Pachaiyappan et al. *Journal of Inequalities and Applications* (2022) 2022:60
https://doi.org/10.1186/s13660-022-02795-z

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Relation between electrical resistance and conductance using multifarious functional equations and applications to parallel circuit

Divyakumari Pachaiyappan[1], Ramdoss Murali[2], Choonkil Park[3*] and Jung Rye Lee[4*]

*Correspondence:
baak@hanyang.ac.kr;
jrlee@daejin.ac.kr
[3]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
[4]Department of Data Science, Daejin University, Kyunggi 11159, Korea
Full list of author information is available at the end of the article

Abstract

In this paper, we introduce a new generalized *p*-dimensional multifarious radical reciprocal functional equation combining three classical means: arithmetic, geometric, and harmonic. Mainly, we find its general solution and stability related to the Ulam problem in modular spaces by using the fixed point method with suitable counterexamples. Importantly, in this paper, we illustrate the geometrical interpretation and applications of the introduced Pythagorean means multifarious functional equation in connection with the parallel circuit. Furthermore, we provide a formula for finding the equivalent resistance $R_{\mathbf{eq}}$ of parallel electrical circuit using functional equations, which relates the electrical resistances and conductances with suitable examples.

MSC: 39B82; 39B62; 39B52

Keywords: Hyers–Ulam stability; Functional equation; Fixed point method; Fuzzy modular space

1 Introduction and preliminaries

In the development of broad field functional equations, we come acrossing various types like additive, quadratic, cubic and so on. In recent research many researchers modeled functional equations from physical phenomena. In particular, by geometrical construction, many authors introduced remarkable reciprocal-type functional equations. In 2010, Ravi and Senthil Kumar [1] introduced the reciprocal-type functional equation

$$s(z+w)=\frac{s(z)s(w)}{s(z)+s(w)} \tag{1.1}$$

with solution $s(z)=\frac{c}{z}$.

In 2015, Narasimman, Ravi, and Pinelas [2] introduced the radical reciprocal quadratic functional equation

$$s\left(\sqrt[2]{z^2+w^2}\right)=\frac{s(z)s(w)}{s(z)+s(w)}, \quad z,w\in(0,\infty), \tag{1.2}$$

which is satisfied by $s(z) = \frac{c}{z^2}$. Also, they provided the solution and stability of (1.2) with geometrical interpretation and application. Equation (1.2) was extended by Pinelas et al. [3] as follows:

$$s\left(\sqrt[2]{z_1^2+z_2^2}\right)+s\left(\sqrt[2]{z_2^2+z_3^2}\right)+s\left(\sqrt[2]{z_1^2+z_3^2}\right) \\ = \frac{s(z_1)s(z_2)}{s(z_1)+s(z_2)}+\frac{s(z_2)s(z_3)}{s(z_2)+s(z_3)}+\frac{s(z_1)s(z_3)}{s(z_1)+s(z_3)}, \quad z_1,z_2,z_3 \in (0,\infty), \tag{1.3}$$

which is satisfied by $s(z) = \frac{c}{z^2}$.

In 2014, Bodaghi and Kim [4] introduced the quadratic reciprocal functional equation, which was generalized by Song and Song [5].

Definition 1.1 A reciprocal functional equation is a functional equation with solution of the form $\frac{1}{s(z)}$ where $s(z) = z^m$, $m \in \mathbb{N}$. Such functional equations were studied in the recent papers; we refer to [6–11].

Definition 1.2 ([12]) The three classical Pythagorean means are the arithmetic mean (AM), the geometric mean (GM), and the harmonic mean (HM) defined by

$$AM(a_1,a_2,\dots,a_n) = \frac{1}{n}(a_1+\cdots+a_n),$$
$$GM(a_1,a_2,\dots,a_n) = \sqrt[n]{a_1+\cdots+a_n},$$
$$HM(a_1,a_2,\dots,a_n) = \frac{n}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}}.$$

Definition 1.3 Functional equations that arise from the relations between three Pythagorean means (arithmetic, geometric, and harmonic) are known as Pythagorean mean functional equations.

Definition 1.4 A reciprocal Pythagorean mean functional equation is said to be multifarious if it has various nature like additive, quadratic, cubic, and so on with respect to its dimensions.

For the necessary introduction on stability related to the Ulam problem and the notion of modular spaces, we refer to [13–21].

Nakano [22] in 1959 introduced modular spaces with an abstract functional. This abstract functional is called modular, and it forms the basis of the modular space theory. Musielak and Orlicz [23] redefined and generalized the notion of a modular space.

Definition 1.5 Let Z be a vector space. A real function ξ on Z is said to be a modular if it fulfills the following conditions:

(i) $\xi(z_1) = 0$ if and only if $z_1 = \Gamma$ (the null vector),
(ii) $\xi(z_1) = \xi(-z_1)$.
(iii) $\xi(\alpha z_1 + \beta z_2) \le \xi(z_1) + \xi(z_2)$ for all $z_1, z_2 \in Z$ and $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$.

If

(iii)' $\xi(\alpha z_1 + \beta z_2) \le \alpha\xi(z_1) + \beta\xi(z_2)$ for all $z_1, z_2 \in Z$ and $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$,

then ξ is called a convex modular.

Note that $\xi(t\cdot z_1)$ is an increasing function in t for each $z_1 \in Z$. Suppose $0 < a < b$. Put $z_2 = 0$ in property (iii) of Definition 1.5. Then $\xi(az_1) = \xi(\frac{a}{b}bz_1) \leq \xi(bz_1)$ for all $z_1 \in Z$. Moreover, if ξ is a convex modular on Z and $|\alpha| \leq 1$, then $\xi(\alpha z_1) = \alpha\xi(z_1)$ and also $\xi(z_1) \leq \frac{1}{2}\xi(2z_1)$ for all $z_1 \in Z$.

Definition 1.6 A modular space Z_ξ is defined by a corresponding modular ξ, that is, $Z_\xi = \{z_1 \in Z : \xi(\lambda z_1) \to 0 \text{ as } \lambda \to 0\}$.

Definition 1.7 A modular is said to satisfy the Δ_2-condition if there exists $\kappa > 0$ such that $\xi(2z) \leq \kappa\xi(z)$ for all $z \in Z_\xi$.

Definition 1.8 Let $\{z_i\}$ and z be in Z_ξ. Then the modular ξ has the Fatou property if $\xi(z) \leq \liminf_{i\to\infty} \xi(z_i)$ whenever the sequence $\{z_i\}$ is ξ-convergent to z.

Definition 1.9 ([24]) Let A be a C^*-algebra, and let $x \in A$ be a self-adjoint element, i.e., $x^* = x$. Then x is said to be *positive* if it is of the form yy^* for some $y \in A$. The set of positive elements of A is denoted by A^+.

Note that A^+ is a closed convex cone (see [24]). It is well known that for a positive element x and a positive integer n, there exists a unique positive element $y \in A^+$ such that $x = y^n$. We denote y by $x^{\frac{1}{n}}$ (see [25]).

In Sect. 1, we provide the necessary introduction and preliminaries. Also, we give the geometrical construction and geometrical interpretation of p-dimensional multifarious radical reciprocal functional equations. In Sect. 2, we find general solutions of (1.7) and (1.8). In Sect. 3, we investigate the Hyers–Ulam stability of (1.7) and (1.8). We illustrate applications of (1.7) in Sect. 4, and the conclusion is given in Sect. 5.

1.1 Main results

In this paper, using Pythagorean means, we introduce new generalized two-, three-, and p-dimensional multifarious radical reciprocal functional equations.

The following two- and three-dimensional multifarious radical reciprocal functional equations are obtained by (1.1) and (1.2):

$$s\left(\sqrt[m]{z^m + w^m}\right) = \frac{s(z)s(w)}{s(z) + s(w)}, \tag{1.4}$$

$$s\left(\sqrt[m]{z_1^m + z_2^m + z_3^m}\right) = \frac{s(z_1)s(z_2)s(z_3)}{s(z_1)s(z_2) + s(z_2)s(z_3) + s(z_1)s(z_3)}, \tag{1.5}$$

which are satisfied by $s(z) = \frac{c}{z^m}$ for all $z, w, z_1, z_2, z_3 \in (0,\infty)$, $m \in \mathbb{N}$. Observe that if $m = 1$ and $m = 2$ in (1.4), then we get (1.1) and (1.2), respectively. Hence the functional equation (1.4) is known as a two-dimensional multifarious radical reciprocal functional equation. By a similar argument (1.5) is known as a three-dimensional multifarious radical reciprocal functional equation.

Moreover, by (1.4) we have

$$\begin{aligned} &s\left(\sqrt[m]{z_1^m + z_2^m}\right) + s\left(\sqrt[m]{z_2^m + z_3^m}\right) + s\left(\sqrt[m]{z_1^m + z_3^m}\right) \\ &\quad = \frac{s(z_1)s(z_2)}{s(z_1) + s(z_2)} + \frac{s(z_2)s(z_3)}{s(z_2) + s(z_3)} + \frac{s(z_1)s(z_3)}{s(z_1) + s(z_3)}, \end{aligned} \tag{1.6}$$

which is satisfied by $s(z_1) = \frac{c}{z_1^m}$, $z_1, z_2, z_3 \in (0, \infty)$, $m \in \mathbb{N}$, and the denominators are not equal to zero. We may observe that if $m = 2$, then we get (1.3).

Importantly, we generalize the above two- and three-dimensional multifarious radical reciprocal functional equations (1.4), (1.5), and (1.6) into p-dimensional multifarious radical reciprocal functional equations, as follows:

$$s\left(\sqrt[m]{\sum_{i=1}^{p} z_i^m}\right) = \frac{\prod_{i=1}^{p} s(z_i)}{\frac{1}{s(z_1)}\prod_{i=1}^{p} s(z_i) + \frac{1}{s(z_2)}\prod_{i=1}^{p} s(z_i) + \cdots + \frac{1}{s(z_p)}\prod_{i=1}^{p} s(z_i)}, \tag{1.7}$$

$$\begin{aligned} &\sum_{i=1}^{p}\left(s\left(\sqrt[m]{z_i^m + z_j^m}\right)\right) + s\left(\sqrt[m]{z_1^m + z_{p+1}^m}\right) \\ &\quad = \sum_{i=1}^{p}\left(\frac{s(z_i)s(z_j)}{s(z_i) + s(z_j)}\right) + \frac{s(z_1)s(z_{p+1})}{s(z_1) + s(z_{p+1})}, \end{aligned} \tag{1.8}$$

which are satisfied by $s(z) = \frac{c}{z^m}$, $j = i + 1$, for all $m, p \in \mathbb{N}$, and the denominators of (1.7) and (1.8) are not equal to zero. We may observe that if $p = 2$ in (1.7), then we get (1.4). Also, if $p = 1$ and $p = 2$ in (1.8), then we get (1.4) and (1.6), respectively.

1.2 Geometrical construction of p-dimensional multifarious radical reciprocal functional equations

Geometric construction of three Pythagorean means of two variables can be constructed geometrically as shown in Fig. 1. Geometric construction of geometric mean of three variables are not possible, but the other Pythagorean means can be constructed for any number of variables; we refer to [26–28].

The relations between three Pythagorean means of p-objects $z_1, z_2, \ldots, z_p$ are represented by the equation

$$H(z_1, z_2, \ldots, z_p) = \frac{G(z_1, z_2, \ldots, z_p)^p}{A(\frac{1}{z_1}\prod_{i=1}^{p} z_i, \frac{1}{z_2}\prod_{i=1}^{p} z_i, \ldots, \frac{1}{z_p}\prod_{i=1}^{p} z_i)}. \tag{1.9}$$

Consider two spheres S_1 and S_2 of radii $r_1 > r_2$ located along the x-axis and centered at $C_1(0,0,0)$ and $C_2(d,0,0)$, respectively.

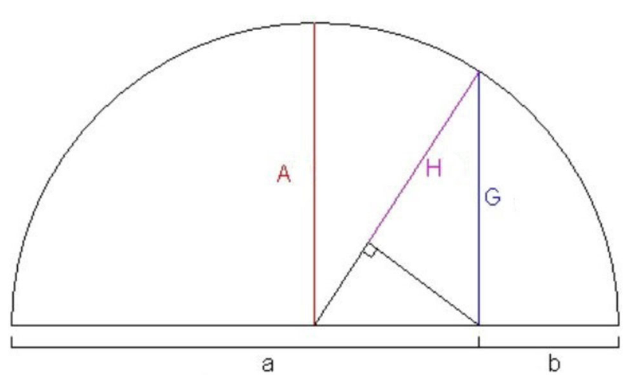


Figure 1 Pythagorean means of a and b: A is the arithmetic mean, H is the harmonic mean, and G is the geometric mean

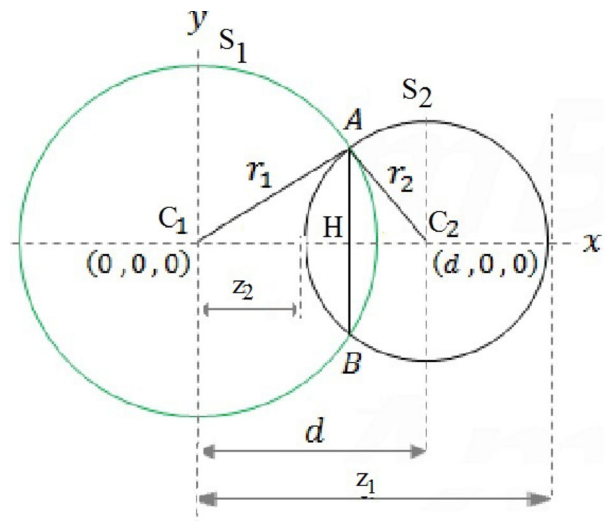


Figure 2 Two intersecting spheres S_1 and S_2

We can show the length of C_2C_1 is $\frac{z_1+z_2}{2}$, which is the arithmetic mean of z_1 and z_2. Using Pythagoras' theorem, we can find that the length of AC_1 is the geometric mean $\sqrt{z_1z_2}$ of z_1 and z_2. Also, we can obtain that the length of HC_1 is $\frac{2z_1z_2}{z_1+z_2}$, which the harmonic mean of z_1 and z_2, since C_2AC_1 and AHC_1 are similar.

From Fig. 2, we have the equality $HC_1 = \frac{AC_1^2}{C_2C_1}$, that is,

$$H(z_1,z_2) = \frac{G(z_1,z_2)^2}{A(\frac{1}{z_1}\prod_{i=1}^{2} z_i, \frac{1}{z_2}\prod_{i=1}^{2} z_i)},$$

which is a particular case of (1.9) for $p = 2$, which implies

$$\frac{1}{\frac{1}{z_1} + \frac{1}{z_2}} = \frac{z_1z_2}{z_1 + z_2}. \tag{1.10}$$

Letting $z_1 = \frac{1}{z}$ and $z_2 = \frac{1}{w}$ in (1.10), we get

$$\frac{1}{z + w} = \frac{\frac{1}{z}\frac{1}{w}}{\frac{1}{z} + \frac{1}{w}}. \tag{1.11}$$

In that case, (1.1) is valid by (1.11), which is satisfied by $s(z) = \frac{c}{z}$.

Letting $z_1 = \frac{1}{z^2}$ and $z_2 = \frac{1}{w^2}$ in (1.10), we get

$$\frac{1}{z^2 + w^2} = \frac{\frac{1}{z^2}\frac{1}{w^2}}{\frac{1}{z^2} + \frac{1}{w^2}}. \tag{1.12}$$

In that case, (1.2) is valid by (1.12), which is satisfied by $s(z) = \frac{c}{z^2}$. In general, letting $z_1 = \frac{1}{z^m}$ and $z_2 = \frac{1}{w^m}$ in (1.10), we get

$$\frac{1}{z^m + w^m} = \frac{\frac{1}{z^m}\frac{1}{w^m}}{\frac{1}{z^m} + \frac{1}{w^m}}. \tag{1.13}$$

In that case, (1.4) is valid by (1.13), which is satisfied by $s(z) = \frac{c}{z^m}$.

In Fig. 2, AB is the diameter of the common circle. The common circle is the solution of the system

$$\begin{aligned} &z_1^2+z_2^2+z_3^2=r_1^2,\\ &(z_1-d)^2+z_2^2+z_3^2=r_2^2, \end{aligned}$$

which imply

$$\begin{aligned} &\frac{1}{z_1^2+z_2^2+z_3^2}=\frac{1}{r_1^2},\\ &\frac{1}{(z_1-d)^2+z_2^2+z_3^2}=\frac{1}{r_2^2}. \end{aligned} \tag{1.14}$$

The system of equations (1.14) can be expressed by radical reciprocal quadratic functional equations of the form

$$\begin{aligned} &s\bigl(r_1^2\bigr)=\frac{s(z_1)s(z_2)s(z_3)}{s(z_1)s(z_2)+s(z_2)s(z_3)+s(z_1)s(z_3)},\\ &s\bigl(r_2^2\bigr)=\frac{s(z_1-d)s(z_2)s(z_3)}{s(z_1-d)s(z_2)+s(z_2)s(z_3)+s(z_1-d)s(z_3)} \end{aligned} \tag{1.15}$$

for $z_1,z_2,z_3,r_1,r_2\in(0,\infty)$, which are satisfied by $s(z_1)=\frac{c}{z_1^2}$, and the denominators are not equal to zero. Also, we observe that equation (1.15) is a particular case of (1.5) for $m=2$. Letting $p=3$ in (1.9), we get

$$H(z_1,z_2,z_3)=\frac{G(z_1,z_2,z_3)^3}{A(\frac{1}{z_1}\prod_{i=1}^3 z_i,\frac{1}{z_2}\prod_{i=1}^3 z_i,\frac{1}{z_3}\prod_{i=1}^3 z_i)},$$

which gives

$$\frac{1}{\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}}=\frac{z_1z_2z_3}{z_2z_3+z_1z_3+z_1z_2}. \tag{1.16}$$

Letting $z_1=\frac{1}{z_1^m}$, $z_2=\frac{1}{z_2^m}$ and $z_3=\frac{1}{z_3^m}$ in (1.16), we get

$$\frac{1}{z_1^m+z_2^m+z_3^m}=\frac{\frac{1}{z_1^m}\frac{1}{z_2^m}\frac{1}{z_3^m}}{\frac{1}{z_1^m}+\frac{1}{z_2^m}+\frac{1}{z_3^m}}. \tag{1.17}$$

In that case, (1.5) is valid by (1.17), which is satisfied by $s(z_1)=\frac{c}{z_1^m}$. By the same process for p-objects, (1.9) becomes

$$\frac{1}{\frac{1}{z_1}+\frac{1}{z_2}+\cdots+\frac{1}{z_p}}=\frac{z_1z_2z_3\cdots z_p}{z_2z_3\cdots z_p+z_1z_3\cdots z_p+\cdots+z_1z_2\cdots z_{p-1}}. \tag{1.18}$$

Letting $z_1=\frac{1}{z_1^m}$, $z_2=\frac{1}{z_2^m},\ldots,z_{p-1}=\frac{1}{z_{p-1}^m}$ in (1.18), we get

$$\frac{1}{z_1^m+z_2^m+\cdots+z_p^m}=\frac{\frac{1}{z_1^m}\frac{1}{z_2^m}\cdots\frac{1}{z_p^m}}{\frac{1}{z_2^m}\frac{1}{z_3^m}\cdots\frac{1}{z_p^m}+\frac{1}{z_1^m}\frac{1}{z_3^m}\cdots\frac{1}{z_p^m}+\cdots+\frac{1}{z_1^m}\frac{1}{z_2^m}\cdots\frac{1}{z_{p-1}^m}}. \tag{1.19}$$

Clearly, equation (1.7) holds by (1.19) with solution $s(z_1) = \frac{c}{z_1^m}$. By a similar process and by (1.9), if we choose pairwise from p-objects $z_1, z_2, \dots, z_p$ as (z_i, z_j), end with (z_1, z_{p+1}) for $j = i + 1$, $i = 1, 2, \dots, p$, and sum all the resultants, then we obtain (1.8).

2 General solutions of *p*-dimensional multifarious radical reciprocal functional equations

The following theorems give the solutions of (1.7) and (1.8), which are motivated by the work of Ger [26].

Theorem 2.1 *A general solution of* (1.7) *is* $s(z) = \frac{c}{z^m}$, $z \in (0, \infty)$, *with* $\frac{s(z)}{\frac{1}{z^m}}$ *a quotient at zero.*

Proof Letting $z_1 = z_2 = \cdots = z_p = z$ in (1.7), we obtain

$$s\left(\sqrt[m]{p}z\right) = \frac{1}{p}s(z). \tag{2.1}$$

Assuming that

$$g(z) = \frac{s(z)}{\frac{1}{z^{\frac{m}{2}}}}, \tag{2.2}$$

we have

$$\lim_{z \to 0^+} \frac{g(z)}{\frac{1}{z^{\frac{m}{2}}}} =: c \in \mathbb{R}.$$

Dividing (2.1) by $\frac{1}{z^{\frac{m}{2}}}$, we have

$$\frac{s(\sqrt[m]{p}z)}{\frac{\sqrt{p}}{\sqrt{p}z^{\frac{m}{2}}}} = \frac{\frac{1}{p}s(z)}{\frac{1}{z^{\frac{m}{2}}}}. \tag{2.3}$$

By (2.2) and (2.3) we have

$$g\left(\sqrt[m]{p}z\right) = \frac{1}{\sqrt{p}}g(z). \tag{2.4}$$

Replacing z by $\frac{z}{\sqrt[m]{p}}$ in (2.4), we get

$$\sqrt{p}g(z) = g\left(\frac{z}{\sqrt[m]{p}}\right). \tag{2.5}$$

Again, replacing z by $\frac{z}{\sqrt[m]{p}}$ in (2.5), we obtain

$$(\sqrt{p})^2 g(z) = g\left(\frac{z}{(\sqrt[m]{p})^2}\right).$$

Continuing the same process k times, we have

$$(\sqrt{p})^k g(z) = g\left(\frac{z}{(\sqrt[m]{p})^k}\right).$$

Now

$$\frac{g(z)}{\frac{1}{z^{\frac{m}{2}}}} = \frac{(\sqrt{p})^k g(z)}{(\sqrt{p})^k \frac{1}{z^{\frac{m}{2}}}} = \frac{g(\frac{1}{(\sqrt[m]{p})^k} z)}{\frac{(\sqrt{p})^k}{z^{\frac{m}{2}}}} \to c \quad \text{as } k \to \infty,$$

and (2.2) implies that

$$s(z) = \frac{1}{z^{\frac{m}{2}}} g(z) = \frac{1}{z^{\frac{m}{2}}} \frac{1}{z^{\frac{m}{2}}} c = \frac{c}{z^m}.$$

This completes the proof. □

Theorem 2.2 *A general solution of* (1.8) *is* $s(z) = \frac{c}{z^m}$, $z \in (0, \infty)$, *with* $\frac{s(z)}{\frac{1}{z^m}}$ *a quotient at zero.*

Proof Letting $z_1 = z_2 = \cdots = z_{p+1} = z$ in (1.8), we get

$$s(\sqrt[m]{2}z) = \frac{1}{2} s(z), \tag{2.6}$$

and assuming that

$$h(z) = \frac{s(z)}{\frac{1}{z^{\frac{m}{2}}}}, \tag{2.7}$$

we obtain

$$\lim_{z \to 0^+} \frac{h(z)}{\frac{1}{z^{\frac{m}{2}}}} =: c \in \mathbb{R}.$$

Dividing (2.6) by $\frac{1}{z^{\frac{m}{2}}}$, we get

$$\frac{s(\sqrt[m]{2}z)}{\frac{\sqrt{2}}{\sqrt{2}z^{\frac{m}{2}}}} = \frac{\frac{1}{2}s(z)}{\frac{1}{z^{\frac{m}{2}}}}, \tag{2.8}$$

and by (2.7) and (2.8) we have

$$h(\sqrt[m]{2}z) = \frac{1}{\sqrt{2}} h(z). \tag{2.9}$$

Replacing z by $\frac{z}{\sqrt[m]{2}}$ in (2.9) and (2.10), we get

$$\sqrt{2}h(z) = h\left(\frac{z}{\sqrt[m]{2}}\right) \tag{2.10}$$

and

$$(\sqrt{2})^2 h(z) = h\left(\frac{z}{(\sqrt[m]{2})^2}\right).$$

respectively. Continuing the same process k times, we get

$$(\sqrt{2})^k h(z) = h\left(\frac{z}{(\sqrt[m]{2})^k}\right). \tag{2.11}$$

Since

$$\frac{h(z)}{\frac{1}{z^{\frac{m}{2}}}} = \frac{(\sqrt{2})^k h(z)}{(\sqrt{2})^k \frac{1}{z^{\frac{m}{2}}}} = \frac{h(\frac{1}{(\sqrt[m]{2})^k} z)}{\frac{(\sqrt{2})^k}{z^{\frac{m}{2}}}} \to c \quad \text{as } k \to \infty,$$

equations (2.7) and (2.11) imply that

$$s(z) = \frac{1}{z^{\frac{m}{2}}} h(z) = \frac{1}{z^{\frac{m}{2}}} \frac{1}{z^{\frac{m}{2}}} c = \frac{c}{z^m}.$$

This completes the proof. □

In the following theorem, we obtain general solutions of (1.7) and (1.8) by the derivative method.

Theorem 2.3 *Let $s : (0,\infty) \to \mathbb{R}$ be a continuously differentiable function with nowhere vanishing derivative s'. Then s has a solution of the functional equation (1.7) if and only if there exists a nonzero real constant c such that $s(z) = \frac{c}{z^m}$, $z \in (0,\infty)$.*

Proof Differentiating (1.7) with respect to z_1 on both sides, we have

$$\begin{aligned} &s'\left(\sqrt[m]{z_1^m + z_2^m + \cdots + z_p^m}\right) \frac{(z_1)^{m-1}}{(\sqrt[m]{z_1^m + z_2^m \cdots + z_p^m})^{m-1}} \\ &\quad = \frac{\begin{array}{c}(s'(z_1)s(z_2)\cdots s(z_p))[s(z_2)s(z_3)\cdots s(z_p)+\cdots+s(z_1)s(z_2)\cdots s(z_{p-1})] \\ -(s(z_1)s(z_2)\cdots s(z_p))[s'(z_1)s(z_3)\cdots s(z_p)+\cdots+s'(z_1)s(z_2)\cdots s(z_{p-1})]\end{array}}{(s(z_2)s(z_3)\cdots s(z_p) + \cdots + s(z_1)s(z_2)\cdots s(z_{p-1}))^2}. \end{aligned} \tag{2.12}$$

Letting $z_1 = z_2 = \cdots = z_p = z$ in (2.12), we have

$$s'\left(\sqrt[m]{p}z\right) = \frac{1}{p\sqrt[m]{p}} s'(z), \tag{2.13}$$

and setting $z_1 = \sqrt[m]{2}z$ and $z_2 = z_3 = \cdots = z_p = z$ in (2.12) and using (2.1) and (2.13), we obtain

$$s'\left(\sqrt[m]{p+1}z\right) = \frac{1}{(p+1)\sqrt[m]{p+1}} s'(z). \tag{2.14}$$

By (2.13) and (2.14) we get

$$s'\left(\left(\sqrt[m]{p}\right)^k \left(\sqrt[m]{p+1}\right)^l z\right) = \frac{1}{p^k(\sqrt[m]{p})^k} \frac{1}{(p+1)^l(\sqrt[m]{p+1})^l} s'(z)$$

for all integers k, l. We derive its linearity by assuming $\lambda = (\sqrt[m]{p})^k(\sqrt[m]{p+1})^l$ and $z = 1$:

$$s'(\lambda) = s'(1)\frac{1}{(\lambda)^{m+1}}$$

for $\lambda \in (0,\infty)$. Therefore there exist real numbers $c \neq 0$ and d such that $s(z) = \frac{c}{z^m} + d$ for $z \in (0,\infty)$. Note that we have $d = 0$, since the equality $s(\sqrt[m]{p}z) = \frac{1}{p}s(z)$ is valid for all positive real numbers z. This completes the proof. □

Theorem 2.4 *Let $s : (0,\infty) \to \mathbb{R}$ be a continuously differentiable function with nowhere vanishing derivative s'. Then s is a solution of the functional equation* (1.8) *if and only if there exists a nonzero real constant c such that $s(z) = \frac{c}{z^m}$, $z \in (0,\infty)$.*

Proof Differentiating (1.8) with respect to z_1 on both sides, we obtain

$$\begin{aligned} &s'\left(\sqrt[m]{z_1^m + z_2^m}\right)\frac{(z_1)^{m-1}}{(\sqrt[m]{z_1^m + z_2^m})^{m-1}} + s'\left(\sqrt[m]{z_1^m + z_{p+1}^m}\right)\frac{(z_1)^{m-1}}{(\sqrt[m]{z_1^m + z_{p+1}^m})^{m-1}} \\ &\quad = \frac{s'(z_1)(s(z_2))^2}{(s(z_1) + s(z_2))^2} + \frac{s'(z_1)(s(z_{p+1}))^2}{(s(z_1) + s(z_{p+1}))^2}, \end{aligned} \tag{2.15}$$

and by (2.6) we have

$$s'\left(\sqrt[m]{2}z\right) = \frac{1}{2\sqrt[m]{2}}s'(z). \tag{2.16}$$

Letting $z_1 = z$ and $z_2 = z_{p+1} = \sqrt[m]{2}z$ in (2.15) and using (2.6) and (2.16), we get

$$s'\left(\sqrt[m]{3}z\right) = \frac{1}{3\sqrt[m]{3}}s'(z), \tag{2.17}$$

and from (2.16) and (2.17) we get

$$s'\left(\left(\sqrt[m]{2}\right)^k\left(\sqrt[m]{3}\right)^l z\right) = \frac{1}{2^k(\sqrt[m]{2})^k}\frac{1}{3^l(\sqrt[m]{3})^l}s'(z),$$

for all integers k, l. We derive its linearity by assuming $\lambda = (\sqrt[m]{2})^k(\sqrt[m]{3})^l$ and $z = 1$,

$$s'(\lambda) = s'(1)\frac{1}{(\lambda)^{m+1}}$$

for $\lambda \in (0,\infty)$. Therefore, there exist real numbers $c \neq 0$ and d such that $s(z) = \frac{c}{z^m} + d$ for $z \in (0,\infty)$. Note that we have $d = 0$, since the equality $s(\sqrt[m]{2}z) = \frac{1}{2}s(z)$ is valid for all positive real numbers z. This completes the proof. □

Consider $M = A^+$, the positive cone of a C^*-algebra A, and let Z be the real field $\mathbb{R}$. Let Z_ξ be the ξ-complete modular space where ξ is a convex modular on Z. For convenience, let us define the difference operators $D_1 s(z_1, z_2, \dots, z_p) : \underbrace{M \times \cdots \times M}_{p \text{ times}} \to Z_\xi$ and

$D_2 s(z_1, z_2, \dots, z_{p+1}) : \underbrace{M \times \cdots \times M}_{p+1 \text{ times}} \to Z_\xi$ as follows:

$$D_1 s(z_1, z_2, \dots, z_p) = s\left(\sqrt[m]{\sum_{i=1}^{p} z_i^m}\right) - \frac{\prod_{i=1}^{p} s(z_i)}{\frac{1}{s(z_1)}\prod_{i=1}^{p} s(z_i) + \frac{1}{s(z_2)}\prod_{i=1}^{p} s(z_i) + \cdots + \frac{1}{s(z_p)}\prod_{i=1}^{p} s(z_i)},$$

$$D_2 s(z_1, z_2, \dots, z_p, z_{p+1}) = \sum_{i=1}^{p}\left(s\left(\sqrt[m]{z_i^m + z_j^m}\right)\right) + s\left(\sqrt[m]{z_1^m + z_{p+1}^m}\right) - \sum_{i=1}^{p}\left(\frac{s(z_i)s(z_j)}{s(z_i) + s(z_j)}\right) - \frac{s(z_1)s(z_{p+1})}{s(z_1) + s(z_{p+1})}$$

for $z_1, z_2, \dots, z_p, z_{p+1} \in M$.

3 Hyers–Ulam stability of *p*-dimensional multifarious radical reciprocal functional equations

In this section, we prove the Hyers–Ulam stability of (1.7) and (1.8) in modular spaces by the fixed point method.

Theorem 3.1 *Let $\eta : M^p \to [0, +\infty)$ be a function satisfying*

$$\eta\left((p)^{\frac{1}{m}} z_1, (p)^{\frac{1}{m}} z_2, \dots, (p)^{\frac{1}{m}} z_p\right) \leq \frac{1}{p}\psi\,\eta(z_1, z_2, \dots, z_p) \tag{3.1}$$

for all $z_1, z_2, \dots, z_n \in M$ and $\psi < 1$. Assume that $s : M \to Z_\xi$ fulfills

$$\xi\left(D_1 s(z_1, z_2, \dots z_p)\right) \leq \eta(z_1, z_2, \dots, z_p) \tag{3.2}$$

for all $z_1, z_2, \dots, z_p \in M$. Then there exists a unique multifarious radical reciprocal mapping $R : M \to Z_\xi$ such that

$$\xi\left(R(z) - s(z)\right) \leq \frac{1}{\frac{1}{p}(1-\psi)}\eta(z, z, \dots, z) \quad \forall z \in M. \tag{3.3}$$

Proof Let $N = \{h : M \to Z_\xi\}$ and define ξ' on N as

$$\xi'(h) =: \inf\left\{(p)^{\frac{1}{m}} > 0 : \xi\left(h(z)\right) \leq (p)^{\frac{1}{m}}\eta(z, z, \dots, z)\right\}.$$

We can easily prove that ξ' is a convex modular with Fatou property on N. Let $\{h_i\}$ be a ξ'-Cauchy sequence in $N_{\xi'}$, and let $\epsilon > 0$. There exists a positive integer $i_0 \in \mathbb{N}$ such that $\xi'(h_i - h_j) \leq \epsilon$ for all $i, j \geq i_0$, and we can easily prove that $\{h_i\}$ is a ξ'-convergent sequence in $N_{\xi'}$. Therefore $N_{\xi'}$ is ξ'-complete; see [29, 30]. Consider the function $\sigma : N_{\xi'} \to N_{\xi'}$ defined by

$$\sigma h(z) = \frac{1}{p} h\left(p^{\frac{1}{m}} z\right) \tag{3.4}$$

for $h \in N_{\xi'}$. Let $h, r \in N_{\xi'}$ and $(p)^{\frac{1}{m}} \in [0,1]$ with $\xi'(h-r) < (p)^{\frac{1}{m}}$. By the definition of ξ' we get

$$\xi\big(h(z) - r(z)\big) \leq (p)^{\frac{1}{m}} \eta(z_1, z_2, \ldots, z_p) \quad \forall z_1, z_2, \ldots, z_p \in M. \tag{3.5}$$

By (3.1) and (3.5) we obtain

$$\begin{aligned}
\xi\left(\frac{h((p)^{\frac{1}{m}}z)}{\frac{1}{p}} - \frac{r((p)^{\frac{1}{m}}z)}{\frac{1}{p}}\right) &\leq \frac{1}{\frac{1}{p}}\xi\big(h\big((p)^{\frac{1}{m}}z\big) - r\big((p)^{\frac{1}{m}}z\big)\big) \\
&\leq \frac{1}{\frac{1}{p}}(p)^{\frac{1}{m}}\eta\big((p)^{\frac{1}{m}}z_1, (p)^{\frac{1}{m}}z_2, \ldots, (p)^{\frac{1}{m}}z_p\big) \\
&\leq (p)^{\frac{1}{m}}\psi\eta(z_1, z_2, \ldots, z_p)
\end{aligned}$$

for all $z_1, z_2, \ldots, z_p \in M$. In that case, σ is a ξ'-contraction, and (3.2) implies

$$\xi\left(\frac{s((p)^{\frac{1}{m}}z)}{\frac{1}{p}} - s(z)\right) \leq \frac{1}{\frac{1}{p}}\eta(z, z, \ldots, z) \quad \forall z \in M, \tag{3.6}$$

and replacing z by $(p)^{\frac{1}{m}}z$ in (3.6), we get

$$\xi\left(\frac{s((p)^{\frac{2}{m}}z)}{\frac{1}{p}} - s\big((p)^{\frac{1}{m}}z\big)\right) \leq \frac{\eta((p)^{\frac{1}{m}}z, (p)^{\frac{1}{m}}z, \ldots, (p)^{\frac{1}{m}}z)}{\frac{1}{p}}, \tag{3.7}$$

and by (3.6) and (3.7) we get

$$\xi\left(\frac{s((p)^{\frac{2}{m}}z)}{\frac{1}{p^2}} - s(z)\right) \leq \frac{1}{\frac{1}{p^2}}\eta\big((p)^{\frac{1}{m}}z, (p)^{\frac{1}{m}}z, \ldots, (p)^{\frac{1}{m}}z\big) + \frac{1}{\frac{1}{p}}\eta(z, z, \ldots, z).$$

By generalization we get

$$\begin{aligned}
\xi\left(\frac{s((p)^{\frac{k}{m}}z)}{\frac{1}{p^k}} - s(z)\right) &\leq \sum_{i=1}^{k}\frac{1}{\frac{1}{p^i}}\eta\Big(\big((p)^{\frac{1}{m}}\big)^{i-1}z, \big((p)^{\frac{1}{m}}\big)^{i-1}z, \ldots, \big((p)^{\frac{1}{m}}\big)^{i-1}z\Big) \\
&\leq \frac{1}{\psi\frac{1}{p}}\eta(z, z, \ldots, z)\sum_{i=1}^{k}\psi^i \\
&\leq \frac{1}{\frac{1}{p}(1-\psi)}\eta(z, z, \ldots, z) \quad \forall z \in M.
\end{aligned} \tag{3.8}$$

We obtain from (3.8) that

$$\begin{aligned}
\xi\left(\frac{s((p)^{\frac{k}{m}}z)}{\frac{1}{p^k}} - \frac{s((p)^{\frac{u}{m}}z)}{\frac{1}{p^u}}\right) &\leq \frac{1}{2}\xi\left(2\frac{s((p)^{\frac{k}{m}}z)}{\frac{1}{p^k}} - 2s(z)\right) + \frac{1}{2}\xi\left(2\frac{s((p)^{\frac{u}{m}}z)}{\frac{1}{p^u}} - 2s(z)\right) \\
&\leq \frac{\kappa}{2}\xi\left(\frac{s((p)^{\frac{k}{m}}z)}{\frac{1}{p^k}} - s(z)\right) + \frac{\kappa}{2}\xi\left(\frac{s((p)^{\frac{u}{m}}z)}{\frac{1}{p^u}} - s(z)\right) \\
&\leq \frac{\kappa}{\frac{1}{p}(1-\psi)}\eta(z, z, \ldots z) \quad \forall z \in M,
\end{aligned}$$

where $k,u\in\mathbb{N}$. Thus

$$\xi'\left(\sigma^k s-\sigma^u s\right)\leq\frac{\kappa}{\frac{1}{p}(1-\psi)},$$

and hence the boundedness of an orbit of σ at s implies that $\{\tau^k s\}$ is ξ'-convergent to $R\in N_{\xi'}$ by [29, Theorem 1.5]. By the ξ'-contractivity of σ we get

$$\xi'\left(\sigma^k s-\sigma R\right)\leq\psi\xi'\left(\sigma^{k-1}s-R\right).$$

Taking the limit as $k\to\infty$, by the Fatou property of ξ' we get

$$\xi'(\sigma R-R)\leq\lim_{p\to\infty}\inf\xi'\left(\sigma R-\sigma^k s\right)\leq\psi\lim_{k\to\infty}\inf\xi'\left(R-\sigma^{k-1}s\right)=0.$$

Hence R is a fixed point of σ. In (3.2), replacing $(z_1,z_2,\dots,z_p)$ by $((p)^{\frac{k}{m}}z_1,(p)^{\frac{k}{m}}z_2,\dots,(p)^{\frac{k}{m}}z_p)$, we get

$$\xi\left(\frac{1}{\frac{1}{p^k}}M_1 s\left((p)^{\frac{k}{m}}z_1,(p)^{\frac{k}{m}}z_2,\dots,(p)^{\frac{k}{m}}z_p\right)\right)\leq\frac{1}{\frac{1}{p^k}}\eta\left((p)^{\frac{k}{m}}z_1,(p)^{\frac{k}{m}}z_2,\dots,(p)^{\frac{k}{m}}z_p\right).$$

By Theorems 2.1 and 2.3, taking the limit as $k\to\infty$, R is a multifarious radical reciprocal mapping, and using (3.8), we obtain (3.3). For the uniqueness of R, consider another multifarious radical reciprocal mapping $T:M\to Z_\xi$ satisfying (3.3). Then T is a fixed point of σ. So

$$\xi'(R-T)=\xi'(\sigma R-\sigma T)\leq\psi\xi'(R-T).\tag{3.9}$$

From (3.9) we get $R=T$. Hence the proof is complete. □

Since each normed space is a modular space with modular $\xi(z)=\|z\|$, we can obtain the following corollaries.

Corollary 3.2 *Let η be a function from M^p to $[0,+\infty)$ satisfying*

$$\eta\left\{\left(p^{\frac{1}{m}}\right)z_1,\left(p^{\frac{1}{m}}\right)z_2,\dots,\left(p^{\frac{1}{m}}\right)z_p\right\}\leq\frac{1}{p}\psi\eta(z_1,z_2,\dots,z_p)$$

for all $z_1,z_2,\dots,z_n\in M$ and $\psi<1$. Assume that $s:M\to Z$ satisfies the condition

$$\left\|D_1 s(z_1,z_2,\dots,z_p)\right\|\leq\eta(z_1,z_2,\dots,z_p)$$

for all $z_1,z_2,\dots,z_p\in M$. Then there is a unique multifarious radical reciprocal mapping $R:M\to Z$ such that

$$\left\|R(z)-s(z)\right\|\leq\frac{\eta(z,z,\dots,z)}{\frac{1}{p}(1-\psi)}$$

for all $z\in M$.

Theorem 3.3 *Let η be a function from M^p to $[0,+\infty)$ satisfying*

$$\eta\left(\frac{z_1}{(p)^{\frac{1}{m}}},\frac{z_2}{(p)^{\frac{1}{m}}},\ldots,\frac{z_p}{(p)^{\frac{1}{m}}}\right)\leq\frac{\psi}{\frac{1}{p}}\rho(z_1,z_2,\ldots,z_p)$$

for all $z_1,z_2,\ldots,z_p\in M$ and $\psi<1$. Assume that $s:M\to Z_\xi$ fulfills

$$\xi\big(D_1s(z_1,z_2,\ldots,z_p)\big)\leq\eta(z_1,z_2,\ldots,z_p)$$

for all $z_1,z_2,\ldots,z_p\in M$. Then there is a unique multifarious radical reciprocal mapping $R:M\to Z_\xi$ such that

$$\xi\big(R(z)-s(z)\big)\leq\frac{p\psi}{1-\psi}\eta(z,z,\ldots,z)\quad\forall z\in M.$$

Proof Replacing z by $\frac{z}{(p)^{\frac{1}{m}}}$ in (3.4) of Theorem 3.1 and proceeding similarly, we complete the proof. □

Corollary 3.4 *Let η be a function from M^p to $[0,+\infty)$ satisfying*

$$\eta\left(\frac{z_1}{(p)^{\frac{1}{m}}},\frac{z_2}{(p)^{\frac{1}{m}}},\ldots,\frac{z_p}{(p)^{\frac{1}{m}}}\right)\leq\frac{\psi}{\frac{1}{p}}\eta(z_1,z_2,\ldots,z_p)$$

for all $z_1,z_2,\ldots,z_p\in M$ and $\psi<1$. Assume that $s:M\to Z$ fulfills the inequality

$$\big\|D_1s(z_1,z_2,\ldots,z_p)\big\|\leq\eta(z_1,z_2,\ldots,z_p)$$

for all $z_1,z_2,\ldots,z_p\in M$. Then there is a unique multifarious radical reciprocal mapping $R:M\to Z$ such that

$$\big\|R(z)-s(z)\big\|\leq\frac{p\psi}{1-\psi}\eta(z,z,\ldots,z)$$

for all $z\in M$.

Using Corollaries 3.2 and 3.4, we prove the Hyers–Ulam stability of (1.7).

Corollary 3.5 *Let η be a function from M^p to $[0,+\infty)$ such that*

$$\eta\big((p)^{\frac{1}{m}}z_1,(p)^{\frac{1}{m}}z_2,\ldots,(p)^{\frac{1}{m}}z_p\big)\leq\frac{1}{p}\psi\eta(z_1,z_2,\ldots,z_p)$$

for all $z_1,z_2,\ldots,z_p\in M$ and $\psi<1$. Suppose that for some $\epsilon\geq 0$, $s:M\to Z$ fulfills

$$\big\|D_1s(z_1,z_2,\ldots,z_n)\big\|\leq\epsilon$$

for all $z_1,z_2,\ldots,z_p\in M$. Then there is a unique multifarious radical reciprocal mapping $R:M\to Z$, defined by $R(z)=\lim_{k\to\infty}\frac{s((p)^{\frac{k}{m}}z)}{\frac{1}{p^k}}$, such that

$$\big\|R(z)-s(z)\big\|\leq\frac{p\epsilon}{|1-p|}$$

for all $z\in M$ and $p\neq 0,\pm 1$.

Proof Assume that $\eta(z_1,z_2,\dots,z_p)=\epsilon$ for all $z_1,z_2,\dots,z_p\in M$. Then Corollary 3.2 implies that

$$\|R(z)-s(z)\|\le\frac{p\epsilon}{1-p}$$

for all $z\in M$ and $p\neq 0,\pm 1$, and by Corollary 3.4 we get

$$\|R(z)-s(z)\|\le\frac{p\epsilon}{p-1}$$

for all $z\in M$ and $p\neq 0,\pm 1$. □

Corollary 3.6 *Suppose that for some $\epsilon\ge 0$, $s:M\to Z$ fulfills the inequality*

$$\|D_1 s(z_1,z_2,\dots,z_p)\|\le\epsilon\bigl(\|z_1\|^u+\|z_2\|^u+\dots+\|z_p\|^u\bigr) \tag{3.10}$$

for all $z_1,z_2,\dots,z_p\in M$ with $0\le u<-m$ or $u>-m$. Then there is a multifarious radical reciprocal mapping $R:M\to Z$, defined by $R(z)=\lim_{k\to\infty}\frac{s((p)^{\frac{k}{m}}z)}{\frac{1}{p^k}}$, such that

$$\|R(z)-s(z)\|\le\frac{p^2\epsilon}{|1-p^{\frac{m+u}{m}}|}\|z\|^u\quad\forall z\in M,p\neq 0,\pm 1.$$

Proof If we choose $\eta(z_1,z_2,\dots,z_p)=\epsilon(\|z_1\|^u+\|z_2\|^u+\dots+\|z_p\|^u)$, then by Corollary 3.2

$$\|R(z)-s(z)\|\le\frac{p^2\epsilon}{1-p^{\frac{m+u}{m}}}\|z\|^u$$

for all $z\in M$ and $u<-m$. Using Corollary 3.4, we have

$$\|R(z)-s(z)\|\le\frac{p^2\epsilon}{p^{\frac{m+u}{m}}-1}\|z\|^u$$

for all $z\in M$ and $u>-m$. □

The following is an example to elucidate that (1.7) is not stable for $u=-m$ in Corollary 3.6 using the method introduced by Gajda [31].

Example 3.7 Ffor some $a>0$, define $\phi:M\to Z$ by

$$\phi(z)=\begin{cases}\frac{a}{z^m} & \text{if } z\in(1,\infty),\\ a & \text{otherwise,}\end{cases}$$

and $s:M\to Z$ by $s(z)=\sum_{k=0}^{\infty}\frac{\phi(p^{-k}z)}{p^{mk}}$. Then s fulfills

$$\|D_1 s(z_1,z_2,\dots,z_p)\|\le\frac{ap^{2m}(p+1)}{p(p^m-1)}\times\left(\left|\frac{1}{z_1^m}\right|+\left|\frac{1}{z_2^m}\right|+\dots+\left|\frac{1}{z_p^m}\right|\right) \tag{3.11}$$

for all $z_1, z_2, \dots, z_p \in M$. In that case, there does not exist a multifarious radical reciprocal mapping $R : M \to Z$ such that

$$\left|s(z) - R(z)\right| \leq \beta \left|\frac{1}{z^m}\right|, \quad \beta > 0, \forall z \in \mathbb{R}. \tag{3.12}$$

Proof We have

$$\left|s(z)\right| \leq \sum_{k=0}^{\infty} \frac{|\phi(p^{-k}z)|}{|p^{mk}|} = \sum_{k=0}^{\infty} \frac{a}{p^{mk}} = \frac{ap^m}{p^m - 1}.$$

Therefore we see that s is bounded. We are going to prove that s satisfies (3.11).

If $|\frac{1}{z_1^m}| + |\frac{1}{z_2^m}| + \cdots + |\frac{1}{z_p^m}| \geq 1$, then the left-hand side of (3.11) is less than $\frac{ap^m(p+1)}{p(p^m-1)}$. Now suppose that $0 < |\frac{1}{z_1^m}| + |\frac{1}{z_2^m}| + \cdots + |\frac{1}{z_p^m}| < 1$. Then there exists a positive integer r such that

$$\frac{1}{p^{m(r+1)}} \leq \left|\frac{1}{z_1^m}\right| + \left|\frac{1}{z_2^m}\right| + \cdots + \left|\frac{1}{z_p^m}\right| < \frac{1}{p^{mr}}, \tag{3.13}$$

and so

$$\begin{aligned} & p^{mr}\frac{1}{z_1^m} < 1, \qquad p^{mr}\frac{1}{z_2^m} < 1, \qquad \dots, \qquad p^{mr}\frac{1}{z_p^m} < 1, \\ \text{or} \quad & \frac{z_1^m}{p^{mr}} > 1, \qquad \frac{z_2^m}{z^{mr}} > 1, \qquad \dots, \qquad \frac{z_p^m}{p^{mr}} > 1, \\ \text{or} \quad & \frac{z_1}{p^r} > 1, \qquad \frac{z_2}{p^r} > 1, \qquad \dots, \qquad \frac{z_p}{p^r} > 1. \end{aligned} \tag{3.14}$$

Consequently, $\frac{z_1}{p^{r-1}} > p > 1$, $\frac{z_2}{p^{r-1}} > p > 1, \dots, \frac{z_p}{p^{r-1}} > p > 1$. Again from (3.14) we get

$$\frac{z_1^m}{p^{m(r-1)}} > p^m > 1, \qquad \frac{z_2^m}{p^{m(r-1)}} > p^m > 1, \qquad \dots, \qquad \frac{z_p^m}{p^{m(r-1)}} > p^m > 1.$$

Consequently, $\frac{1}{p^{m(r-1)}}(z_1^m + z_2^m + \cdots + z_p^m) > 1$, $\frac{1}{p^{r-1}} \sqrt[m]{z_1^m + z_2^m + \cdots + z_p^m} > 1$. Hence

$$\frac{z_1}{p^{r-1}} > 1, \qquad \frac{z_2}{p^{r-1}} > 1, \qquad \dots, \qquad \frac{z_p}{p^{r-1}} > 1, \qquad \frac{1}{p^{r-1}} \sqrt[m]{z_1^m + z_2^m + \cdots + z_p^m} > 1.$$

Therefore, for each $k = 0, 1, \dots, r-1$, we have

$$\frac{z_1}{p^k} > 1, \qquad \frac{z_2}{p^k} > 1, \qquad \dots, \qquad \frac{z_p}{p^k} > 1, \qquad \frac{1}{p^k} \sqrt[m]{z_1^m + z_2^m + \cdots + z_p^m} > 1$$

and

$$\begin{aligned} & \phi\left(\frac{1}{p^k}\left(\sqrt[m]{z_1^m + z_2^m + \cdots + z_p^m}\right)\right) \\ & \quad - \frac{\phi(\frac{z_1}{p^k})\phi(\frac{z_2}{p^k})\cdots\phi(\frac{z_p}{p^k})}{\phi(\frac{z_2}{p^k})\phi(\frac{z_3}{p^k})\cdots\phi(\frac{z_p}{p^k}) + \cdots + \phi(\frac{z_1}{p^k})\phi(\frac{z_2}{p^k})\cdots\phi(\frac{z_{p-1}}{p^k})} = 0 \end{aligned}$$

for $k = 0, 1, \ldots, r-1$. From the definition of s and (3.13) we obtain that

$$\begin{aligned}
&\left| s\left(\sqrt[m]{z_1^m + z_2^m + \cdots + z_p^m}\right) - \frac{s(z_1)s(z_2)\cdots s(z_p)}{s(z_2)s(z_3)\cdots s(z_p) + \cdots + s(z_1)s(z_2)\cdots s(z_{p-1})}\right| \\
&\quad \le \sum_{k=0}^{\infty} \frac{1}{p^{mk}} \left| \phi\left(\frac{1}{p^k}\left(\sqrt[m]{z_1^m + z_2^m + \cdots + z_p^m}\right)\right)\right. \\
&\qquad \left. + \frac{\phi(\frac{z_1}{p^k})\phi(\frac{z_2}{p^k})\cdots\phi(\frac{z_p}{p^k})}{\phi(\frac{z_2}{p^k})\phi(\frac{z_3}{p^k})\cdots\phi(\frac{z_p}{p^k}) + \cdots + \phi(\frac{z_1}{p^k})\phi(\frac{z_2}{p^k})\cdots\phi(\frac{z_{p-1}}{p^k})}\right| \\
&\quad \le \sum_{k=r}^{\infty} \frac{1}{p^{mk}}\left(a + \frac{a^p}{pa^{p-1}}\right) \\
&\quad \le \sum_{k=r}^{\infty} \frac{1}{p^{mk}}\left(a + \frac{a^p}{pa^{p-1}}\right) = \frac{ap^m}{p^m - 1}\left(\frac{p+1}{p}\right) \times \frac{1}{p^{mr}} \\
&\quad = \frac{ap^m}{p^m - 1}\left(\frac{p+1}{p}\right) \times p^m\left(\left|\frac{1}{z_1^m}\right| + \left|\frac{1}{z_2^m}\right| + \cdots + \left|\frac{1}{z_p^m}\right|\right) \\
&\quad = \frac{ap^{2m}}{p^m - 1}\left(\frac{p+1}{p}\right) \times \left(\left|\frac{1}{z_1^m}\right| + \left|\frac{1}{z_2^m}\right| + \cdots + \left|\frac{1}{z_p^m}\right|\right).
\end{aligned}$$

Thus s satisfies (3.11) for all $z_1, z_1, \ldots, z_p \in M$ with $0 < |\frac{1}{z_1^m}| + |\frac{1}{z_2^m}| + \cdots + |\frac{1}{z_p^m}| < 1$.

We claim that the multifarious radical reciprocal functional equation (1.7) is not stable for $u = -m$ in Corollary 3.6. Suppose on the contrary that there exist a multifarious radical reciprocal mapping $R : M \to Z$ and a constant $\beta > 0$ satisfying (3.12). Then we have

$$\left|s(z)\right| \le (\beta + 1)\left|\frac{1}{z^m}\right|. \tag{3.15}$$

However, we can choose a positive integer q such that $qa > \beta + 1$.

If $z \in (1, p^{q-1})$, then $p^{-k}z \in (1, \infty)$ for all $k = 0, 1, \ldots, q-1$. For this z, we get

$$s(z) = \sum_{k=0}^{\infty} \frac{\phi(p^{-k}z)}{p^{mk}} \ge \sum_{k=0}^{q-1} \frac{\frac{a}{(p^{-k}z)^m}}{p^{mk}} = q\frac{a}{z^m} > (\beta + 1)\frac{1}{z^m},$$

which contradicts (3.15). Therefore the multifarious radical reciprocal functional equation (1.7) is not stable in sense of Ulam, Hyers, and Rassias if $p = -m$ in (3.10). □

Corollary 3.8 *Let $s : M \to Z$. Suppose that there exists $\epsilon \ge 0$ such that*

$$\left\|D_1 s(z_1, z_2, \ldots, z_p)\right\| \le \epsilon\left(\|z_1\|^{\frac{u}{p}} \|z_2\|^{\frac{u}{p}} \cdots \|z_p\|^{\frac{u}{p}}\right)$$

for all $z_1, z_2, \ldots, z_p \in M$. Then there exists a unique multifarious radical reciprocal mapping $R : M \to Z$ satisfying (1.7) and

$$\left\|R(z) - s(z)\right\| \le \begin{cases} \frac{p\epsilon}{1 - p^{\frac{m+u}{m}}}\|z\|^u, & u < -m, \\ \frac{p\epsilon}{p^{\frac{m+u}{m}} - 1}\|z\|^u, & u > -m, \end{cases}$$

for all $z \in M$.

Proof Replacing $\eta(z_1,z_2,\dots,z_p)$ by $\epsilon(\|z_1\|^{\frac{u}{p}}\|z_2\|^{\frac{u}{p}}\cdots\|z_p\|^{\frac{u}{p}})$ in Corollary 3.2, we obtain

$$\|R(z)-s(z)\| \le \frac{p\epsilon}{1-p^{\frac{m+u}{m}}}\|z\|^p$$

for $u<-m$ and all $z\in M$, and by Corollary 3.4 we get

$$\|R(z)-s(z)\| \le \frac{p\epsilon}{p^{\frac{m+u}{m}}-1}\|z\|^p$$

for $u>-m$ and all $z\in M$. □

Corollary 3.9 *Let $\epsilon>0$, and let $\alpha<-\frac{m}{p}$ or $\alpha>-\frac{m}{p}$ be real numbers, and let $s:M\to Z$ be a mapping satisfying the functional inequality*

$$\|D_1 s(z_1,z_2,\dots,z_p)\| \le \epsilon\big\{\|z_1\|^{p\alpha}+\|z_2\|^{p\alpha}+\cdots+\|z_p\|^{p\alpha}+\big(\|z_1\|^{\alpha}\|z_2\|^{\alpha}\cdots\|z_p\|^{\alpha}\big)\big\}.$$

Then there exists a unique multifarious radical reciprocal mapping $R:M\to Z$ satisfying (1.7) *and*

$$\|R(z)-s(z)\| \le \begin{cases} \frac{p(p+1)\epsilon}{1-p^{\frac{p\alpha+m}{m}}}\|z\|^{p\alpha} & \text{for } \alpha<-\frac{m}{p}, \\ \frac{p(p+1)\epsilon}{p^{\frac{p\alpha+m}{m}}-1}\|z\|^{p\alpha} & \text{for } \alpha>-\frac{m}{p}, \end{cases}$$

for all $z\in M$.

Proof Replacing $\eta(z_1,z_2,\dots,z_p)$ by $\epsilon\{\|z_1\|^{p\alpha}+\|z_2\|^{p\alpha}+\cdots+\|z_p\|^{p\alpha}+(\|z_1\|^{\alpha}\|z_2\|^{\alpha}\cdots\|z_p\|^{\alpha})\}$, by Corollary 3.4 we have

$$\|R(z)-s(z)\| \le \frac{p(p+1)\epsilon}{1-p^{\frac{p\alpha+m}{m}}}\|z\|^{p\alpha}$$

for $\alpha<-\frac{m}{p}$ and all $z\in M$, and by Corollary 3.4 we get

$$\|R(z)-s(z)\| \le \frac{p(p+1)\epsilon}{p^{\frac{p\alpha+m}{m}}-1}\|z\|^{p\alpha}$$

for $\alpha>-\frac{m}{p}$ and all $z\in M$. □

The following example elucidates that (1.7) is not stable for $\alpha=-\frac{m}{p}$ in Corollary 3.9 using the method introduced by Gajda [31].

Example 3.10 Let the function $\phi:M\to Z$ be defined as

$$\phi(z) = \begin{cases} \frac{l}{z^m}, & z\in(1,\infty), \\ l & \text{otherwise}, \end{cases}$$

with a constant $l > 0$, and let the function $s : M \to Z$ be defined as $s(z) = \sum_{k=0}^{\infty} \frac{\phi(p^{-k}z)}{p^{mk}}$. Then s fulfills

$$\begin{aligned} &\left\| D_1 s(z_1, z_2, \dots, z_p) \right\| \\ &\quad \le \frac{ap^{2m}(p+1)}{p(p^m-1)} \times \left(\left|\frac{1}{z_1^m}\right| + \left|\frac{1}{z_2^m}\right| + \dots + \left|\frac{1}{z_p^m}\right| + \left|\frac{1}{z_1^m}\right| \left|\frac{1}{z_2^m}\right| \cdots \left|\frac{1}{z_p^m}\right| \right) \end{aligned} \tag{3.16}$$

for all $z_1, z_2, \dots, z_p \in M$. In that case, there does not exist a multifarious radical reciprocal mapping $R : M \to Z$ such that

$$\left| s(z) - R(z) \right| \le \beta \left| \frac{1}{z^m} \right|, \quad \beta > 0, \forall z \in M.$$

Proof We have $|s(z)| \le \sum_{k=0}^{\infty} \frac{|\phi(p^{-k}z)|}{|p^{mk}|} = \sum_{k=0}^{\infty} \frac{l}{p^{mk}} = \frac{lp^m}{p^m-1}$. Therefore we see that s is bounded. We are going to prove that s satisfies (3.16).

If $|\frac{1}{z_1^m}| + |\frac{1}{z_2^m}| + \dots + |\frac{1}{z_p^m}| + |\frac{1}{z_1^m}||\frac{1}{z_2^m}| \cdots |\frac{1}{z_p^m}| \ge 1$, then the left-hand side of (3.16) is less than $\frac{ap^m(p+1)}{p(p^m-1)}$. Now suppose that $0 < |\frac{1}{z_1^m}| + |\frac{1}{z_2^m}| + \dots + |\frac{1}{z_p^m}| + |\frac{1}{z_1^m}||\frac{1}{z_2^m}| \cdots |\frac{1}{z_p^m}| < 1$. Then there exists a positive integer r such that

$$\frac{1}{p^{m(r+1)}} \le \left|\frac{1}{z_1^m}\right| + \left|\frac{1}{z_2^m}\right| + \dots + \left|\frac{1}{z_p^m}\right| + \left|\frac{1}{z_1^m}\right| \left|\frac{1}{z_2^m}\right| \cdots \left|\frac{1}{z_p^m}\right| < \frac{1}{p^{mr}},$$

and the rest of the proof is the same as in that of Example 3.7. □

Theorem 3.11 *Let $\eta : M^{p+1} \to [0, +\infty)$ be a function such that*

$$\eta\left((2)^{\frac{1}{m}} z_1, (2)^{\frac{1}{m}} z_2, \dots, (2)^{\frac{1}{m}} z_{p+1}\right) \le \frac{1}{2} \psi \eta(z_1, z_2, \dots, z_{p+1}) \tag{3.17}$$

for all $z_1, z_2, \dots, z_{p+1} \in M$ and $\psi < 1$. Suppose that $s : M \to Z_\xi$ fulfills the inequality

$$\xi\left(D_2 s(z_1, z_2, \dots z_{p+1})\right) \le \eta(z_1, z_2, \dots, z_{p+1}) \tag{3.18}$$

for all $z_1, z_2, \dots, z_{p+1} \in M$. Then there is a unique multifarious radical reciprocal mapping $R : M \to Z_\xi$ such that

$$\xi\left(R(z) - s(z)\right) \le \frac{1}{\frac{1}{2}(p+1)(1-\psi)} \eta(z, z, \dots, z) \quad \forall z \in M. \tag{3.19}$$

Proof Consider $N = \{h : M \to Z_\xi\}$ and define ξ' on N as

$$\xi'(h) =: \inf\left\{(2)^{\frac{1}{m}} > 0 : \xi\left(h(z)\right) \le (2)^{\frac{1}{m}} \eta(z, z, \dots, z)\right\}.$$

We can easily prove that ξ' is a convex modular with Fatou property on N. Let $\{h_i\}$ be a ξ'-Cauchy sequence in $N_{\xi'}$. For any $\epsilon > 0$, there exists a positive integer $i_0 \in \mathbb{N}$ such that $\xi'(h_i - h_j) \le \epsilon$ for all $i, j \ge i_0$, and we can easily prove that $\{h_i\}$ is a ξ'-convergent sequence in $N_{\xi'}$. Therefore $N_{\xi'}$ is ξ'-complete; see [29, 30]. Consider the function $\sigma : N_{\xi'} \to N_{\xi'}$ defined by

$$\sigma h(z) = \frac{1}{2} h\bigl(2^{\frac{1}{m}} z\bigr) \tag{3.20}$$

for $h \in N_{\xi'}$. Let $h, r \in N_{\xi'}$ and $(2)^{\frac{1}{m}} \in [0,1]$ with $\xi'(h-r) < (2)^{\frac{1}{m}}$. By the definition of ξ' we get

$$\xi\bigl(h(z) - r(z)\bigr) \le (2)^{\frac{1}{m}} \eta(z_1, z_2, \ldots, z_p) \tag{3.21}$$

for all $z_1, z_2, \ldots, z_{p+1} \in M$. By (3.17) and (3.21) we obtain

$$\begin{aligned} \xi\left(\frac{h((2)^{\frac{1}{m}} z)}{\frac{1}{2}} - \frac{r((2)^{\frac{1}{m}} z)}{\frac{1}{2}}\right) &\le \frac{1}{\frac{1}{2}} \xi\bigl(h\bigl((2)^{\frac{1}{m}} z\bigr) - r\bigl((2)^{\frac{1}{m}} z\bigr)\bigr) \\ &\le \frac{1}{\frac{1}{2}} (2)^{\frac{1}{m}} \eta\bigl((2)^{\frac{1}{m}} z_1, (2)^{\frac{1}{m}} z_2, \ldots, (2)^{\frac{1}{m}} z_{p+1}\bigr) \\ &\le (2)^{\frac{1}{m}} \psi \eta(z_1, z_2, \ldots, z_{p+1}) \end{aligned}$$

for all $z_1, z_2, \ldots, z_{p+1} \in M$. Hence σ is a ξ'-contraction. From (3.18) we obtain

$$\xi\left(\frac{s((2)^{\frac{1}{m}} z)}{\frac{1}{2}} - s(z)\right) \le \frac{1}{\frac{1}{2}(p+1)} \eta(z, z, \ldots, z). \tag{3.22}$$

Replacing z by $(2)^{\frac{1}{m}} z$ in (3.22), we get

$$\xi\left(\frac{s((2)^{\frac{2}{m}} z)}{\frac{1}{2}} - s\bigl((2)^{\frac{1}{m}} z\bigr)\right) \le \frac{\eta((2)^{\frac{1}{m}} z, (2)^{\frac{1}{m}} z, \ldots, (2)^{\frac{1}{m}} z)}{\frac{1}{2}(p+1)}. \tag{3.23}$$

It follows from (3.22) and (3.23) that

$$\begin{aligned} \xi\left(\frac{s((2)^{\frac{2}{m}} z)}{\frac{1}{2^2}} - s(z)\right) &\le \xi\left(\frac{s((2)^{\frac{2}{m}} z)}{\frac{1}{2^2}} - \frac{s((2)^{\frac{1}{m}} z)}{\frac{1}{2}}\right) + \xi\left(\frac{s((2)^{\frac{1}{m}} z)}{\frac{1}{2}} - s(z)\right) \\ &\le \frac{1}{\frac{1}{2^2}(p+1)} \eta\bigl((2)^{\frac{1}{m}} z, (2)^{\frac{1}{m}} z, \ldots, (2)^{\frac{1}{m}} z\bigr) + \frac{1}{\frac{1}{2}(p+1)} \eta(z, z, \ldots, z). \end{aligned}$$

By induction we get

$$\begin{aligned} \xi\left(\frac{s((2)^{\frac{k}{m}} z)}{\frac{1}{2^k}} - s(z)\right) &\le \frac{1}{p+1} \sum_{i=1}^{k} \frac{1}{\frac{1}{2^i}} \eta\bigl(\bigl((2)^{\frac{1}{m}}\bigr)^{i-1} z, \bigl((2)^{\frac{1}{m}}\bigr)^{i-1} z, \ldots, \bigl((2)^{\frac{1}{m}}\bigr)^{i-1} z\bigr) \\ &\le \frac{1}{\psi \frac{1}{2}(p+1)} \eta(z, z, \ldots, z) \sum_{i=1}^{k} \psi^i \\ &\le \frac{1}{\frac{1}{2}(p+1)(1-\psi)} \eta(z, z, \ldots, z). \end{aligned} \tag{3.24}$$

It follows from (3.24) that

$$\begin{aligned}\xi\left(\frac{s((2)^{\frac{k}{m}}z)}{\frac{1}{2^k}}-\frac{s((2)^{\frac{u}{m}}z)}{\frac{1}{2^u}}\right) &\leq \frac{1}{2}\xi\left(2\frac{s((2)^{\frac{k}{m}}z)}{\frac{1}{2^k}}-2s(z)\right)+\frac{1}{2}\xi\left(2\frac{s((2)^{\frac{u}{m}}z)}{\frac{1}{2^u}}-2s(z)\right)\\ &\leq \frac{\kappa}{2}\xi\left(\frac{s((2)^{\frac{k}{m}}z)}{\frac{1}{2^k}}-s(z)\right)+\frac{\kappa}{2}\xi\left(\frac{s((2)^{\frac{u}{m}}z)}{\frac{1}{2^u}}-s(z)\right)\\ &\leq \frac{\kappa}{\frac{1}{2}(p+1)(1-\psi)}\eta(z,z,\dots z),\end{aligned}$$

where $k,u\in\mathbb{N}$. Thus

$$\xi'\left(\sigma^k s-\sigma^u s\right)\leq \frac{\kappa}{\frac{1}{2}(p+1)(1-\psi)},$$

and hence by the boundedness of an orbit of σ at s, $\{\tau^k s\}$ is ξ'-convergent to $R\in N_{\xi'}$ by [29, Theorem 1.5]. By the ξ'-contractivity of σ we get

$$\xi'\left(\sigma^k s-\sigma R\right)\leq \psi\xi'\left(\sigma^{k-1}s-R\right).$$

Taking the limit as $k\to\infty$, by the Fatou property of ξ' we get

$$\xi'(\sigma R-R)\leq \liminf_{k\to\infty}\xi'\left(\sigma R-\sigma^k s\right)\leq \psi\liminf_{k\to\infty}\xi'\left(R-\sigma^{k-1}s\right)=0.$$

Hence R is a fixed point of σ. In (3.18), replacing $(z_1,z_2,\dots,z_{p+1})$ by $((2)^{\frac{k}{m}}z_1,(2)^{\frac{k}{m}}z_2,\dots,(2)^{\frac{k}{m}}z_{p+1})$, we get

$$\xi\left(\frac{1}{\frac{1}{2^k}}D_2 s\left((2)^{\frac{k}{m}}z_1,(2)^{\frac{k}{m}}z_2,\dots,(2)^{\frac{k}{m}}z_{p+1}\right)\right)\leq \frac{1}{\frac{1}{2^k}}\eta\left((2)^{\frac{k}{m}}z_1,(2)^{\frac{k}{m}}z_2,\dots,(2)^{\frac{k}{m}}z_{p+1}\right).$$

By Theorems 2.2 and 2.4, letting $k\to\infty$, R satisfies the reciprocal functional equation, and using (3.24), we obtain (3.19). For the uniqueness of R, consider another multifarious radical reciprocal mapping $T:M\to Z_\xi$ satisfying (3.19). Then T is a fixed point of σ. So

$$\xi'(R-T)=\xi'(\sigma R-\sigma T)\leq \psi\xi'(R-T). \tag{3.25}$$

From (3.25) we get $R=T$. Hence the proof is complete. □

Since each normed space is a modular space with modular $\xi(z)=\|z\|$, we get the following:

Corollary 3.12 *Let η be a function from M^{p+1} to $[0,+\infty)$ such that*

$$\eta\left(\left(2^{\frac{1}{m}}\right)z_1,\left(2^{\frac{1}{m}}\right)z_2,\dots,\left(2^{\frac{1}{m}}\right)z_{p+1}\right)\leq \frac{1}{2}\psi\eta(z_1,z_2,\dots,z_{p+1})$$

for all $z_1,z_2,\dots,z_{p+1}\in M$ and $\psi<1$. Let Z be an arbitrary vector space. Suppose that $s:M\to Z$ fulfills the condition

$$\left\|D_2 s(z_1,z_2,\dots,z_{p+1})\right\|\leq \eta(z_1,z_2,\dots,z_{p+1})$$

for all $z_1, z_2, \ldots, z_{p+1} \in M$. Then there is a unique multifarious radical reciprocal mapping $R : M \to Z$ such that

$$\left\|R(z) - s(z)\right\| \leq \frac{\eta(z, z, \ldots, z)}{\frac{1}{2}(p+1)(1-\psi)}$$

for all $z \in M$.

Theorem 3.13 *Let η be a function from M^{p+1} to $[0, +\infty)$ satisfying*

$$\eta\left(\frac{z_1}{(2)^{\frac{1}{m}}}, \frac{z_2}{(2)^{\frac{1}{m}}}, \ldots, \frac{z_{p+1}}{(2)^{\frac{1}{m}}}\right) \leq \frac{\psi}{\frac{1}{2}} \eta(z_1, z_2, \ldots, z_{p+1})$$

for all $z_1, z_2, \ldots, z_{p+1} \in M$ and $\psi < 1$. Suppose that $s : M \to Z_\xi$ fulfills

$$\xi\left(D_2 s(z_1, z_2, \ldots, z_{p+1})\right) \leq \eta(z_1, z_2, \ldots, z_{p+1})$$

for all $z_1, z_2, \ldots, z_{p+1} \in M$. Then there is a unique multifarious radical reciprocal mapping $R : M \to Z_\xi$ such that

$$\xi\left(R(z) - s(z)\right) \leq \frac{2\psi}{(1-\psi)(p+1)} \eta(z, z, \ldots, z) \quad \forall z \in M.$$

Proof Replacing z by $\frac{z}{(2)^{\frac{1}{m}}}$ in (3.20) of Theorem 3.11 and proceeding similarly, we complete the proof. □

Corollary 3.14 *Let η be a function from M^{p+1} to $[0, +\infty)$ satisfying*

$$\eta\left(\frac{z_1}{(2)^{\frac{1}{m}}}, \frac{z_2}{(2)^{\frac{1}{m}}}, \ldots, \frac{z_{p+1}}{(2)^{\frac{1}{m}}}\right) \leq \frac{\psi}{\frac{1}{2}} \eta(x_1, z_2, \ldots, z_{p+1})$$

for all $z_1, z_2, \ldots, z_{p+1} \in M$ and $\psi < 1$. Suppose that $s : M \to Z$ fulfills the inequality

$$\left\|D_2 s(z_1, z_2, \ldots, z_{p+1})\right\| \leq \eta(z_1, z_2, \ldots, z_{p+1})$$

for all $z_1, z_2, \ldots, z_{p+1} \in M$. Then there is a unique multifarious radical reciprocal mapping $R : M \to Z$ such that

$$\left\|R(z) - s(z)\right\| \leq \frac{2\psi}{(1-\psi)(p+1)} \eta(z, z, \ldots, z)$$

for all $z \in M$.

Using Corollaries 3.12 and 3.14, we prove the Hyers–Ulam stability of (1.8).

Corollary 3.15 *Let η be a function from M^{p+1} to $[0, +\infty)$ such that*

$$\eta\left((2)^{\frac{1}{m}} z_1, (2)^{\frac{1}{m}} z_2, \ldots, (2)^{\frac{1}{m}} z_{p+1}\right) \leq \frac{1}{2} \psi \eta(z_1, z_2, \ldots, z_{p+1})$$

for all $z_1, z_2, \dots, z_{p+1} \in M$ *and* $\psi < 1$. *Suppose that for some real number* $\epsilon \geq 0$, $s : M \to Z$ *fulfills*

$$\|D_2 s(z_1, z_2, \dots, z_{p+1})\| \leq \epsilon$$

for all $z_1, z_2, \dots, z_{n+1} \in M$. *Then there is a unique multifarious radical reciprocal mapping* $R : M \to Z$, *defined by* $R(z) = \lim_{k\to\infty} \frac{s((2)^{\frac{k}{m}} z)}{\frac{1}{2^k}}$, *such that*

$$\|R(z) - s(z)\| \leq \frac{2\epsilon}{p+1}$$

for all $z \in M$.

Proof Setting $\epsilon = \eta(z_1, z_2, \dots, z_{p+1})$ in Corollaries 3.12 and 3.14, we obtain

$$\|R(z) - s(z)\| \leq \frac{2\epsilon}{p+1}$$

for all $z \in M$ and $p \neq 0, \pm 1$. □

Corollary 3.16 *If for some* $\epsilon \geq 0$, $s : M \to Z$ *fulfills the inequality,*

$$\|D_2 s(z_1, z_2, \dots, z_{s+1})\| \leq \epsilon \left(\|z_1\|^u + \|z_2\|^u + \cdots + \|z_{p+1}\|^u\right) \tag{3.26}$$

for all $z_1, z_2, \dots, z_{u+1} \in M$ *with* $0 \leq u < -m$ *or* $u > -m$, *then there is a multifarious radical reciprocal mapping* $R : M \to Z$, *defined by* $R(z) = \lim_{k\to\infty} \frac{s((2)^{\frac{k}{m}} z)}{\frac{1}{2^k}}$, *such that*

$$\|R(z) - s(z)\| \leq \frac{2\epsilon}{|1 - 2^{\frac{m+u}{m}}|} \|z\|^u \quad \forall z \in M.$$

Proof Setting $\epsilon(\|z_1\|^u + \|z_2\|^u + \cdots + \|z_{p+1}\|^u) = \eta(z_1, z_2, \dots, z_{p+1})$ in Corollary 3.12, we obtain

$$\|R(z) - s(z)\| \leq \frac{2\epsilon}{1 - 2^{\frac{m+u}{m}}} \|z\|^u$$

for $u < -m$, and by Corollary 3.14 we get

$$\|R(z) - s(z)\| \leq \frac{2\epsilon}{2^{\frac{m+u}{m}} - 1} \|z\|^u$$

for $u > -m$. □

The following example elucidates that (1.8) is not stable in Corollary 3.16 for $u = -m$.

Example 3.17 For a constant $a > 0$, define $\phi : M \to Z$ by

$$\phi(z) = \begin{cases} \frac{a}{z^m} & \text{if } z \in (1, \infty), \\ a & \text{otherwise,} \end{cases}$$

and $s : M \to Z$ by $s(z) = \sum_{k=0}^{\infty} \frac{\phi(2^{-k}z)}{2^{mk}}$. Then s fulfills

$$\|D_2 s(z_1, z_2, \ldots, z_p)\| \leq \frac{a2^{2m}}{2(2^m - 1)} \times \left(\left|\frac{1}{z_1^m}\right| + \left|\frac{1}{z_2^m}\right| + \cdots + \left|\frac{1}{z_{p+1}^m}\right| \right) \tag{3.27}$$

for all $z_1, z_2, \ldots, z_{p+1} \in M$. In that case, there does not exist a multifarious radical reciprocal mapping $R : M \to Z$ such that

$$|s(z) - R(z)| \leq \beta \left|\frac{1}{z^m}\right|, \quad \beta > 0, \forall z \in M. \tag{3.28}$$

Proof We have $|s(z)| \leq \sum_{k=0}^{\infty} \frac{|\phi(2^{-k}z)|}{|2^{mk}|} = \sum_{k=0}^{\infty} \frac{a}{2^{mk}} = \frac{a2^m}{2^m-1}$. Therefore we see that s is bounded. Now the aim is to prove that s satisfies (3.27).

If $|\frac{1}{z_1^m}| + |\frac{1}{z_2^m}| + \cdots + |\frac{1}{z_{p+1}^m}| \geq 1$, then the left-hand side of (3.27) is less than $\frac{3(p+1)a2^m}{2^m-1}$. Now suppose that $0 < |\frac{1}{z_1^m}| + |\frac{1}{z_2^m}| + \cdots + |\frac{1}{z_{p+1}^m}| < 1$. Then there exists a positive integer r such that

$$\frac{1}{2^{m(r+1)}} \leq \left|\frac{1}{z_1^m}\right| + \left|\frac{1}{z_2^m}\right| + \cdots + \left|\frac{1}{z_{p+1}^m}\right| < \frac{1}{2^{mr}}, \tag{3.29}$$

and so

$$\begin{aligned} &2^{mr}\frac{1}{z_1^m} < 1, \qquad 2^{mr}\frac{1}{z_2^m} < 1, \qquad \ldots, \qquad 2^{mr}\frac{1}{z_{p+1}^m} < 1, \\ \text{or}\quad &\frac{z_1^m}{2^{mr}} > 1, \qquad \frac{z_2^m}{2^{mr}} > 1, \qquad \ldots, \qquad \frac{z_{p+1}^m}{2^{mr}} > 1, \\ \text{or}\quad &\frac{z_1}{2^r} > 1, \qquad \frac{z_2}{2^r} > 1, \qquad \ldots, \qquad \frac{z_{p+1}}{2^r} > 1. \end{aligned} \tag{3.30}$$

Consequently, $\frac{z_1}{2^{r-1}} > 2 > 1$, $\frac{z_2}{2^{r-1}} > 2 > 1, \ldots, \frac{z_{p+1}}{2^{r-1}} > 2 > 1$. Again from (3.30) we get

$$\frac{z_1^m}{2^{m(r-1)}} > 2^m > 1, \qquad \frac{z_2^m}{2^{m(r-1)}} > 2^m > 1, \qquad \ldots, \qquad \frac{z_{p+1}^m}{2^{m(r-1)}} > 2^m > 1.$$

Consequently,

$$\begin{aligned} &\frac{1}{2^{m(r-1)}}\left(z_1^m + z_2^m\right) > 1, \qquad \frac{1}{2^{r-1}}\sqrt[m]{z_1^m + z_2^m} > 1, \\ &\frac{1}{2^{m(r-1)}}\left(z_2^m + z_3^m\right) > 1, \qquad \frac{1}{2^{r-1}}\sqrt[m]{z_2^m + z_3^m} > 1, \qquad \ldots, \\ &\frac{1}{2^{m(r-1)}}\left(z_p^m + z_{p+1}^m\right) > 1, \qquad \frac{1}{2^{r-1}}\sqrt[m]{z_p^m + z_{p+1}^m} > 1, \\ &\frac{1}{2^{m(r-1)}}\left(z_1^m + z_{p+1}^m\right) > 1, \qquad \frac{1}{2^{r-1}}\sqrt[m]{z_1^m + z_{p+1}^m} > 1. \end{aligned}$$

Hence we get

$$\begin{aligned} &\frac{z_1}{2^{r-1}} > 1, \qquad \frac{z_2}{2^{r-1}} > 1, \qquad \ldots, \qquad \frac{z_{p+1}}{2^{r-1}} > 1, \\ &\frac{1}{2^{r-1}}\sqrt[m]{z_1^m + z_2^m} > 1, \qquad \frac{1}{2^{r-1}}\sqrt[m]{z_2^m + z_3^m} > 1, \qquad \ldots, \end{aligned}$$

$$\frac{1}{2^{r-1}}\sqrt[m]{z_p^m+z_{p+1}^m}>1,\qquad \frac{1}{2^{r-1}}\sqrt[m]{z_1^m+z_{p+1}^m}>1.$$

Therefore for each $k=0,1,\dots,r-1$, we have

$$\begin{aligned}&\frac{z_1}{2^k}>1,\qquad \frac{z_2}{2^k}>1,\qquad \dots,\qquad \frac{z_{p+1}}{2^k}>1,\\&\frac{1}{2^k}\sqrt[m]{z_1^m+z_2^m}>1,\qquad \frac{1}{2^k}\sqrt[m]{z_2^m+z_3^m}>1,\qquad \dots,\\&\frac{1}{2^k}\sqrt[m]{z_n^m+z_{n+1}^m}>1,\qquad \frac{1}{2^k}\sqrt[m]{z_1^m+z_{p+1}^m}>1,\end{aligned}$$

and

$$\begin{aligned}&\phi\biggl(\frac{1}{2^k}\Bigl(\bigl(\sqrt[m]{z_1^m+z_2^m}\bigr)+\bigl(\sqrt[m]{z_2^m+z_3^m}\bigr)+\dots+\bigl(\sqrt[m]{z_p^m+z_{p+1}^m}\bigr)\Bigr)\biggr)\\&\quad+\phi\biggl(\frac{1}{2^k}\bigl(\sqrt[m]{z_1^m+z_{p+1}^m}\bigr)\biggr)-\biggl(\frac{\phi(\frac{z_1}{2^k})\phi(\frac{z_2}{2^k})}{\phi(\frac{z_1}{2^k})+\phi(\frac{z_2}{2^k})}+\dots+\frac{\phi(\frac{z_p}{2^k})\phi(\frac{z_{p+1}}{2^k})}{\phi(\frac{z_p}{2^k})+\phi(\frac{z_{p+1}}{2^k})}\biggr)\\&\quad-\frac{\phi(\frac{z_1}{2^k})\phi(\frac{z_{p+1}}{2^k})}{\phi(\frac{z_1}{2^k})+\phi(\frac{z_{p+1}}{2^k})}=0\end{aligned}$$

for $k=0,1,\dots,r-1$. From the definition of s and (3.29) we obtain that

$$\begin{aligned}&\left|\sum_{i=1}^{p}\bigl(s\bigl(\sqrt[m]{z_i^m+z_j^m}\bigr)\bigr)+s\bigl(\sqrt[m]{z_1^m+z_{p+1}^m}\bigr)-\sum_{i=1}^{p}\biggl(\frac{s(z_i)s(z_j)}{s(z_i)+s(z_j)}\biggr)-\frac{s(z_1)s(z_{p+1})}{s(z_1)+s(z_{p+1})}\right|\\&\quad\le\sum_{k=0}^{\infty}\frac{1}{2^{mk}}\left|\sum_{i=1}^{p}\biggl(\phi\biggl(\frac{1}{2^k}\sqrt[m]{z_i^m+z_j^m}\biggr)\biggr)+\phi\biggl(\frac{1}{2^k}\sqrt[m]{z_1^m+z_{p+1}^m}\biggr)\right.\\&\qquad\left.-\sum_{i=1}^{p}\biggl(\frac{\phi(\frac{z_i}{2^k})\phi(\frac{z_j}{2^k})}{\phi(\frac{z_i}{2^k})+\phi(\frac{z_j}{2^k})}\biggr)-\frac{\phi(\frac{z_1}{2^k})\phi(\frac{z_{p+1}}{2^k})}{\phi(\frac{z_1}{2^k})+\phi(\frac{z_{p+1}}{2^k})}\right|\\&\quad\le\sum_{k=r}^{\infty}\frac{1}{2^{mk}}\biggl(\frac{a(p+1)}{2}\biggr)=\frac{3(p+1)a2^m}{2^m-1}\times\frac{1}{2^{mr}}\\&\quad=\frac{3(p+1)a2^m}{2^m-1}\times 2^m\biggl(\left|\frac{1}{z_1^m}\right|+\left|\frac{1}{z_2^m}\right|+\dots+\left|\frac{1}{z_{p+1}^m}\right|\biggr)\\&\quad=\frac{3(p+1)a4^m}{2^m-1}\times\biggl(\left|\frac{1}{z_1^m}\right|+\left|\frac{1}{z_2^m}\right|+\dots+\left|\frac{1}{z_{p+1}^m}\right|\biggr).\end{aligned}$$

Thus s satisfies (3.27) for all $z_1,z_2,\dots,z_{p+1}\in M$ such that $0<|\frac{1}{z_1^m}|+|\frac{1}{z_2^m}|+\dots+|\frac{1}{z_{p+1}^m}|<1$.

We claim that the multifarious radical reciprocal functional equation (1.8) is not stable for $u=-m$ in Corollary 3.16. Suppose on the contrary that there exist a multifarious radical reciprocal mapping $R:M\to Z$ and a constant $\beta>0$ satisfying (3.28). Then we have

$$\bigl|s(z)\bigr|\le(\beta+1)\left|\frac{1}{z^m}\right|.\tag{3.31}$$

We can choose a positive integer q such that $qa>\beta+1$.

If $z \in (1, 2^{q-1})$, then $2^{-k}z \in (1, \infty)$ for all $k = 0, 1, \ldots, q-1$. For this z, we get

$$s(z) = \sum_{k=0}^{\infty} \frac{\phi(2^{-k}z)}{2^{mk}} \geq \sum_{k=0}^{q-1} \frac{\frac{a}{(2^{-k}z)^m}}{2^{mk}} = q\frac{a}{z^m} > (\beta + 1)\frac{1}{z^m},$$

which contradicts (3.31). Therefore the multifarious radical reciprocal functional equation (1.8) is not stable in the sense of Ulam, Hyers, and Rassias if $u = -m$ in (3.26). □

Corollary 3.18 *Let $s : M \to Z$ and suppose that there exists $\epsilon \geq 0$ such that*

$$\left\| D_2 s(z_1, z_2, \ldots, z_{p+1}) \right\| \leq \epsilon \left(\|z_1\|^{\frac{u}{p+1}} \|z_2\|^{\frac{u}{p+1}} \cdots \|z_{p+1}\|^{\frac{u}{p+1}} \right)$$

for all $z_1, z_2, \ldots, z_{p+1} \in M$. Then there exists a unique multifarious radical reciprocal mapping $R : M \to Z$ satisfying (1.8) *and*

$$\left\| R(z) - s(z) \right\| \leq \begin{cases} \frac{2\epsilon}{(p+1)(1-2^{\frac{m+u}{m}})} \|z\|^u & \text{for } u < -m, \\ \frac{2\epsilon}{(p+1)(2^{\frac{m+u}{m}}-1)} \|z\|^u & \text{for } u > -m, \end{cases}$$

for all $z \in M$.

Proof Replacing $\eta(z_1, z_2, \ldots, z_{p+1})$ by $\epsilon(\|z_1\|^{\frac{u}{p}} \|z_2\|^{\frac{u}{p}} \cdots \|z_{p+1}\|^{\frac{u}{p}})$ in Corollary 3.12, we get

$$\left\| R(z) - s(z) \right\| \leq \frac{2\epsilon}{(p+1)(1 - 2^{\frac{m+u}{m}})} \|z\|^u$$

for $u < -m$ and for all $z \in M$, and by Corollary 3.14 we have

$$\left\| R(z) - s(z) \right\| \leq \frac{2\epsilon}{(p+1)(2^{\frac{m+u}{m}} - 1)} \|z\|^u$$

for $u > -m$. □

Corollary 3.19 *Let $s : M \to Z$ and suppose that there exists $\epsilon \geq 0$ such that*

$$\begin{aligned} &\left\| D_2 s(z_1, z_2, \ldots, z_{p+1}) \right\| \\ &\quad \leq \epsilon \left\{ \|z_1\|^{(p+1)\alpha} + \|z_2\|^{(p+1)\alpha} + \cdots + \|z_{p+1}\|^{(p+1)\alpha} + \left(\|z_1\|^{\alpha} \|z_2\|^{\alpha} \cdots \|z_{p+1}\|^{\alpha} \right) \right\} \end{aligned}$$

for all $z_1, z_2, \ldots, z_{p+1} \in M$. Then there exists a unique multifarious radical reciprocal mapping $R : M \to Z$ satisfying (1.8) *and*

$$\left\| R(z) - s(z) \right\| \leq \begin{cases} \frac{2(p+2)\epsilon}{(p+1)(1-2^{\frac{(p+1)\alpha+m}{m}})} \|z\|^{(p+1)\alpha} & \text{for } \alpha < -\frac{m}{p+1}, \\ \frac{2(p+2)\epsilon}{(p+1)(2^{\frac{(p+1)\alpha+m}{m}}-1)} \|z\|^{(p+1)\alpha} & \text{for } \alpha > -\frac{m}{p+1}, \end{cases}$$

for all $z \in M$.

Proof Replacing $\eta(z_1, z_2, \dots, z_{p+1})$ by

$$\epsilon\Big\{\|z_1\|^{(p+1)\alpha} + \|z_2\|^{(p+1)\alpha} + \cdots + \|z_{p+1}\|^{(p+1)\alpha} + \big(\|z_1\|^{\alpha}\|z_2\|^{\alpha}\cdots\|z_{p+1}\|^{\alpha}\big)\Big\}$$

in Corollary 3.14, we obtain

$$\big\|R(z) - s(z)\big\| \le \frac{2(p+2)\epsilon}{(p+1)(1 - 2^{\frac{(p+1)\alpha+m}{m}})}\|z\|^{(p+1)\alpha}$$

for $\alpha < -\frac{m}{p+1}$, and by Corollary 3.14 we get

$$\big\|R(z) - s(z)\big\| \le \frac{2(p+2)\epsilon}{(p+1)(2^{\frac{(p+1)\alpha+m}{m}} - 1)}\|z\|^{(p+1)\alpha}$$

for $\alpha > -\frac{m}{p+1}$. □

The following example elucidates that (1.8) is not stable in Corollary 3.19 for $\alpha = -\frac{m}{p+1}$.

Example 3.20 For a constant $l > 0$, define $\phi : M \to Z$ by

$$\phi(z) = \begin{cases} \frac{l}{z^m} & \text{if } z \in (1, \infty), \\ l & \text{otherwise,} \end{cases}$$

and $s : M \to Z$ by $s(z) = \sum_{k=0}^{\infty} \frac{\phi(2^{-k}z)}{2^{mk}}$. Then s fulfills

$$\begin{aligned} &\big\|D_2 s(z_1, z_2, \dots, z_{p+1})\big\| \\ &\quad \le \frac{l2^{2m}}{2(2^m - 1)} \times \left(\left|\frac{1}{z_1^m}\right| + \left|\frac{1}{z_2^m}\right| + \cdots + \left|\frac{1}{z_{p+1}^m}\right| + \left|\frac{1}{z_1^m}\right|\left|\frac{1}{z_2^m}\right|\cdots\left|\frac{1}{z_{p+1}^m}\right|\right) \end{aligned} \tag{3.32}$$

for all $z_1, z_2, \dots, z_{p+1} \in M$. In that cas,e there does not exist a multifarious radical reciprocal mapping $R : M \to Z$ such that

$$\big|s(z) - R(z)\big| \le \beta\left|\frac{1}{z^m}\right|, \quad \beta > 0, \forall z \in M.$$

Proof We have $|s(z)| \le \sum_{k=0}^{\infty} \frac{|\phi(2^{-k}z)|}{|2^{mk}|} = \sum_{k=0}^{\infty} \frac{l}{2^{mk}} = \frac{l2^m}{2^m-1}$. Therefore we see that s is bounded. The next aim is to prove that s satisfies (3.32). If $|\frac{1}{z_1^m}| + |\frac{1}{z_2^m}| + \cdots + |\frac{1}{z_{p+1}^m}| + |\frac{1}{z_1^m}||\frac{1}{z_2^m}|\cdots|\frac{1}{z_{p+1}^m}| \ge 1$, then the left-hand side of (3.32) is less than $\frac{3(p+1)a2^m}{2^m-1}$. Now suppose that $0 < |\frac{1}{z_1^m}| + |\frac{1}{z_2^m}| + \cdots + |\frac{1}{z_{p+1}^m}| + |\frac{1}{z_1^m}||\frac{1}{z_2^m}|\cdots|\frac{1}{z_{p+1}^m}| < 1$. Then there exists a positive integer r such that

$$\frac{1}{2^{m(r+1)}} \le \left|\frac{1}{z_1^m}\right| + \left|\frac{1}{z_2^m}\right| + \cdots + \left|\frac{1}{z_{p+1}^m}\right| + \left|\frac{1}{z_1^m}\right|\left|\frac{1}{z_2^m}\right|\cdots\left|\frac{1}{z_{p+1}^m}\right| < \frac{1}{2^{mr}},$$

and the rest of the proof is the same as in that of Example 3.17. □

4 Applications of *p*-dimensional multifarious radical reciprocal functional equations

4.1 The parallel circuit and the *p*-dimensional multifarious radical reciprocal functional equation

A parallel circuit has more than one resistor and gets its name from having multiple paths to move along. Also, we know that the following rule applies to a parallel circuit.

The sum of inverses of individual resistances is equal to the inverse of the total resistance r_t of the circuit, that is,

$$\frac{1}{r_t} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \cdots .$$

In Fig. 3, $i_s = i_1 + i_2 + \cdots + i_p$ or $i_s = \frac{v}{r_1} + \frac{v}{r_2} + \cdots + \frac{v}{r_p}$, where $r_1, r_2, \ldots, r_p$ are the p individual resistances of the parallel circuit, and r_t is the total resistance. Then

$$\frac{1}{r_t} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \cdots + \frac{1}{r_p}. \tag{4.1}$$

If we do not want more resistors $r_1, r_2, \ldots, r_p$ as in the circuit in Fig. 3, then they can be replaced by a single equivalent resistor r_t (see Fig. 4). Now equality (4.1) simplifies to

$$r_t = \frac{\prod_{i=1}^{p} r_i}{\frac{1}{r_1}\prod_{i=1}^{p} r_i + \frac{1}{r_2}\prod_{i=1}^{p} r_i + \cdots + \frac{1}{r_p}\prod_{i=1}^{p} r_i}.$$

Considering $r_1 = \frac{c}{z_1^m}, r_2 = \frac{c}{z_2^m}, \ldots, r_p = \frac{c}{z_p^m}$ and Fig. 3, we get

$$r_t = \frac{\prod_{i=1}^{p} \frac{c}{z_i^m}}{\frac{z_1^m}{c}\prod_{i=1}^{p} \frac{c}{z_i^m} + \frac{z_2^m}{c}\prod_{i=1}^{p} \frac{c}{z_i^m} + \cdots + \frac{z_p^m}{c}\prod_{i=1}^{p} \frac{c}{z_i^m}}. \tag{4.2}$$

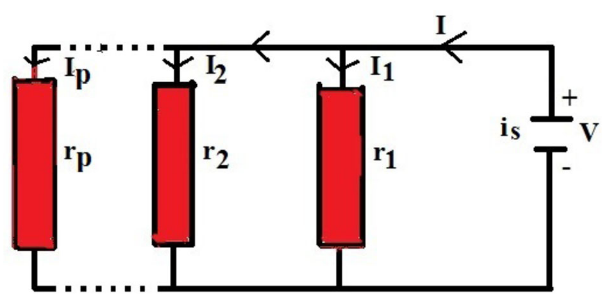
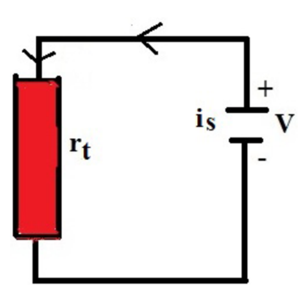


Figure 3 Parallel circuit with *p* resistances

Figure 4 Equivalent circuit



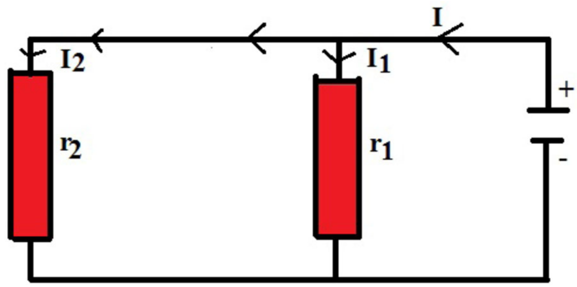


Figure 5 Parallel circuit with two resistances

The total conductance t_d of a circuit in Fig. 3 is $t_d = z_1^m + z_2^m + \cdots + z_p^m$, since r_t is reciprocal to t_d, that is, $r_t = \frac{c}{t_d}$, where c is an arbitrary constant. Equation (4.2) implies that

$$\frac{c}{t_d} = \frac{\prod_{i=1}^{p} \frac{c}{z_i^m}}{\frac{z_1^m}{c} \prod_{i=1}^{p} \frac{c}{z_i^m} + \frac{z_2^m}{c} \prod_{i=1}^{p} \frac{c}{z_i^m} + \cdots + \frac{z_p^m}{c} \prod_{i=1}^{p} \frac{c}{z_i^m}},$$

which implies

$$\frac{c}{z_1^m + z_2^m + \cdots + z_p^m} = \frac{\prod_{i=1}^{p} \frac{c}{z_i^m}}{\frac{z_1^m}{c} \prod_{i=1}^{p} \frac{c}{z_i^m} + \frac{z_2^m}{c} \prod_{i=1}^{p} \frac{c}{z_i^m} + \cdots + \frac{z_p^m}{c} \prod_{i=1}^{p} \frac{c}{z_i^m}}. \tag{4.3}$$

We may observe that equation (4.3) is our introduced functional equation (1.7) with solution $s(z) = \frac{c}{z^m}$. Hence for a circuit in Fig. 3, we have the functional equation (1.7). Also, the left-hand side of equation (4.3) corresponds the circuit in Fig. 4 with only one resistor, and the right-hand side of equation (4.3) corresponds to the circuit in Fig. 3 with p resistors.

Hence we may conclude that our introduced functional equation (1.7) means that a parallel circuit with p resistors is equivalent to a parallel circuit with only one resistor, which is a combination of all the single resistors.

If we consider only two resistances in the parallel circuit, then equation (4.3) with $p = 2$ is our introduced functional equation (1.4) with solution $s(z) = \frac{c}{z^m}$. Hence for a circuit in Fig. 5, we have the functional equation (1.4).

If we consider only three resistances in the parallel circuit, equation (4.3) with $p = 3$ is our introduced functional equation (1.5) with solution $s(z) = \frac{c}{z^m}$. Hence for a circuit in Fig. 6, we have the functional equation (1.5).

4.2 Relation between electrical resistance and conductance using functional equation

In this subsection, we provide a formula for finding the equivalent resistance R_{eq} of parallel electrical circuit using the functional equation that relates the electrical resistances and conductances:

$$s\left(\sqrt[m]{\text{Total conductance}}\right) = R_{\text{eq}}.$$

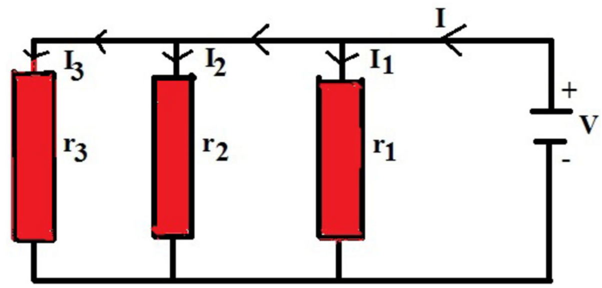


Figure 6 Parallel circuit with three resistances

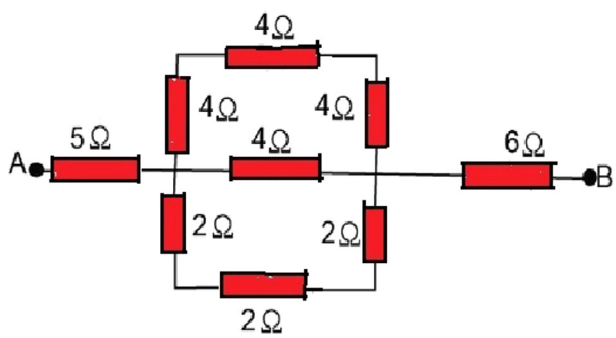


Figure 7 Parallel and series circuit

Let $s(z_1), s(z_2), \dots, s(z_p)$ be the resistances of a parallel electrical circuit, and let $z_1^m, z_2^m, \dots, z_p^m$ be the conductances corresponding to the resistances. Then we have

$$s\left(\sqrt[m]{z_1^m + z_2^m + \cdots + z_p^m}\right) = R_{\text{eq}}, \tag{4.4}$$

where R_{eq} is the equivalent resistance of p resistors. Then

$$R_{eq} = \frac{\prod_{i=1}^{p} s(z_i)}{\frac{1}{s(z_1)} \prod_{i=1}^{p} s(z_i) + \frac{1}{s(z_2)} \prod_{i=1}^{p} s(z_i) + \cdots + \frac{1}{s(z_p)} \prod_{i=1}^{p} s(z_i)}. \tag{4.5}$$

Also, we may observe that by equations (4.4) and (4.5) we have our introduced functional equation (1.7) with solution $s(z) = \frac{1}{z^m}$.

4.3 Counterexamples

In this subsection, we obtain the equivalent resistance using the functional equation (4.4).

Example 4.1 Find the equivalent resistance for the following combination of resistors between A and B in Fig. 7.

Solution. The resistances 4 Ω, 4 Ω, and 4 Ω are in series, and thus

$$s(z_1) = 4 + 4 + 4 = 12\ \Omega,$$

and also 2 Ω, 2 Ω, and 2 Ω are in series, and so

$$s(z_2) = 2 + 2 + 2 = 6\ \Omega.$$

At present, $s(z_1)$, $s(z_2)$, and 4 Ω are in parallel. For the three parallel resistances, by (4.4) we have

$$R_{eq} = f\left(\sqrt[m]{z_1^m + z_2^m + z_3^m}\right), \tag{4.6}$$

where $R_{\text{eq}} = \frac{s(z_1)s(z_2)s(z_3)}{s(z_2)s(z_3)+s(z_1)s(z_3)+s(z_1)s(z_2)}$. Let $s(z_1) = \frac{1}{z_1^m} = 12$, $s(z_2) = \frac{1}{z_2^m} = 6$, and $s(z_3) = \frac{1}{z_3^m} = 4$. Then (4.6) implies that

$$R_{eq} = s\left(\sqrt[m]{\frac{1}{12} + \frac{1}{6} + \frac{1}{4}}\right) = s\left(\sqrt[m]{\frac{1}{2}}\right) = 2\ \Omega.$$

Let $R_{\text{eq}} = s(z_4)$. Now, from A to B, 5 Ω, $s(z_4) = 2$ Ω, and 6 Ω are in series, and hence

$$R_{\text{eq}} = 5\ \Omega + 2\ \Omega + 6\ \Omega = 13\ \Omega.$$

Therefore the equivalent resistance between A and B is 13 Ω.

Example 4.2 Find the equivalent resistance for the parallel circuit in Fig. 8.

Solution. Let $s(z_1) = 6$ Ω, $s(z_2) = 12$ Ω, $s(z_3) = 24$ Ω, $s(z_4) = 12$ Ω, $s(z_5) = 6$ Ω, $s(z_6) = 24$ Ω, $s(z_7) = 24$ Ω, $s(z_8) = 12$ Ω, $s(z_9) = 6$ Ω, and $s(z_{10}) = 8$ Ω be the parallel resistors. Then for the ten parallel resistances by using functional equation (4.4), we have

$$R_{eq} = s\left(\sqrt[m]{z_1^m + z_2^m + z_3^m + \cdots + z_{10}^m}\right), \tag{4.7}$$

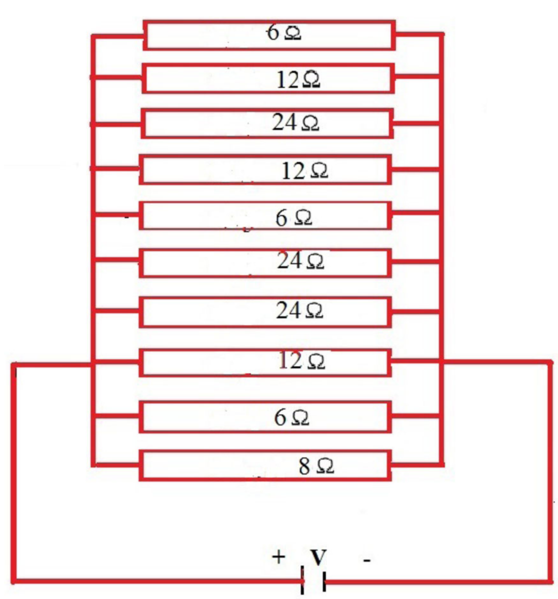


Figure 8 Parallel circuit with 10 resistances

where $R_{eq} = \frac{\prod_{i=1}^{10} s(z_i)}{\frac{1}{s(z_1)}\prod_{i=1}^{10} s(z_i)+\frac{1}{s(z_2)}\prod_{i=1}^{10} s(z_i)+\cdots+\frac{1}{s(z_{10})}\prod_{i=1}^{10} s(z_i)}$. Now (4.7) implies that

$$\begin{aligned} R_{eq} &= s\left(\sqrt[m]{\frac{1}{6}+\frac{1}{12}+\frac{1}{24}+\frac{1}{12}+\frac{1}{6}+\frac{1}{24}+\frac{1}{24}+\frac{1}{12}+\frac{1}{6}+\frac{1}{8}}\right) \\ &= s\left(\sqrt[m]{\frac{24}{24}}\right) = 1\ \Omega. \end{aligned}$$

Hence the equivalent resistance of a given circuit is 1 Ω.

5 Conclusions

In this work, we introduced new generalized multifarious radical reciprocal functional equations combining three classical Pythagorean means: arithmetic, geometric, and harmonic. Importantly, we obtained their general solution and stability related to the Ulam problem with suitable counterexamples in modular spaces by using the fixed point method. Also, we illustrated their geometrical interpretation and applications in connection with the parallel circuit. Importantly, we provided a formula for finding the equivalent resistance R_{eq} of parallel electrical circuit using functional equations, which relates the electrical resistances and conductances with suitable examples.

Acknowledgements
The authors are highly grateful to the referees for their valuable comments and suggestions enriching the contents of the paper.

Funding
The authors declare that there is no funding available for this paper.

Availability of data and materials
Not applicable to this paper.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
DP and RM together studied and prepared the manuscript. CP analyzed all the results and made necessary improvements. JRL is the major contributor in writing the paper. All authors read and approved the final maniscript.

Author details
[1]Department of Mathematics, Don Bosco College (Co-Ed), Yelagiri Hills 635 853, TamilNadu, India. [2]PG and Research Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur 635 601, TamilNadu, India. [3]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea. [4]Department of Data Science, Daejin University, Kyunggi 11159, Korea.

Received: 30 October 2021 Accepted: 4 May 2022 Published online: 21 May 2022

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