furthermore, we present some numerical examples and compare the results:

$$\operatorname{diag}[0.5, 0.2, 1] \odot_M \operatorname{diag}[0.3, 0.7, 0] = \begin{bmatrix} 0.5 & & \\ & 0.2 & \\ & & 1 \end{bmatrix} \odot_M \begin{bmatrix} 0.3 & & \\ & 0.7 & \\ & & 0 \end{bmatrix} = \begin{bmatrix} 0.3 & & \\ & 0.2 & \\ & & 0 \end{bmatrix},$$

$$\operatorname{diag}[0.5, 0.2, 1] \odot_P \operatorname{diag}[0.3, 0.7, 0] = \begin{bmatrix} 0.5 & & \\ & 0.2 & \\ & & 1 \end{bmatrix} \odot_P \begin{bmatrix} 0.3 & & \\ & 0.7 & \\ & & 0 \end{bmatrix} = \begin{bmatrix} 0.15 & & \\ & 0.14 & \\ & & 0 \end{bmatrix},$$

$$\operatorname{diag}[0.5, 0.2, 1] \odot_L \operatorname{diag}[0.3, 0.7, 0] = \begin{bmatrix} 0.5 & & \\ & 0.2 & \\ & & 1 \end{bmatrix} \odot_L \begin{bmatrix} 0.3 & & \\ & 0.7 & \\ & & 0 \end{bmatrix} = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}.$$

Also, since

$$\operatorname{diag}[0.3, 0.2, 0] \succeq \operatorname{diag}[0.15, 0.14, 0] \succeq \operatorname{diag}[0, 0, 0],$$

we get

$$\operatorname{diag}[0.5, 0.2, 1] \odot_M \operatorname{diag}[0.3, 0.7, 0]$$
$$\succeq \operatorname{diag}[0.5, 0.2, 1] \odot_P \operatorname{diag}[0.3, 0.7, 0]$$
$$\succeq \operatorname{diag}[0.5, 0.2, 1] \odot_L \operatorname{diag}[0.3, 0.7, 0].$$

Consider $\mathcal{E}^+$, the set of matrix distribution functions, including left continuous and increasing maps $\Phi : \mathbb{R} \cup \{-\infty, \infty\} \to \operatorname{diag} \mathsf{N}_n(\mho)$ such that $\Phi_0 = \mathbf{0}$ and $\Phi_{+\infty} = \mathbf{1}$. Now $\Delta^+ \subseteq \mathcal{E}^+$ are all (proper) mappings $\Phi \in \mathcal{E}^+$ for which $\ell^- \Phi_\Theta = \lim_{\sigma \to \Theta^-} \Phi_\sigma = \mathbf{1}$. Notice that proper matrix distribution functions are the matrix distribution functions of real random variables q such that $P(|q| = \infty) = 0$.

On $\mathcal{E}^+$, we define "$\preceq$" as follows:

$$\Psi \preceq \Phi \quad \Longleftrightarrow \quad \Psi_\Theta \preceq \Phi_\Theta, \quad \forall \Theta \in \mathbb{R}.$$

Also

$$\nabla_r^s = \begin{cases} \mathbf{0}, & \text{if } r \leq s, \\ \mathbf{1}, & \text{if } r > s \end{cases}$$

belongs to $\mathcal{E}^+$ and for any matrix distribution function Φ, $\Phi \preceq \nabla^0$ [11, 13–15]. For example,

$$\Phi_\Theta = \begin{cases} \mathbf{0}, & \Theta \leq 0, \\ \operatorname{diag}[1 - \frac{1}{e^\Theta}, \frac{1}{e^{\frac{1}{\Theta}}}, \frac{1}{\frac{1}{\Theta}+1}], & \Theta > 0 \end{cases}$$

is a matrix distribution function in $\operatorname{diag} M_3(\mho)$. Notice that $\Phi_\Theta = \operatorname{diag}[\Phi_{1,\Theta}, \ldots, \Phi_{n,\Theta}]$, where $\Phi_{i,\Theta}$ are distribution functions, is a matrix distribution function.

Definition 1.2 Let $\mathbb{S}$ be a linear space, $\odot$ be a continuous generalized t-norm, and $\Phi : \mathbb{S} \to \Delta^+$ be a matrix distribution function. The triple $(\mathbb{S}, \Phi, \odot)$ is said to be a matrix Menger normed space if we have

(D-1) $\Phi_\Theta^s = \nabla_\Theta^0$ for any $\Theta > 0$ if and only if $s = 0$;
(D-2) $\Phi_\Theta^{\rho s} = \Phi_{\frac{\Theta}{|\rho|}}^s$ for all $s \in \mathbb{S}$ and $\rho \in \mathbb{C}$ with $\rho \neq 0$;
(D-3) $\Phi_{\Theta+\varrho}^{s+s'} \succeq \Phi_\Theta^s \odot \Phi_\varrho^{s'}$ for all $s, s' \in \mathbb{S}$ and $\Theta, \varrho \geq 0$.

For example, the matrix distribution function Φ given by

$$\Phi_\Theta^s = \begin{cases} \mathbf{0}, & \text{if } \Theta \leq 0, \\ \operatorname{diag}[\exp(-\frac{\|s\|}{\Theta}), \frac{\Theta}{\Theta+\|s\|}], & \text{if } \Theta > 0, \end{cases}$$

is a matrix Menger norm and $(\mathbb{S}, \Phi, \odot_M)$ and $(\mathbb{S}, \|\cdot\|)$ are a matrix Menger normed space and a linear normed space, respectively.

Definition 1.3 Let $(\mathbb{S}, \Phi, \odot)$ be a matrix Menger normed space and $\odot, \oslash$ be continuous generalized t-norms. If

(D-4) $\Phi_{\Theta\Theta'}^{ss'} \succeq \Phi_\Theta^s \odot \Phi_{\Theta'}^{s'}$ for any $s, s' \in \mathbb{S}$ and any $\Theta', \Theta > 0$,

then $(\mathbb{S}, \Phi, \odot, \oslash)$ is called a matrix Menger normed algebra.
If

$$\|ss'\| \leq \|s\|\|s'\| + \Theta'\|s'\| + \Theta\|s\| \quad (s, s' \in (\mathbb{S}, \|\cdot\|); \Theta, \Theta' > 0),$$

then with

$$\Phi_\Theta^s = \begin{cases} \mathbf{0}, & \Theta \leq 0, \\ \operatorname{diag}[\frac{1}{e^{\frac{\|s\|}{\Theta}}}, \frac{1}{1+\frac{\|s\|}{\Theta}}], & \Theta > 0, \end{cases}$$

$(\mathbb{S}, \Phi, \odot_M, \odot_P)$ is a matrix Menger normed algebra, and vice versa. A complete matrix Menger normed algebra is called a matrix Menger Banach algebra.

Consider matrix Menger Banach algebras Ω_1 and Ω_2. Let $(\mathcal{J}, \Pi, \Phi)$ be a probability measure space. Let $(\Omega_1, \mathfrak{B}_{\Omega_1})$ and $(\Omega_2, \mathfrak{B}_{\Omega_2})$ be Borel measurable spaces. Then a map $\mathcal{Q} : \mathcal{J} \times \Omega_1 \to \Omega_2$ is a random operator if $\{j : \mathcal{Q}(j, \mathsf{S}) \in \mathcal{C}\} \in \Pi$ for all S in Ω_1 and $\mathcal{C} \in \mathfrak{B}_{\Omega_2}$. Also $\mathcal{Q}$ is linear if

$$\mathcal{Q}(j, d\mathsf{S} + b\mathsf{R}) = d\mathcal{Q}(j, \mathsf{S}) + b\mathcal{Q}(j, \mathsf{R}), \quad \forall \mathsf{S}, \mathsf{R} \in \Omega_1, b, d \in \mathbb{R},$$

and $\mathcal{Q}$ is bounded if there is a $\mathsf{Q}(j) > 0$ such that

$$\Phi_{\mathsf{Q}(j)\Theta}^{\mathcal{Q}(j,\mathsf{S}) - \mathcal{Q}(j,\mathsf{R})} \succeq \Phi_{\Theta}^{\mathsf{S}-\mathsf{R}}, \quad \forall \mathsf{S}, \mathsf{R} \in \Omega_1, \Theta > 0.$$

Theorem 1.4 ([16]) *Consider a complete generalized metric space (ξ, ζ) and a strictly contractive function $\mathfrak{L} : \xi \to \xi$ with Lipschitz constant $\mathfrak{P} < 1$. Then, for every given element $j \in \xi$, either*

$$\zeta\left(\mathfrak{L}^\iota j, \mathfrak{L}^{\iota+1} j\right) = \infty$$

for each $\iota \in \mathbb{N}$ or there is $\iota_0 \in \mathbb{N}$ such that

(1) *$\zeta(\mathfrak{L}^\iota j, \mathfrak{L}^{\iota+1} j) < \infty, \forall m \geq m_0$;*
(2) *the limit point of the sequence $\{\mathfrak{L}^\iota j\}$ is the fixed point $\mathfrak{Z}^*$ of $\mathfrak{L}$;*
(3) *$\mathfrak{Z}^*$ is the unique fixed point of $\mathfrak{L}$ in the set $\mathsf{D} = \{\mathfrak{Z} \in \xi \mid \zeta(\mathfrak{L}^{\iota_0} j, \mathfrak{Z}) < \infty\}$;*
(4) *$(1 - \mathfrak{P})\zeta(\mathfrak{Z}, \mathfrak{Z}^*) \leq \zeta(\mathfrak{Z}, \mathfrak{L}\mathfrak{Z})$ for every $\mathfrak{Z} \in \mathsf{D}$.*

2 Direct method approximation of inequality (1.1)

In this section, we improve and get a generalization of results in [9] yielding a better approximation (see also [17–28]).

Lemma 2.1 *Suppose that $\mathcal{Q} : \mathcal{J} \times \Omega_1 \to \Omega_2$ is a random operator satisfying (1.1) for each $\mathsf{S}, \mathsf{R}, \mathsf{A} \in \Omega_1$ and $j \in \mathcal{J}$. Then $\mathcal{Q} : \mathcal{J} \times \Omega_1 \to \Omega_2$ is additive.*

Proof Putting $\mathsf{S} = \mathsf{R} = \mathsf{A} = 0$ in (1.1), we have

$$\Phi_{\Theta}^{2\mathcal{Q}(j,0)} \succeq \Phi_{\Theta}^{\mathcal{G}_1 \mathcal{Q}(j,0)} \circledast_M \Phi_{\Theta}^{\mathcal{G}_2(\mathcal{Q}j,0)},$$

and so

$$\Phi_{\frac{\Theta}{2}}^{\mathcal{Q}(j,0)} \succeq \Phi_{\frac{\Theta}{\max\{|\mathcal{G}_1|,|\mathcal{G}_2|\}}}^{\mathcal{Q}(j,0)}, \tag{2.1}$$

since $\max\{|\mathcal{G}_1|, |\mathcal{G}_2|\} < 2$, $\mathcal{Q}(j, 0) = 0$ for each $j \in \mathcal{J}$.

Now, putting $\mathsf{R} = 0$ in (1.1), we have

$$\Phi_{\Theta}^{\mathcal{Q}(j,\mathsf{S}+\mathsf{A}) - \mathcal{Q}(j,\mathsf{S}) - \mathcal{Q}(j,\mathsf{A})} \succeq \Phi_{\Theta}^{\mathcal{G}_1[\mathcal{Q}(j,\mathsf{S}+\mathsf{A}) - \mathcal{Q}(j,\mathsf{S}) - \mathcal{Q}(j,\mathsf{A})]} \tag{2.2}$$

and

$$\mathcal{Q}(j, S + A) - \mathcal{Q}(j, S) - \mathcal{Q}(j, A) = 0,$$

or $\mathcal{Q}(j, S + A) = \mathcal{Q}(j, S) + \mathcal{Q}(j, A)$ for all $S, A \in \Omega_1$, since $|\mathcal{G}_1| < 2$. So $\mathcal{Q}$ is additive. $\qquad \square$

Theorem 2.2 *Suppose the following assumptions hold*:

- $(\Omega_1, \Phi, \odot_M, \oslash_M)$ *is a matrix Menger Banach algebra*,
- $\phi : \Omega_1^3 \to \Delta^+$ *is a matrix distribution function*,
- *there exists a* $\mathfrak{P} < 1$ *such that* $\phi_\Theta^{\frac{S}{2}, \frac{R}{2}, \frac{A}{2}} \succeq \phi_{\frac{2\Theta}{\mathfrak{P}}}^{S,R,A}$ *for all* $S, R, A \in \Omega_1$ *and* $\Theta > 0$,
- *for all* $S, R, A \in \Omega_1$ *and* $\Theta > 0$,

$$\lim_{n \to \infty} \phi_{\frac{\Theta}{2^n}}^{\frac{S}{2^n}, \frac{R}{2^n}, \frac{A}{2^n}} = \nabla_\Theta^0, \tag{2.3}$$

- *the random operator* $\mathcal{Q} : \mathcal{J} \times \Omega_1 \to \Omega_2$ *satisfies* $\mathcal{Q}(j, 0) = 0$ *and*

$$\begin{aligned}
&\Phi_\Theta^{\mathcal{Q}(j,S+R+A)-\mathcal{Q}(j,S)-\mathcal{Q}(j,R)-\mathcal{Q}(j,A)} \\
&\quad \succeq \Phi_\Theta^{\mathcal{G}_1[\mathcal{Q}(j,S+A)-\mathcal{Q}(j,S)-\mathcal{Q}(j,A)]} \odot_M \Phi_\Theta^{\mathcal{G}_2[\mathcal{Q}(j,R+A)-\mathcal{Q}(j,R)-\mathcal{Q}(j,A)]} \odot_M \phi_\Theta^{S,R,A},
\end{aligned} \tag{2.4}$$

for all $S, R, A \in \Omega_1$, $j \in \mathcal{J}$, *and* $\Theta > 0$.

Then we can find a unique additive random operator $\mathcal{V} : \mathcal{J} \times \Omega_1 \to \Omega_2$ *such that*

$$\Phi_\Theta^{\mathcal{Q}(j,S)-\mathcal{V}(j,S)} \succeq \phi_{\frac{2(1-\mathfrak{P})\Theta}{\mathfrak{P}}}^{S,S,0}, \tag{2.5}$$

for all $S, R, A \in \Omega_1$, $j \in \mathcal{J}$, *and* $\Theta > 0$.

Proof Putting $A = 0$ and $S = R$ in (2.4), we get

$$\Phi_\Theta^{\mathcal{Q}(j,2S)-2\mathcal{Q}(j,S)} \succeq \phi_\Theta^{S,S,0}, \tag{2.6}$$

for all $S \in \Omega_1$, $j \in \mathcal{J}$, and $\Theta > 0$. Thus

$$\Phi_\Theta^{\mathcal{Q}(j,S)-2\mathcal{Q}(j,\frac{S}{2})} \succeq \phi_\Theta^{\frac{S}{2}, \frac{S}{2}, 0} \succeq \phi_{\frac{2\Theta}{\mathfrak{P}}}^{S,S,0}, \tag{2.7}$$

for all $S \in \Omega_1$, $j \in \mathcal{J}$, and $\Theta > 0$. Replacing S by $\frac{S}{2^n}$ in (2.7), we get

$$\begin{aligned}
\Phi_\Theta^{2^n \mathcal{Q}(j, \frac{S}{2^n}) - 2^{n+1} \mathcal{Q}(j, \frac{S}{2^{n+1}})} &\succeq \phi_{\frac{\Theta}{2^{n-1}\mathfrak{P}}}^{\frac{S}{2^n}, \frac{S}{2^n}, 0} \\
&\succeq \phi_{\frac{2}{\mathfrak{P}}(\frac{\Theta}{2^{n-1}\mathfrak{P}})}^{\frac{S}{2^{n-1}}, \frac{S}{2^{n-1}}, 0} \\
&\succeq \cdots \\
&\succeq \phi_{\frac{2}{\mathfrak{P}^{n+1}}\Theta}^{S,S,0}.
\end{aligned} \tag{2.8}$$

It follows from

$$2^n \mathcal{Q}\left(j, \frac{\mathsf{S}}{2^n}\right) - \mathcal{Q}(j, \mathsf{S}) = \sum_{k=1}^{n} \left(2^k \mathcal{Q}\left(j, \frac{\mathsf{S}}{2^k}\right) - 2^{k-1} \mathcal{Q}\left(j, \frac{\mathsf{S}}{2^{k-1}}\right)\right),$$

and (2.8) that

$$\Phi^{2^n \mathcal{Q}(j, \frac{\mathsf{S}}{2^n}) - \mathcal{Q}(j, \mathsf{S})}_{\sum_{k=1}^{n} \frac{\mathfrak{P}^k}{2} \Theta} \succeq \phi_\tau^{\mathsf{S},\mathsf{S},0} \odot_M \cdots \odot_M \phi_\Theta^{\mathsf{S},\mathsf{S},0} = \phi_\Theta^{\mathsf{S},\mathsf{S},0},$$

for all $\mathsf{S} \in \Omega_1, j \in \mathcal{J}, \Theta > 0$. That is,

$$\Phi_\Theta^{2^n \mathcal{Q}(j, \frac{\mathsf{S}}{2^n}) - \mathcal{Q}(j, \mathsf{S})} \succeq \phi^{\mathsf{S},\mathsf{S},0}_{\frac{\Theta}{\sum_{k=1}^{n} \frac{\mathfrak{P}^k}{2}}}. \tag{2.9}$$

Replacing S with $\frac{\mathsf{S}}{2^m}$ in (2.9), we get

$$\Phi_\Theta^{2^{n+m} \mathcal{Q}(j, \frac{\mathsf{S}}{2^{n+m}}) - 2^m \mathcal{Q}(j, \frac{\mathsf{S}}{2^m})} \succeq \phi^{\mathsf{S},\mathsf{S},0}_{\frac{\Theta}{\sum_{k=m+1}^{n+m} \frac{\mathfrak{P}^k}{2}}}. \tag{2.10}$$

Since $\phi^{\mathsf{S},\mathsf{S},0}_{\frac{\Theta}{\sum_{k=m+1}^{n+m} \frac{\mathfrak{P}^k}{2}}}$ tends to ∇_Θ^0 as $n, m \to \infty$, we conclude that the sequence $\{2^n \mathcal{Q}(j, \frac{\mathsf{S}}{2^n})\}$ is Cauchy for all $\mathsf{S} \in \Omega_1, j \in \mathcal{J}$. Since Ω_2 is a matrix Menger Banach algebra, the sequence $\{2^n \mathcal{Q}(j, \frac{\mathsf{S}}{2^n})\}$ is convergent. Consider the random operator $\mathcal{V} : \mathcal{J} \times \Omega_1 \to \Omega_2$ defined by

$$\mathcal{V}(j, \mathsf{S}) := \lim_{k \to \infty} 2^k \mathcal{Q}\left(j, \frac{\mathsf{S}}{2^k}\right),$$

for all $\mathsf{S} \in \Omega_1, j \in \mathcal{J}$. Putting $m = 0$ and letting $n \to \infty$ in (2.10), we obtain

$$\Phi_\Theta^{\mathcal{Q}(j, \mathsf{S}) - \mathcal{V}(j, \mathsf{S})} \succeq \phi^{\mathsf{S},\mathsf{S},0}_{\frac{2(1-\mathfrak{P})\Theta}{\mathfrak{P}}}, \tag{2.11}$$

for all $\mathsf{S} \in \Omega_1, j \in \mathcal{J}$, and $\Theta > 0$.

Now, (2.4) implies that

$$\Phi_\Theta^{\mathcal{V}(j, \mathsf{S}+\mathsf{R}+\mathsf{A}) - \mathcal{V}(j, \mathsf{S}) - \mathcal{V}(j, \mathsf{R}) + \mathcal{V}(j, \mathsf{A})}$$

$$= \lim_{n \to \infty} \Phi_\Theta^{2^n(\mathcal{Q}(j, \frac{\mathsf{S}+\mathsf{R}+\mathsf{A}}{2^n}) - \mathcal{Q}(j, \frac{\mathsf{S}}{2^n}) - \mathcal{Q}(j, \frac{\mathsf{R}}{2^n}) - \mathcal{Q}(j, \frac{\mathsf{A}}{2^n}))}$$

$$\succeq \lim_{n \to \infty} \Phi_\Theta^{2^n \mathcal{G}_1 [\mathcal{Q}(j, \frac{\mathsf{S}+\mathsf{A}}{2^n}) - \mathcal{Q}(j, \frac{\mathsf{S}}{2^n}) - \mathcal{Q}(j, \frac{\mathsf{A}}{2^n})]}$$

$$\odot_M \Phi_\Theta^{2^n \mathcal{G}_2 [\mathcal{Q}(j, \frac{\mathsf{R}+\mathsf{A}}{2^n}) - \mathcal{Q}(j, \frac{\mathsf{R}}{2^n}) - \mathcal{Q}(j, \frac{\mathsf{A}}{2^n})]} \odot_M \lim_{n \to \infty} \phi^{\frac{\mathsf{S}}{2^n}, \frac{\mathsf{S}}{2^n}, 0}_{\frac{\Theta}{2^n}}$$

$$\succeq \Phi_\Theta^{\mathcal{G}_1 [\mathcal{V}(j, \mathsf{S}+\mathsf{A}) - \mathcal{V}(j, \mathsf{S}) - \mathcal{V}(j, \mathsf{A})]} \circledast_M \Phi_\Theta^{\mathcal{G}_2 [\mathcal{V}(j, \mathsf{R}+\mathsf{A}) - \mathcal{V}(j, \mathsf{R}) - \mathcal{V}(j, \mathsf{A})]},$$

for all $\mathsf{S}, \mathsf{R}, \mathsf{A} \in \Omega_1, j \in \mathcal{J}, \Theta > 0$, since $\phi^{\frac{x}{2^n}, \frac{\mathsf{S}}{2^n}, 0}_{\frac{\Theta}{2^n}}$ tends to ∇_Θ^0 as $n \to \infty$. Thus

$$\Phi_\Theta^{\mathcal{V}(j, \mathsf{S}+\mathsf{R}+\mathsf{A}) - \mathcal{V}(j, \mathsf{S}) - \mathcal{V}(j, \mathsf{R}) - \mathcal{V}(j, \mathsf{A})} \succeq \Phi_\Theta^{\mathcal{G}_1 [\mathcal{V}(j, \mathsf{S}+\mathsf{A}) - \mathcal{V}(j, \mathsf{S}) - \mathcal{V}(j, \mathsf{A})]} \circledast_M \Phi_\Theta^{\mathcal{G}_2 [\mathcal{V}(j, \mathsf{R}+\mathsf{A}) - \mathcal{V}(j, \mathsf{R}) - \mathcal{V}(j, \mathsf{A})]},$$

for all $\mathsf{S}, \mathsf{R}, \mathsf{A} \in \Omega_1, j \in \mathcal{J}, \Theta > 0$. Lemma 2.1 implies that the random operator $\mathcal{V} : \mathcal{J} \times \Omega_1 \to \Omega_2$ is stochastic additive.

Now, to prove the uniqueness of the random operator $\mathcal{V}$, suppose that there exists a stochastic additive random operator $\mathcal{V}' : \mathcal{J} \times \Omega_1 \to \Omega_2$ which satisfies (2.5). Then

$$\Phi_\Theta^{\mathcal{V}(j,\mathsf{S})-\mathcal{V}'(j,\mathsf{S})} = \lim_{n \to \infty} \Phi_\Theta^{2^n \mathcal{V}(j,\frac{\mathsf{S}}{2^n})-2^n \mathcal{V}'(j,\frac{\mathsf{S}}{2^n})},$$

$$\Phi_\Theta^{2^n \mathcal{V}(j,\frac{\mathsf{S}}{2^n})-2^n \mathcal{V}'(j,\frac{\mathsf{S}}{2^n})} \succeq \Phi_{\frac{\Theta}{2}}^{2^n \mathcal{V}(j,\frac{\mathsf{S}}{2^n})-2^n \mathcal{Q}(j,\frac{\mathsf{S}}{2^n})} \odot_M \Phi_{\frac{\Theta}{2}}^{2^n \mathcal{V}'(j,\frac{\mathsf{S}}{2^n})-2^n \mathcal{Q}(j,\frac{\mathsf{S}}{2^n})}$$

$$\succeq \phi_{\frac{2(1-\mathfrak{P})\Theta}{2^n \mathfrak{P}}}^{\frac{\mathsf{S}}{2^n},\frac{\mathsf{S}}{2^n},0}$$

$$\succeq \phi_{\frac{2(1-\mathfrak{P})\Theta}{\mathfrak{P}^{n+1}}}^{\mathsf{S},\mathsf{S},0}.$$

Since $\lim_{n \to \infty} \frac{2(1-\mathfrak{P})}{\mathfrak{P}^{n+1}} = \infty$, we get that $\phi_{\frac{2(1-\mathfrak{P})\Theta}{\mathfrak{P}^{n+1}}}^{\mathsf{S},\mathsf{S},0}$ tends to ∇_Θ^0 as $n \to \infty$.

Thus we conclude that $\Phi_\Theta^{2^n \mathcal{V}(j,\frac{\mathsf{S}}{2^n})-2^n \mathcal{V}'(j,\frac{\mathsf{S}}{2^n})} = 1$ for all $\mathsf{S} \in \Omega_1, j \in \mathcal{J}, \Theta > 0$. So $\mathcal{V}(j,\mathsf{S}) = \mathcal{V}'(j,\mathsf{S})$ for all $\mathsf{S} \in \Omega_1$, and $j \in \mathcal{J}$. $\qquad\square$

Theorem 2.3 *Suppose* $(\Omega_1, \Phi, \odot_M, \oslash_M)$ *is a matrix Menger Banach algebra and* $\phi : \Omega_1^3 \to \Delta^+$ *is a matrix distribution function such that there exists a* $\mathfrak{P} < 1$ *with* $\phi_\Theta^{\mathsf{S},\mathsf{R},0} \succeq \phi_{\frac{\Theta}{2\mathfrak{P}}}^{\frac{\mathsf{S}}{2},\frac{\mathsf{R}}{2},0}$ *for all* $\mathsf{S}, \mathsf{R} \in \Omega_1, \lim_{n \to \infty} \phi_{2^n \Theta}^{2^n \mathsf{S}, 2^n \mathsf{R}, 0} = \nabla_\Theta^0$ *for any* $\mathsf{S}, \mathsf{R} \in \Omega_1, \Theta > 0$. *Assume that a random operator* $\mathcal{Q} : \mathcal{J} \times \Omega_1 \to \Omega_2$ *satisfies* (2.4) *and* $\mathcal{Q}(j,\mathsf{S}) = 0$ *for all* $\mathsf{S}, \mathsf{R} \in \Omega_1$ *and* $j \in \mathcal{J}$. *Then there exists a unique additive random operator* $\mathcal{V} : \mathcal{J} \times \Omega_1 \to \Omega_2$ *such that*

$$\Phi_\Theta^{\mathcal{Q}(j,\mathsf{S})-\mathcal{V}(j,\mathsf{S})} \succeq \phi_{2(1-\mathfrak{P})\Theta}^{\mathsf{S},\mathsf{S},0}, \tag{2.12}$$

for all $\mathsf{S} \in \Omega_1, j \in \mathcal{J}$, *and* $\Theta > 0$.

Proof Putting $\mathsf{A} = 0$ and $\mathsf{S} = \mathsf{R}$ in (2.4), we have

$$\Phi_\Theta^{\frac{1}{2}\mathcal{Q}(j,2\mathsf{S})-\mathcal{Q}(j,\mathsf{S})} \succeq \phi_{2\Theta}^{\mathsf{S},\mathsf{S},0}, \tag{2.13}$$

for all $\mathsf{S} \in \Omega_1, j \in \mathcal{J}$, and $\Theta > 0$. Thus

$$\Phi_\Theta^{\frac{1}{2}\mathcal{Q}(j,\mathsf{S})-\mathcal{Q}(j,2\mathsf{S})} \succeq \phi_\Theta^{2t,2t,0} \succeq \phi_{\frac{\Theta}{2\mathfrak{P}}}^{\mathsf{S},\mathsf{S},0}, \tag{2.14}$$

for all $\mathsf{S} \in \Omega_1, j \in \mathcal{J}$, and $\Theta > 0$. Replacing S by $2^n \mathsf{S}$ in (2.14), we have

$$\Phi_\Theta^{\frac{1}{2^n}\mathcal{Q}(j,2^n\mathsf{S})-\frac{1}{2^{n+1}}\mathcal{Q}(j,2^{n+1}\mathsf{S})} \succeq \phi_{2 \times 2^n \Theta}^{2^n \mathsf{S}, 2^n \mathsf{S}, 0} \succeq \phi_{\frac{2 \times 2^n}{(2\mathfrak{P})^n}\Theta}^{\mathsf{S},\mathsf{S},0}. \tag{2.15}$$

From

$$\frac{1}{2^n}\mathcal{Q}(j,2^n\mathsf{S}) - \mathcal{Q}(j,\mathsf{S}) = \sum_{k=0}^{n-1}\left(\frac{1}{2^{k+1}}\mathcal{Q}(j,2^{k+1}\mathsf{S}) - \frac{1}{2^k}\mathcal{Q}(j,2^k\mathsf{S})\right)$$

and (2.15), we get

$$\Phi^{\frac{1}{2^n}\mathcal{Q}(j,2^n\mathsf{S})-\mathcal{Q}(j,\mathsf{S})}_{\sum_{k=0}^{n-1}\frac{(2\mathfrak{P})^k}{2\times 2^k}\Theta} \succeq \phi^{\mathsf{S},\mathsf{S},0}_{\Theta},$$

for each $\mathsf{S} \in \Omega_1$, $j \in \mathcal{J}$, and $\Theta > 0$. That is,

$$\Phi^{\frac{1}{2^n}\mathcal{Q}(j,2^n\mathsf{S})-\mathcal{Q}(j,\mathsf{S})}_{\Theta} \succeq \phi^{\mathsf{S},\mathsf{S},0}_{\frac{\Theta}{\sum_{k=0}^{n-1}\frac{(2\mathfrak{P})^k}{2\times 2^k}}}. \tag{2.16}$$

Replacing S by $2^m\mathsf{S}$ in (2.16), we get

$$\Phi^{\frac{1}{2^{n+m}}\mathcal{Q}(j,2^{n+m}\mathsf{S})-\frac{1}{2^m}\mathcal{Q}(j,2^m\mathsf{S})}_{\Theta} \succeq \phi^{\mathsf{S},\mathsf{S},0}_{\frac{\Theta}{\sum_{k=m}^{n+m}\frac{(2\mathfrak{P})^k}{2\times 2^k}}}. \tag{2.17}$$

As $m, n \to \infty$, $\phi^{\mathsf{S},\mathsf{S},0}_{\frac{\Theta}{\sum_{k=m}^{n+m}\frac{(2\mathfrak{P})^k}{2\times 2^k}}}$ tends to ∇^0_Θ. It implies that the sequence $\{\frac{1}{2^n}\mathcal{Q}(j,2^n\mathsf{S})\}$ is Cauchy for any $\mathsf{S} \in \Omega_1$ and $j \in \mathcal{J}$. Since Ω_2 is a matrix Menger Banach algebra, the sequence $\{\frac{1}{2^n}\mathcal{Q}(j,2^n\mathsf{S})\}$ converges.

Now, we determine the random operator $\mathcal{V} : \mathcal{J} \times \Omega_1 \to \Omega_2$ as follows:

$$\mathcal{V}(j,\mathsf{S}) := \lim_{k\to\infty} \frac{1}{2^k}\mathcal{Q}(j,2^k\mathsf{S}),$$

for each $\mathsf{S} \in \Omega_1$ and $j \in \mathcal{J}$. Putting $m = 0$ and letting $n \to \infty$ in (2.17), we get

$$\Phi^{\mathcal{Q}(j,\mathsf{S})-\mathcal{V}(j,\mathsf{S})}_{\Theta} \succeq \phi^{\mathsf{S},\mathsf{S},0}_{2(1-\mathfrak{P})\Theta}, \tag{2.18}$$

for all $\mathsf{S} \in \Omega_1$, $j \in \mathcal{J}$, and $\Theta > 0$. Using Theorem 2.2 completes the proof. □

3 Fixed point method for approximating inequality (1.1)
We use the fixed point technique to get an approximation of the additive $(\mathcal{G}_1, \mathcal{G}_2)$-random operator inequality (1.1) in matrix Menger Banach algebras.

Theorem 3.1 *Suppose the following assumptions hold*:

- $(\Omega_1, \Phi, \odot_M, \oslash_M)$ *is a matrix Menger Banach algebra*,
- $\phi : \Omega_1^3 \to \Delta^+$ *is a matrix distribution function, satisfying* (2.3) *for all* $\mathsf{S}, \mathsf{R}, \mathsf{A} \in \Omega_1$, *and* $\Theta > 0$,
- *there exists a* $\mathfrak{P} < 1$, *such that* $\phi^{\frac{\mathsf{S}}{2},\frac{\mathsf{R}}{2},\frac{\mathsf{A}}{2}}_{\Theta} \succeq \phi^{\mathsf{S},\mathsf{R},\mathsf{A}}_{\frac{2\Theta}{\mathfrak{P}}}$ *for all* $\mathsf{S}, \mathsf{R}, \mathsf{A} \in \Omega_1$ *and* $\Theta > 0$,
- *the random operator* $\mathcal{Q} : \mathcal{J} \times \Omega_1 \to \Omega_2$ *satisfies* $\mathcal{Q}(j,0) = 0$ *and* (2.4) *for all* $\mathsf{S}, \mathsf{R}, \mathsf{A} \in \Omega_1$, $j \in \mathcal{J}$, *and* $\Theta > 0$,

Then we can find a unique additive random operator $\mathcal{V} : \mathcal{J} \times \Omega_1 \to \Omega_2$ *satisfying* (2.5) *for all* $\mathsf{S}, \mathsf{R}, \mathsf{A} \in \Omega_1$, $j \in \mathcal{J}$, *and* $\Theta > 0$.

Proof Putting $\mathsf{S} = \mathsf{R}$ and $\mathsf{A} = 0$ in (2.4), we have

$$\Phi^{2\mathcal{Q}(j,\frac{\mathsf{S}}{2})-\mathcal{Q}(j,\mathsf{S})}_{\Theta} \geq \phi^{\frac{\mathsf{S}}{2},\frac{\mathsf{S}}{2},0}_{\Theta} \tag{3.1}$$

almost everywhere for each $\mathsf{S} \in \Omega_1$ and $\Theta > 0$.

Consider

$$\xi := \big\{ \mathcal{Z} : \mathcal{J} \times \Omega_1 \to \Omega_2, \mathcal{Z}(j,0) = 0 \big\}$$

and the generalized metric defined as follows:

$$\zeta(\mathcal{P}, \mathcal{Z}) = \inf\big\{ \hbar \in \mathbb{R}_+ : \Phi_{\hbar\Theta}^{\mathcal{P}(j,S)-\mathcal{Z}(j,S)} \geq \phi_{\Theta}^{S,S,0}, \forall S \in \Omega_1, \Theta > 0 \big\}.$$

In [29], Miheţ and Radu showed that (ξ, ζ) is complete.

We define the linear function $\mathfrak{L} : \xi \to \xi$ as

$$\mathfrak{L}\mathcal{H}(j,S) := 2\mathcal{H}\left(j, \frac{S}{2}\right)$$

almost everywhere for each $S \in \Omega_1$. Consider $\mathcal{H}, \mathcal{K} \in \xi$ such that $\zeta(\mathcal{H}, \mathcal{K}) = \varepsilon$. Then

$$\Phi_{\varepsilon\Theta}^{\mathcal{H}(j,S)-\mathcal{K}(j,S)} \geq \phi_{\Theta}^{S,S,0}$$

almost everywhere for any $S \in \Omega_1$ and $\Theta > 0$, and also

$$\begin{aligned} \Phi_{\mathfrak{P}\varepsilon\Theta}^{\mathfrak{L}\mathcal{H}(j,S)-\mathfrak{L}\mathcal{K}(j,S)} &= \Phi_{\frac{\mathfrak{P}\varepsilon\Theta}{2}}^{\mathcal{H}(j,\frac{S}{2})-\mathcal{K}(j,\frac{S}{2})} \\ &\geq \phi_{\frac{\mathfrak{P}\Theta}{2}}^{\frac{S}{2},\frac{S}{2},0} \\ &\geq \phi_{\Theta}^{S,S,0} \end{aligned}$$

almost everywhere for each $S \in \Omega_1$ and $\Theta > 0$. Thus, from $\zeta(\mathcal{H}, \mathcal{K}) = \varepsilon$, we conclude that $\zeta(\mathfrak{L}\mathcal{H}, \mathfrak{L}\mathcal{K}) \leq \mathfrak{P}\varepsilon$, and so

$$\zeta(\mathfrak{L}\mathcal{H}, \mathfrak{L}\mathcal{K}) \leq \mathfrak{P}\zeta(\mathcal{H}, \mathcal{K}),$$

for each $\mathcal{H}, \mathcal{K} \in \xi$.

By (3.1), we have that

$$\Phi_{\frac{\mathfrak{P}\Theta}{2}}^{2\mathcal{Q}(j,\frac{S}{2})-\mathcal{Q}(j,S)} \geq \phi_{\Theta}^{S,S,0}$$

almost everywhere for each $S \in \Omega_1$ and $\Theta > 0$, which implies that $\zeta(\mathcal{Q}, \mathfrak{L}\mathcal{Q}) \leq \frac{\mathfrak{P}}{2}$.

Theorem 1.4 implies that there is a random operator $\mathcal{V} : \mathcal{J} \times \Omega_1 \to \Omega_2$ such that:

(1) A fixed point for the function $\mathfrak{L}$ is $\mathcal{V}$,

$$\mathcal{V}(j,x) = 2\mathcal{V}\left(j, \frac{x}{2}\right) \tag{3.2}$$

almost everywhere for each $x \in \Omega_1$, which is unique in the set

$$D = \big\{ \mathcal{H} \in \xi : \zeta(\mathcal{H}, \mathcal{K}) < \infty \big\};$$

(2)　$\zeta(\mathfrak{L}^p \mathcal{Q}, \mathcal{V}) \to 0$ as $p \to \infty$, which implies that

$$\lim_{p \to \infty} 2^p \mathcal{Q}\left(j, \frac{\mathsf{S}}{2^p}\right) = \mathcal{V}(j, \mathsf{S}) \tag{3.3}$$

almost everywhere for each $\mathsf{S} \in \Omega_1$;

(3)　$\zeta(\mathcal{Q}, \mathcal{V}) \leq \frac{1}{1 - \mathfrak{P}} \zeta(\mathcal{Q}, \mathfrak{L}\mathcal{Q})$, which implies that

$$\Phi_\Theta^{\mathcal{Q}(j,\mathsf{S}) - \mathcal{V}(j,\mathsf{S})} \geq \phi_{\frac{2(1-\mathfrak{P})}{\mathfrak{P}}\Theta}^{\mathsf{S},\mathsf{S},0}$$

almost everywhere for each $\mathsf{S} \in \Omega_1$ and $\Theta > 0$.

Using (2.4) and (3.3), we have

$$\Phi_\Theta^{\mathcal{V}(j,\mathsf{S}+\mathsf{R}+\mathsf{A}) - \mathcal{V}(j,\mathsf{S}) - \mathcal{V}(j,\mathsf{R}) + \mathcal{V}(j,\mathsf{A})}$$

$$= \lim_{n \to \infty} \Phi_\Theta^{2^n(\mathcal{Q}(j,\frac{\mathsf{S}+\mathsf{R}+\mathsf{A}}{2^n}) - \mathcal{Q}(j,\frac{\mathsf{S}}{2^n}) - \mathcal{Q}(j,\frac{\mathsf{R}}{2^n}) - \mathcal{Q}(j,\frac{\mathsf{A}}{2^n}))}$$

$$\succeq \lim_{n \to \infty} \Phi_\Theta^{2^n \mathcal{G}_1[\mathcal{Q}(j,\frac{\mathsf{S}+\mathsf{A}}{2^n}) - \mathcal{Q}(j,\frac{\mathsf{S}}{2^n}) - \mathcal{Q}(j,\frac{\mathsf{A}}{2^n})]}$$

$$\odot_M \Phi_\Theta^{2^n \mathcal{G}_2[\mathcal{Q}(j,\frac{\mathsf{R}+\mathsf{A}}{2^n}) - \mathcal{Q}(j,\frac{\mathsf{R}}{2^n}) - \mathcal{Q}(j,\frac{\mathsf{A}}{2^n})]} \odot_M \lim_{n \to \infty} \phi_{\frac{\Theta}{2^n}}^{\frac{\mathsf{S}}{2^n},\frac{\mathsf{S}}{2^n},0}$$

$$\succeq \Phi_\Theta^{\mathcal{G}_1[\mathcal{V}(j,\mathsf{S}+\mathsf{A}) - \mathcal{V}(j,\mathsf{S}) - \mathcal{V}(j,\mathsf{A})]} \circledast_M \Phi_\Theta^{\mathcal{G}_2[\mathcal{V}(j,\mathsf{R}+\mathsf{A}) - \mathcal{V}(j,\mathsf{R}) - \mathcal{V}(j,\mathsf{A})]}$$

almost everywhere for any $\mathsf{S}, \mathsf{R}, \mathsf{A} \in \Omega_1$ and $\Theta > 0$. Thus

$$\Phi_\Theta^{\mathcal{V}(j,\mathsf{S}+\mathsf{R}+\mathsf{A}) - \mathcal{V}(j,\mathsf{S}) - \mathcal{V}(j,\mathsf{R}) - \mathcal{V}(j,\mathsf{A})} \succeq \Phi_\Theta^{\mathcal{G}_1[\mathcal{V}(j,\mathsf{S}+\mathsf{A}) - \mathcal{V}(j,\mathsf{S}) - \mathcal{V}(j,\mathsf{A})]} \circledast_M \Phi_\Theta^{\mathcal{G}_2[\mathcal{V}(j,\mathsf{R}+\mathsf{A}) - \mathcal{V}(j,\mathsf{R}) - \mathcal{V}(j,\mathsf{A})]}$$

almost everywhere for each $\mathsf{S}, \mathsf{R}, \mathsf{A} \in \Omega_1$ and $\Theta > 0$. Now, Lemma 2.1 implies that the $\mathcal{V}$ is an additive random operator. □

Theorem 3.2 *Suppose $(\Omega_1, \Phi, \odot_M, \oslash_M)$ is a matrix Menger Banach algebra and $\phi : \Omega_1^3 \to \Delta^+$ is a matrix distribution function such that there exists a $\mathfrak{P} < 1$ with $\phi_\Theta^{\mathsf{S},\mathsf{R},0} \succeq \phi_{\frac{\Theta}{2\mathfrak{P}}}^{\frac{\mathsf{S}}{2},\frac{\mathsf{R}}{2},0}$ for all $\mathsf{S}, \mathsf{R} \in \Omega_1$, $\lim_{n \to \infty} \phi_{2^n \Theta}^{2^n \mathsf{S}, 2^n \mathsf{R}, 0} = \nabla_\Theta^0$ for any $\mathsf{S}, \mathsf{R} \in \Omega_1$, $\Theta > 0$. Assume that a random operator $\mathcal{Q} : \mathcal{J} \times \Omega_1 \to \Omega_2$ satisfies (2.4) and $\mathcal{Q}(j, \mathsf{S}) = 0$ for all $\mathsf{S}, \mathsf{R} \in \Omega_1$ and $j \in \mathcal{J}$. Then there exists a unique additive random operator $\mathcal{V} : \mathcal{J} \times \Omega_1 \to \Omega_2$ satisfying (2.12) for all $\mathsf{S} \in \Omega_1, j \in \mathcal{J}$, and $\Theta > 0$.*

Proof Let (ξ, ζ) be the same as in the proof of Theorem 2.2. We define the linear function $\mathfrak{L} : \xi \to \xi$ as

$$\mathfrak{L}\mathcal{H}(j, \mathsf{S}) := \frac{1}{2}\mathcal{H}(j, 2\mathsf{S})$$

almost everywhere for each $\mathsf{S} \in \Omega_1$. Using (2.5), we obtain

$$\Phi_\Theta^{\frac{\mathcal{Q}(j,2\mathsf{S})}{2} - \mathcal{Q}(j,\mathsf{S})} \geq \phi_{\frac{\Theta}{\mathfrak{P}}}^{\frac{\mathsf{S}}{2},\frac{\mathsf{S}}{2},0}$$

almost everywhere for each $\mathsf{S} \in \Omega_1$ and $\Theta > 0$.

By a similar method as in the proof of Theorem 3.1, the proof will be completed. □

4 Application with stochastic Mittag-Leffler and Fox's $\mathbb{H}$-control functions

In this section, we apply stochastic Mittag-Leffler control functions and stochastic Fox $\mathbb{H}$-control functions to get a better approximation in the random operator inequality (1.1). Now, we introduce the concepts of the above stochastic control functions.

Suppose $\mathbb{T}$ is a vector space and $\Theta_\bullet > 0$.

We will present an example of a stochastic normed space by means of Mittag-Leffler function, but before that we introduce Mittag-Leffler function itself.

The special function

$$\Xi_\sigma(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\sigma k)}, \quad \sigma \in \mathbb{C}, \Re(\sigma) > 0, z \in \mathbb{C} \tag{4.1}$$

is called Mittag-Leffler function [3], where $\mathbb{C}$ and Γ are respectively the set of complex numbers and the gamma function.

Consider the one-parameter Mittag-Leffler function

$$\Xi_\sigma\left(-\frac{\|\mathbf{T}_\circ\|}{\Theta_\bullet}\right) = \sum_{k=0}^{\infty} \frac{(-\frac{\|\mathbf{T}_\circ\|}{\Theta_\bullet})^k}{\Gamma(1+\sigma k)}, \quad \sigma \in (0,1], \mathbf{T}_\circ \in \mathbb{T}, \Theta_\bullet > 0, \|\mathbf{T}_\circ\| < \frac{1}{n}, n \in \mathbb{N}.$$

Now we want to show in the following four steps that $(\mathbb{T}, \Xi_\sigma(-\frac{\|\mathbf{T}_\circ\|}{\Theta_\bullet}), \min)$ is a random normed space.

(1) If $\sigma \in (0,1]$, then $\Xi_\sigma(0) = 1$ and $\lim_{\mathbf{T}_\circ \to -\infty} \Xi_\sigma(\mathbf{T}_\circ) = 0$. Hence we can conclude that Ξ_σ is an increasing function for all $\sigma \in (0,1]$, and also we have $\Xi_\sigma \in (0,1]$.

(2) It is straightforward to show that $\Xi_\sigma(-\frac{\|\mathbf{T}_\circ\|}{\Theta_\bullet}) = 1$ for every $\Theta_\bullet > 0$, if and only if $\mathbf{T}_\circ = 0$.

(3) For any $\mathbf{T}_\circ \in \mathbb{T}$ and $\Theta_\bullet > 0$, we have

$$\begin{aligned}
\Xi_\sigma\left(-\frac{\|\hbar\mathbf{T}_\circ\|}{\Theta_\bullet}\right) &= \sum_{k=0}^{\infty} \frac{(-\frac{\|\hbar\mathbf{T}_\circ\|}{\Theta_\bullet})^k}{\Gamma(1+\sigma k)} \\
&= \sum_{k=0}^{\infty} \frac{(-\frac{\|\mathbf{T}_\circ\|}{\frac{\Theta_\bullet}{|\hbar|}})^k}{\Gamma(1+\sigma k)} \\
&= \Xi_\sigma\left(-\frac{\|\mathbf{T}_\circ\|}{\frac{\Theta_\bullet}{|\hbar|}}\right).
\end{aligned}$$

(4) Let $\Xi_\sigma(-\frac{\|\mathbf{T}_\circ\|}{\Theta_\bullet}) \leq \Xi_\sigma(-\frac{\|\mathbf{T}_\circ'\|}{\Theta_\bullet'})$. Then we have $\frac{\|\mathbf{T}_\circ'\|}{\Theta_\bullet'} \leq \frac{\|\mathbf{T}_\circ\|}{\Theta_\bullet}$ for all $\mathbf{T}_\circ, \mathbf{T}_\circ' \in \mathbb{T}$ and $\Theta_\bullet, \Theta_\bullet' > 0$. Now, if $\mathbf{T}_\circ = \mathbf{T}_\circ'$, then we have $\Theta_\bullet \leq \Theta_\bullet'$. Otherwise, we have

$$\begin{aligned}
\frac{\|\mathbf{T}_\circ\|}{\Theta_\bullet} + \frac{\|\mathbf{T}_\circ\|}{\Theta_\bullet} &\geq \frac{\|\mathbf{T}_\circ\|}{\Theta_\bullet} + \frac{\|\mathbf{T}_\circ'\|}{\Theta_\bullet'} \\
&\geq 2\frac{\|\mathbf{T}_\circ\|}{\Theta_\bullet + \Theta_\bullet'} + 2\frac{\|\mathbf{T}_\circ'\|}{\Theta_\bullet + \Theta_\bullet'} \\
&\geq 2\frac{\|\mathbf{T}_\circ + \mathbf{T}_\circ'\|}{\Theta_\bullet + \Theta_\bullet'},
\end{aligned}$$

and so $\frac{\|\mathbf{T}_\circ\|}{\Theta_\bullet} \ge \frac{\|\mathbf{T}_\circ + \mathbf{T}'_\circ\|}{\Theta_\bullet + \Theta'_\bullet}$. But $-\frac{\|\mathbf{T}_\circ\|}{\Theta_\bullet} \le -\frac{\|\mathbf{T}_\circ + \mathbf{T}'_\circ\|}{\Theta_\bullet + \Theta'_\bullet}$ and also

$$\sum_{k=0}^{\infty} \frac{(-\frac{\|\mathbf{T}_\circ\|}{\Theta_\bullet})^k}{\Gamma(1 + \sigma k)} \le \sum_{k=0}^{\infty} \frac{(-\frac{\|\mathbf{T}_\circ + \mathbf{T}'_\circ\|}{\Theta_\bullet + \Theta'_\bullet})^k}{\Gamma(1 + \sigma k)}, \tag{4.2}$$

which implies that

$$\Xi_\sigma\left(-\frac{\|\mathbf{T}_\circ\|}{\Theta_\bullet}\right) \le \Xi_\sigma\left(-\frac{\|\mathbf{T}_\circ + \mathbf{T}'_\circ\|}{\Theta_\bullet + \Theta'_\bullet}\right).$$

Hence we have

$$\Xi_\sigma\left(-\frac{\|\mathbf{T}_\circ + \mathbf{T}'_\circ\|}{\Theta_\bullet + \Theta'_\bullet}\right) \ge \min\left\{\Xi_\sigma\left(-\frac{\|\mathbf{T}_\circ\|}{\Theta_\bullet}\right), \Xi_\sigma\left(-\frac{\|\mathbf{T}'_\circ\|}{\Theta'_\bullet}\right)\right\},$$

for all $\mathbf{T}_\circ, \mathbf{T}'_\circ \in \mathbb{T}$ and $\Theta_\bullet, \Theta'_\bullet > 0$. Therefore, $\Xi_\sigma(-\frac{\|\mathbf{T}_\circ\|}{\Theta_\bullet})$ is a stochastic Mittag-Leffler control function.

Now, we introduce the Fox $\mathbb{H}$-function [30] as follows:

$$\mathbb{H}_{p,q}^{m,n}\left[z\big|_{(b_j,B_j)_{1,q}}^{(a_j,A_j)_{1,p}}\right] := \frac{1}{2\pi i} \int_{\mathfrak{L}} \theta(\xi) z^\xi \, d\xi,$$
$$i^2 = 1, \quad z \in \mathbb{C}\backslash\{0\}, \quad z^\xi = \exp\big(\xi\big[\log|z| + i\arg(z)\big]\big), \tag{4.3}$$

in which $\log|z|$ denotes the natural logarithm of $|z|$ and $\arg(z)$ is not necessarily the principal value. For convenience, let

$$\theta(\xi) := \frac{\prod_{j=1}^{m} \Gamma(b_j - B_j\xi) \prod_{j=1}^{n} \Gamma(1 - a_j + A_j\xi)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + B_j\xi) \prod_{j=n+1}^{p} \Gamma(a_j - A_j\xi)},$$

with an empty product interpreted as 1, and the integers m, n, p, q satisfy the inequalities

$$0 \le n \le p \quad \text{and} \quad 1 \le m \le q,$$

where the coefficients

$$A_j > 0 \quad (j = 1, \ldots, p) \quad \text{and} \quad B_j > 0 \quad (j = 1, \ldots, q)$$

and the complex parameters

$$a_j \quad (j = 1, \ldots, p) \quad \text{and} \quad b_j \quad (j = 1, \ldots, q)$$

are so constrained that no poles of integrand in (4.3) coincide, and $\mathfrak{L}$ is a suitable contour of the Mellin–Barnes type (in the complex ξ-plane) which separates the poles of one product from those of the other. In addition, if we let

$$\ell := \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0,$$

then the integral in (4.3) converges absolutely and defines the $\mathbb{H}$-function, analytic in the sector

$$\left| \arg(z) \right| < \frac{1}{2} \ell \pi,$$

the point $z = 0$ being tacitly excluded. In fact, the $\mathbb{H}$-function makes sense and defines an analytic function of z also when either

$$\epsilon := \sum_{j=1}^{p} A_j - \sum_{j=1}^{q} B_j < 0 \quad \text{and} \quad 0 < |z| < \infty,$$

or

$$\epsilon = 0 \quad \text{and} \quad 0 < |z| < R := \prod_{j=1}^{p} A_j^{-A_j} \prod_{j=1}^{q} B_j^{B_j}.$$

In a similar way, we can show that $\mathbb{H}_{p,q}^{m,n}\left[-\frac{\|\mathbf{T}_\circ\|}{\Theta_\bullet}\big|_{(b_j,B_j)_{1,q}}^{(a_j,A_j)_{1,p}}\right]$ is a stochastic Fox $\mathbb{H}$-control function, for all $\mathbf{T}_\circ \in \mathbb{T}$ and $\Theta_\bullet > 0$, $A_j, a_j > 0$ $(j = 1,\ldots,p)$, $B_j, b_j > 0$ $(j = 1,\ldots,q)$ and $p, q \in \mathbb{N}$.

Corollary 4.1 *Let $(\Omega_1, \Phi, \odot_M, \odot_M)$ be a matrix Menger Banach algebra. Suppose that $\mathsf{M} > 1$ and W is a nonnegative real number, and $\mathcal{Q} : \mathcal{J} \times \Omega_1 \to \Omega_2$ is a random operator satisfying $\mathcal{Q}(j,0) = 0$ and*

$$\Phi_\Theta^{\mathcal{Q}(j,\mathsf{S}+\mathsf{R}+\mathsf{A})-\mathcal{Q}(j,\mathsf{S})-\mathcal{Q}(j,\mathsf{R})-\mathcal{Q}(j,\mathsf{A})}$$

$$\succeq \Phi_\Theta^{\mathcal{G}_1[\mathcal{Q}(j,\mathsf{S}+\mathsf{A})-\mathcal{Q}(j,\mathsf{S})-\mathcal{Q}(j,\mathsf{A})]} \odot_M \Phi_\Theta^{\mathcal{G}_2[\mathcal{Q}(j,\mathsf{R}+\mathsf{A})-\mathcal{Q}(j,\mathsf{R})-\mathcal{Q}(j,\mathsf{A})]}$$

$$\odot_M \operatorname{diag}\left[\exp\left(-\frac{\mathsf{W}(\|\mathsf{S}\|^{\mathsf{M}} + \|\mathsf{R}\|^{\mathsf{M}} + \|\mathsf{A}\|^{\mathsf{M}})}{\Theta} \right), \right.$$

$$\frac{\Theta}{\Theta + \mathsf{W}(\|\mathsf{S}\|^{\mathsf{M}} + \|\mathsf{R}\|^{\mathsf{M}} + \|\mathsf{A}\|^{\mathsf{M}})} \Xi_\sigma\left(-\frac{\mathsf{W}(\|\mathsf{S}\|^{\mathsf{M}} + \|\mathsf{R}\|^{\mathsf{M}} + \|\mathsf{A}\|^{\mathsf{M}})}{\Theta} \right),$$

$$\left. \mathbb{H}_{p,q}^{m,n}\left[-\frac{\mathsf{W}(\|\mathsf{S}\|^{\mathsf{M}} + \|\mathsf{R}\|^{\mathsf{M}} + \|\mathsf{A}\|^{\mathsf{M}})}{\Theta} \Big|_{(b_j,B_j)_{1,q}}^{(a_j,A_j)_{1,p}} \right] \right] \tag{4.4}$$

for all $\mathsf{S}, \mathsf{R}, \mathsf{A} \in \Omega_1$, $j \in \mathcal{J}$, $0 < \sigma \leq 1$, $\Theta > 0$, $A_j, a_j > 0$ $(j = 1,\ldots,p)$, $B_j, b_j > 0$ $(j = 1,\ldots,q)$, and $p, q \in \mathbb{N}$. Then there exists a unique additive random operator $\mathcal{V} : \mathcal{J} \times \Omega_1 \to \Omega_2$ such that

$$\Phi_\Theta^{\mathcal{Q}(j,\mathsf{S})-\mathcal{V}(j,\mathsf{S})} \succeq \operatorname{diag}\left[\exp\left(-\frac{2^{\mathsf{M}+2}\mathsf{W}\|\mathsf{S}\|^{\mathsf{M}}}{2(2^{\mathsf{M}} - 2)\Theta} \right), \right.$$

$$\frac{2(2^{\mathsf{M}} - 2)\Theta}{2(2^{\mathsf{M}} - 2)\Theta + 2^{\mathsf{M}+2}\mathsf{W}\|\mathsf{S}\|^{\mathsf{M}}} \Xi_\sigma\left(-\frac{2^{\mathsf{M}+2}\mathsf{W}\|\mathsf{S}\|^{\mathsf{M}}}{2(2^{\mathsf{M}} - 2)\Theta} \right),$$

$$\left. \mathbb{H}_{p,q}^{m,n}\left[-\frac{2^{\mathsf{M}+2}\mathsf{W}\|\mathsf{S}\|^{\mathsf{M}}}{2(2^{\mathsf{M}} - 2)\Theta} \Big|_{(b_j,B_j)_{1,q}}^{(a_j,A_j)_{1,p}} \right] \right], \tag{4.5}$$

for all $\mathsf{S} \in \Omega_1$, $j \in \mathcal{J}$, $0 < \sigma \leq 1$, $\Theta > 0$, $A_j, a_j > 0$ $(j = 1,\ldots,p)$, $B_j, b_j > 0$ $(j = 1,\ldots,q)$, and $p, q \in \mathbb{N}$.

Proof It follows from Theorem 2.2 by putting

$$
\phi_\Theta^{\mathsf{S},\mathsf{R},\mathsf{A}} = \mathrm{diag}\Bigg[\exp\bigg(-\frac{\mathsf{W}(\|\mathsf{S}\|^{\mathsf{M}} + \|\mathsf{R}\|^{\mathsf{M}} + \|\mathsf{A}\|^{\mathsf{M}})}{\Theta}\bigg), \frac{\Theta}{\Theta + \mathsf{W}(\|\mathsf{S}\|^{\mathsf{M}} + \|\mathsf{R}\|^{\mathsf{M}} + \|\mathsf{A}\|^{\mathsf{M}})},
$$
$$
\boldsymbol{\Xi}_\sigma\bigg(-\frac{\mathsf{W}(\|\mathsf{S}\|^{\mathsf{M}} + \|\mathsf{R}\|^{\mathsf{M}} + \|\mathsf{A}\|^{\mathsf{M}})}{\Theta}\bigg),
$$
$$
\mathbb{H}_{p,q}^{m,n}\bigg[-\frac{\mathsf{W}(\|\mathsf{S}\|^{\mathsf{M}} + \|\mathsf{R}\|^{\mathsf{M}} + \|\mathsf{A}\|^{\mathsf{M}})}{\Theta}\,\bigg|_{(b_j,B_j)_{1,q}}^{(a_j,A_j)_{1,p}}\bigg]\Bigg],
$$

for all $\mathsf{S},\mathsf{R},\mathsf{A} \in \Omega_1$, $j \in \mathcal{J}$, $\Theta > 0$, $0 < \sigma \leq 1$, $A_j, a_j > 0$ ($j = 1,\dots,p$), $B_j, b_j > 0$ ($j = 1,\dots,q$), $p,q \in \mathbb{N}$, and $\mathfrak{P} = 2^{1-\mathsf{M}}$. $\qquad\square$

Corollary 4.2 *Assume* $(\Omega_1, \Phi, \odot_M, \oslash_M)$ *is a matrix Menger Banach algebra. Suppose that* $\mathsf{M} < 1, \mathsf{W} \geq 0$, *and* $\mathcal{Q} : \mathcal{J} \times \Omega_1 \to \Omega_2$ *is a random operator satisfying* (4.4) *and* $\mathcal{Q}(j,\mathsf{S}) = 0$ *for all* $\mathsf{S} \in \Omega_1$ *and* $j \in \mathcal{J}$. *Then there exists a unique additive random operator* $\mathcal{V} : \mathcal{J} \times \Omega_1 \to \Omega_2$ *such that*

$$
\Phi_\Theta^{\mathcal{Q}(j,\mathsf{S})-\mathcal{V}(j,\mathsf{S})} \succeq \mathrm{diag}\Bigg[\exp\bigg(-\frac{2\mathsf{W}\|\mathsf{S}\|^{\mathsf{M}}}{(2-2^{\mathsf{M}})\Theta}\bigg), \frac{(2-2^{\mathsf{M}})\Theta}{(2-2^{\mathsf{M}})\Theta + 2\mathsf{W}\|\mathsf{S}\|^{\mathsf{M}}},
$$
$$
\boldsymbol{\Xi}_\sigma\bigg(-\frac{2\mathsf{W}\|\mathsf{S}\|^{\mathsf{M}}}{(2-2^{\mathsf{M}})\Theta}\bigg), \mathbb{H}_{p,q}^{m,n}\bigg[-\frac{2^{\mathsf{M}+2}\mathsf{W}\|\mathsf{S}\|^{\mathsf{M}}}{2(2^{\mathsf{M}}-2)\Theta}\,\bigg|_{(b_j,B_j)_{1,q}}^{(a_j,A_j)_{1,p}}\bigg]\Bigg], \tag{4.6}
$$

for all $\mathsf{S} \in \Omega_1$, $j \in \mathcal{J}$, $0 < \sigma \leq 1$, $\Theta > 0$, $A_j, a_j > 0$ ($j = 1,\dots,p$), $B_j, b_j > 0$ ($j = 1,\dots,q$), *and* $p,q \in \mathbb{N}$.

Proof The claim follows from Theorem 2.3 by putting

$$
\phi_\Theta^{\mathsf{S},\mathsf{R},\mathsf{A}} = \mathrm{diag}\Bigg[\exp\bigg(-\frac{\mathsf{W}(\|\mathsf{S}\|^{\mathsf{M}} + \|\mathsf{R}\|^{\mathsf{M}} + \|\mathsf{A}\|^{\mathsf{M}})}{\Theta}\bigg), \frac{\Theta}{\Theta + \mathsf{W}(\|\mathsf{S}\|^{\mathsf{M}} + \|\mathsf{R}\|^{\mathsf{M}} + \|\mathsf{A}\|^{\mathsf{M}})},
$$
$$
\boldsymbol{\Xi}_\sigma\bigg(-\frac{\mathsf{W}(\|\mathsf{S}\|^{\mathsf{M}} + \|\mathsf{R}\|^{\mathsf{M}} + \|\mathsf{A}\|^{\mathsf{M}})}{\Theta}\bigg),
$$
$$
\mathbb{H}_{p,q}^{m,n}\bigg[-\frac{\mathsf{W}(\|\mathsf{S}\|^{\mathsf{M}} + \|\mathsf{R}\|^{\mathsf{M}} + \|\mathsf{A}\|^{\mathsf{M}})}{\Theta}\,\bigg|_{(b_j,B_j)_{1,q}}^{(a_j,A_j)_{1,p}}\bigg]\Bigg],
$$

for all $\mathsf{S} \in \Omega_1$, $j \in \mathcal{J}$, $\Theta > 0$, $0 < \sigma \leq 1$, $A_j, a_j > 0$ ($j = 1,\dots,p$), $B_j, b_j > 0$ ($j = 1,\dots,q$), $p,q \in \mathbb{N}$, and $\mathfrak{P} = 2^{\mathsf{M}-1}$. $\qquad\square$

5 Conclusions

By means of both direct and fixed point methods, we investigated an approximation for additive $(\mathcal{G}_1, \mathcal{G}_2)$-random operator inequality using a class of stochastic matrix control functions in matrix Menger normed algebras. As an application, we applied stochastic Mittag-Leffler control functions and the $\mathbb{H}$-fox control function to get a better approximation in the random operator inequality.

As can be seen in the previous section, using a new method (an alternative fixed point method), we get a new stability approximation result that, compared to the direct method, yields a better approximation.

As for the future research directions, we can replace the above control functions with hypergeometric function, Wright function, Fox–Wright function, and so on. Also, we can use matrix-valued fuzzy control functions instead of a class of stochastic matrix control functions.

Abbreviations
$\odot$, $\oslash$, Continuous generalized t-norm; $\mathcal{E}^+$, The set of matrix distribution functions; Φ, ϕ, Matrix distribution function; $\mathbb{S}$, Linear space; $(\mathbb{S}, \Phi, \odot)$, Matrix Menger norm space; $(\mathbb{S}, \Phi, \odot, \oslash)$, Matrix Menger normed algebra; Ω_1, Ω_2, Matrix Menger Banach algebra; $\mho$, $[0, 1]$; $\mathcal{Q}$, Random operator; ζ, Generalized metric; Ξ_σ, Mittag-Leffler function; $\mathbb{H}_{p,q}^{m,n}$, Fox $\mathbb{H}$-function; $\mathbb{T}$, Vector space.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, as well as read and approved the final manuscript.

Author details
[1]School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran. [2]Department of Mathematics, National Technical University of Athens, Zografou Campus, 157 80, Athens, Greece. [3]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References
1. Paris, R.B., Consiglio, A., Mainardi, F.: On the asymptotics of Wright functions of the second kind. Fract. Calc. Appl. Anal. **24**(1), 54–72 (2021)
2. Mehrez, K.: Positivity of certain classes of functions related to the Fox H-functions with applications. Anal. Math. Phys. **11**(3), Article ID 114 (2021)
3. Rezaei Aderyani, S., Saadati, R.: Best approximations of the φ-Hadamard fractional Volterra integro-differential equation by matrix valued fuzzy control functions. Adv. Differ. Equ. **2021**, Article ID 154 (2021)
4. Rezaei Aderyani, S., Saadati, R., Feckan, M.: The Cadariu–Radu method for existence, uniqueness and Gauss hypergeometric stability of Ω-Hilfer fractional differential equations. Mathematics **9**(12), 1408 (2021)
5. Rahman, G., Saboor, A., Anjum, Z., Nisar, K.S., Abdeljawad, T.: An extension of the Mittag-Leffler function and its associated properties. Adv. Math. Phys. **2020**, Article ID 5792853 (2020)
6. Fernandez, A., Kurt, C., Ozarslan, M.A.: A naturally emerging bivariate Mittag-Leffler function and associated fractional-calculus operators. Comput. Appl. Math. **39**(3), Article ID 200 (2020)
7. Karpa, D.B.: A Note on Fox's H Function in the Light of Braaksma's Results. Chapman & Hall, London (2019)
8. Meena, S., Bhatter, S., Jangid, K., Purohit, S.D.: Some expansion formulas for incomplete H- and $\overline{H}$-functions involving Bessel functions. Adv. Differ. Equ. **2020**, Article ID 562 (2020)
9. Jin, Y., Park, C., Rassias, M.T.: Hom-derivations in C^*-ternary algebras. Acta Math. Sin. Engl. Ser. **36**, 1025–1038 (2020)
10. Madadi, M., O'Regan, D., Rassias, T.M., Saadati, R.: Best approximation of κ-random operator inequalities in matrix MB-algebras. J. Inequal. Appl. **2021**, Article ID 13 (2021)
11. Schweizer, B., Sklar, A.: Probabilistic Metric Spaces. North-Holland Series in Probability and Applied Mathematics. North-Holland, New York (1983)
12. Hadzic, O., Pap, E.: Fixed Point Theory in Probabilistic Metric Spaces. Mathematics and Its Applications, vol. 536. Kluwer Academic, Dordrecht (2001)
13. Jang, S.Y., Saadati, R.: Approximation of an additive (ϱ_1, ϱ_2)-random operator inequality. J. Funct. Spaces **2020**, Article ID 7540303 (2020)
14. Saadati, R.: Random Operator Theory. Elsevier/Academic Press, London (2016)
15. Nšdăban, S., Bînzar, T., Pater, F.: Some fixed point theorems for φ-contractive mappings in fuzzy normed linear spaces. J. Nonlinear Sci. Appl. **10**, 5668–5676 (2017)

16. Diaz, J.B., Margolis, B.: A fixed point theorem of the alternative, for contractions on a generalized complete metric space. Bull. Am. Math. Soc. **74**, 305–309 (1968)
17. Rezaei Aderyani, S., Saadati, R., Yang, X.J.: Radu–Mihet method for UHML stability for a class of ξ-Hilfer fractional differential equations in matrix valued fuzzy Banach spaces. Math. Methods Appl. Sci. (2021)
18. Brzdek, J., Ciepliński, K.: A fixed point theorem in n-Banach spaces and Ulam stability. J. Math. Anal. Appl. **470**, 632–646 (2019)
19. Srivastava, H.M., Saadati, R., Jang, S.Y.: Bi-additive σ-random operator inequalities and random quasi-∗-multipliers on MB-algebras. Math. Sci. **15**(4), 325–336 (2021)
20. Narasimman, P.: Solution and stability of a generalized k-additive functional equation. J. Interdiscip. Math. (2018)
21. Sene, N.: Global asymptotic stability of the fractional differential equations. J. Nonlinear Sci. Appl. **13**(3), 171–175 (2020)
22. Park, C., Jin, Y., Zhang, X.: Bi-additive s-functional inequalities and quasi-multipliers on Banach algebras. Rocky Mt. J. Math. **49**, 593–607 (2019)
23. Park, C., Lee, J., Zhang, X.: Additive s-functional inequality and hom-derivations in Banach algebras. J. Fixed Point Theory Appl. **21**, Article ID 18 (2019)
24. Murali, R., Park, C.: Hyers–Ulam stability for an nth order differential equation using fixed point approach. J. Appl. Anal. Comput. **11**, 614–631 (2021)
25. Kim, G.H., Lee, Y.-H.: Stability of an additive-quadratic-quartic functional equation. Demonstr. Math. **53**(1), 1–7 (2020)
26. Thanyacharoen, A., Sintunavarat, W.: The new investigation of the stability of mixed type additive-quartic functional equations in non-Archimedean spaces. Demonstr. Math. **53**(1), 174–192 (2020)
27. Fripertinger, H., Schwaiger, J.: Some remarks on the stability of the Cauchy equation and completeness. Aequ. Math. (2021)
28. Tamilvanan, K., Lee, J.R., Park, C.: Hyers–Ulam stability of a finite variable mixed type quadratic-additive functional equation in quasi-Banach spaces. AIMS Math. **5**(6), 5993–6005 (2020)
29. Mihet, D., Radu, V.: On the stability of the additive Cauchy functional equation in random normed spaces. J. Math. Anal. Appl. **343**, 567–572 (2009)
30. Srivastava, H.M., Hussain, M.A.: Fractional integration of the H-function of several variables. Comput. Math. Appl. **30**(9), 73–85 (1995)