Continuous controlled cone metric-type spaces over real Banach algebras and fixed-point results

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Abstract

In this article, we introduce a new geometrical structure that is the hybrid of a cone metric space over Banach algebra and a controlled metric-type space. We introduce a new metric space and prove analogs of Banach-, Kannan- and Reich-type fixed-point theorems. We also furnish various concrete examples to establish the validity of our results. The obtained results generalize many well-known results in the literature.

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1 Introduction

In 2018, Mlaiki *et al.* [1] proposed the idea of controlled metric-type spaces that is the generalization of a b-metric space [2]. For more information on b-metric spaces and more generalizations, see [3–6]. They used $\theta : \mathcal{U} \times \mathcal{U} \to [1, \infty)$ instead of b in the triangular inequality condition by a different approach from Kamran *et al.* [7] who proposed the idea of extended b-metric spaces. They established a Banach-like contraction and proved some fixed-point results in such spaces. This shows that the class of such type of spaces is much more larger than the class of b-metric spaces and the class of metric spaces. Using the idea of a controlled metric-type space, several authors have published a number of papers in different directions (see [8–10]).

In 2007, Huang and Zhang [11] initiated the concept of a cone metric space over a Banach space as the generalization of metric spaces. They used an ordered Banach space E instead of $\mathbb{R}$ as the range set of metric d, i.e., they used $d : X \times X \to E$. They also discussed Banach-type contraction and proved some fixed-point results. Then, many researchers gave attention to the study of fixed-point results in such spaces. Rough estimates show that, more than six hundred papers have been published dealing with the cone metric spaces [12]. However, recently, some scholars obtained that a cone metric space (X, d) is equivalent to the usual metric space (X, d^*), where they defined the real-valued metric function d^* as the nonlinear scalarization function ξ_e [13]. However, the current situation altered when Liu and Xu [14] introduced cone metric spaces over a real Banach algebra.

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They defined generalized Lipschitz mapping and proved some fixed theorems. They established a concrete example to elucidate that the fixed-point results in cone metric spaces over Banach algebras are not equivalent to those in metric spaces. Later, in 2016, Huang and Radenovic [15] extended the idea of a cone metric space over Banach algebras to cone b-metric spaces over Banach algebras. They proved Banach- and Kannan-type theorems for such spaces (see also [16]).

In 1971, Reich [17] introduced a new type of contraction that we call Reich contraction. It was a generalization of the well-known Banach contraction and Kannan contraction (see [18, 19]). On the other hand, Samet *et al.* [20] initiated the idea of α-admissible mapping in metric spaces. Recently, Malhotra *et al.* [21, 22] used the idea of α-admissibility in cone metric spaces over Banach algebras and proved Banach-type and Kannan-type theorems. Later, in 2017, Hussain *et al.* [23] used the concept of α-admissible mapping in cone b-metric spaces over Banach algebras and proved Banach-type results in such spaces.

In this paper, we introduce the definition of a controlled cone metric-type space over a Banach algebra and then we prove some fixed-point results. We also furnish an example to show the validity of our obtained results. The last section of this paper consists of some important consequences of our results. Throughout the paper, we will use only real Banach algebras with identity.

2 Preliminaries

Let $(\mathcal{A}, \|\cdot\|)$ be a real Banach algebra with zero element θ. A cone $\mathcal{K}$ in $\mathcal{A}$ is a nonempty closed subset of $\mathcal{A}$ such that $\mathcal{P}^2 = \mathcal{P} \cdot \mathcal{P} \subseteq \mathcal{P}, \mathcal{P} \cap (-\mathcal{P}) = \theta, \mathcal{P} + \mathcal{P} \subseteq \mathcal{P}$ and $\lambda \mathcal{P} \subseteq \mathcal{P}$ for all $\lambda \geq 0$. If the interior of $\mathcal{K}$ denoted by $\mathrm{int}\mathcal{K}$ is nonempty, then the cone $\mathcal{K}$ is called a solid cone. If we define a relation $\preceq$ on $\mathcal{A}$ by $t \preceq v$ if and only if $v - t \in \mathcal{K}$, then $\preceq$ is a partial order on $\mathcal{A}$. We write $t \ll cv$ if and only if $v - t \in \mathcal{K}$ and $t \neq v$. Define another partial order $\ll$ on $\mathcal{A}$ by $t \ll v$ if and only if $v - t \in \mathrm{int}\mathcal{K}$. A cone $\mathcal{K}$ in $\mathcal{A}$ is said to be a normal cone if for all $t, v \in \mathcal{A}$ with $\theta \preceq t \preceq v$, there exists a real number $M > 0$ such that $\|t\| \leq M\|v\|$. The normal constant of $\mathcal{K}$ is the least positive constant M for which the above inequality holds.

Let $(\mathcal{A}, \|\cdot\|)$ be a unital Banach algebra with identity element e. An element $t \in \mathcal{A}$ is said to be invertible if there exists $v \in \mathcal{A}$ such that $tv = vt = e$. A complex number $\lambda \in \mathbb{C}$ is said to be the spectral value of $v \in \mathcal{A}$ if $v - \lambda e$ is noninvertible in $\mathcal{A}$. The set of all spectral values of $v \in \mathcal{A}$ that is denoted by $\sigma(v)$ is called the spectrum of v. The number $r_\sigma(v)$ (or $r(v)$) defined by $r_\sigma(v) = \sup\{|\lambda| : \lambda \in \sigma(v)\}$ is called the spectral radius of $v \in \mathcal{A}$.

Lemma 2.1 ([24]) *Let $\mathcal{A}$ be a Banach algebra with identity e. Then, the spectral radius $r(v)$ of $v \in \mathcal{A}$ satisfies*

$$r(v) = \lim_{n \to \infty} \|v^n\|^{1/n}. \tag{2.1}$$

Furthermore, if $r(v) < |\lambda|$ for some $v \in \mathcal{A}$, then $(\lambda e - v)$ is invertible,

$$(\lambda e - v)^{-1} = \sum_{i=0}^{\infty} \frac{v^i}{\lambda^{i+1}} \quad and \quad r\big[(\lambda e - v)^{-1}\big] \leq \frac{1}{|\lambda| - r(v)}.$$

Lemma 2.2 ([24]) *Let $\mathcal{A}$ be a Banach algebra and $u, v \in \mathcal{A}$. If u and v commute, then*

$$r(u + v) \leq r(u) + r(v) \quad and \quad r(uv) \leq r(u)r(v).$$

Definition 2.3 ([25]) Consider a Banach algebra $\mathcal{A}$ with solid cone $\mathcal{K}$. A c-sequence is a sequence $\{u_n\}$ in $\mathcal{K}$ such that for every $c \gg \theta$, there exists $N \in \mathbb{N}$ such that $u_n \ll c$ for all $n \geq N$.

Lemma 2.4 ([15]) *Let $\alpha, \beta \in \mathcal{K}$ be two arbitrary vectors and $\{u_n\}$, $\{q_n\}$ be two sequences in a solid cone $\mathcal{K}$ of a Banach algebra $\mathcal{A}$. Then, $\{\alpha u_n + \beta q_n\}$ is a c-sequence.*

Lemma 2.5 ([26]) *Let $\mathcal{A}$ be a Banach algebra and $\mathcal{K}$ be a cone in $\mathcal{A}$ (it need not be a normal cone). Then, the following hold:*

(u_1) *If for each $c \in \operatorname{int}\mathcal{K}, \theta \preceq p \ll c$, then $p = \theta$.*

(u_2) *If $p \in \mathcal{K}$ is such that $r(p) < 1$, then $\|p^n\| \to 0$ as $n \to \infty$.*

(u_3) *Let $c \in \operatorname{int}\mathcal{K}$ and $q_n \to \theta$ in $\mathcal{A}$ as $n \to \infty$. Then, there exists $N \in \mathbb{N}$ such that $q_n \ll c$ for all $n \geq N$.*

(u_4) *If $p \preceq pk$, where $p, k \in \mathcal{K}$ and $r(k) < 1$, then $p = \theta$.*

Definition 2.6 ([15]) Let $\mathfrak{X}$ be a nonempty set and $b \geq 1$ be a constant. A mapping $d_b : \mathfrak{X} \times \mathfrak{X} \to \mathcal{A}$ is called a cone b-metric over a Banach algebra $\mathcal{A}$ if the following axioms hold:

(B_1) for all $p, t \in \mathfrak{X}, d_b(p, t) \succeq \theta$ and $d_b(p, t) = \theta$ if and only if $p = t$;

(B_2) for all $p, t \in \mathfrak{X}, d_b(p, t) = d_b(t, p)$;

(B_3) for all $p, t, u \in \mathfrak{X}$, there exists $b \geq 1$ such that $d_b(p, u) \preceq b[d_b(p, t) + d_b(t, u)]$.

The pair $(\mathfrak{X}, d_b)$ is called a cone b-metric space over a Banach algebra $\mathcal{A}$.

Remark 2.7 If $b = 1$, then we say that d_1 is a cone metric over a Banach algebra $\mathcal{A}$. Hence, we can say that a cone b-metric is a generalization of a cone metric.

Example 2.8 Consider the Banach algebra $\mathcal{A} = C([0, 1])$ with unit element $e(t) = 1$ and supremum norm where multiplication is defined point-wise. Let $\mathfrak{X} = \mathbb{R}$ and $\mathcal{K} = \{f \in \mathcal{A} : f(l) \geq 0; \forall l \in [0, 1]\}$. Define $d_b : \mathfrak{X} \times \mathfrak{X} \to \mathcal{A}$ by

$$d_b(p, q)(t) = |p - q|^a e^t, \quad \forall x, y \in \mathfrak{X} \ \& \ a > 1.$$

Then, d_b is a cone b-metric on $\mathfrak{X}$ over a Banach algebra $\mathcal{A}$ with $b = 2^{a-1}$, but it is not a cone metric on $\mathfrak{X}$.

Definition 2.9 ([15]) Let $\{u_k\}$ be a sequence in $\mathfrak{X}$ where $(\mathfrak{X}, d_b)$ is a cone b-metric space over a Banach algebra $\mathcal{A}$. Then, $\{u_k\}$ is said to be

(i) a convergent sequence that converges to $p \in \mathfrak{X}$ if for every $c \in \operatorname{int}\mathcal{K}(i.e., \theta \ll c)$, there exists a natural number N such that $d_b(u_k, p) \ll c$ for all $k \geq N$;

(ii) a Cauchy sequence if for every $c \in \operatorname{int}\mathcal{K}(i.e., \theta \ll c)$, there exists a natural number N such that $d_b(u_k, u_i) \ll c$ for all $k, i \geq N$.

The pair $(\mathfrak{X}, d_b)$ is said to be a complete cone b-metric space if every Cauchy sequence in $\mathfrak{X}$ is convergent in $\mathfrak{X}$.

Remark 2.10 ([15, 26]) (i) If $\{p_n\}$ converges to p in $\mathfrak{X}$, then $\{d_b(p_k, p)\}$ and $\{d_b(p_k, p_{k+i})\}$ are c-sequences for any $i \in \mathbb{N}$.

(ii) If $\|p_k\| \to 0$ as $k \to \infty$, then for any $c \gg \theta$, there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $p_k \ll c$.

Definition 2.11 ([1]) Let $\mathfrak{X}$ be a nonempty set and $s : \mathfrak{X} \times \mathfrak{X} \to [1, \infty)$. A function $d_s :$ $\mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ is called a controlled metric type if for all $p, v, t \in \mathfrak{X}$ it satisfies

(i) $d_s(p, v) = 0$ iff $p = v$;

(ii) $d_s(p, v) = d_s(v, p)$;

(iii) $d_s(p, t) \le s(p, v)d_s(p, v) + s(v, t)d_s(v, t)$.

The pair $(\mathfrak{X}, d_s)$ is then called a controlled metric-type space.

Remark 2.12 If for all $p, q \in \mathfrak{X}, s(p, q) = b$ for some $b \ge 1$, then Definition 2.11 coincides with the definition of a b-metric space.

Theorem 2.13 ([1]) *Let $(\mathfrak{X}, d_s)$ be a complete controlled metric-type space where d_s is continuous. Let $F : \mathfrak{X} \to \mathfrak{X}$ be a given mapping such that*

$$d_s(Fp, Fq) \le \kappa d_s(p, q) \quad \text{for all } p, q \in \mathfrak{X}, \tag{2.2}$$

where $\kappa \in [0, 1)$, and for each $p_0 \in \mathfrak{X}$

$$\sup_{m \ge 1} \lim_{i \to \infty} \frac{s(p_{i+1}, p_{i+2})}{s(p_i, p_{i+1})} s(p_{i+1}, p_m) < \frac{1}{\kappa}, \tag{2.3}$$

where $p_i = T^i p_0$. In addition, assume that for every $x \in \mathfrak{X}$, we have

$$\lim_{n \to \infty} s(x_n, x) \quad \text{and} \quad \lim_{n \to \infty} s(x, x_n) \quad \text{exist and are finite.} \tag{2.4}$$

Then, F has a unique fixed point ϱ and for each $y \in \mathfrak{X}, F^k y \to \varrho$.

Now, we want to recall the definitions of generalized α-admissible, α-regular, and generalized Reich-type mappings in the setting of cone b-metric spaces over a Banach algebra.

Definition 2.14 ([27]) Let $(\mathfrak{X}, d_b)$ be a cone b-metric space over a Banach algebra $\mathcal{A}$ with $\mathcal{K}$ an underlying solid cone. Let $\alpha : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ and $F : \mathfrak{X} \to \mathfrak{X}$ be given mappings. Then,

(i) F is said to be a generalized α-admissible mapping if for all $p, q \in \mathfrak{X}, \alpha(p, q) \ge b$ implies that $\alpha(Fp, Fq) \ge b$;

(ii) $(\mathfrak{X}, d_b)$ is said to be α-regular if for every sequence $\{u_k\} \in \mathfrak{X}$ with $\alpha(u_k, u_{k+1}) \ge b$ for all $k \in \mathbb{N}$ so that $u_k \to p, \alpha(u_k, p) \ge b$.

Definition 2.15 ([27]) Let $(\mathfrak{X}, d_b)$ be a cone b-metric space over a Banach algebra $\mathcal{A}$ with coefficient b, $\mathcal{K}$ an underlying solid cone, and $\alpha : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ be a function. Then, a mapping $F : \mathfrak{X} \to \mathfrak{X}$ is called a generalized Reich-type contraction if there exist $v_1, v_2, v_3 \in \mathcal{K}$ such that for all $p, q \in \mathfrak{X}$ with $\alpha(p, q) \ge b$,

(i) $2br(v_1) + (b + 1)r(v_2 + v_3) < 2$;

(ii) $d(Fp, Fq) \preceq v_1 d(p, q) + v_2 d(p, Fp) + v_3 d(q, Fq)$.

3 Main results

In the following, we introduce a new type of metric space over a real Banach algebra that is called a controlled cone metric-type space over a Banach algebra. Using such spaces we

prove some fixed-point theorems for generalized Reich-type contractions and generalized Lipschitz mappings. Our results extend and generalize some previous well-known results in the literature.

Definition 3.1 Let $\mathcal{A}$ be a real Banach algebra with cone $\mathcal{K}$, $\mathfrak{X}$ be a nonempty set, and $s : \mathfrak{X} \times \mathfrak{X} \to [1, \infty)$ be a function. A controlled cone metric type on $\mathfrak{X}$ over a Banach algebra $\mathcal{A}$ is a mapping $d_s : \mathfrak{X} \times \mathfrak{X} \to \mathcal{A}$ such that

(E_1) $d_s(p, q) \succeq \theta$ for all $p, q \in \mathfrak{X}$ and $d_s(p, q) = \theta$ iff $p = q$;

(E_2) $d_s(p, q) = d_s(q, p)$ for all $p, q \in \mathfrak{X}$;

(E_3) $d_s(p, v) \preceq s(p, q) d_s(p, q) + s(q, v) d_s(q, v)$ for all $p, q, v \in \mathfrak{X}$.

The pair $(\mathfrak{X}, d_s)$ is then called a controlled cone metric-type space over a Banach algebra $\mathcal{A}$.

Remark 3.2 It is clear that the class of controlled cone metric-type spaces over Banach algebra $\mathcal{A}$ is larger than the classes of b-metric spaces and metric spaces over Banach algebras.

The definition of Cauchy sequences, convergent sequences, and completeness for a controlled cone metric-type space over a Banach algebra are the same as cone b-metric spaces over a Banach algebra defined in Definition 2.9.

In general, d_s is not necessarily a continuous function, but in this paper d_s will always mean a continuous function $d_s : \mathfrak{X} \times \mathfrak{X} \to \mathcal{A}$.

Example 3.3 Let $\mathfrak{X} = \{1, 2, 3\}$ and $s : \mathfrak{X} \times \mathfrak{X} \to [1, \infty)$ be defined as $s(p, q) = 1 + p + q$. Consider the real Banach algebra $\mathcal{A} = \mathbb{R}^2$ with solid cone $\mathcal{K} = \{(a, b) \in \mathbb{R}^2 : a, b \geq 0\}$. Assume that we define $d_s : \mathfrak{X} \times \mathfrak{X} \to \mathcal{A}$ by

$$d_s(1, 2) = d_s(2, 1) = (100, 100);$$

$$d_s(1, 3) = d_s(3, 1) = (1200, 1200);$$

$$d_s(3, 2) = d_s(2, 3) = (800, 800);$$

$$d_s(1, 1) = d_s(2, 2) = d_s(3, 3) = (0, 0) = \theta.$$

Clearly, the first and second conditions of a controlled cone metric-type space over a Banach algebra $\mathcal{A}$ are satisfied. For the third condition, we have

$$s(1, 3) d_s(1, 3) + s(3, 2) d_s(3, 2) - d_s(1, 2)$$
$$= 5(1200, 1200) + 6(800, 800) - (100, 100) = (10{,}700, 10{,}700) \in \mathcal{K};$$
$$s(1, 2) d_s(1, 2) + s(2, 3) d_s(2, 3) - d_s(1, 3)$$
$$= 4(100, 100) + 6(800, 800) - (1200, 1200) = (4000, 4000) \in \mathcal{K};$$
$$s(2, 1) d_s(2, 1) + s(1, 3) d_s(1, 3) - d_s(2, 3)$$
$$= 4(100, 100) + 5(1200, 1200) - (600, 600) = (5800, 5800) \in \mathcal{K}.$$

Hence, for all $p, q, v \in \mathfrak{X}$,

$$d_s(p, v) \preceq s(p, q) d_s(p, q) + s(q, v) d_s(q, v).$$

Thus, $(\mathfrak{X}, d_s)$ is a controlled cone metric-type space over a Banach algebra $\mathcal{A} = \mathbb{R}^2$.

Remark 3.4 Let $(\mathfrak{X}, d_s)$ be a controlled cone metric-type space over Banach algebra $\mathcal{A}$ with $s : \mathfrak{X} \times \mathfrak{X} \to [1, \infty)$. If $\mathcal{A} = \mathbb{R}$ and $\mathcal{K} = [0, \infty)$, then $(\mathfrak{X}, d_s)$ is a controlled metric-type space.

We now define a generalized α-admissible mapping and an α-regular space in terms of controlled cone metric-type spaces over Banach algebras.

Definition 3.5 Consider a controlled cone metric space $(\mathfrak{X}, d_s)$ over a Banach algebra $\mathcal{A}$ and $\mathcal{K}$ an underlying solid cone in $\mathcal{A}$. Let $F : \mathfrak{X} \to \mathfrak{X}$ and $\alpha : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ be mappings. Then,

   (i) F is said to be a generalized α-admissible mapping if for $p, q \in \mathfrak{X}, \alpha(p, q) \geq s(p, q)$ implies that $\alpha(Fp, Fq) \geq s(Fp, Fq)$;
  (ii) $(\mathfrak{X}, d_s)$ is said to be α-regular if for every sequence $\{u_k\} \in \mathfrak{X}$ with $\alpha(u_k, u_{k+1}) \geq s(u_k, u_{k+1})$ for all $k \in \mathbb{N}$ so that $u_k \to q, \alpha(u_k, q) \geq s(u_k, q)$.

We are now able to define a generalized Reich-type contraction in the setting of controlled cone metric-type spaces over a Banach algebra.

Definition 3.6 Let $(\mathfrak{X}, d_s)$ be a controlled cone metric-type space over Banach algebra $\mathcal{A}$ with $\mathcal{K}$ an underlying solid cone and $\alpha : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ be a function. Then, a mapping $F : \mathfrak{X} \to \mathfrak{X}$ is called a generalized Reich-type contraction if there exist three vectors v_1, v_2, v_3 in $\mathcal{K}$ such that for all $p, q \in \mathfrak{X}$ with $\alpha(p, q) \geq s(p, q)$,

   (i) $2s(p, q)r(v_1) + (s(p, q) + 1)r(v_2 + v_3) < 2$ and for each $u_0 \in \mathfrak{X}$ with $u_m = F^m u_0$,

$$\lim_{m, i \to \infty} \frac{s(u_{i+1}, u_{i+2})}{s(u_i, u_{i+1})} s(u_{i+1}, u_m) < \frac{1}{\|\kappa\|},$$

        where $\kappa = (2e - v)^{-1}(2v_1 + v)$ for $v = v_2 + v_3$;
  (ii) $d_s(Fp, Fq) \preceq v_1 d_s(p, q) + v_2 d_s(p, Fp) + v_3 d_s(q, Fq)$.

The main result of our paper is given as follows:

Theorem 3.7 *Let $(\mathfrak{X}, d_s)$ be a complete controlled cone metric-type space over a Banach algebra $\mathcal{A}$ with $\mathcal{K}$ an underlying solid cone and $\alpha : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ a mapping. Suppose that the mapping $F : \mathfrak{X} \to \mathfrak{X}$ is a generalized Reich-type contraction with vectors $v_1, v_2, v_3 \in \mathcal{K}$ such that*

   1. *F is a generalized α-admissible mapping;*
   2. *there exists $u_0 \in \mathfrak{X}$ such that $\alpha(u_0, Fu_0) \geq s(u_0, Fu_0)$;*
   3. *$(\mathfrak{X}, d_s)$ is regular or F is continuous.*

Then, there exists a point ϱ in $\mathfrak{X}$ that is fixed under the mapping F.

Proof Let u_0 be a point in $\mathfrak{X}$ such that $\alpha(u_0, Fu_0) \geq s(u_0, Fu_0)$. For $u_0 \in \mathfrak{X}$, if we define $u_1 = Fu_0, u_2 = Fu_1 = F(Fu_0) = F^2 u_0, \ldots, u_{n+1} = Fu_n = F^{n+1} u_0$, then

$$\alpha(u_0, u_1) \geq s(u_0, u_1).$$

However, F is generalized α-admissible and thus

$$\alpha(Fu_0, Fu_1) = \alpha(u_1, u_2) \geq s(u_1, u_2),$$

and so by induction we obtain

$$\alpha(u_n, u_{n+1}) \geq s(u_n, u_{n+1}).$$

By Definition 3.6, we have

$$d_s(u_n, u_{n+1}) = d_s(Fu_{n-1}, Fu_n)$$
$$\preceq v_1 d_s(u_{n-1}, u_n) + v_2 d_s(u_{n-1}, Fu_{n-1}) + v_3 d_s(u_n, Fu_n),$$

that is,

$$(e - v_3) d_s(u_n, u_{n+1}) \preceq (v_1 + v_2) d_s(u_{n-1}, u_n). \tag{3.1}$$

Similarly,

$$d_s(u_{n+1}, u_n) = d_s(Fu_n, Fu_{n-1})$$
$$\preceq v_1 d_s(u_n, u_{n-1}) + v_2 d_s(u_n, Fu_n) + v_3 d_s(u_{n-1}, Fu_{n-1}),$$

i.e.,

$$(e - v_2) d_s(u_{n+1}, u_n) \preceq (v_1 + v_3) d_s(u_{n-1}, u_n). \tag{3.2}$$

Adding (3.1) and (3.2), we obtain

$$(2e - v_2 - v_3) d_s(u_n, u_{n+1}) \preceq (2v_1 + v_2 + v_3) d_s(u_{n-1}, u_n).$$

If we take $v = v_2 + v_3$, then we obtain

$$(2e - v) d_s(u_{n+1}, u_n) \preceq (2v_1 + v) d_s(u_{n-1}, u_n). \tag{3.3}$$

Note that

$$2r(v) \leq \big(s(u_n, u_{n+1}) + 1\big) r(v) \leq 2r(v_1) + \big(s(u_n, u_{n+1}) + 1\big) r(v) < 2.$$

Hence, $r(v) < 1 < 2$ and $r(v) < 2$. Thus, by Lemma 2.1, we deduce that $2e - v$ is invertible and $(2e - v)^{-1} = \sum_{n=0}^{\infty} \frac{v^n}{2^{n+1}}, r((2e - v)^{-1}) < \frac{1}{2 - r(v)}$.
 Thus, (3.3) becomes

$$d_s(u_n, u_{n+1}) \preceq \kappa d_s(u_{n-1}, u_n), \tag{3.4}$$

where $\kappa = (2e - v)^{-1}(2v_1 + v)$. The inequality (3.4) implies that for all $n \in \mathbb{N}$

$$d_s(u_n, u_{n+1}) \preceq \kappa d_s(u_{n-1}, u_n) \preceq \kappa^2 d_s(u_{n-2}, u_{n-1}) \preceq \cdots \preceq \kappa^n d_s(u_0, u_1). \tag{3.5}$$

Now, if we take $m > n$, then by (3.5), Definition 3.1 and (E_3), we have

$$d_s(u_n, u_m)$$

$$\preceq s(u_n, u_{n+1})d_s(u_n, u_{n+1}) + s(u_{n+1}, u_m)d_s(u_{n+1}, u_m)$$

$$\preceq s(u_n, u_{n+1})d_s(u_n, u_{n+1}) + s(u_{n+1}, u_m)s(u_{n+1}, u_{n+2})d_s(u_{n+1}, u_{n+2})$$

$$+ s(u_{n+1}, u_m)s(u_{n+2}, u_m)d_s(u_{n+2}, u_m)$$

$$\preceq s(u_n, u_{n+1})d_s(u_n, u_{n+1}) + s(u_{n+1}, u_m)s(u_{n+1}, u_{n+2})d_s(u_{n+1}, u_{n+2})$$

$$+ s(u_{n+1}, u_m)s(u_{n+2}, u_m)s(u_{n+2}, u_{n+3})d_s(u_{n+2}, u_{n+3})$$

$$+ s(u_{n+1}, u_m)s(u_{n+2}, u_m)s(u_{n+3}, u_m)d_s(u_{n+3}, u_m)$$

$$\preceq \cdots$$

$$\preceq s(u_n, u_{n+1})d_s(u_n, u_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i} s(u_j, u_m)\right)s(u_i, u_{i+1})d_s(u_i, u_{i+1})$$

$$+ \left(\prod_{k=n+1}^{m-1} s(u_k, u_m)\right)d_s(u_{m-1}, u_m)$$

$$\preceq s(u_n, u_{n+1})\kappa^n d_s(u_0, u_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i} s(u_j, u_m)\right)s(u_i, u_{i+1})\kappa^i d_s(u_0, u_1)$$

$$+ \left(\prod_{k=n+1}^{m-1} s(u_k, u_m)\right)\kappa^{m-1} d_s(u_0, u_1)$$

$$\preceq s(u_n, u_{n+1})\kappa^n d_s(u_0, u_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i} s(u_j, u_m)\right)s(u_i, u_{i+1})\kappa^i d_s(u_0, u_1)$$

$$+ \left(\prod_{k=n+1}^{m-1} s(u_k, u_m)\right)s(u_{m-1}, u_m)\kappa^{m-1} d_s(u_0, u_1)$$

$$\preceq s(u_n, u_{n+1})\kappa^n d_s(u_0, u_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i} s(u_j, u_m)\right)s(u_i, u_{i+1})\kappa^i d_s(u_0, u_1)$$

$$\preceq \left(\prod_{j=0}^{n} s(u_n, u_{n+1})\right)\kappa^n d_s(u_0, u_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i} s(u_j, u_m)\right)s(u_i, u_{i+1})\kappa^i d_s(u_0, u_1)$$

$$= d_s(u_0, u_1) \sum_{i=n}^{m-1}\left(\prod_{j=0}^{i} s(u_j, u_m)\right)s(u_i, u_{i+1})\kappa^i.$$

In the above steps, we use the fact that $s(p, q) \geq 1$ and thus $x \preceq s(p, q)x$ for any $x \in \mathcal{A}$. Let

$$a_n = \left(\prod_{j=0}^{n} s(u_j, u_m)\right)\kappa^n s(u_n, u_{n+1}) \quad \text{and} \quad S = \sum_{n=1}^{\infty} a_n.$$

Since by Definition 3.6, $\|\kappa\| \lim_{m,i\to\infty} \frac{s(u_{i+1}, u_{i+2})}{s(u_i, u_{i+1})} s(u_{i+1}, u_m) < 1$, the series S converges absolutely. Using the ratio test, we have

$$\lim_{n\to\infty} \frac{\|a_{n+1}\|}{\|a_n\|} \leq \lim_{n\to\infty} \frac{\|\kappa\|\|\kappa^n\|(\prod_{j=1}^{n+1} s(u_j, u_m))s(u_{n+1}, u_{n+2})}{\|\kappa^n\|(\prod_{j=1}^{n} s(u_j, u_m))s(u_n, u_{n+1})}$$

$$= \|\kappa\| \lim_{n,m\to\infty} \frac{s(u_{n+1}, u_{n+2})}{s(u_{n+1}, u_{n+1})} s(u_{n+1}, u_m) < 1.$$

However, $\mathcal{A}$ is a Banach algebra and the series S is absolutely convergent and so converges in $\mathcal{A}$. Thus, $S_{m-1} - S_n = [\sum_{i=n}^{m-1} (\prod_{j=0}^{i} s(u_j, u_m)) s(u_i, u_{i+1}) \kappa^i] \to \theta$ as $n, m \to \infty$ and so is $d_s(u_0, u_1)(S_{m-1} - S_n)$. Hence, by Lemma 2.5, for every $c \gg \theta$, there exists a natural number n_0 such that for all $n \geq n_0, d_s(u_n, u_m) \ll c$. Thus, by Definition 2.9, $\{u_n\}$ is a Cauchy sequence in $\mathfrak{X}$. However, $\mathfrak{X}$ is complete and so there exists $\varrho \in \mathfrak{X}$ such that $u_n \to \varrho$ as $n \to \infty$.

We show that ϱ is fixed under the mapping F.

Suppose that F is continuous. It follows that $u_{n+1} = F u_n \to F \varrho$ as $n \to \infty$. However, the limit of a sequence is unique and so we have $F \varrho = \varrho$. Hence, ϱ is fixed under the mapping F in this case.

However, if $(\mathfrak{X}, d_s)$ is α-regular, then by Definition 3.5, we obtain

$$\alpha(u_n, \varrho) \geq s(u_n, \varrho) \quad \text{for all } n \in \mathbb{N}$$

and

$$
\begin{aligned}
d_s(\varrho, F\varrho) \\
\preceq s(\varrho, F u_n) d_s(\varrho, F u_n) + s(F u_n, F \varrho) d_s(F u_n, F \varrho) \\
\preceq s(\varrho, F u_n) d_s(\varrho, F u_n) + s(F u_n, F \varrho)[v_1 d_s(u_n, \varrho) + v_2 d_s(u_n, F u_n) + v_3 d_s(\varrho, F \varrho)] \\
\preceq s(\varrho, F u_n) d_s(\varrho, F u_n) + s(F u_n, F \varrho) v_1 d_s(u_n, \varrho) + s(F u_n, F \varrho) v_3 d_s(\varrho, F \varrho) \\
\quad + s(F u_n, F \varrho) v_2 [s(u_n, \varrho) d_s(u_n, \varrho) + s(\varrho, u_{n+1}) d_s(\varrho, u_{n+1})] \\
= s(\varrho, F F u_n)(e + s(u_{n+1}, F \varrho) v_2) d_s(\varrho, u_{n+1}) + s(u_{n+1}, F \varrho) v_3 d_s(\varrho, F \varrho) \\
\quad + s(u_{n+1}, F \varrho)(v_1 + s(u_n, \varrho) v_2) d_s(u_n, \varrho),
\end{aligned}
$$

which further implies that

$$
\begin{aligned}
(e - s(u_{n+1}, F \varrho) v_3) d_s(\varrho, F \varrho) &\preceq s(\varrho, F u_n)(e + s(u_{n+1}, F \varrho) v_2) d_s(u_{n+1}, \varrho) \\
&\quad + s(u_{n+1}, F \varrho)(v_1 + s(u_n, \varrho) v_2) d_s(u_n, \varrho).
\end{aligned} \tag{3.6}
$$

Similarly,

$$
\begin{aligned}
d_s(\varrho, F\varrho) \\
\preceq s(\varrho, F u_n) d_s(\varrho, F u_n) + s(F u_n, F \varrho) d_s(F u_n, F \varrho) \\
= s(\varrho, F u_n) d_s(\varrho, F u_n) + s(F u_n, F \varrho) d_s(F \varrho, F u_n) \\
\preceq s(\varrho, F u_n) d_s(\varrho, F u_n) + s(F u_n, F \varrho)[v_1 d_s(\varrho, u_n) + v_2 d_s(\varrho, F \varrho) + v_3 d_s(u_n, F u_n)] \\
\preceq s(\varrho, F u_n) d_s(\varrho, F u_n) + s(F u_n, F \varrho) v_1 d_s(u_n, \varrho) + s(F u_n, F F \varrho) v_2 d_s(\varrho, F \varrho) \\
\quad + s(F u_n, F \varrho) v_3 [s(u_n, \varrho) d_s(u_n, \varrho) + s(\varrho, u_{n+1}) d_s(\varrho, u_{n+1})] \\
= s(\varrho, F u_n)(e + s(u_{n+1}, F \varrho) v_3) d_s(\varrho, u_{n+1}) + s(u_{n+1}, F \varrho) v_2 d_s(\varrho, F \varrho)
\end{aligned}
$$

$$+ s(u_{n+1}, F\varrho)\big(v_1 + s(u_n, \varrho)v_2\big)d_s(u_n, \varrho),$$

which further implies that

$$\big(e - s(u_{n+1}, F\varrho)v_2\big)d_s(\varrho, F\varrho) \preceq s(\varrho, Fu_n)\big(e + s(u_{n+1}, F\varrho)v_3\big)d_s(u_{n+1}, \varrho)$$
$$+ s(u_{n+1}, F\varrho)\big(v_1 + s(u_n, \varrho)v_3\big)d_s(u_n, \varrho). \tag{3.7}$$

Therefore, by adding (3.6) and (3.7), we obtain

$$\big(2e - s(u_{n+1}, F\varrho)v_2 - s(u_{n+1}, F\varrho)v_3\big)d_s(\varrho, F\varrho)$$
$$\preceq s(\varrho, u_{n+1})\big(2e + s(u_{n+1}, F\varrho)v_2 + s(u_{n+1}, F\varrho)v_3\big)d_s(u_{n+1}, \varrho)$$
$$+ s(\varrho, F\varrho)\big(2v_1 + s(\varrho, F\varrho)v_2 + s(\varrho, F\varrho)v_3\big)d_s(u_n, \varrho),$$

i.e.,

$$\big(2e - s(u_{n+1}, F\varrho)v\big)d_s(\varrho, F\varrho) \preceq s(\varrho, u_{n+1})\big(2e + s(u_{n+1}, F\varrho)v\big)d_s(u_{n+1}, \varrho)$$
$$+ s(\varrho, F\varrho)\big(2v_1 + s(\varrho, F\varrho)v\big)d_s(u_n, \varrho). \tag{3.8}$$

We also note by Definition 3.6 that

$$2r\big(s(u_{n+1}, F\varrho)v\big) = 2s(u_{n+1}, F\varrho)r(v) \le 2s(u_{n+1}, F\varrho)r(v_1) + \big(s(u_{n+1}, F\varrho) + 1\big)r(v) < 2,$$

i.e., $r(s(u_{n+1}, F\varrho)v) < 1 < 2$. Thus, by Lemma 2.1, $2e - s(u_{n+1}, F\varrho)v$ is invertible and so (3.8) implies that

$$d_s(\varrho, F\varrho) \preceq \big(2e - s(u_{n+1}, F\varrho)v\big)^{-1}\big[s(\varrho, u_{n+1})\big(2e + s(u_{n+1}, F\varrho)v\big)d_s(u_{n+1}, \varrho)$$
$$+ s(\varrho, F\varrho)\big(2v_1 + s(\varrho, F\varrho)v\big)d_s(u_n, \varrho)\big]. \tag{3.9}$$

By Remark 2.10, the sequences $\{d_s(u_{n+1}, \varrho)\}$ and $\{d_s(u_n, \varrho)\}$ are c-sequences. Hence, by Lemma 2.4, the sequence $\{\tau_1 d_s(u_{n+1}, \varrho) + \tau_2 d_s(u_n, \varrho)\}$ is a c-sequence (where $\tau_1 = (2e - s(u_{n+1}, F\varrho)v)^{-1}s(\varrho, u_{n+1})(2e + s(u_{n+1}, F\varrho)v)$ and $\tau_2 = (2e - s(u_{n+1}, F\varrho)v)^{-1}s(\varrho, F\varrho)(2v_1 + s(\varrho, F\varrho)v))$. Therefore, for any $c \in \mathcal{A}$ with $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that

$$d_s(\varrho, F\varrho) \preceq \tau_1 d_s(u_{n+1}, \varrho) + \tau_2 d_s(u_n, \varrho) \ll c,$$

which further implies by Lemma 2.5 that $d_s(\varrho, F\varrho) = \theta$. Therefore, $F\varrho = \varrho$ and this completes the proof. □

Example 3.8 Let $\mathcal{A} = C^1_{\mathbb{R}}[0, 1]$ and $\|f\| = \|f\|_\infty + \|f'\|_\infty$. If we define point-wise multiplication of functions on $\mathcal{A}$, then $\mathcal{A}$ becomes a real Banach algebra with identity $e(t) = 1$. If we take $\mathcal{K} = \{f \in \mathcal{A} : f(t) \ge 0, t \in [0, 1]\}$, then $\mathcal{K}$ is a nonnormal cone (see [28]). Let $\mathfrak{X} = [0, \infty)$ and $s : \mathfrak{X} \times \mathfrak{X} \to [1, \infty)$ be defined as $s(p, q) = 2 + p + q$. Define $d_s : \mathfrak{X} \times \mathfrak{X} \to \mathcal{A}$ by

$$d_s(p, q)(t) = (p - q)^2 e^t.$$

Then, d_s is a controlled-type cone metric over $\mathcal{A}$. Also, note that $\mathfrak{X}$ is complete with respect to d_s. We define $\alpha : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ and $F : \mathfrak{X} \to \mathfrak{X}$ by

$$
\alpha(p,q) = \begin{cases} s(p,q) & \text{if } p, q \in [0,1], \\ 0 & \text{elsewhere,} \end{cases}
$$

and

$$
F(p) = \begin{cases} \frac{\sqrt{5}}{3}p & \text{if } p \in [0,1], \\ p + 1 & \text{if } p > 1. \end{cases}
$$

Note that for every $p \in [0,1]$, $Fp \in [0,1]$. Choosing $v_1(t) = \frac{1}{9} + \frac{1}{9}t$, $v_2(t) = \frac{1}{18} + \frac{1}{18}t$ and $v_3(t) = \frac{1}{24} + \frac{1}{24}t$, we obtain that $r(v_1) = \frac{2}{9}$, $r(v) = r(v_2 + v_3) = \frac{7}{36}$. Simple calculations show that $2(2)r(v_1) + (2 + 1)r(v) = \frac{53}{36}$ and so F is a generalized Reich-type contraction as

$$
\frac{1}{2(p + q + 2)r(v_1) + ((p + q + 2) + 1)r(v)} \preceq \frac{1}{2(2)r(v_1) + (2 + 1)r(v)} = \frac{36}{53},
$$

which further implies that $2s(p,q)r(v_1) + (s(p,q) + 1)r(v) \leq \frac{53}{36} < 2$. Also,

$$
\lim_{m,i \to \infty} \frac{s(p_{i+1}, p_{i+2})}{s(p_i, p_{i+1})} s(p_{i+1}, p_m) = 2
$$

and

$$
\begin{aligned}
\|\kappa\| &= \left\| (2e - v)^{-1}(2v_1 + v) \right\| \\
&\leq \left(\frac{72}{130} \right)\left(\frac{46}{72} \right) = \frac{23}{65} < \frac{1}{2} \\
&= \lim_{m,i \to \infty} \frac{s(p_i, p_{i+1})}{s(p_{i+1}, p_{i+2})} s(p_{i+1}, p_m).
\end{aligned}
$$

Similarly, by a simple calculation, one can show that

$$
d_s(Fp, Fq) \preceq v_1 d_s(p,q) + v_2 d_s(p, Fp) + v_3 d_s(q, Fq).
$$

Next, we show that there is a point u_0 in $\mathfrak{X}$ such that $\alpha(u_0, Fu_0) \geq s(u_0, Fu_0)$. Indeed, for $u_0 = 1$, we have

$$
\alpha(1, F1) = \alpha\left(1, \frac{\sqrt{5}}{3}\right) \geq s\left(1, \frac{\sqrt{5}}{3}\right) = s(1, F1).
$$

Next, we show that F is a generalized α-admissible mapping. In fact, if $p, q \in \mathfrak{X}$ are such that $\alpha(p,q) \geq s(p,q)$, then by definition of α, $p, q \in [0,1]$. Therefore, $Fp, Fq \in [0,1]$ and so

$$
\alpha(Fp, Fq) \geq s(Fp, Fq).
$$

Finally, we show that $(\mathfrak{X}, d_s)$ is α-regular. If we assume that a sequence $\{p_n\}$ in $\mathfrak{X}$ such that $\alpha(p_n, p_{n+1}) \geq s(p_n, p_{n+1})$ for all $n \in \mathbb{N}$ and $p_n \to q \in \mathfrak{X}$ (as $n \to \infty$), then $\{p_n\} \subseteq [0,1]$.

However, $[0, 1]$ is closed and so $q \in [0, 1]$. This implies that $\alpha(p_n, q) \geq s(p_n, q)$ for all $n \in \mathbb{N}$. Hence, all the conditions of Theorem 3.7 are satisfied and so F has a fixed point $\varrho = 0$.

Theorem 3.9 *Let $\mathcal{A}$ be a Banach algebra and $\mathcal{K}$ be a solid cone in $\mathcal{A}$. Let $(\mathfrak{X}, d_s)$ be a complete controlled cone metric-type space over $\mathcal{A}$ and $\alpha : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ be a function. Suppose that the mapping $F : \mathfrak{X} \to \mathfrak{X}$ is a generalized Reich-type contraction with vectors v_1, v_2, v_3 in $\mathcal{K}$ such that v_1 commutes with $v_2 + v_3$ and*
1. *F is a generalized α-admissible;*
2. *there exists $u_0 \in \mathfrak{X}$ such that $\alpha(u_0, F u_0) \geq s(u_0, F u_0)$;*
3. *F is continuous or $(\mathfrak{X}, d_s)$ is regular;*
4. *for any two fixed points ϖ, ζ of F, there exists $z \in \mathfrak{X}$ such that $\alpha(\varpi, z) \geq s(\varpi, z)$ and $\alpha(\zeta, z) \geq s(\zeta, z)$.*

Then, there exists a unique point ϱ in $\mathfrak{X}$ that is fixed under the mapping F.

Proof Using Theorem 3.7 and the first three given conditions, we can say that there exists a point $\varrho \in \mathfrak{X}$ that is fixed under the mapping F. We show that this point is unique and for this, let $\zeta \in \mathrm{Fix}(F)$ such that $\varrho \neq \zeta$. Then, by Condition 4, there exists $z \in \mathfrak{X}$ such that

$$\alpha(\varrho, z) \geq s(\varrho, z) \quad \text{and} \quad \alpha(\zeta, z) \geq s(\zeta, z). \tag{3.10}$$

Since F is a generalized α-admissible mapping and $\varrho, \zeta \in \mathrm{Fix}(F)$, by (3.10), we obtain

$$\alpha(\varrho, F^i z) \geq s(\varrho, F^i z) \quad \text{and} \quad \alpha(\zeta, F^i z) \geq s(\zeta, F^i z), \quad \text{for all } i \in \mathbb{N}. \tag{3.11}$$

By Definition 3.6 and (3.11), we obtain

$$\begin{aligned}
d_s(\varrho, F^i z) &= d_s(F\varrho, F(F^{i-1}z)) \\
&\leq v_1 d_s(\varrho, F^{i-1}z) + v_2 d_s(\varrho, F\varrho) + v_3 d_s(F^{i-1}z, F^i z) \\
&\leq v_1 d_s(\varrho, F^{i-1}z) + v_3 s(F^{i-1}z, \varrho) d_s(F^{i-1}z, \varrho) + v_3 s(\varrho, F^i z) d_s(\varrho, F^i z),
\end{aligned}$$

which further implies that

$$(e - d_s(\varrho, F^i z) v_3) d_s(\varrho, F^i z) \preceq (v_1 + s(F^{i-1}z, \varrho) v_3) d_s(\varrho, F^{i-1}z). \tag{3.12}$$

Similarly,

$$\begin{aligned}
d_s(F^i z, \varrho) &= d_s(F(F^{i-1}z), F\varrho) \\
&\leq v_1 d_s(F^{i-1}z, \varrho) + v_2 d_s(F^{i-1}z, F^i z)) + v_3 d_s(\varrho, F\varrho) \\
&\leq v_1 d_s(F^{i-1}z, \varrho) + v_2 s(F^{i-1}z, \varrho) d_s(F^{i-1}z, \varrho) + (\varrho, F^i z) v_2 d_s(\varrho, F^i z),
\end{aligned}$$

which further implies that

$$(e - d_s(F^i z, \varrho) v_2) d_s(F^i z, \varrho) \preceq (v_1 + s(F^{i-1}z, \varrho)) v_2) d_s(F^{i-1}z, \varrho). \tag{3.13}$$

Adding (3.12) and (3.13), we have

$$\left(2e - s(\varrho, F^i z)v_2 - s(\varrho, F^i z)v_3\right)d_s(\varrho, F^i z)$$
$$\preceq \left(2v_1 + s(F^{i-1}z, \varrho)v_2 + s(F^{i-1}z, \varrho)v_3\right)d_s(\varrho, F^{i-1}z),$$

i.e.,

$$\left(2e - s(\varrho, F^i z)v\right)d_s(\varrho, F^i z) \preceq \left(2v_1 + s(F^{i-1}z, \varrho)v\right)d_s(\varrho, F^{i-1}z). \tag{3.14}$$

Note that

$$2r\left(s(\varrho, F^i z)v\right) \le \left(s(\varrho, F^i z) + 1\right)r(v)$$
$$\le 2s(\varrho, F^i z)r(v_1) + \left(s(\varrho, F^i z) + 1\right)r(v) < 2.$$

Thus, $r(s(\varrho, F^i z)v) < 1 < 2$ and by Lemma 2.1, we can say that $2e - s(\varrho, F^i z)v$ is invertible and $(2e - s(\varrho, F^i z)v)^{-1} = \sum_{n=0}^{\infty} \frac{(s(\varrho, F^i z)v)^n}{2^{n+1}}, r((2e - s(\varrho, F^i z)v)^{-1}) < \frac{1}{2 - r(s(\varrho, F^i z)v)}$. Thus, by (3.14), we have

$$d_s(\varrho, F^i z) \preceq \left(2e - s(\varrho, F^i z)v\right)^{-1}\left(2v_1 + s(F^{i-1}z, \varrho)v\right)d_s(\varrho, F^{i-1}z),$$

i.e.,

$$d_s(\varrho, F^i z) \preceq \tau d_s(\varrho, F^{i-1}z), \tag{3.15}$$

where $\tau = (2e - s(\varrho, F^i z)v)^{-1}(2v_1 + s(F^{i-1}z, \varrho)v)$. Therefore, we have

$$d_s(\varrho, F^i z) \preceq \tau d_s(\varrho, F^{i-1}z)$$
$$\preceq \tau^2 d_s(\varrho, F^{i-2}z)$$
$$\vdots$$
$$\preceq \tau^i d_s(\varrho, z), \quad \text{for all } i \in \mathbb{N}.$$

Since v_1 commutes with $v_2 + v_3 = v$,

$$\left(2e - s(\varrho, F^i z)v\right)^{-1}\left(2v_1 + s(F^{i-1}z, \varrho)v\right)$$
$$= \left(\sum_{n=0}^{\infty} \frac{(s(\varrho, F^i z)v)^n}{2^{n+1}}\right)\left(2v_1 + s(F^{i-1}z, \varrho)v\right)$$
$$= 2v_1\left(\sum_{n=0}^{\infty} \frac{(s(\varrho, F^i z)v)^n}{2^{n+1}}\right) + s(F^{i-1}z, \varrho)v\left(\sum_{n=0}^{\infty} \frac{(s(\varrho, F^i z)v)^n}{2^{n+1}}\right)$$
$$= \left(2v_1 + s(F^{i-1}z, \varrho)v\right)\left(2e - s(\varrho, F^i z)v\right)^{-1},$$

which shows that $(2e - s(\varrho, F^i z)v)^{-1}$ commutes with $(2v_1 + s(F^{i-1}z, \varrho)v)$. Hence, by Lemmas 2.1 and 2.2, we obtain

$$r(\tau) = r\left(\left(2e - s(\varrho, F^i z)v\right)^{-1}\left(2v_1 + s(F^{i-1}z, \varrho)v\right)\right)$$

$$\leq r\big((2e - s(\varrho, F^i z)v)^{-1}\big) \cdot r\big((2v_1 + s(F^{i-1}z, \varrho)v)\big)$$

$$\leq \frac{1}{2 - s(F^{i-1}z, \varrho)r(v)}\big(2r(v_1) + s(F^{i-1}z, \varrho)r(v)\big)$$

$$< 1.$$

Then,

$$2\big(r(v_1) + s(F^{i-1}z, \varrho)r(v)\big) \leq 2s(F^{i-1}z, \varrho)r(v_1) + \big(s(F^{i-1}z, \varrho) + 1\big)r(v) < 2$$

implies that

$$2r(v_1) + s(F^{i-1}z, \varrho)r(v) < 2 - s(F^{i-1}z, \varrho)r(v).$$

From Lemma 2.5, it follows that $\|\tau^i\| \to 0$ as $i \to \infty$ and hence

$$\big\|\tau^i d_s(\varrho, z)\big\| \leq \big\|\tau^i\big\|\big\|d_s(\varrho, z)\big\| \to 0 \quad (i \to \infty).$$

By Remark 2.10, we conclude that for any $c \in \mathcal{A}$ with $c \gg \theta$, there exists $N \in \mathbb{N}$ such that

$$d_s\big(\varrho, F^i z\big) \preceq \tau^i d_s(\varrho, z) \preceq c, \quad \forall i \geq N.$$

Thus, by Lemma 2.5, $F^i z \to \varrho$ as $i \to \infty$. Similarly, we obtain that $F^i z \to \zeta$ as $i \to \infty$. Now, by uniqueness of limit, we conclude that $\varrho = \zeta$, which completes the proof. $\square$

Theorem 3.10 *Let $(\mathfrak{X}, d_s)$ be a complete controlled cone metric-type space over a Banach algebra $\mathcal{A}$ and $\mathcal{K}$ be the associated cone in $\mathcal{A}$. Let $F : \mathfrak{X} \to \mathfrak{X}$ be a mapping such that for all $p, q \in \mathfrak{X}$,*

$$d_s(Fp, Fq) \preceq \kappa d_s(p, q), \tag{3.16}$$

where $\kappa \in \mathcal{K}$ with $r(\kappa) < 1$ and for each $p_0 \in \mathfrak{X}$,

$$\lim_{m,i \to \infty} \frac{s(p_{i+1}, p_{i+2})}{s(p_i, p_{i+1})} s(p_{i+1}, p_m) < \frac{1}{\|\kappa\|}.$$

Then, there exists a unique point $\varrho \in \mathfrak{X}$ that is fixed under the mapping F. Furthermore, for each $u_0 \in \mathfrak{X}$, the iterative sequence $u_n = F(u_{n-1}) = F^n u_0$ converges to ϱ.

Proof If we take $v_1 = \kappa$, $v_2 = v_3 = \theta$ and $\alpha(p, q) = s(p, q)$, then all the conditions of Theorem 3.7 are satisfied, i.e., F satisfies the condition of Definition 3.6. That is, F is generalized α-admissible, $(\mathfrak{X}, d_s)$ is regular, and for every $u_0 \in \mathfrak{X}$, $\alpha(u_0, Fu_0) \succeq s(u_0, Fu_0)$. Hence, there exists ϱ in $\mathfrak{X}$ that is fixed under the mapping F. Now, it remains only to show that this fixed point is unique. Suppose that there is $\zeta \in \mathfrak{X}$ such that $F\zeta = \zeta$. Then, we have

$$d_s(\varrho, \zeta) = d_s(F\varrho, F\zeta) \preceq \kappa d_s(\varrho, \zeta).$$

However, $r(\kappa) < 1$ and so by Lemma 2.1, $e - \kappa$ is invertible. Thus, by Lemma 2.5 $d_s(\varrho, \zeta) = \theta$. $\square$

Theorem 3.11 *Let* $(\mathfrak{X}, d_s)$ *be a complete controlled cone metric-type space over a Banach algebra* $\mathcal{A}$ *and* $\mathcal{K}$ *be the associated cone in* $\mathcal{A}$. *Suppose that* $F : \mathfrak{X} \to \mathfrak{X}$ *satisfies the generalized Lipschitz condition, i.e., for all* $p, q \in \mathfrak{X}$,

$$d_s(Fp, Fq) \preceq \kappa\big[d_s(Fp, p) + d_s(Fq, q)\big], \tag{3.17}$$

where $\kappa \in \mathcal{K}$ *with* $r(\kappa) < \frac{1}{s(p,q)+1}$ *and for each* $p_0 \in \mathfrak{X}$,

$$\lim_{m,i \to \infty} \frac{s(p_{i+1}, p_{i+2})}{s(p_i, p_{i+1})} s(p_{i+1}, p_m) < \frac{1}{\|\tau\|},$$

with $\tau = (e - \kappa)^{-1}\kappa$. *Then, there exists a unique point* $\varrho \in \mathfrak{X}$ *that is fixed under the mapping* F.

Proof If we take $\nu_1 = \theta$, $\nu_2 = \nu_3 = \kappa$ and $\alpha(p, q) = s(p, q)$, then all the conditions of Theorem 3.7 are satisfied. Hence, there exists ϱ in $\mathfrak{X}$ that is fixed under the mapping F. Finally, we show that ϱ is a unique fixed point of F. For this, if ζ is another fixed point of F, then

$$d_s(\varrho, \zeta) = d_s(F\varrho, F\zeta) \preceq \kappa[d_s(\varrho, F\varrho) + d_s(\zeta, F\zeta)] = \theta.$$

Therefore, $\varrho = \zeta$. □

The following is the result of generalized Lipschitz mappings on cone b-metric spaces over a Banach algebra [15] that becomes a special case of Theorems 3.10 and 3.11 when we define $s(p, q) = b$ for some $b \geq 1$.

Theorem 3.12 ([15]) *Let* $(\mathfrak{X}, d)$ *be a complete cone b-metric space over a Banach algebra* $\mathcal{A}$ *with coefficient* $b \geq 1$ *and* $\mathcal{K}$ *be the associated solid cone* (*not necessarily normal*) *in* $\mathcal{A}$. *Suppose that* $F : \mathfrak{X} \to \mathfrak{X}$ *is a mapping such that for all* $p, q \in \mathfrak{X}$ *one of the following conditions holds*:
  (i) $d(Fp, Fq) \preceq \kappa d(p, q)$ *where* $\kappa \in \mathcal{K}$ *with* $r(\kappa) < \frac{1}{b}$.
  (ii) $d(Fp, Fq) \preceq \kappa(d(Fp, p) + d(Fq, q))$ *where* $\kappa \in \mathcal{K}$ *with* $r(\kappa) < \frac{1}{1+b}$.
Then, there exists a unique point $\varrho \in \mathfrak{X}$ *that is fixed under the mapping* F.

Corollary 3.13 *Let* $(\mathfrak{X}, d_s)$ *be a complete cone metric space over a Banach algebra* $\mathcal{A}$ *and* $\mathcal{K}$ *be the associated cone in* $\mathcal{A}$. *Let* $F : \mathfrak{X} \to \mathfrak{X}$ *be a mapping such that for all* $p, q \in \mathfrak{X}$,

$$d_s(Fp, Fq) \preceq \kappa d_s(p, q), \tag{3.18}$$

where $\kappa \in \mathcal{K}$ *with* $r(\kappa) < 1$. *Then, for each* $u_0 \in \mathfrak{X}$, *the iterative sequence* $u_n = F(u_{n-1}) = F^n u_0$ *converges to a unique fixed point of* F.

Proof Taking $b = 1$ in Theorem 3.12, we obtain the required result. □

Remark 3.14 (i) If we take $s(x, y) = b$ for some $b \geq 1$ in Theorems 3.10 and 3.11, we obtain the main results of [15] for cone b-metric spaces over a Banach algebra.
   (ii) By Remark 3.4, we obtain that Theorem 2.13 as a corollary of Theorem 3.10.
   (iii) If we take $s(x, y) = b$ for some $b \geq 1$ in Theorems 3.7 and 3.9, we obtain the main results of [27] for cone b-metric spaces over a Banach algebra.

4 Consequences

In this section, we list some important consequences of our results that generalize Theorems 3.1, 3.2, and 3.4 in Hussain *et al.* [23], Theorems 3.1 and 3.2 in Xu and Radenovic [26], Theorems 3.1, 3.2, and 3.5 in Malhotra *et al.* [21], Theorems 3.1, 3.2, and 3.5 in Malhotra *et al.* [22], and Theorems 2.1 and 2.2 in Liu and Xu [14].

Definition 4.1 ([20]) Let $\mathfrak{X}$ be a nonempty set and $\alpha : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ be a function. A mapping $F : \mathfrak{X} \to \mathfrak{X}$ is said to be an α-admissible mapping if $\alpha(p, q) \geq 1 \implies \alpha(Fp, Fq) \geq 1$.

Definition 4.2 Let $(\mathfrak{X}, d_s)$ be a complete controlled cone metric-type space over a Banach algebra $\mathcal{A}$ and $\mathcal{K}$ be the underlying solid cone. A mapping $F : \mathfrak{X} \to \mathfrak{X}$ is said to be a generalized α-Lipschitz contraction if for all $p, q \in \mathfrak{X}$ with $\alpha(p, q) \geq 1$,

$$d_s(Fp, Fq) \preceq \kappa d_s(p, q),$$

where $\kappa \in \mathcal{K}$ with $r(\kappa) < \frac{1}{s(p,q)}$ and for each $p_0 \in \mathfrak{X}$,

$$\lim_{m, i \to \infty} \frac{s(p_{i+1}, p_{i+2})}{s(p_i, p_{i+1})} s(p_{i+1}, p_m) < \frac{1}{\|\kappa\|}.$$

The following theorem becomes a special case of Theorem 3.7 if we define $\alpha : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ by $\alpha(p, q) = s(p, q) \geq 1$ for all $p, q \in \mathfrak{X}$ and take $\kappa = \nu_1, \nu_2 = \nu_3 = \theta$.

Theorem 4.3 *Let $(\mathfrak{X}, d_s)$ be a complete controlled cone metric-type space over a Banach algebra $\mathcal{A}$ and $\mathcal{K}$ be the associated solid cone. Suppose that $F : \mathfrak{X} \to \mathfrak{X}$ is a generalized α-Lipschitz contraction with Lipschitz constant κ such that*

  (i) *F is α-admissible;*
 (ii) *there exists $u_0 \in \mathfrak{X}$ such that $\alpha(u_0, Fu_0) \geq 1$;*
(iii) *F is continuous or if a sequence $\{u_n\} \in \mathfrak{X}$ so that $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $u_n \to q$ implies that $\alpha(u_n, q) \geq 1$ for all $n \in \mathbb{N}$.*

Then, there is a point ϱ in $\mathfrak{X}$ that is fixed under F.

For the uniqueness of the fixed point, we use the following extra condition:

$$\forall \varrho, \zeta \in \text{Fix}(F), \quad \text{there exists} z \in \mathfrak{X} \text{ such that } \alpha(\varrho, z) \geq 1 \text{ and } \alpha(\zeta, z) \geq 1. \tag{4.1}$$

Theorem 4.4 *If we add the condition (4.1) in the assumptions of Theorem 4.3, then the fixed point is unique.*

Proof The assertion follows simply from Theorems 4.3 and 3.9. □

Remark 4.5 (i) If we take $s(p, q) = b$ for some $b \geq 1$, then we obtain the main results of Hussain *et al.* [23, Theorems 3.1 and 3.2].

(ii) Theorems 3.1, 3.2, and 3.5 in Malhotra *et al.* [21] become special cases of Theorems 4.3 and 4.4, respectively, with $s(x, y) = 1, \nu_1 = 1$ and $\nu_2 = \nu_3 = \theta$.

(iii) If we define $s(p, q) = 1, \nu_1 = \theta$ and $\nu_2 = \nu_3$, then Theorems 3.1, 3.2, and 3.3 in Malhotra *et al.* [22] become special cases of Theorems 4.3 and 4.4, respectively.

If the given controlled cone metric-type space $(\mathfrak{X}, d_s)$ over a Banach algebra $\mathcal{A}$ is partially ordered, then we can use the following theorem.

Theorem 4.6 *Let $(\mathfrak{X}, \unrhd)$ be a partially ordered set and $s : \mathfrak{X} \times \mathfrak{X} \to [1, \infty)$. Let $(\mathfrak{X}, d_s)$ be a complete controlled cone metric-type space over a Banach algebra $\mathcal{A}$ with underlying solid cone $\mathcal{K}$. Assume that a nondecreasing mapping $F : \mathfrak{X} \to \mathfrak{X}$ with respect to $\unrhd$ satisfies the following conditions*:

(1) *there exist vectors $v_1, v_2, v_3 \in \mathcal{K}$ such that $2s(p,q)r(v_1) + (s(p,q) + 1)r(v_2 + v_3) < 2$, $d_s(Fp, Fq) \preceq v_1 d_s(p,q) + v_2 d_s(p, Fp) + v_3 d_s(q, Fq)$ for all $p, q \in \mathfrak{X}$ with $p \unrhd q$ and for each $p_0 \in \mathfrak{X}$ with $p_n = F^n p_0$,*

$$\lim_{m,i \to \infty} \frac{s(p_{i+1}, p_{i+2})}{s(p_i, p_{i+1})} s(p_{i+1}, p_m) < \frac{1}{\|\kappa\|},$$

$$\text{where } \kappa = (2e - v)^{-1}(2v_1 + v) \text{ for } v = v_2 + v_3;$$

(2) *there exists $u_0 \in \mathfrak{X}$ such that $u_0 \unrhd F u_0$;*

(3) *F is continuous or if $\{u_n\}$ is a nondecreasing sequence in $\mathfrak{X}$ with respect to $\unrhd$ such that $u_n \to q \in \mathfrak{X}$ as $(n \to \infty)$, then $u_n \unrhd q$ for all $n \in \mathbb{N}$.*

Then, there exists a point ζ in $\mathfrak{X}$ that is fixed under the mapping F.

Proof Define a function $\alpha : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ by

$$\alpha(p,q) = \begin{cases} s(p,q) & \text{if } p \unrhd q, \\ 0 & \text{elsewhere.} \end{cases}$$

Then, F is a generalized Reich-type contraction by Condition (1). Now, since F is a nondecreasing mapping, F is a generalized α-admissible mapping. Condition (2) implies that there exists $u_0 \in \mathfrak{X}$ such that $\alpha(u_0, F u_0) = s(u_0, F u_0)$. By Condition (3), we can see that either F is continuous or $(\mathfrak{X}, d_s)$ is regular. It follows that all the conditions of Theorem 3.7 are satisfied and so we conclude that there exists a point in $\mathfrak{X}$ that is fixed under the mapping F. $\qquad \square$

Corollary 4.7 *Let $(\mathfrak{X}, \unrhd)$ be a partially ordered set and $s : \mathfrak{X} \times \mathfrak{X} \to [1, \infty)$. Let $(\mathfrak{X}, d_s)$ be a complete controlled cone metric-type space over a Banach algebra $\mathcal{A}$ with underlying solid cone $\mathcal{K}$. Suppose $F : \mathfrak{X} \to \mathfrak{X}$ is a nondecreasing mapping with respect to $\unrhd$ and the following assumptions hold*:

(1) *there exists $\kappa \in \mathcal{K}$ with $r(\kappa) < \frac{1}{s(p,q)}$ so that $d_s(Fp, Fq) \preceq \kappa d_s(p,q)$ for all $p, q \in \mathfrak{X}$ with $p \unrhd q$ and for each $p_0 \in \mathfrak{X}$ with $p_n = F^n p_0$,*

$$\lim_{m,i \to \infty} \frac{s(p_{i+1}, p_{i+2})}{s(p_i, p_{i+1})} s(p_{i+1}, p_m) < \frac{1}{\|\kappa\|};$$

(2) *there exists $u_0 \in \mathfrak{X}$ such that $u_0 \unrhd F u_0$;*

(3) *T is continuous or if $\{u_n\}$ is a nondecreasing sequence in $\mathfrak{X}$ with respect to $\unrhd$ such that $u_n \to q \in \mathfrak{X}$ as $(n \to \infty)$, then $u_n \unrhd q$ for all $n \in \mathbb{N}$.*

Then, there exists a unique point ϱ in $\mathfrak{X}$ that is fixed under the mapping F.

Proof The assertion follows directly, from Theorem 4.6 if we take $\nu_1 = \kappa$ and $\nu_2 = \nu_3 = \theta$. □

Remark 4.8 (i) Theorem 4.6 becomes [27, Theorem 3.6] if we take $s(p,q) = b$ for some $b \geq 1$.

(ii) If we take $s(p,q) = b$ for some $b \geq 1$ in Corollary 4.7, then we obtain Theorems 4.2 and 4.3 of Hussain *et al.* [23].

(iii) Theorems 2.1 and 2.2 in Nieto and Rodreguez-Lopez [29] are special cases of Corollary 4.7 with $s(p,q) = 1$ and $\mathcal{A} = \mathbb{R}$.

5 Conclusion

In this article, we introduced a new geometrical structure that is the hybrid of a cone metric space over Banach algebra and a controlled metric-type space. We have defined a metric space and we proved analogs of Banach-, Kannan- and Reich-type fixed-point theorems. We also furnished various concrete examples to establish the validity of our results. The obtained results generalize many well-known results in the literature.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
WU and HI carried out the mathematical studies, participated in their design and coordination, drafted the manuscript, and participated in the sequence alignment. NA, CP, and JRL participated in the design of the study and performed the mathematical and functional analysis. All the authors read and approved the final manuscript.

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