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Performance of Stochastic Inventory System with a Fresh Item, Returned Item, Refurbished Item, and Multi-Class Customers

K. Jeganathan [1], S. Selvakumar [1], S. Saravanan [2], N. Anbazhagan [3], S. Amutha [4], Woong Cho [5], Gyanendra Prasad Joshi [6,*] and Joohan Ryoo [7,*]

[1] Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chepauk, Chennai 600005, India; kjeganathan@unom.ac.in (K.J.); illoduselvakumar@gmail.com (S.S.)
[2] Madras School of Economics, Chennai 600025, India; profsaran11@gmail.com
[3] Department of Mathematics, Alagappa University, Karaikudi 630003, India; anbazhagann@alagappauniversity.ac.in
[4] Ramanujan Center for Higher Mathematics, Alagappa University, Karaikudi 630003, India; amuthas@alagappauniversity.ac.in
[5] Department of Software Convergence, Daegu Catholic University, Gyeongsan 38430, Korea; wcho@cu.ac.kr
[6] Department of Computer Science and Engineering, Sejong University, Seoul 05006, Korea
[7] Division of International Studies, Hanyang University, Seoul 04763, Korea
* Correspondence: joshi@sejong.ac.kr (G.P.J.); jhryoo@hanyang.ac.kr (J.R.); Tel.: +82-2-6935-2481 (G.P.J.)

Abstract: This paper deals with an integrated and interconnected stochastic queuing-inventory system with a fresh item, a returned item, and a refurbished item. This system provides a multi-type service facility to an arriving multi-class customer through a dedicated channel. It sells fresh and refurbished items, buys used items from customers, refurbishes the used items for resale, and provides a repair service for defective items. The assumption of purchasing a used item from the customer and allowing them to buy a fresh item is a new idea in stochastic queuing-inventory modeling. To do so, this system has four parallel queues to receive four classes of customers and five dedicated servers to provide a multi-type service facility. Customers are classified according to the type of service they require. Each class of arrival follows an independent Poisson process. The service time of each dedicated server is assumed to be exponentially distributed and independent. This system assumes an instantaneous ordering policy for the replenishment of a fresh item. In the long run of this considered system, the joint probability distribution of the seven-dimensional stochastic process, significant system performance measures, and the optimum total cost are to be derived using the Neuts matrix geometric technique. The main objective of the system was to increase the occurrence of all kinds of customers by providing a multi-type service facility in one place. Buying a used item is unavoidable in an emerging society because it helps form a green society. Furthermore, the numerical result shows that the assumption of a system that allows a customer to sell their used item and purchase a new item will increase the number of customers approaching the system.

Keywords: multi-type service; parallel queues; resales; repair; refurbishment; instantaneous ordering policy

MSC: 60K25

Citation: Jeganathan, K.; Selvakumar, S.; Saravanan, S.; Anbazhagan, N.; Amutha, S.; Cho, W.; Joshi, G.P.; Ryoo, J. Performance of Stochastic Inventory System with a Fresh Item, Returned Item, Refurbished Item, and Multi-Class Customers. *Mathematics* **2022**, *10*, 1137. https://doi.org/10.3390/math10071137

Academic Editors: Xu Chen, Xiaojun Wang and Zheng Luo

Received: 24 February 2022
Accepted: 26 March 2022
Published: 1 April 2022

1. Introduction

The use of refurbished (second-hand) products might sound strange to people all over the world. Let us see why many people show their interest in purchasing a refurbished product. Firstly, its price will be a considerable fact for people in an economically low category. Secondly, some of them want to use a refurbished product for temporary use. The purchase of a refurbished product from the second-hand market is shown in Figure 1. The refurbishment process not only concentrates on business points of view, but also helps to form a green society. For example, every refurbished laptop, printer, scanner, server, or desktop computer you buy is one less piece of equipment going to the toxic waste dump. This is the main objective of discussing the refurbished products.

Generally, the second-hand markets are doing well with the sales of refurbished products. One might ask a question: why can a retailer, wholesaler, or company not sell the refurbished product? Yes, all of us can do it. Are the refurbished products available only on the second-hand market? No, this sale process has been going on smoothly in small businesses too. To conduct such business, the business owner must first be ready to purchase a used product from a customer; however, the used product need not be defective. For example, a person who is already using an old mobile phone. It is working well. They want to buy the new model of mobile phone, but at the same time, they are not willing to keep it. So, they sell their used mobile phone at some fixed price and buy the new one by paying an adequate charge on the same system. The articles discussed earlier on the refurbished products are based on the failed products, and at the same time, they must be purchased in the same system as before (see [1]).

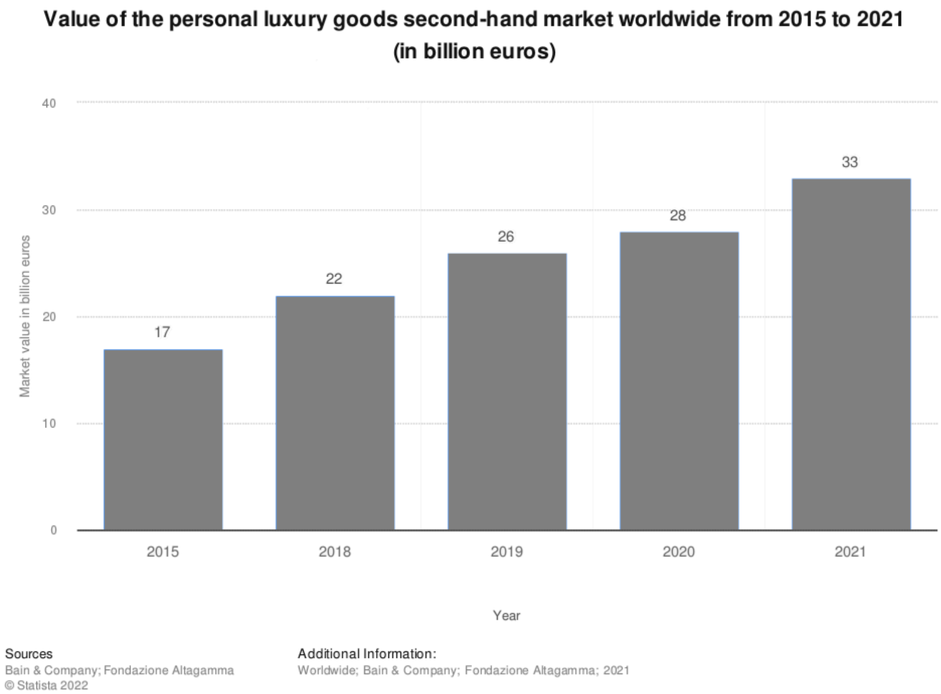

Figure 1. The use of a refurbished product in the second-hand market.

To improvise a small business, having only one type of service option is not very helpful, so we need a multi-type service facility. When observing inventory and supply chain management, we understand that they are the foundations of any company's operations. As a result of technological advancements and the availability of process-driven software systems, inventory management has experienced revolutionary changes. All functions are interconnected and integrated with any business or organization, and they frequently overlap. The backbone of the business delivery function is comprised of crucial factors such as supply chain management, logistics, and inventories. As a result, both marketing managers and financial controllers place a high value on these functions (see [2]). For example, it can be easily seen that from a motorcycle showroom that sells new motor vehicles, used motor vehicles (refurbished), provides repair work on the defective motorcycle, and purchases a used motorcycle from the customer all in one place. All these functions are integrated and interconnected in that type of business. These factors impress us enough to propose this stochastic modeling.

1.1. Motivation

The author was inspired to create this mathematical model by a real-life experience. The author recently intended to buy a new Hewlett-Packard (HP) laptop by returning his old one (which he had previously purchased) to a branded laptop showroom. The author noticed something intriguing about the functioning of that exhibit room while there. This showroom uses different types of dedicated servers for different types of queues. Actually, the showroom provided service to the consumer by selling new laptops, receiving old laptops and selling new ones, and also selling used laptops through the appropriate queues. They also offered re-service of the repaired laptop at a separate service station with different servers. One server was restoring an acquired old laptop on the inside of a showroom. The author was inspired to make mathematical models of this showroom in the MQIS because of how well it worked and how well it looked.

1.2. Purchasing Strategy

However, it is far from certain that every consumer in each queue will purchase the thing they desire. Some clients enter the system and proceed through the service, but at the point when the service is completed, they may elect not to purchase the product. Impulse customers are those who buy the product on the spur of the moment. Although they have no intention of purchasing a product from the shop, there is a potential that they will do so if they are satisfied with the system's service. When looking at the laptop showroom, one can see that not every arriving consumer purchases a laptop owing to a lack of funds, dissatisfaction with the service, product features, and so on. Despite the fact that the suggested MQIS is designed for impulse customers, it will be a generalized model for both customers who purchase a product compulsorily when the probability is 1, and customers who may purchase with a probability value in the range of $[0,1)$.

1.3. Return Strategy

A customer returns their old stuff, which does not need to be purchased in the same system. The system assumes that returned products always satisfy the terms and conditions of the system. At the end of the return procedures, the customer can choose to sell the old stuff and buy the new stuff, or leave the system with or without selling the old stuff, based on the probability p_1, p_2, p_3, where $p_1 + p_2 + p_3 = 1$. The old product that the customer returns is called a "returned product". Further, the system assumes that the returned product need not be defective.

1.4. Refurbishment Strategy

The returned products are refurbished to their original quality and resold to the market at a markdown price as refurbished products.

1.5. Repair Strategy

Customers arrive at the system with their own defective products. The server identifies the fault in the product and starts repair work on it. The system assumes that the server can do repair work for any type of defectiveness that occurs in the product.

1.6. Contribution of the Model

The contribution of the paper is listed as follows:

1. This paper concentrates on multi-type service facilities provided by dedicated servers.
2. It analyses the sales of a new product or fresh product, purchases the old or used product from the customer, conducts refurbishing work on the returned product, sells the refurbished product, and repairs the defective product.
3. There are four classes of customers that arrive at the system and they are classified according to their needs. To receive those customers, the system allocates three finite queues and one infinite queue.

4. As in the normal lifestyle, this paper assumes that a customer will purchase the product (fresh or refurbished) if they are satisfied with the service with respect to the Bernoulli schedule.
5. It assumes the instantaneous ordering principle for the replenishment process. The Neuts [3] matrix geometric approach and the logarithmic reduction algorithm are used to derive the stationary probability vector.
6. The numerical illustrations investigate the impact of each queue, server busy, or idle period, and the cost analysis according to the parameter variation.

1.7. Novelty of the Model

A customer selling their old product and buying the new product from the system is introduced in the stochastic queuing-inventory modeling. Many articles consider the return of failed products; however, we assume that the returned product need not be defective. In addition, we also encountered the sales service of fresh and refurbished products, defective product repair work, and refurbished work on returned products through the multi-type service facility.

Design of the Paper

This paper is organized as follows: Reviews of related work are presented in Section 1.8 and the research gap is given in Section 1.9. Section 2 explains the mathematical formulation of the model. The process of the system states is explained in Section 3. Further, it investigates the stability analysis of the model. Following that, Section 4 derives the characteristic metrics of the model, and Section 5 interprets the numerical illustration. Furthermore, in Section 6, a conclusion is presented.

1.8. Review of the Related Work

Queuing-inventory theory, one of the fields of operations research, can help businesses make better judgments about how to construct more efficient and cost-effective workflow systems. Since then, to the present, many authors have presented their discussions in the queuing-inventory system (QIS). Very few authors started their analysis of QIS without a service facility. In this connection, the readers can refer to the following papers to know more details regarding the analysis of instantaneous service: Paul Manual et al. [4], Sivakumar [5], Sivakumar [6], Jeganathan et al. [7], and Abdul Reiyas and Jeganathan [8].

This aside, many authors explored their QIS with positive service time or service facilities; however, when observing the practical situation, an instantaneous service facility is not the most suitable one because many supply chain manufacturers, traders, and retailers provide their services to the customers with a positive service time. Not only that, every customer requires a demonstration of the product, a warranty and guarantee on the product, a price and offer on the product, and so on. These customer needs allow us to consider positive service time in this paper. Actually, the work involving service facilities was introduced by Melikov and Molchanov [9] and Sigman and Simchi -Levi [10] in the QIS.

The QIS's single-server service station is the familiar model where most of the researchers developed their academic knowledge. Amirthakodi [11] studied feedback from customers who required a feedback service after completion of the main service with a positive service time in the QIS. Those who required feedback were able to enter an orbit based on the Bernoulli schedule. Jeganathan et al. [12] investigated a single-server QIS with a queue-dependent service rate that was expected to reduce customer waiting time. According to their model, retrial customers are not permitted to obtain their services directly from the orbit. The orbital customer can obtain service through the waiting hall, and their retrial procedure follows the classical retrial policy.

When performing the replenishment process, there are two different approaches to make the order-up-to level: periodic review and continuous review. In such a way, to learn more about periodic review replenishment policies the reader can refer [13,14]. This paper deals with a continuous review ordering policy. Krishnamoorthy et al. [15] discussed a QIS in which they analyzed two control policies: (s, Q) and (s, S) ordering principles. Aside from that, they assumed that the inventory provided to the customer at the end of service completion was not necessary. According to the Bernoulli schedule, the customer has either left the system with or without an item. The central assumption of this paper is that an arriving customer is not permitted to enter the system if the current stock level is zero. The reader can also refer [7] to know more about (s, Q) and (s, S) ordering principles.

Up to this point, a review of the literature works is performed on the single server with a single commodity in a QIS. The readers can refer to the recent papers on a single server in [16,17]. Nevertheless, the queuing-inventory literature has expanded with some other extensions: (i) single commodity, multi-class customers, single server; (ii) single commodity, single-class customers, multi-server; (iii) multi-commodity, single-class customers, single server; (iv) multi-commodity, single-class customers, multi-server. Vinitha et al. [18] investigated a QIS with a cancellation of sold items. In addition, the considered system allowed two classes of customers, i.e., ordinary and impulsed customers. Both classes of customers approached the system to purchase the same product. The cancellation of items is accepted when there is at most less than their maximum capacity. Fong-Fan Wang [19] determined the approximation and optimization of multi-server QIS with two types of customers. They are classified as high and low-priority customers whose arrivals occur according to the MAP. They assumed that the low-priority customer had left the system with a Bernoulli reneging probability if they became impatient.

Valentina et al. [20] considered the in-homogeneous customers in the queuing system with a single server under the assumption of priority. Jeganathan and Abdul Reiyas [21] discussed a two-parallel heterogeneous server QIS. The servers are exclusively dedicated to high and low-priority customers, respectively. Among these assumptions, in this paper, the dedicated server 1 and server 2 had offered that they could choose the modified and delayed working vacation options, respectively, upon the interruption. Kingsly et al. [22] provided the study about two server QIS. Even though the system has a dedicated server for the high-priority queue and a flexible server that is able to deliver the service for both queues. Jeganathan et al. [23] gave the compared discussion on the Markovian QIS with server interruption. The distinguished results are given on the basis of two homogeneous and two heterogeneous servers. In addition, the customers from a retrial group approached the system using the classical and constant retrial policies.

Vishnevsky et al. [24] investigated the performance of a priority multi-server queuing system with heterogeneous customers, whereas Klimenok et al. [25] worked on a multi-server queuing system for the retrial queue in which they used a phase-type for the retrial process if the number of customers in orbit is less than the threshold level. Recently, Jeganathan et al. [26] explored the two multi-server service channels in the retrial QIS with homogeneous customers. In this study, they assumed the interconnected arrival would enter the system. This system provided product sales service to the customer via one multi-server service channel. On the other hand, the products' repair service is performed by another multi-server service channel. Any arriving customer who bought a fresh product and required additional service is sent to the second service station. Apart from that, customers who only require repair services for the product can also approach the second service station. Rajkumar et al. [27] looked into an infinite queue multi-server and single-commodity QIS. In this work, the first passage and waiting time analysis are derived analytically for an infinite queue. The reader can refer to the recent papers [28,29] to know more details about the multi-server queuing-inventory system.

Many authors concentrate on single-commodity QIS, while only a few go on to investigate a two-commodity QIS. Sivakumar [5] investigated the two commodity inventory sales with a single server QIS in this regard. In this study, arriving consumers are treated as single-class customers, and both goods are presumed to be interchangeable. Serife Ozkar and Umay Uzunoglu Kocer [30] assumed two classes of clients for the two commodity QIS with a single-server service station in a recent paper. Customers in different classes (priority and ordinary) require distinct products from the system. The system then replenished both commodities according to the individual ordering policy. The system's service facility is maintained by a single server. Binitha Benny et al. [31] discussed the two commodity QIS with a single server service facility. This system took into account single-class arrivals when purchasing the product. A customer who purchased a product is determined by a certain probability.

In recent days, many customers have focused their attention on purchasing refurbished products instead of purchasing the most expensive fresh products in the consumer market. Zhang et al. [32] studied the refurbishment and quality recovery of returned defective products in a closed-loop supply chain. Zhang et al. [33] determined the retailers' and suppliers' equilibrium decisions on the new product and refurbished product by the divide-and-conquer method. They also discussed how an arriving customer could choose the product when faced with both new and refurbished products at the same time. Consumer returns an unsatisfied product to the retailer with a full refund is considered in He et al. [34]. It is to be resold as a refurbished product to consumers. Tseng-Fung Ho et al. [35] analyzed a problem involving re-manufacturing products in a three-echelon supply chain. At the time of the screening test, some products were found to have failed. These failed products are packed for the re-manufacturer.

Rani et al. [36] studied a refurbished deteriorating item with cannibalization of the green supply chain. Cárdenas-Barrón et al. [37] studied a rework on imperfect quality products with non-linear demand. Very recently, Saranya et al. [1] investigated an inventory problem with a refurbished product. There are two types of customers who arrive in the system. The first type of customers came to the system to return their failed product and obtain a replacement with a new product. The second type of customers arrived to purchase the refurbished product. Sinu Lal et al. [38] examined the multi-type inventory system. The marked Markovian arrivals generated the different types of channels of finite size, which are allocated according to their requirements. Jacob et al. [39] studied the inventory system with one essential item and m optional items in a random environment. The service time of essential and optional items is assumed to be phase-type and exponentially distributed, respectively.

In the queuing literature, there is an interesting paper that is analyzed using multi-class customers with multi-server service facilities. Van Harten and Sleptchenko [40] considered a N type of clients whose service rate is assumed to be heterogeneous. Each server is identical and non-dedicated. They are trained to provide service to all classes of customers. Karumbathil Rasmi et al. [41] explored the heterogeneous QIS with heterogeneous inventory access. This is a single-source inventory system; however, there is K class of customers to purchase the single-source inventory. To accommodate each class of customer, the system has K waiting spaces with dedicated heterogeneous servers. All classes of customer arrival occur according to the marked MAP. Krishnamoorthy et al. [42] investigated the $M/M/2$ and $M/M/3$ QIS with homogeneous servers. This study has crucial assumptions such that the arriving customers' are not allowed when the inventory level is zero. In addition, at the end of service completion, the inventory provided to the customer is not certain. This idea is, of course, to explore a realistic situation in real life. It can be easily seen that not all customers are always ready to purchase the product when they visit the QIS. though some of them may buy it.

Some other interesting papers reported in the literature in a similar domain are [43–45].

1.9. Research Gap

There has been no paper published in the existing queuing-inventory literature with an FP, buying the OP from the customer, performing some refurbishing work on it, and selling it back to those who require such products along with a multi-type service facility. This idea is still incomplete along with these three different levels of the same product. This is because many customers are interested in purchasing used products, such as via online shopping organizations that sell used mobile phones, laptop computers, furniture, and so on. They also offer the customer the opportunity to purchase a new product in exchange for returning a previously purchased product. Some jewelry stores, car dealerships, bike showrooms, and house sales companies, among others, offer sales services for new products, used products, and purchasing old products. So, the concept of buying a new product by selling the old product was not yet discussed in a stochastic QIS. This is also a familiar existing real-life application of the queuing-inventory problem, which has not yet been investigated.

1.10. Model Proposal

We suggest a novel model that investigates the selling of new items, second-hand items (refurbished), and the buying of old products from customers and performs refurbish work on them in order to convert them into refurbished items. The system creates specialized queues for FP, RFP, and OP purchases from customers. Dedicated servers are in charge of all of these diverse tasks. In addition to these tasks, the system also has a separate repair service for people who need their items fixed. They can wait in a separate queue. Figure 2 depicts the graphical representation of the suggested model.

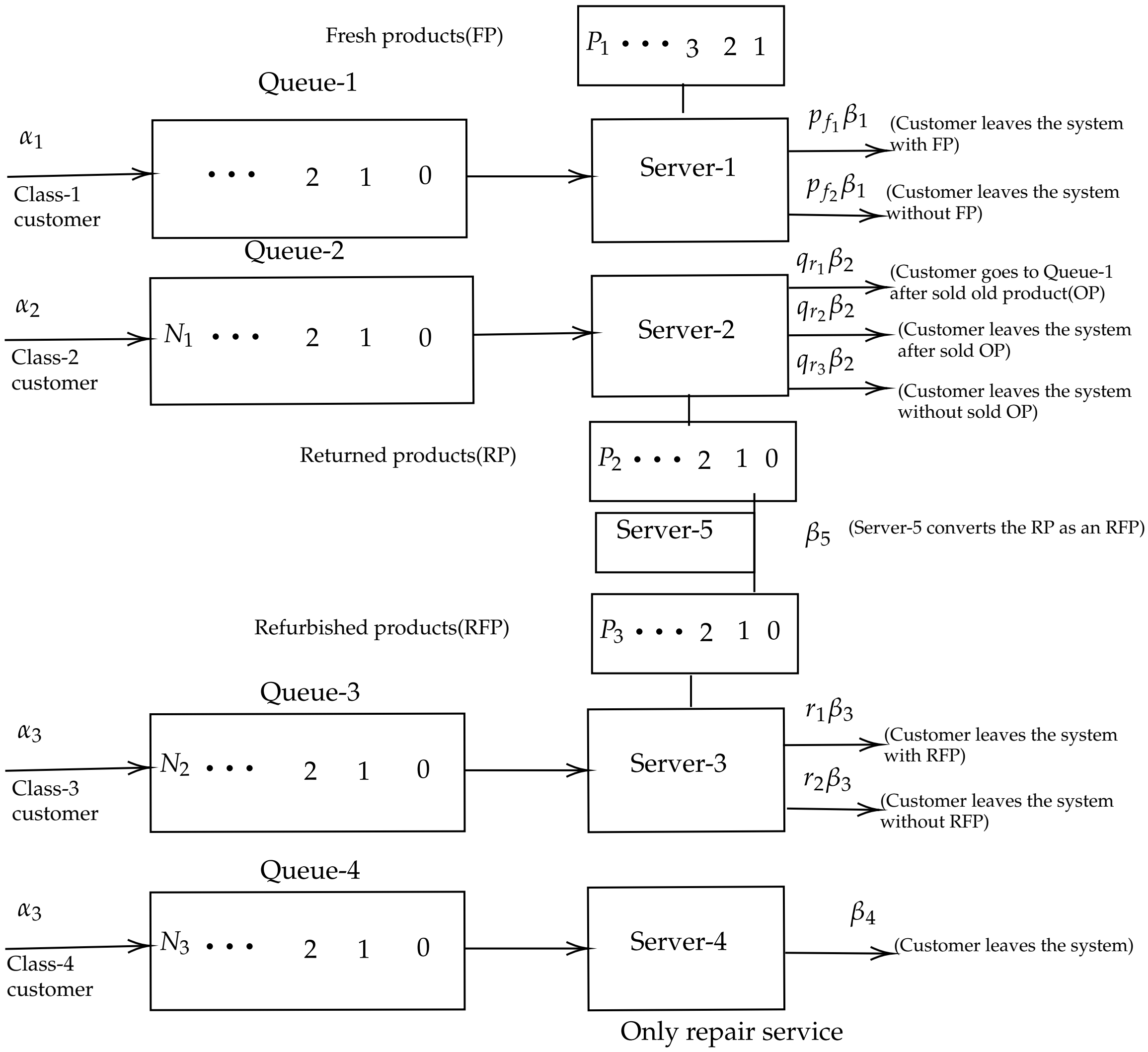

Figure 2. Pictorial representation of the system.

2. The Mathematical Formulation of the Model

This paper considers a multi-type service facility corresponding to multi-class customers with FP, RFP, and OP in an MQIS. In order to keep the products for sales and service procedures, this system divides them into three kinds of storage rooms. Each storage space has a capacity of a maximum of P_1, P_2, and P_3 products and is labeled as FP, RP, and RFP, respectively. The four parallel queues, Queue-i, $i \in S_1^4$, are of unlimited size, N_1, N_2, and N_3, respectively. The following actions are performed by the system under consideration:

1. Sales for both FP and RFP.
2. Purchase of OP from the customer.
3. Refurbishes the purchased OP to resell.
4. Provision of re-service to those who just require repair work on the defective product.

The assumption of purchasing a used item from the customer and allowing them to buy a fresh item is a new idea in stochastic queuing-inventory modeling. TaC will play a crucial role in the system for sales of FP, RFP, and the purchase of OP. Each of these three services will be successful only if the customer accepts the TaC of purchasing the required product. Before beginning to provide FP and RFP services to customers, the server will explain to them what the TaC is and how it is used. A customer may decide whether or not to acquire a product under the Bernoulli schedule after reading and comprehending the TaC. As an example, if an OP does not fulfill the TaC of the system, a customer may choose to return and purchase an FP, to only return and leave the system, or to leave the system without returning, taking into consideration the probabilities whose sum is one; however, there is no TaC for performing repair work, and Bernoulli choices are available for receiving such work instead of TaC. With no need for negotiation, all of the RPs purchased are fixed and rebuilt. The RPs purchased have only minor issues, which will be described in detail in the system's TOM, in order for the refurbishing service procedure to run as smoothly as possible.

According to their requirements, the MQIS welcomes four types of customers, each with their own set of requirements. Depending on their needs, for example, if they want to buy an FP, return an OP, buy an RFP, and only need repair service, they are routed to the appropriate queues, which are designated as Queue-i, $i \in S_1^4$. The arrival patterns of all four classes of customers are also classified as Class-i, $i \in S_1^4$, respectively. A Poisson distribution is used to describe the arrival of customers from each class independently. When a customer belongs to a Class-i, his or her intensity rate is specified as α_i, where $i \in S_1^4$ is the number of customers. As a result of the limited size of Queue-i where $i \in S_2^4$, an arriving customer from the Class-i, $i \in S_2^4$ who finds that Queue-i where $i \in S_2^4$ has reached its maximum capacity is deemed to have lost his or her place in the queue. The following is the system's service description at this point: The system assigns a Server-i where $i \in S_1^4$ to each Queue-i where $i \in S_1^4$ in order to provide better service to an arriving Class-i with $i \in S_1^4$ customers. Due to the fact that each server is a dedicated server, it is expected that they are heterogeneous (i.e., not identical). The performance of each queue is described as follows:

Performance of Queue-1: FPs are being purchased by customers that have arrived at this queue. For this queue, the mean service completion time is defined as $\frac{1}{\beta_1}$. *At the end of the service completion of each customer, they decide whether to purchase the product with probability* p_{f_1} *or leave the system without purchasing under the probability* $p_{f_2}(p_{f_2} = 1 - p_{f_1})$ *according to the satisfaction of the customer.*

Performance of Queue-2: A customer who wants to sell their used product (OP) to the system goes to Queue-2 and meets Server-2. The average service completion time per customer is denoted as $\frac{1}{\beta_2}$. *If a customer's OP requires minor repair, Server-2 will purchase it from them. As an example, Server-2 does not purchase the product since it requires extensive repair work. This is due to the violation of TaC. Perhaps Server-2 agrees to purchase the OP from the customer, but Server-2 is unable to do so because the customer can only choose to return their OP and move to Queue-1 to buy an FP with probability* q_{r_1}, *returns their OP and leaves the system with probability* q_{r_2}, *or leaves the system without returning their OP with probability* q_{r_3}, *where* $q_{r_1} + q_{r_2} + q_{r_3} = 1$. *If the allocated storage space for RP's has reached its maximum size* P_2, *Server-2 will become idle, and Class-2 customers will be forced to wait in Queue-2 until there are less items than* P_2 *available for purchase.*

Performance of Queue-3: To sell the RFP, Server-3 has been assigned to this queue. Aside from that, it is claimed that the RFPs are second-hand items. When a customer is placed in this *queue, the average service time for that customer is given by the parameter $\frac{1}{\beta_3}$. The server provides all the necessary explanations and demonstrations of the product during the servicing procedure. Customers who are happy either purchase the RFP with a probability of r_1 or quit the system without purchasing it with a compliment of $r_2 = 1 - r_1$. Suppose there is no RFP in stock, the customer in Queue-3 has to wait until the new RFP arrives.*

Performance of Queue-4: Customers' defective items are brought to the dedicated Server-4, which is then assigned to repair or re-service them as necessary. The average service time is specified as $\frac{1}{\beta_4}$ for these defective products. If there is no customer in the line at this point, Server-4 goes inactive until the next customer arrives.

Furthermore, the process of reconditioning the things that it has purchased in the past is to be executed. This type of service activity is carried out by Server-5. In order to refurbish each OP, an average service time of $\frac{1}{\beta_5}$ is needed. Server-5 goes inactive until they can find an RFP that is less than P_3 in the storage space. Otherwise, they keep reformulating the OP as RFP. There is no correlation between any of the mean service times and the intensity rate of each server, which is defined as $\beta_i > 0$ for each of the servers with $i \in S_1^5$. All the mean service times are independently exponentially distributed. In addition to that, the replenishment procedure for the FP is to be carried out in accordance with the $(0, P_1)$ re-ordering (replenishment) concept. Whenever the inventory level drops to zero, then P_1 number of FP replenished immediately. There is an exponential distribution in the amount of time that elapses between two successive reorders.

3. Main Results

3.1. Process of the States of the Stochastic Model

According to the assumptions described in Section 2, the process of a proposed model is defined as

$$J_t = \{(J_1(t), J_2(t), J_3(t), J_4(t), J_5(t), J_6(t), J_7(t))\}, \quad t \geq 0.$$

Since $J_t, t \geq 0$ consists of a family of collection of random variables $J_i(t), i \in S_1^7, t \geq 0$ depending on any time t, $J_t, t \geq 0$ is called a seven-dimensional stochastic process with the discrete state space K. In this stochastic modeling, the sequence of possible events describes that the probability of each future event depends only on the present event and not on the past event, $J_t, t \geq 0$ is said to be the Markov process. As $J_t, t \geq 0$ has a continuous time process, it is called a CTMC. This Markov chain holds the property that every state of K is can be reached from every other state of itself. Thus, the process $J_t, t \geq 0$ is also said to be an irreducible CTMC.

3.2. Construction of Transition Matrices of the System

The seven-dimensional irreducible CTMC, $J_t, t \geq 0$ has the infinitesimal generator matrix,

$$\mathbf{B} = \begin{pmatrix} B_0 & B_c & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ B_a & B_b & B_c & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & B_a & B_b & B_c & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & B_a & B_b & B_c & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad (1)$$

where

$$[B_c]_{\gamma_0} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ P_1 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & \dots & P_1 \end{array} \\ \begin{pmatrix} A_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & A_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_1 & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & A_1 \end{pmatrix} \end{array}$$

$$[A_1]_{\gamma_1} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N_1-1 \\ N_1 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \dots & N_1-1 & N_1 \end{array} \\ \begin{pmatrix} \alpha_1 I_{\gamma_2} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ A_2 & \alpha_1 I_{\gamma_2} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_2 & \alpha_1 I_{\gamma_2} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \alpha_1 I_{\gamma_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & A_2 & \alpha_1 I_{\gamma_2} \end{pmatrix} \end{array}$$

$$[A_2]_{\gamma_2} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ P_2-1 \\ P_2 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & \dots & P_2 \end{array} \\ \begin{pmatrix} \mathbf{0} & q_{r_1}\beta_2 I_{\gamma_3} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & q_{r_1}\beta_2 I_{\gamma_3} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & q_{r_1}\beta_2 I_{\gamma_3} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & q_{r_1}\beta_2 I_{\gamma_3} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix} \end{array}$$

$$[B_a]_{\gamma_0} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ P_1-1 \\ P_1 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & \dots & P_1-1 & P_1 \end{array} \\ \begin{pmatrix} p_{f_2}\beta_1 I_{\gamma_1} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & p_{f_1}\beta_1 I_{\gamma_1} \\ p_{f_1}\beta_1 I_{\gamma_1} & p_{f_2}\beta_1 I_{\gamma_1} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & p_{f_1}\beta_1 I_{\gamma_1} & p_{f_2}\beta_1 I_{\gamma_1} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \ddots & \ddots & p_{f_2}\beta_1 I_{\gamma_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & p_{f_1}\beta_1 I_{\gamma_1} & p_{f_2}\beta_1 I_{\gamma_1} \end{pmatrix} \end{array}$$

For $j_1 = 0$ and b.

$$[B_{j_1}]_{\gamma_0} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ P_1 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & \dots & P_1 \end{array} \\ \begin{pmatrix} D_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & D_2 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_3 & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & D_{P_1} \end{pmatrix} \end{array},$$

For $j_2 \in S_1^{P_1}$.

$$
[D_{j_2}]_{\gamma_1} = \begin{array}{c c}
 & \begin{array}{cccccc} 0 & 1 & 2 & \dots & N_1-1 & N_1 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ N_1-1 \\ N_1 \end{array} &
\begin{pmatrix}
D_{j_2}^3 & D_{j_2}^1 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
D_{j_2}^2 & D_{j_2}^4 & D_{j_2}^1 & \dots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & D_{j_2}^2 & D_{j_2}^4 & \ddots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & D_{j_2}^4 & D_{j_2}^1 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & D_{j_2}^2 & D_{j_2}^5
\end{pmatrix}
\end{array}
$$

$$
[D_{j_2}^1]_{\gamma_2} = \begin{array}{c c}
 & \begin{array}{cccccc} 0 & 1 & 2 & \dots & P_2-1 & P_2 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ P_2-1 \\ P_2 \end{array} &
\begin{pmatrix}
\alpha_2 I_{\gamma_3} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \alpha_2 I_{\gamma_3} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \alpha_2 I_{\gamma_3} & \dots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \alpha_2 I_{\gamma_3} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \alpha_2 I_{\gamma_3}
\end{pmatrix}
\end{array}
$$

$$
[D_{j_2}^2]_{\gamma_2} = \begin{array}{c c}
 & \begin{array}{cccccc} 0 & 1 & 2 & \dots & P_2-1 & P_2 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ P_2-1 \\ P_2 \end{array} &
\begin{pmatrix}
q_{r_3}\beta_2 I_{\gamma_3} & q_{r_2}\beta_2 I_{\gamma_3} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & q_{r_3}\beta_2 I_{\gamma_3} & q_{r_2}\beta_2 I_{\gamma_3} & \dots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & q_{r_3}\beta_2 I_{\gamma_3} & \ddots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & q_{r_3}\beta_2 I_{\gamma_3} & q_{r_2}\beta_2 I_{\gamma_3} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0}
\end{pmatrix}
\end{array}
$$

For $i = 3,\ 4$ and 5.

$$
[D_{j_2}^i]_{\gamma_2} = \begin{array}{c c}
 & \begin{array}{cccccc} 0 & 1 & 2 & \dots & P_2-1 & P_2 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ P_2-1 \\ P_2 \end{array} &
\begin{pmatrix}
D_{i1} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
D_{2b} & D_{i2} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & D_{2b} & D_{i2} & \dots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & D_{i2} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & D_{2b} & D_{i3}
\end{pmatrix}
\end{array}
$$

$$
[D_{2b}]_{\gamma_3} = \begin{array}{c c}
 & \begin{array}{cccccc} 0 & 1 & 2 & \dots & N_2-1 & N_2 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ N_2-1 \\ N_2 \end{array} &
\begin{pmatrix}
D_4 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & D_4 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & D_4 & \dots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & D_4 & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & D_4
\end{pmatrix}
\end{array}
$$

$$[D_4]_{\gamma_4} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ P_3-1 \\ P_3 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & \dots & P_3 \end{array} \\ \begin{pmatrix} \mathbf{0} & \beta_5 I_{\gamma_5} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \beta_5 I_{\gamma_5} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \beta_5 I_{\gamma_5} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \beta_5 I_{\gamma_5} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix} \end{array}$$

For $j = 1,\ 2$ and 3.

$$[D_{ij}]_{\gamma_3} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N_2-1 \\ N_2 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \dots & N_2-1 & N_2 \end{array} \\ \begin{pmatrix} D_{ij1} & D_{3a} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ D_{3b} & D_{ij2} & D_{3a} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_{3b} & D_{ij2} & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & D_{ij2} & D_{3a} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & D_{3b} & D_{ij3} \end{pmatrix} \end{array}$$

$$[D_{3a}]_{\gamma_4} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ P_3-1 \\ P_3 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \dots & P_3-1 & P_3 \end{array} \\ \begin{pmatrix} \alpha_3 I_{\gamma_5} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \alpha_3 I_{\gamma_5} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \alpha_3 I_{\gamma_5} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \alpha_3 I_{\gamma_5} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \alpha_3 I_{\gamma_5} \end{pmatrix} \end{array}$$

$$[D_{3b}]_{\gamma_4} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ P_3-1 \\ P_3 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \dots & P_3-1 & P_3 \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ r_1\beta_3 I_{\gamma_5} & r_2\beta_3 I_{\gamma_5} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & r_1\beta_3 I_{\gamma_5} & r_2\beta_3 I_{\gamma_5} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & r_2\beta_3 I_{\gamma_5} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & r_1\beta_3 I_{\gamma_5} & r_2\beta_3 I_{\gamma_5} \end{pmatrix} \end{array}$$

For $k = 1, 2$ and 3.

$$[D_{ijk}]_{\gamma_4} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \dots & P_3 \end{array} \\ \begin{pmatrix} D^1_{ijk} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & D^2_{ijk} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D^2_{ijk} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & D^3_{ijk} \end{pmatrix} \end{array}$$

For $l = 1, 2$ and 3.

$$[D^{l}_{ijk}]_{\gamma_5} = \begin{array}{c c} & \begin{array}{cccccc} 0 & 1 & 2 & \dots & N_3-1 & N_3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ N_3-1 \\ N_3 \end{array} & \begin{pmatrix} d^{l1}_{ijk} & \alpha_4 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \beta_4 & d^{l2}_{ijk} & \alpha_4 & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \beta_4 & d^{l2}_{ijk} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & d^{l2}_{ijk} & \alpha_4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \beta_4 & d^{l3}_{ijk} \end{pmatrix} \end{array}$$

For $m = 1, 2$ and 3.
$d^{lm}_{ijk} = -(\alpha_1 + \bar{\delta}_{i5}\alpha_2 + \bar{\delta}_{k3}\alpha_3 + \bar{\delta}_{m3}\alpha_4 + \bar{\delta}_{j_1 0}\beta_1 + \bar{\delta}_{i3}\bar{\delta}_{j3}\beta_2 + \bar{\delta}_{l1}\bar{\delta}_{k1}\beta_3 + \bar{\delta}_{m1}\beta_4 + \bar{\delta}_{l3}\bar{\delta}_{j1}\beta_5)$.

3.3. Explanation of the Transition Rates of the System

The parameter $\alpha_i, i \in S_1^4$, indicates that the transition of Class-i, $i \in S_1^4$ customer enters into the corresponding Queue-i, $i \in S_1^4$, respectively. The transitions of $\alpha_i, i \in S_1^4$ are defined as follows:

- $(j_1, j_2, j_3, j_4, j_5, j_6, j_7) \xrightarrow{\alpha_1} (j_1 + 1, j_2, j_3, j_4, j_5, j_6, j_7)$, if $j_1 \in \mathbb{Z}^+, j_2 \in S_1^{P_1}, j_3 \in S_0^{N_1}, j_4 \in S_0^{P_2}, j_5 \in S_0^{N_2}, j_6 \in S_0^{P_3}, j_7 \in S_0^{N_3}$.
- $(j_1, j_2, j_3, j_4, j_5, j_6, j_7) \xrightarrow{\alpha_2} (j_1, j_2, j_3 + 1, j_4, j_5, j_6, j_7)$, if $j_1 \in \mathbb{Z}^+, j_2 \in S_1^{P_1}, j_3 \in S_0^{N_1-1}, j_4 \in S_0^{P_2}, j_5 \in S_0^{N_2}, j_6 \in S_0^{P_3}, j_7 \in S_0^{N_3}$.
- $(j_1, j_2, j_3, j_4, j_5, j_6, j_7) \xrightarrow{\alpha_3} (j_1, j_2, j_3, j_4, j_5 + 1, j_6, j_7)$, if $j_1 \in \mathbb{Z}^+, j_2 \in S_1^{P_1}, j_3 \in S_0^{N_1}, j_4 \in S_0^{P_2}, j_5 \in S_0^{N_2-1}, j_6 \in S_0^{P_3}, j_7 \in S_0^{N_3}$.
- $(j_1, j_2, j_3, j_4, j_5, j_6, j_7) \xrightarrow{\alpha_4} (j_1, j_2, j_3, j_4, j_5, j_6, j_7 + 1)$, if $j_1 \in \mathbb{Z}^+, j_2 \in S_1^{P_1}, j_3 \in S_0^{N_1}, j_4 \in S_0^{P_2}, j_5 \in S_0^{N_2}, j_6 \in S_0^{P_3}, j_7 \in S_0^{N_3-1}$.

The transition of a customer in Queue-1 leaves the system after their service completion with an FP or without FP is defined as follows:

- $(j_1, j_2, j_3, j_4, j_5, j_6, j_7) \xrightarrow{p_{f_1}\beta_1} (j_1 - 1, P_1, j_3, j_4, j_5, j_6, j_7)$, if $j_1 \in \mathbb{Z}^+ - \{0\}, j_2 \in S_1^1, j_3 \in S_0^{N_1}, j_4 \in S_0^{P_2}, j_5 \in S_0^{N_2}, j_6 \in S_0^{P_3}, j_7 \in S_0^{N_3}$.
- $(j_1, j_2, j_3, j_4, j_5, j_6, j_7) \xrightarrow{p_{f_1}\beta_1} (j_1 - 1, j_2 - 1, j_3, j_4, j_5, j_6, j_7)$, if $j_1 \in \mathbb{Z}^+ - \{0\}, j_2 \in S_2^{P_1}, j_3 \in S_0^{N_1}, j_4 \in S_0^{P_2}, j_5 \in S_0^{N_2}, j_6 \in S_0^{P_3}, j_7 \in S_0^{N_3}$.
- $(j_1, j_2, j_3, j_4, j_5, j_6, j_7) \xrightarrow{p_{f_2}\beta_1} (j_1 - 1, j_2, j_3, j_4, j_5, j_6, j_7)$, if $j_1 \in \mathbb{Z}^+ - \{0\}, j_2 \in S_1^{P_1}, j_3 \in S_0^{N_1}, j_4 \in S_0^{P_2}, j_5 \in S_0^{N_2}, j_6 \in S_0^{P_3}, j_7 \in S_0^{N_3}$.

The transition of Class-2 customer leaves from Queue-2 after their service completion is calculated as follows:

- $(j_1, j_2, j_3, j_4, j_5, j_6, j_7) \xrightarrow{q_{r_1}\beta_2} (j_1 + 1, j_2, j_3 - 1, j_4 + 1, j_5, j_6, j_7)$, if $j_1 \in \mathbb{Z}^+, j_2 \in S_1^{P_1}, j_3 \in S_1^{N_1}, j_4 \in S_0^{P_2-1}, j_5 \in S_0^{N_2}, j_6 \in S_0^{P_3}, j_7 \in S_0^{N_3}$.
- $(j_1, j_2, j_3, j_4, j_5, j_6, j_7) \xrightarrow{q_{r_2}\beta_2} (j_1, j_2, j_3 - 1, j_4 + 1, j_5, j_6, j_7)$, if $j_1 \in \mathbb{Z}^+, j_2 \in S_1^{P_1}, j_3 \in S_1^{N_1}, j_4 \in S_0^{P_2-1}, j_5 \in S_0^{N_2}, j_6 \in S_0^{P_3}, j_7 \in S_0^{N_3}$.
- $(j_1, j_2, j_3, j_4, j_5, j_6, j_7) \xrightarrow{q_{r_3}\beta_2} (j_1, j_2, j_3 - 1, j_4, j_5, j_6, j_7)$, if $j_1 \in \mathbb{Z}^+, j_2 \in S_1^{P_1}, j_3 \in S_1^{N_1}, j_4 \in S_0^{P_2-1}, j_5 \in S_0^{N_2}, j_6 \in S_0^{P_3}, j_7 \in S_0^{N_3}$.

The transition of Class-3 customer leaves from Queue-3 after their service completion is given by

- $(j_1, j_2, j_3, j_4, j_5, j_6, j_7) \xrightarrow{r_1\beta_3} (j_1, j_2, j_3, j_4, j_5 - 1, j_6 - 1, j_7)$, if $j_1 \in \mathbb{Z}^+, j_2 \in S_1^{P_1}, j_3 \in S_1^{N_1}, j_4 \in S_0^{P_2}, j_5 \in S_1^{N_2}, j_6 \in S_1^{P_3}, j_7 \in S_0^{N_3}$.

- $(j_1, j_2, j_3, j_4, j_5, j_6, j_7) \xrightarrow{r_2\beta_3} (j_1, j_2, j_3, j_4, j_5 - 1, j_6, j_7)$, if $j_1 \in \mathbb{Z}^+, j_2 \in S_1^{P_1}, j_3 \in S_1^{N_1}, j_4 \in S_0^{P_2}, j_5 \in S_1^{N_2}, j_6 \in S_1^{P_3}, j_7 \in S_0^{N_3}$.

The transition of Class-4 customer leaves from Queue-4 after their service completion is given by

- $(j_1, j_2, j_3, j_4, j_5, j_6, j_7) \xrightarrow{\beta_4} (j_1, j_2, j_3, j_4, j_5, j_6, j_7 - 1)$, if $j_1 \in \mathbb{Z}^+, j_2 \in S_1^{P_1}, j_3 \in S_1^{N_1}, j_4 \in S_0^{P_2}, j_5 \in S_0^{N_2}, j_6 \in S_0^{P_3}, j_7 \in S_1^{N_3}$.

The transition of Server-5 refurbishes the RP as the RFP is defined as

- $(j_1, j_2, j_3, j_4, j_5, j_6, j_7) \xrightarrow{\beta_5} (j_1, j_2, j_3, j_4 - 1, j_5, j_6 + 1, j_7)$, if $j_1 \in \mathbb{Z}^+, j_2 \in S_1^{P_1}, j_3 \in S_1^{N_1}, j_4 \in S_1^{P_2}, j_5 \in S_0^{N_2}, j_6 \in S_0^{P_3 - 1}, j_7 \in S_0^{N_3}$.

Along with these transitions, the diagonal element of the diagonal block matrices in the infinitesimal generator matrix is filled by the sum of all elements in the corresponding row of the **B** with a minus sign in order to ensure that the row sum is equal to zero; therefore, combining all the stated transitions in Section 3.2 and collecting them into their corresponding block matrices, we obtain the infinitesimal generator matrix, **B** as in (1).

3.4. Stability Analysis of the Model

Calculation of Stability Condition

According to Neuts [3] the matrix geometric approach to resolve the seven-dimensional Markov chain $J_t, t \geq 0$, we need to compute the stability condition of a proposed system. When observing the structure of infinitesimal generator matrix **B**, the block matrices B_a, B_b, and B_c are remain unaltered from state $j_1 = 1$ on wards. In such a state, we construct a generator matrix, $\mathbb{B} = B_a + B_b + B_c$, which is given by

$$\mathbb{B} = \begin{cases} \bar{D}_{j_2} & j_2' = j_2,\ j_2 \in S_1^{P_1} \\ F_2 & j_2' = j_2,\ j_2 \in S_2^{P_1} \\ F_2 & j_2' = P_1,\ j_2 = 1 \\ \mathbf{0} & \text{Otherwise.} \end{cases}$$

where $\bar{D}_{j_2} = D_{j_2} + A_1, j_2 \in S_1^{P_1}$, and $F_2 = p_{f_1}\beta_1 I_{\gamma_1}$ are used to perform the stability condition of the system. Before computing the stability condition, we require a steady-state probability vector, $\mathbf{y} = (y^{(1)}, y^{(2)}, \cdots, y^{(P_1)})$ to the generator matrix, $\mathbb{B}$. Thus the computation of $\mathbf{y}$ needs the following Lemma:

Lemma 1. *The steady-state probability vector,* $\mathbf{y}$ *to the generator matrix,* $\mathbb{B}$ *is given by*

$$y^{(j_2)} = y^{(1)}\Gamma_{j_2}, \qquad j_2 \in S_1^{P_1}, \tag{2}$$

where,

$$\Gamma_{j_2} = \begin{cases} I, & j_2 = 1 \\ (-1)^{P_1 - j_2 + 1} \prod_{j=j_2}^{P_1} (F_2 \bar{D}_j^{-1}), & j_2 \in S_2^{P_1} \end{cases}$$

and $y^{(1)}$ *is determined by solving the equations*

$$y^{(1)}[\bar{D}_1 + (-1)^{P_1 - 1} \prod_{j=2}^{P_1} (F_2 \bar{D}_j^{-1}) F_2] = \mathbf{0}, \tag{3}$$

$$\sum_{j_2=1}^{P_1} y^{(j_2)}\mathbf{e} = 1. \tag{4}$$

Proof. The steady-state probability vector $\mathbf{y}$ satisfies the following equations:

$$\mathbf{y}\mathbb{B} = \mathbf{0}, \tag{5a}$$
$$\mathbf{y}\mathbf{e} = 1. \tag{5b}$$

In Equation (5a), writing $\mathbf{y}$ and $\mathbb{B}$ explicitly and simplifying it, we obtain the P_1 set of system of homogeneous equations as follows:

$$y^{(j_2)}\bar{D}_{j_2} + y^{(j_2+1)}F_2 = \mathbf{0}, \qquad j_2 \in S_1^{P_1-1} \tag{6}$$
$$y^{(1)}F_2 + y^{(j_2)}\bar{D}_{j_2} = \mathbf{0}, \qquad j_2 = P_1. \tag{7}$$

By solving the system of Equations (6) and (7) recursively from the backward substitution method, we obtain all the steady-state probability vector $y^{(j_2)}, j_2 \in S_2^{P_1}$, in terms of initial steady-state probability vector, $y^{(1)}$ to the generator matrix, and $\mathbb{B}$ as in Equation (2). Further, to compute $y^{(1)}$, we solve the below system of equations $y^{(1)}\bar{D}_1 + y^{(2)}F_2 = \mathbf{0}$ by applying the value of $y^{(2)}$ from (2) and subject to the normalizing condition as stated in (5b), Equations (3) and (4) are obtained, respectively. □

Lemma 2. *The stability condition of the Markov process $J_t, t \geq 0$ is given by*

$$y^{(1)}\mu_1 I_{\gamma_1}\boldsymbol{e} > y^{(1)}A_1\mathbf{e}. \tag{8}$$

Proof. The stability of a proposed system is to be verified by referring Neuts [3] standard results of stability condition

$$\mathbf{y}B_a\mathbf{e} > \mathbf{y}B_c\mathbf{e}. \tag{9}$$

Applying Lemma 1 in the inequality (9) and writing all $\mathbf{y}, B_a, B_c$ and $\mathbf{e}$ explicitly, and simplifying it, the required stability condition stated in the inequality (8) is obtained. □

3.5. Calculation of R-Matrix

After verifying the stability of Markov process $J_t, t \geq 0$, the computation of rate matrix R will have a significant attention to find a steady-state probability vector, $\Omega = (\Omega^{(0)}, \Omega^{(1)}, \ldots)$ to the infinitesimal generator matrix, $\mathbf{B}$. The following lemma gives the R-matrix.

Lemma 3. *The rate matrix R of the seven-dimensional Markov process $J_t, t \geq 0$ can be determined by*

$$B_aR^2 + B_bR + B_c = \boldsymbol{0}, \tag{10}$$

where R is the minimal non-negative solution of the matrix quadratic equation defined by

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ P_1 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & \ldots & P_1 \end{array} \\ \begin{pmatrix} R^{11} & R^{12} & R^{13} & \ldots & R^{1P_1} \\ R^{21} & R^{22} & R^{23} & \ldots & R^{2P_1} \\ R^{31} & R^{32} & R^{33} & \ldots & R^{3P_1} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ R^{P_11} & R^{P_12} & R^{P_13} & \ldots & R^{P_1P_1} \end{pmatrix} \end{array} \tag{11}$$

Proof. Due to the block tridiagonal structure of the infinitesimal generator matrix, $\mathbf{B}$, the rate matrix R satisfies the matrix quadratic Equation (10). Initially, the unknown R-matrix is assumed to be in (11). Indeed, the structure of the R-matrix is identified by observing the number of non-zero rows exist in the B_c matrix. Since the B_c matrix has at-least one non-zero entry in each row, all the rows of R-matrix are to be considered as non-zero rows.

According to these assumptions, the unknown R-matrix is structured as in (11). Now, expanding all the block matrices in Equation (10), we obtain the following set of non-linear systems of homogeneous equations:
if $i = j$, and $i, j \in S_1^{P_1-1}$

$$R^{ij}D_{(j)} + A_1 + \sum_{k=1}^{P_1} R^{ik}R^{kj}p_{f_2}\beta_1 I_{\gamma_1} + \sum_{k=1}^{P_1} R^{ik}R^{kj+1}p_{f_1}\beta_1 I_{\gamma_1} = \mathbf{0}, \tag{12}$$

if $i = j$, and $i, j \in S_{P_1}^{P_1}$

$$R^{ij}D_{(j)} + A_1 + \sum_{k=1}^{P_1} R^{ik}R^{k1}p_{f_1}\beta_1 I_{\gamma_1} + \sum_{k=1}^{P_1} R^{ik}R^{kj}p_{f_2}\beta_1 I_{\gamma_1} = \mathbf{0}, \tag{13}$$

if $i \neq j$, and $i \in S_1^{P_1}$

$$\sum_{k=1}^{P_1} R^{ik}R^{kj}p_{f_2}\beta_1 I_{\gamma_1} + \sum_{k=1}^{P_1} R^{ik}R^{kj+1}p_{f_1}\beta_1 I_{\gamma_1} = \mathbf{0}, \qquad j \in S_1^{P_1-1}, \tag{14}$$

if $i \neq j$, and $i \in S_1^{P_1-1}$

$$\sum_{k=1}^{P_1} R^{ik}R^{k1}p_{f_1}\beta_1 I_{\gamma_1} + \sum_{k=1}^{P_1} R^{ik}R^{kj}p_{f_2}\beta_1 I_{\gamma_1} = \mathbf{0}, \qquad j \in S_{P_1}^{P_1}. \tag{15}$$

The obtained Equations (12)–(15) are solved by Gauss–Seidal iterative process in order to obtain each $R^{ij}, i, j \in S_1^{P_1}$ in the R-matrix. □

Remark 1. *The R-matrix can also be obtained using logarithmic reduction algorithm (LRA), which is referred to by Latouche and Ramaswamy (see [46,47]):*
Step (i): $R_1 \leftarrow -B_b^{-1}B_c, R_2 \leftarrow -B_b^{-1}B_a, R_3 = R_2$ *and* $R_4 = R_1$
Step (ii):

$$\begin{aligned} R_5 &= R_1R_2 + R_2R_1 \\ R_6 &= R_1^2 \\ R_1 &\leftarrow (I - R_5)^{-1}R_6 \\ R_6 &\leftarrow R_2^2 \\ R_2 &\leftarrow (I - R_5)^{-1}R_6 \\ R_3 &\leftarrow R_3 + R_4R_2 \\ R_4 &\leftarrow R_4R_1 \end{aligned}$$

Continue Step (i) until $\|\mathbf{e} - R_3\mathbf{e}\|_\infty < \epsilon$.
Step (iii): $R = -B_c(B_b + B_cR_3)^{-1}$.

Limiting Probability Criterion

From the infinitesimal generator matrix $\mathbf{B}$ as in Equation (1), the seven-dimensional Markov process $J_t, t \geq 0$ with the state space K is regular. Hence, the limiting probability criterion

$$\begin{aligned} \Omega^{(j_1,j_2,j_3,j_4,j_5,j_6,j_7)} = \lim_{t\to\infty} Pr[&J_1(t) = j_1, J_2(t) = j_2, J_3(t) = j_3, J_4(t) = j_4, J_5(t) = j_5, J_6(t) = j_6, \\ &J_7(t) = j_7 | J_1(0), J_2(0), J_3(0), J_4(0), J_5(0), J_6(0), J_7(0)], \end{aligned}$$

exists and it is never dependent on the initial state.

Then $\Omega = (\Omega^{(0)}, \Omega^{(1)}, \ldots,)$ preserves

$$\Omega\mathbf{B} = \mathbf{0} \tag{16a}$$

$$\text{and} \quad \Omega\mathbf{e} = 1. \tag{16b}$$

The partition of $\Omega^{(j_1)}$ is

$$\begin{aligned}
\Omega^{(j_1)} &= (\Omega^{(j_1,1)}, \Omega^{(j_1,2)}, \ldots, \Omega^{(j_1,P_1)}), j_1 \in \mathbb{Z}^+ \\
\Omega^{(j_1,j_2)} &= (\Omega^{(j_1,j_2,0)}, \Omega^{(j_1,j_2,1)}, \ldots, \Omega^{(j_1,j_2,N_1)}), j_1 \in \mathbb{Z}^+; \ j_2 \in S_1^{P_1}. \\
\Omega^{(j_1,j_2,j_3)} &= (\Omega^{(j_1,j_2,j_3,0)}, \Omega^{(j_1,j_2,j_3,1)}, \ldots, \Omega^{(j_1,j_2,j_3,P_2)}), j_1 \in \mathbb{Z}^+; \ j_2 \in S_1^{P_1}; \ j_3 \in S_0^{N_1}. \\
\Omega^{(j_1,j_2,j_3,j_4)} &= (\Omega^{(j_1,j_2,j_3,j_4,0)}, \Omega^{(j_1,j_2,j_3,j_4,1)}, \ldots, \Omega^{(j_1,j_2,j_3,j_4,N_2)}), j_1 \in \mathbb{Z}^+; \ j_2 \in S_1^{P_1}; \\
&\qquad j_3 \in S_0^{N_1}; \ j_4 \in S_0^{P_2}. \\
\Omega^{(j_1,j_2,j_3,j_4,j_5)} &= (\Omega^{(j_1,j_2,j_3,j_4,j_5,0)}, \Omega^{(j_1,j_2,j_3,j_4,j_5,1)}, \ldots, \Omega^{(j_1,j_2,j_3,j_4,j_5,P_3)}), j_1 \in \mathbb{Z}^+; \ j_2 \in S_1^{P_1}; \\
&\qquad j_3 \in S_0^{N_1}; \ j_4 \in S_0^{P_2}; \ j_5 \in S_0^{N_2}. \\
\Omega^{(j_1,j_2,j_3,j_4,j_5,j_6)} &= (\Omega^{(j_1,j_2,j_3,j_4,j_5,j_6,0)}, \Omega^{(j_1,j_2,j_3,j_4,j_5,j_6,1)}, \ldots, \Omega^{(j_1,j_2,j_3,j_4,j_5,j_6,N_3)}), j_1 \in \mathbb{Z}^+; \\
&\qquad j_2 \in S_1^{P_1}; \ j_3 \in S_0^{N_1}; \ j_4 \in S_0^{P_2}; j_5 \in S_0^{N_2}; \ j_6 \in S_0^{P_3}.
\end{aligned}$$

3.6. Calculation of Stationary Probability Vector

Theorem 1. *If the seven-dimensional Markov process $J_t, t \geq 0$ satisfies the stability condition given in Lemma 2, then the steady-state probability vector Ω is given by*

$$\Omega^{(j_1)} = \Omega^{(0)} R^{j_1}, j_1 \in \mathbb{Z}^+ \tag{17}$$

where the matrix R satisfies Equation (10) and the initial steady-state probability vector $\Omega^{(\mathbf{0})}$ satisfies

$$\Omega^{(\mathbf{0})}(B_0 + RB_a) = \mathbf{0} \tag{18a}$$

and subject to the normalizing condition

$$\Omega^{(\mathbf{0})}(I - R)^{-1}\mathbf{e} = 1. \tag{18b}$$

Proof. According to the Neuts [3] Matrix geometric approach, the steady-state probability vector Ω of the seven-dimensional Markov process $J_t, t \geq 0$ satisfies Equations (16a) and (16b). By Equation (16a), we have

$$\Omega^{(j_1)} B_0 + \Omega^{(j_1+1)} B_a = \mathbf{0}, j_1 = 0 \tag{19a}$$

$$\Omega^{(j_1)} B_c + \Omega^{(j_1+1)} B_b + \Omega^{(j_1+2)} B_a = \mathbf{0}, j_1 \in \mathbb{Z}^+. \tag{19b}$$

When observing Equations (19b) and $j_1 \in \mathbb{Z}^+$, the block matrices B_a, B_b, and B_c remain unchanged from their original structure. Thus, the steady-state probability vectors, $\Omega^{(j_1)}, j_1 \in \mathbb{Z}^+$ is dependent only on the initial steady-state probability vector, $\Omega^{(0)}$ and R-matrix where R is the minimal non-negative solution of Equation (10) from Lemma 3; therefore, Equation (17) is achieved. Then, applying the value of $\Omega^{(1)}$ in Equation (19a), we obtain Equation (18a). Further, substituting all the $\Omega^{(j_1)}, j_1 \in \mathbb{Z}^+$ values that are obtained from Equation (17) in Equation (16b), Equation (18b) is to be obtained. To compute the value of $\Omega^{(0)}$, we need the following lemma:

Lemma 4. *The initial steady-state probability vector, $\Omega^{(0)}$ is obtained by solving Equations (18a) and (18b).*

Proof. From Equation (18a) and simplifying it, we obtain the set of non-linear following equations:

$$\sum_{j_1=1}^{P_1} \Omega^{(0,j_1)}[\beta_1 I_{\gamma_1}(R^{j_1 k}p_2 + R^{j_1 k+1}p_1) + \delta_{kj_1} D_{j_1}] \quad = \quad \mathbf{0}, k \in S_1^{P_1-1} \tag{20a}$$

$$\sum_{j_1=1}^{P_1} \Omega^{(0,j_1)}[\beta_1 I_{\gamma_1}(R^{j_1 1}p_2 + R^{j_1 P_1}p_1) + \delta_{P_1 j_1} D_{P_1}] \quad = \quad \mathbf{0}. \tag{20b}$$

By solving the P_1 set of Equations (20a) and (20b) using Gauss–Seidal iterative process with subject to the normalizing condition as in (16b), we obtain the initial steady-state probability vector $\Omega^{(0)}$. □

By Lemmas 3 and 4, all the steady-state probability vector $\Omega^{(j_1)}, j_1 \in \mathbb{Z}^+$ of the seven-dimensional stochastic process $J_t, t \geq 0$ as in (17) is achieved. □

4. Expected Performance Measures of the System (EPMS)

4.1. Computation of Expected Current Number of Products in the System

- The expected current number of fresh items in the system is defined as

$$\mathbf{E_{FP}} \quad = \quad \sum_{j_1=0}^{\infty} \sum_{j_2=1}^{P_1} j_2 \Omega^{(j_1,j_2)} \mathbf{e}.$$

- The expected current number of returned items in the system is defined as

$$\mathbf{E_{RP}} \quad = \quad \sum_{j_1=0}^{\infty} \sum_{j_2=1}^{P_1} \sum_{j_3=0}^{N_1} \sum_{j_4=1}^{P_2} j_4 \Omega^{(j_1,j_2,j_3,j_4)} \mathbf{e}.$$

- The expected current number of refurbished items in the system is defined as

$$\mathbf{E_{RFP}} \quad = \quad \sum_{j_1=0}^{\infty} \sum_{j_2=1}^{P_1} \sum_{j_3=0}^{N_1} \sum_{j_4=0}^{P_2} \sum_{j_5=0}^{N_2} \sum_{j_6=1}^{P_3} \sum_{j_7=0}^{N_3} j_6 \Omega^{(j_1,j_2,j_3,j_4,j_5,j_6,j_7)}.$$

4.2. Computation of Expected Reorder Rate of Fresh Product

The expected reorder rate of fresh products can be defined as follows

$$\mathbf{E_R} \quad = \quad \sum_{j_1=1}^{\infty} \sum_{j_3=0}^{N_1} \sum_{j_4=0}^{P_2} \sum_{j_5=0}^{N_2} \sum_{j_6=0}^{P_3} \sum_{j_7=0}^{N_3} p_{f_1} \beta_1 \Omega^{(j_1,1,j_3,j_4,j_5,j_6,j_7)}.$$

4.3. Computation of Expected Number of Customers in the Queues

- Expected number of customers in Queue-1 is defined as

$$\mathbf{E_{CQ1}} \quad = \quad \sum_{j_1=1}^{\infty} j_1 \Omega^{(j_1)} \mathbf{e} = \Omega^{(0)} R(I-R)^{-2} \mathbf{e}.$$

- Expected number of customers in Queue-2 is defined as

$$\mathbf{E_{CQ2}} \quad = \quad \sum_{j_1=0}^{\infty} \sum_{j_2=1}^{P_1} \sum_{j_3=1}^{N_1} j_3 \Omega^{(j_1,j_2,j_3)} \mathbf{e}.$$

- Expected number of customers in Queue-3 is defined as

$$\mathbf{E_{CQ3}} = \sum_{j_1=0}^{\infty}\sum_{j_2=1}^{P_1}\sum_{j_3=0}^{N_1}\sum_{j_4=0}^{P_2}\sum_{j_5=1}^{N_2} j_5 \Omega^{(j_1,j_2,j_3,j_4,j_5)}\mathbf{e}.$$

- Expected number of customers in Queue-4 is defined as

$$\mathbf{E_{CQ4}} = \sum_{j_1=0}^{\infty}\sum_{j_2=1}^{P_1}\sum_{j_3=0}^{N_1}\sum_{j_4=0}^{P_2}\sum_{j_5=0}^{N_2}\sum_{j_6=0}^{P_3}\sum_{j_7=1}^{N_3} j_7 \Omega^{(j_1,j_2,j_3,j_4,j_5,j_6,j_7)}.$$

4.4. Computation of Expected Number of Lost Customers in the Queues

- Expected number of lost customers in Queue-1 is defined as

$$\mathbf{E_{LCQ1}} = \sum_{j_1=1}^{\infty} p_{f_2}\beta_1 \Omega^{(j_1)}\mathbf{e}.$$

- Expected number of lost customers in Queue-2 is defined as

$$\mathbf{E_{LCQ2}} = \sum_{j_1=0}^{\infty}\sum_{j_2=1}^{P_1}\sum_{j_3=1}^{N_1} q_{r_3}\beta_2 \Omega^{(j_1,j_2,j_3)}\mathbf{e} + \sum_{j_1=0}^{\infty}\sum_{j_2=1}^{P_1} \alpha_2 \Omega^{(j_1,j_2,N_1)}\mathbf{e}.$$

- Expected number of lost customers in Queue-3 is defined as

$$\mathbf{E_{LCQ3}} = \sum_{j_1=0}^{\infty}\sum_{j_2=1}^{P_1}\sum_{j_3=0}^{N_1}\sum_{j_4=0}^{P_2}\sum_{j_5=1}^{N_2} r_2\beta_3 \Omega^{(j_1,j_2,j_3,j_4,j_5)}\mathbf{e} + \sum_{j_1=0}^{\infty}\sum_{j_2=1}^{P_1}\sum_{j_3=0}^{N_1}\sum_{j_4=0}^{P_2} \alpha_3 \Omega^{(j_1,j_2,j_3,j_4,N_2)}\mathbf{e}.$$

- Expected number of lost customers in Queue-4 is defined as

$$\mathbf{E_{LCQ4}} = \sum_{j_1=0}^{\infty}\sum_{j_2=1}^{P_1}\sum_{j_3=0}^{N_1}\sum_{j_4=0}^{P_2}\sum_{j_5=0}^{N_2}\sum_{j_6=0}^{P_3} \alpha_4 \Omega^{(j_1,j_2,j_3,j_4,j_5,j_6,N_3)}.$$

4.5. Computation of Probability That the Servers in the System Are Busy

- Probability that Server-1 becomes busy is given by

$$\mathbf{P_{S_1B}} = \sum_{j_1=1}^{\infty}\sum_{j_2=1}^{P_1} \Omega^{(j_1,j_2)}\mathbf{e}.$$

- Probability that Server-2 becomes busy is given by

$$\mathbf{P_{S_2B}} = \sum_{j_1=0}^{\infty}\sum_{j_2=1}^{P_1}\sum_{j_3=1}^{N_1}\sum_{j_4=0}^{P_2-1} \Omega^{(j_1,j_2,j_3,j_4)}\mathbf{e}.$$

- Probability that Server-3 becomes busy is given by

$$\mathbf{P_{S_3B}} = \sum_{j_1=0}^{\infty}\sum_{j_2=1}^{P_1}\sum_{j_3=0}^{N_1}\sum_{j_4=0}^{P_2}\sum_{j_5=1}^{N_2}\sum_{j_6=1}^{P_3} \Omega^{(j_1,j_2,j_3,j_4,j_5)}\mathbf{e}.$$

- Probability that Server-4 becomes busy is given by

$$\mathbf{P_{S_4B}} = \sum_{j_1=0}^{\infty}\sum_{j_2=1}^{P_1}\sum_{j_3=0}^{N_1}\sum_{j_4=0}^{P_2}\sum_{j_5=0}^{N_2}\sum_{j_6=0}^{P_3}\sum_{j_7=1}^{N_3} \Omega^{(j_1,j_2,j_3,j_4,j_5 j_6,j_7)}.$$

- Probability that Server-5 becomes busy is given by

$$\mathbf{P_{S_5B}} = \sum_{j_1=0}^{\infty}\sum_{j_2=1}^{P_1}\sum_{j_3=0}^{N_1}\sum_{j_4=1}^{P_2}\sum_{j_5=0}^{N_2}\sum_{j_6=0}^{P_3-1}\sum_{j_7=0}^{N_3}\Omega^{(j_1,j_2,j_3,j_4,j_5j_6,j_7)}.$$

4.6. Computation of Miscellaneous Expected Measures of the System

- Expected rate at which the Class-1 customer who did not purchase an FP is defined as

$$\mathbf{E_{CNFP}} = \sum_{j_1=1}^{\infty}\sum_{j_2=1}^{P_1} p_{f_2}\beta_1\Omega^{(j_1,j_2)}\mathbf{e}.$$

- Expected rate at which the Class-2 customer who only sold an OP and does not go to Queue-1 is defined as

$$\mathbf{E_{COSRP}} = \sum_{j_1=0}^{\infty}\sum_{j_2=1}^{P_1}\sum_{j_3=1}^{N_1}\sum_{j_4=0}^{P_2-1} q_{r_2}\beta_2\Omega^{(j_1,j_2,j_3)}\mathbf{e}.$$

- Expected rate at which the Class-2 customer who did not return the OP is defined as

$$\mathbf{E_{CNRP}} = \sum_{j_1=0}^{\infty}\sum_{j_2=1}^{P_1}\sum_{j_3=1}^{N_1}\sum_{j_4=0}^{P_2-1} q_{r_3}\beta_2\Omega^{(j_1,j_2,j_3)}\mathbf{e}.$$

- Expected rate at which the Class-3 customer who purchases an RFP is defined as

$$\mathbf{E_{PRFP}} = \sum_{j_1=0}^{\infty}\sum_{j_2=1}^{P_1}\sum_{j_3=0}^{N_1}\sum_{j_4=0}^{P_2}\sum_{j_5=1}^{N_2}\sum_{j_6=1}^{P_3} r_1\beta_3\Omega^{(j_1,j_2,j_3,j_4,j_5,j_6)}\mathbf{e}.$$

4.7. Computation of Expected Total Cost Value

The total cost value (TCV) of the seven-dimensional stochastic process, $J_t, t \geq 0$, is defined as given below.

$$\begin{aligned} E_{TCV} = {} & HC_{FP}E_{FP} + HC_{RP}E_{RP} + HC_{RFP}E_{RFP} + SC_{FP}E_R + WC_{Q1}E_{CQ1} + WC_{Q2}E_{CQ2} \\ & + WC_{Q3}E_{CQ3} + WC_{Q4}E_{CQ4} + LC_{Q1}E_{LCQ1} + LC_{Q2}E_{LCQ2} + LC_{Q3}E_{LCQ3} + \\ & LC_{Q4}E_{LCQ4}. \end{aligned}$$

5. Numerical Interpretation of Parameter Analysis of the System

The considered seven-dimensional stochastic queuing-inventory problem is to be investigated with a few numerical illustrations using the cost values and parameter values of the system. This section provide insights to the reader about the proposed model and its practical life application related to the society. Here, we conduct an investigation into each queue, product, server, and customer, and, of course, the expected total cost of the system. The discussion of each queue explains how the service provided by the server and the respective probabilities influence the total cost of the system and significant system performance measures. For the interpretation of numerical discussions, the parameter and cost values of the Markov process, $J_t, t \geq 0$ are to be assumed as follows: $P_1 = 3$, $P_2 = 3$, $P_3 = 4, N_1 = 3, N_2 = 3, N_3 = 3, \alpha_1 = 0.5, \alpha_2 = 0.02, \alpha_3 = 0.01, \alpha_4 = 0.8, \beta_1 = 3.7$, $\beta_2 = 2.5$, $\beta_3 = 2.7, \beta_4 = 1.6, \beta_5 = 1.9, p_{f_1} = 0.8, p_{f_2} = 0.2; q_{r_1} = 0.7; q_{r_2} = 0.2, q_{r_3} = 0.1$, $r_1 = 0.7$, $r_2 = 0.3, HC_{FP} = 0.2, HC_{RP} = 0.2, HC_{RFP} = 0.2, SC_{FP} = 10, WC_{Q1} = 5$, $WC_{Q2} = 5$, $WC_{Q3} = 5, WC_{Q4} = 5, LC_{Q1} = 0.5, LC_{Q2} = 0.2, LC_{Q3} = 0.2, LC_{Q4} = 0.1$.

5.1. Interpretation and Remarks on Queues

Example 1. *Using this example, one can see how the activity on Queue-1 and Server-1 is represented graphically. In Queue-1, the system accepts Class-1 customers who wish to purchase the FP. Class-2 customers are also taken into consideration when they wish to purchase the FP after returning their old product in Queue 2.*

1. *Figure 3 depicts the predicted number of customers lost in Queue-1 as a result of increasing both the service rate, β_1, and the probability that customers will be satisfied at the end of service completion to purchase the FP, p_{f_1}, with Server-1.*
2. *As the average service time of Server-1 decreases, the value of E_{LCQ1} rises as well. In general, when the service completion time decreases, the number of lost customers or the total number of customers in the system decreases logically as well. According to the premise that an entering Class-1 customer is a member of the impulse customer category, the likelihood that they will acquire the product is dependent on their level of satisfaction with the service offered to them.*
3. *As a result, each service completion has the option of purchasing or not purchasing the FP. Because of the customer's decision to oscillate, the value of E_{LCQ1} increases when the value of β_1 increases. It is interesting to note that when the p_{f_1} is increased, (i.e., when the E_{LCQ1} is reduced), the situation is different. Indeed, the likelihood that a Class-1 consumer will be satisfied suggests that the supply of fresh products will increase.*
4. *As predicted, an increase in the arrival rate always resulted in Server-1 remaining busy when the rate was increased. In order for the server to be busy, there should be a large number of customers in front of the server at any given time. On the other hand, because Server-1 completes the operation as fast as possible, the likelihood of a server becoming available increases. This is shown in Figure 4.*
5. *As opposed to this, Figure 5 depicts an expected rate at which a customer who does not purchase an FP when the p_{f_2} and the α_1 are put together. It has been shown that when p_{f_2} and α_1 are elevated concurrently, they have a direct impact on E_{CNFP}.*

In order to be a successful business person, one must always improve their service facilities in order to ensure that their customers are completely satisfied.

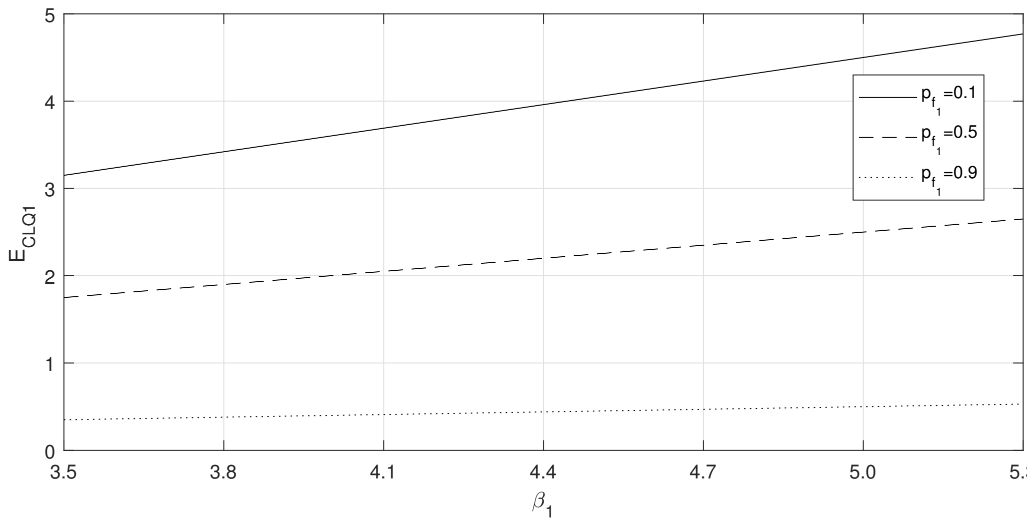

Figure 3. Expected customers lost in Queue-1 on p_{f_1} vs. β_1.

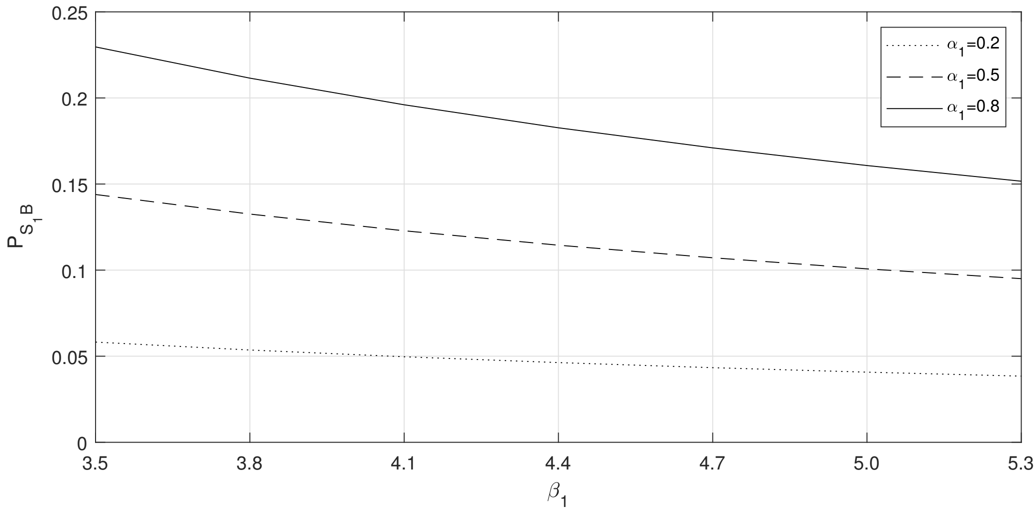

Figure 4. Probability that Server-1 is busy on α_1 vs. β_1.

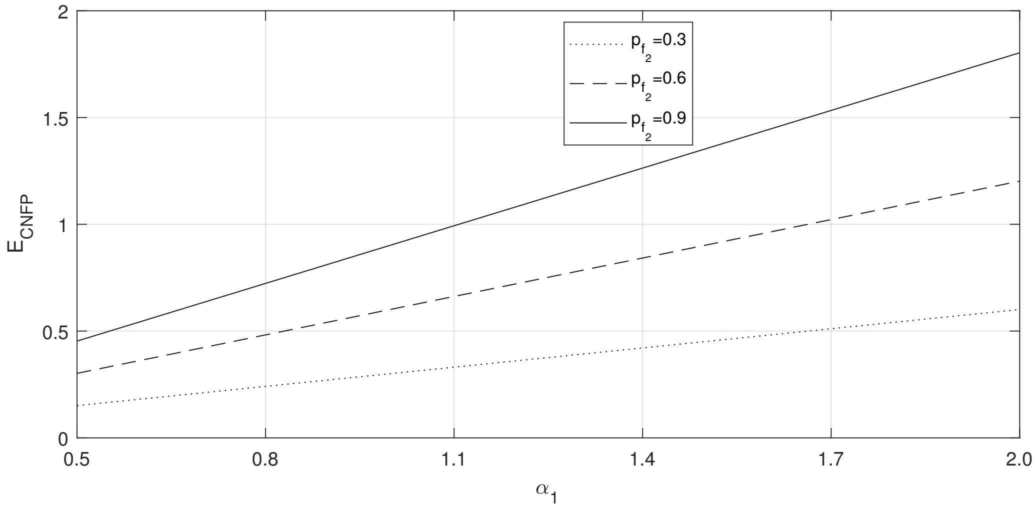

Figure 5. Expected rate at which the customer not purchasing an FP on p_{f_1} vs. α_1.

Example 2. *Table 1 shows the complete in and out activities of Queue-2. In this example, we see the influence of parameters* α_2 *and* β_2 *and the probabilities* $q_{r_1}, q_{r_2}, and q_{r_3}$ *on the* E_{RP}, E_{CQ1}, E_{COSRP}, E_{CNRP}, *and* P_{S_2F}.

1. *If the arrival rate,* α_2 *increases (meaning that the number of Class-2 customers in Queue-2 has increased), the expected number of RP, average arrivals in Queue-1, and expected rate at which a customer who sells only the OP and leaves the system without returning are increased. The probability of Server-2 being free is decreased if* α_2 *is increased, because all the other components (*E_{RP}, E_{CQ1}, E_{COSRP}, E_{CNRP}*) are increased when* α_2 *increases.*
2. *The assumption defined for purchasing the old products from the customer causes the stated changes. This is because when Server-2 is attending a customer in Queue-2, the TaC of the old product is first checked and clearly explained to the customer. Finally, all the checking formalities are over, the customer may agree to the TaC. If they agree to the sale of the old product, then the old product is immediately purchased by Server-2. In this situation, the Class-2 customer may decide whether to buy an FP or not. Suppose they want an FP, immediately they go to Queue-1 with probability* q_{r1} *or else leave the system with* q_{r2}. *If the Class-2 customer is not willing to sell their old product, they leave the system without returning the old product with probability* q_{r3}. *These are the reasons that the following changes happen:*
   - *If the probability* q_{r1} *increases, then* E_{CQ1} *increases where as* E_{CNRP}, P_{S_2F} *are decreased.*

- *If the probability q_{r2} increases, then E_{COSRP} increases where as E_{CNRP}, P_{S_2F} are decreased.*
- *If the probability q_{r3} decreases, then E_{CNRP}, P_{S_2F} are decreased.*

3. *The service rate of Server-2, β_2 also causes the increase in the following measures E_{CQ1}, E_{COSRP}, E_{CNRP}, P_{S_2F}. As the service rate of Server-2 increases, the probability of Server-2 being free is increased. When varying the probabilities in Table 1 q_{r_1}, q_{r_2} and q_{r_3}, which is selected by the customers' own choice will have a great impact on the E_{RP}, E_{CQ1}, E_{COSRP}, E_{CNRP}, and P_{S_2F} as we predicted.*
4. *Furthermore, Figure 6 depicts that Server-5's is busy when α_2 vs. β_2. The job of Server-5 is to recondition the returned OP into an RFP to sell it. To do so, Server-5 needs enough RPs in the storage place, or else Server-5 become free. When both parameters α_2 and β_2 are increased, the number of RP increases; as such, P_{S_5B}s length of the busy period will increase.*

From this interpretation, the reader can conclude that the sales of new products when buying the customers old products will be the new and best business approach to increase the sales of new products.

Table 1. Interpretation of parameters on Queue-2.

α_2	q_{r_1}	q_{r_2}	q_{r_3}	β_2	E_{RP}	E_{CQ1}	E_{COSRP}	E_{CNRP}	P_{S_2F}
0.02	0.4	0.3	0.3	2.5	0.963864	0.611457	0.458592	0.458592	0.388543
				2.8	0.963943	0.684484	0.513363	0.513363	0.388854
				3.1	0.964008	0.757510	0.568132	0.568132	0.389105
		0.4	0.2	2.5	0.963059	0.733650	0.733650	0.366825	0.26635
				2.8	0.963156	0.821505	0.821505	0.410752	0.266514
				3.1	0.963234	0.909359	0.909359	0.454680	0.266646
	0.5	0.3	0.2	2.5	0.966208	0.865863	0.519518	0.346345	0.307309
				2.8	0.966324	0.969414	0.581649	0.387766	0.307561
				3.1	0.966418	1.072964	0.643779	0.429186	0.307765
		0.4	0.1	2.5	0.951715	0.995803	0.796642	0.199161	0.203358
				2.8	0.951819	1.115138	0.892111	0.223028	0.203473
				3.1	0.951903	1.234473	0.987578	0.246895	0.203566
0.05	0.4	0.3	0.3	2.5	0.982526	0.893242	0.669932	0.669932	0.106758
				2.8	0.982736	1.000199	0.750149	0.750149	0.106965
				3.1	0.982906	1.107155	0.830366	0.830366	0.107133
		0.4	0.2	2.5	0.966112	0.935305	0.935305	0.467652	0.064695
				2.8	0.966270	1.047433	1.047433	0.523716	0.064792
				3.1	0.966398	1.159560	1.159560	0.579780	0.064871
	0.5	0.3	0.2	2.5	0.971655	1.151891	0.691135	0.460756	0.078487
				2.8	0.971866	1.289898	0.773939	0.515959	0.078645
				3.1	0.972036	1.427903	0.856742	0.571161	0.078772
		0.4	0.1	2.5	0.956308	1.195612	0.956490	0.239122	0.04351
				2.8	0.956462	1.339005	1.071204	0.267801	0.043568
				3.1	0.956587	1.482398	1.185918	0.296480	0.043614

Table 1. *Cont.*

α_2	q_{r_1}	q_{r_2}	q_{r_3}	β_2	E_{RP}	E_{CQ1}	E_{COSRP}	E_{CNRP}	P_{S_2F}
0.08	0.4	0.3	0.3	2.5	0.983992	0.941741	0.706306	0.706306	0.058259
				2.8	0.984222	1.054541	0.790906	0.790906	0.058445
				3.1	0.984409	1.167340	0.875505	0.875505	0.058596
		0.4	0.2	2.5	0.973038	0.966402	0.966402	0.483201	0.033598
				2.8	0.973210	1.082275	1.082275	0.541137	0.033683
				3.1	0.973349	1.198146	1.198146	0.599073	0.033753
	0.5	0.3	0.2	2.5	0.976472	1.197981	0.718789	0.479193	0.041615
				2.8	0.976702	1.341544	0.804926	0.536618	0.041754
				3.1	0.976888	1.485105	0.891063	0.594042	0.041868
		0.4	0.1	2.5	0.966819	1.223582	0.978865	0.244716	0.021135
				2.8	0.966989	1.370343	1.096275	0.274069	0.021183
				3.1	0.967125	1.517105	1.213684	0.303421	0.021223

Example 3. *This illustration provides the complete analysis of Class-3 customers in Queue-3 with response to Server-3 and Server-5 according to result obtained in Table 2. To explore the actions that occur when the Class-3 customer enters Queue-3 until they were about to leave it, the parameters* α_3*,* β_3*,* β_5*, and* p_{rf_1} *are incorporated into Table 2, which shows the average number of Class-3 customers present and lost in Queue-3.*

1. *First, the arrival rate of Class-3 customers is always directly proportional to* E_{CQ3} *and* E_{LCQ3} *where is the service rate, which is always inversely proportional to* E_{CQ3} *and* E_{LCQ3}*. This is because the number of existing customers in Queue-3 is increased when* α_3 *increases. Since the size of Queue-3 is finite and current number of Class-3 customer increases, newly arrived Class-3 customers are considered lost—this is why the arrival rate,* α_3*, causes the increase in the loss of Class-3 customers when it is increased.*
2. *Simultaneously, the service process of Server-3 will have a crucial role in controlling the loss of Class-3 customers. As the average service time of Server-3 reduces, the number of present and lost customers also reduced.*
3. *The contribution of Server-5 is a remarkable one to determine the* E_{CQ3} *and* E_{LCQ3} *because Server-5 continuously performs the refurbished work on the RP if it is available. Suppose the refurbished products are not available, the Class-3 customer has to wait in Queue-3 and at one stage they will be lost. So, the mean service time of Server-3 causes the decrease in* E_{CQ3} *and* E_{LCQ3} *when it is decreased—this means that the sales of RFP is increased.*
4. *Finally, the probability of a Class-3 customer buying the RFP or not also determines the* E_{CQ3} *and* E_{LCQ3}*. This reflects the exact real-life application of a customer's mindset. Generally, not all customers want to purchase the RFP. So many customers will have an oscillation mindset when they buy an RFP; therefore, when a Class-3 customer purchasing probability increases, the loss of them is to be reduced.*
5. *Figure 7 explores the probability of Server-3 becoming busy when* α_3 *and* β_3 *are incorporated. As the average service time reduces,* P_{S_3B} *also reduces because of the quick service completion, whereas the increase in the number of customers in Queue-3 raises, and the server being busy time is increased.*

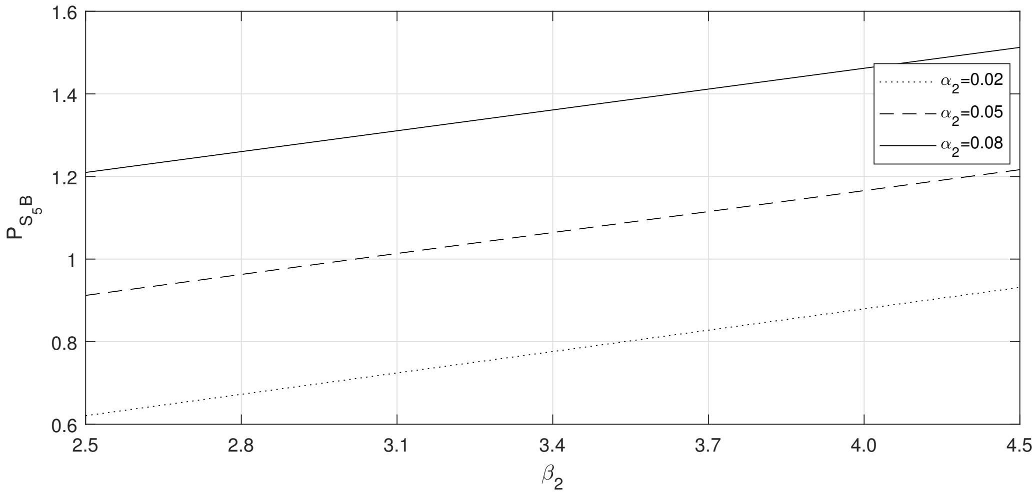

Figure 6. Probability that Server-5 is busy on α_2 vs. β_2.

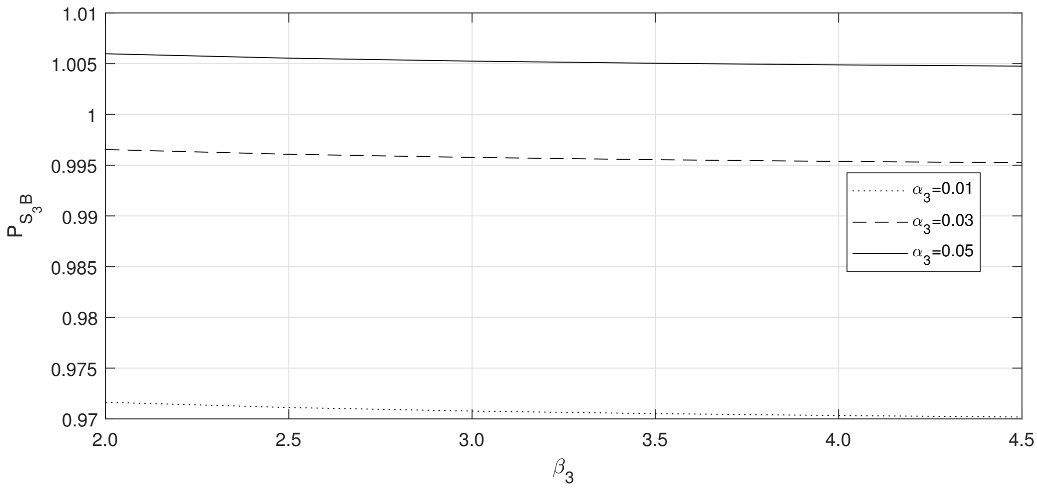

Figure 7. Probability that Server-3 is busy on α_3 vs. β_3.

Table 2. Interpretation of parameters on Queue-3.

			E_{CQ3}			E_{LCQ3}		
β_5	β_3	p_{rf_1}	$\alpha_3 = 0.1$	$\alpha_3 = 0.2$	$\alpha_3 = 0.3$	$\alpha_3 = 0.1$	$\alpha_3 = 0.2$	$\alpha_3 = 0.3$
1	2.5	0.2	1.069820	1.245375	1.329015	1.572298	1.812185	1.944769
		0.4	1.270535	1.389019	1.427365	1.360707	1.498965	1.577249
		0.6	1.377906	1.443120	1.462050	0.977132	1.059310	1.118390
	2.8	0.2	1.068205	1.243361	1.327096	1.756267	2.019233	2.162050
		0.4	1.269631	1.388214	1.426629	1.519199	1.668268	1.750071
		0.6	1.377481	1.442759	1.461722	1.089535	1.175675	1.235914
	3.1	0.2	1.066895	1.241715	1.325517	1.940227	2.226255	2.379294
		0.4	1.268897	1.387555	1.426024	1.677687	1.837563	1.922880
		0.6	1.377135	1.442463	1.461452	1.201936	1.292037	1.353434

Table 2. *Cont.*

			E_{CQ3}			E_{LCQ3}		
β_5	β_3	p_{rf_1}	$\alpha_3 = 0.1$	$\alpha_3 = 0.2$	$\alpha_3 = 0.3$	$\alpha_3 = 0.1$	$\alpha_3 = 0.2$	$\alpha_3 = 0.3$
1.2	2.5	0.2	1.068865	1.244698	1.328570	1.571344	1.811443	1.944249
		0.4	1.269938	1.388746	1.427194	1.360228	1.498724	1.577086
		0.6	1.377606	1.442992	1.461969	0.976958	1.059223	1.118327
	2.8	0.2	1.067245	1.242680	1.326647	1.755200	2.018405	2.161472
		0.4	1.269033	1.387939	1.426457	1.518666	1.668002	1.749892
		0.6	1.377180	1.442629	1.461641	1.089343	1.175580	1.235846
	3.1	0.2	1.065931	1.241029	1.325063	1.939046	2.225342	2.378659
		0.4	1.268296	1.387278	1.425850	1.677099	1.837272	1.922685
		0.6	1.376834	1.442333	1.461370	1.201726	1.291935	1.353362
1.4	2.5	0.2	1.068181	1.244214	1.328252	1.570661	1.810911	1.943876
		0.4	1.269512	1.388550	1.427072	1.359886	1.498552	1.576970
		0.6	1.377392	1.442900	1.461912	0.976834	1.059161	1.118282
	2.8	0.2	1.066558	1.242191	1.326324	1.754435	2.017812	2.161058
		0.4	1.268604	1.387741	1.426334	1.518285	1.667812	1.749763
		0.6	1.376965	1.442537	1.461582	1.089206	1.175512	1.235798
	3.1	0.2	1.065241	1.240538	1.324738	1.938199	2.224687	2.378203
		0.4	1.267867	1.387079	1.425725	1.676679	1.837063	1.922545
		0.6	1.376618	1.442240	1.461311	1.201576	1.291861	1.353310

Example 4. *This example graphically investigates the activities of Queue-4.*

1. *Figure 8 shows the expected loss of Class-4 customers in Queue-4 when both α_4 and β_4 varied together. As the results show, the reader can understand that both parameters influence E_{LCQ4} opposite to each other as predicted. In Figure 9, the size of Queue-4, N_3 is incorporated with α_4 to obtain E_{LCQ4}, whereas in Figure 10, N_3 is connected to β_4.*
2. *Since α_4 causes the increase in customer arrivals in Queue-4, the overflow of Queue-4 leads to the loss of them. On the other side, as β_4 reduces the wait time of Class-4 customers in Queue-4, the loss will be controlled and as N_3 expands the size of Queue-4, a greater number of Class-4 customers can be allowed in Queue-4; thus, the loss of Class-4 customers can be reduced.*
3. *Figure 11 depicts the number of customers, E_{CQ4} in Queue-4 when α_4 and β_4 act together. The measure E_{LCQ4} is increased if α_4 is raised and is decreased if β_4 is decreased due to the influence of corresponding parameters.*
4. *The parameters α_4, β_4, and N_3 are involved to discuss the probability of Server-4 becoming busy, P_{S_4B}. In this analysis, α_4 and N_3 always keeps Server-4 busy when it is increased; however, β_4 always reduces the busy period of Server-4 when it is to be increased. This is graphically shown in Figures 12–14.*
5. *Suppose there are a greater number of Class-4 customers waiting for repair service of their product, Server-4 cannot take a rest because if Server-4 wants to take a rest, the Class-4 customers in Queue-4 will increase. This will also cause an overload of work for Server-4; this is why Server-4 is always busy when both α_4 and N_3 are increased. On the contrary, as we reduce the mean service time of Server-4, the number of Class-4 customers in Queue-4 also decreased; thus, the probability of Server-4 being busy is low when β_4 is high.*

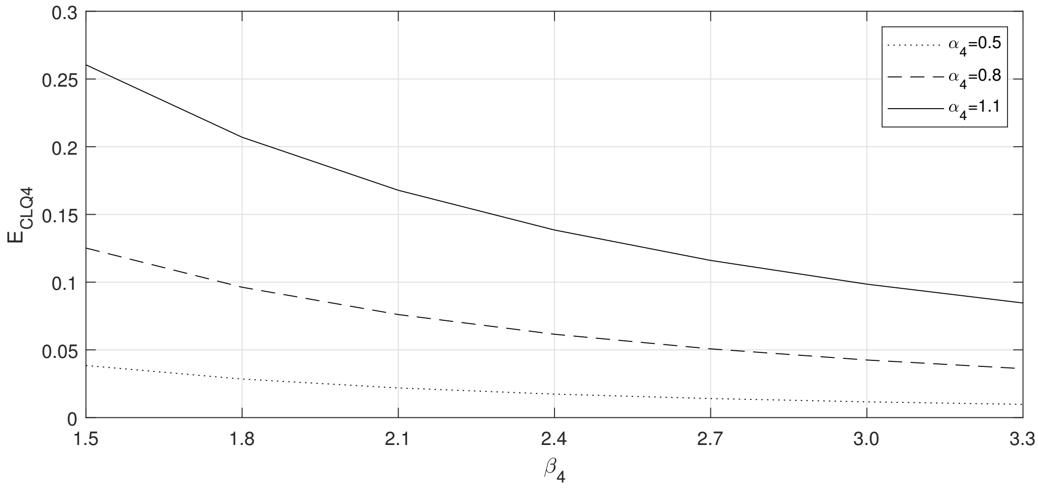

Figure 8. Expected customer loss in Queue-4 on β_4 vs. α_4.

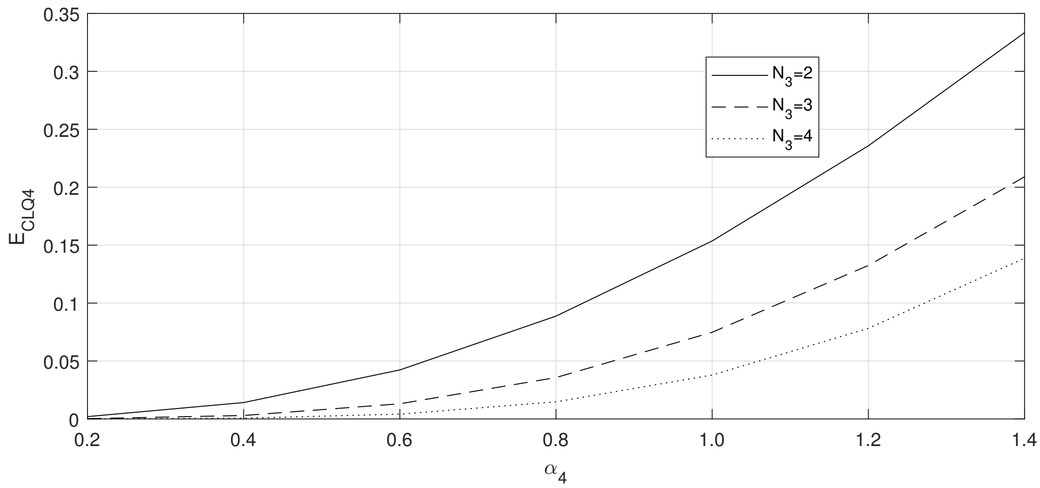

Figure 9. Expected customer loss in Queue-4 on α_4 vs. N_3.

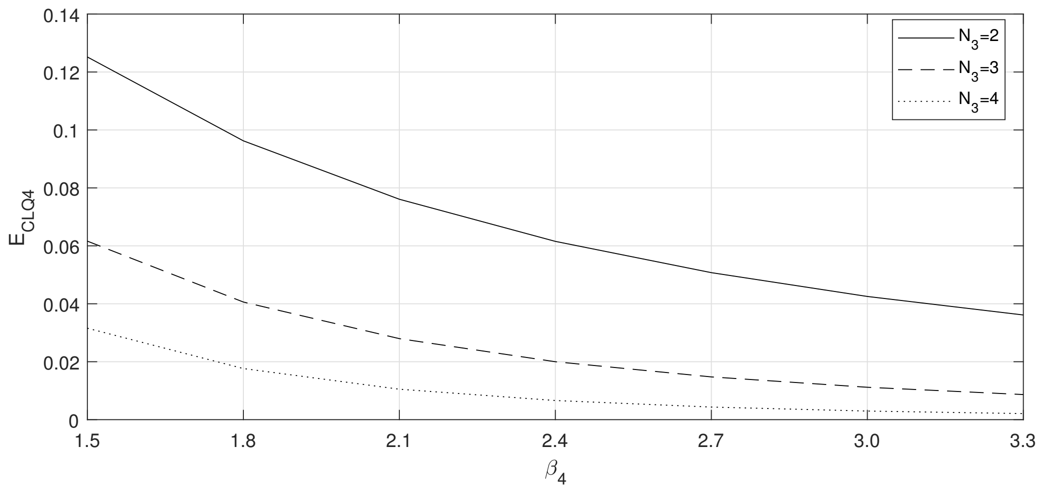

Figure 10. Expected customer loss in Queue-4 on β_4 vs. N_3.

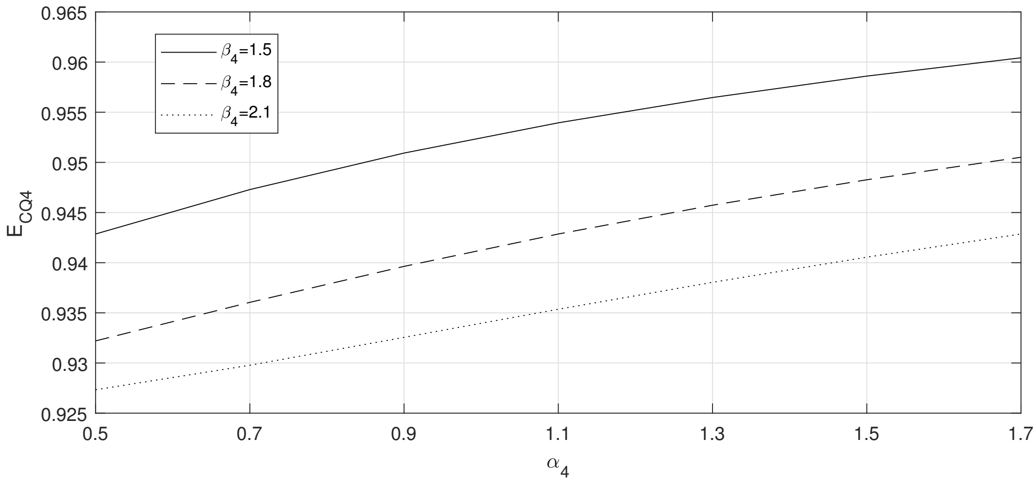

Figure 11. Expected customer loss in Queue-4 on α_4 vs. β_4.

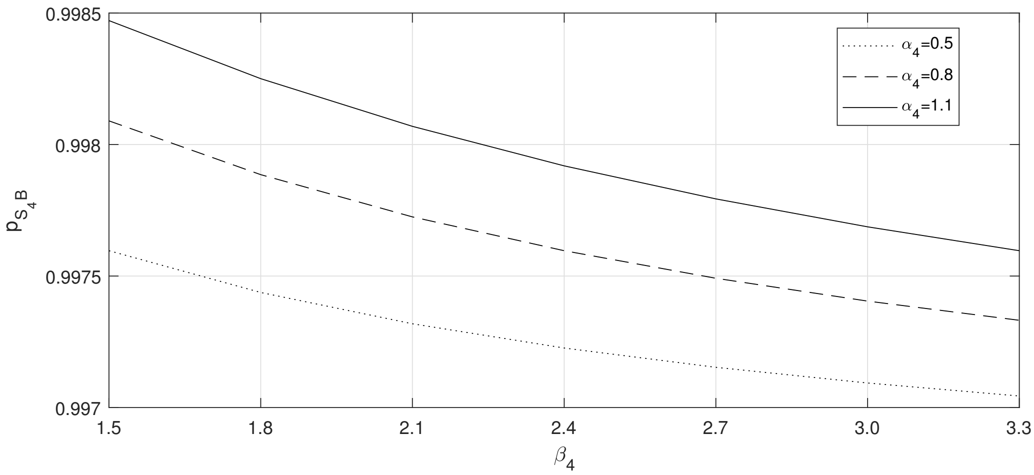

Figure 12. Probability that Server-4 is busy on α_4 vs. β_4.

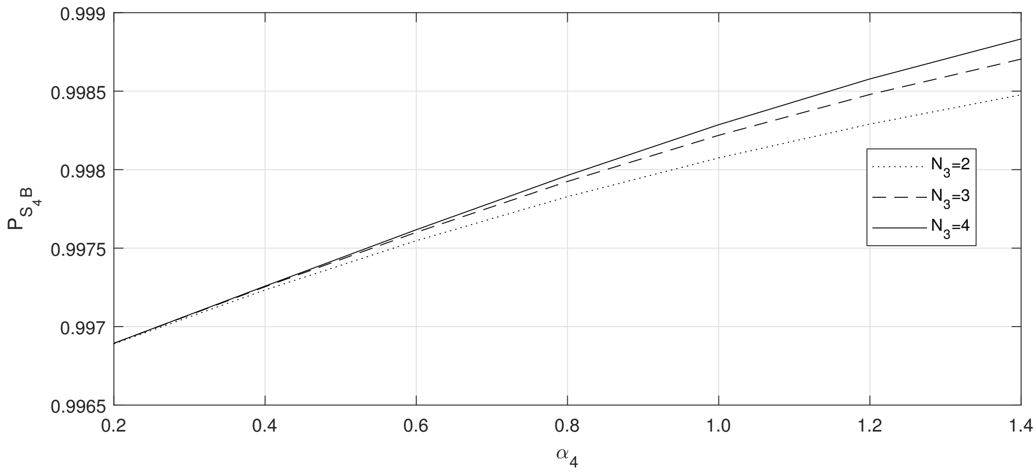

Figure 13. Probability that Server-4 is busy on α_4 vs. N_3.

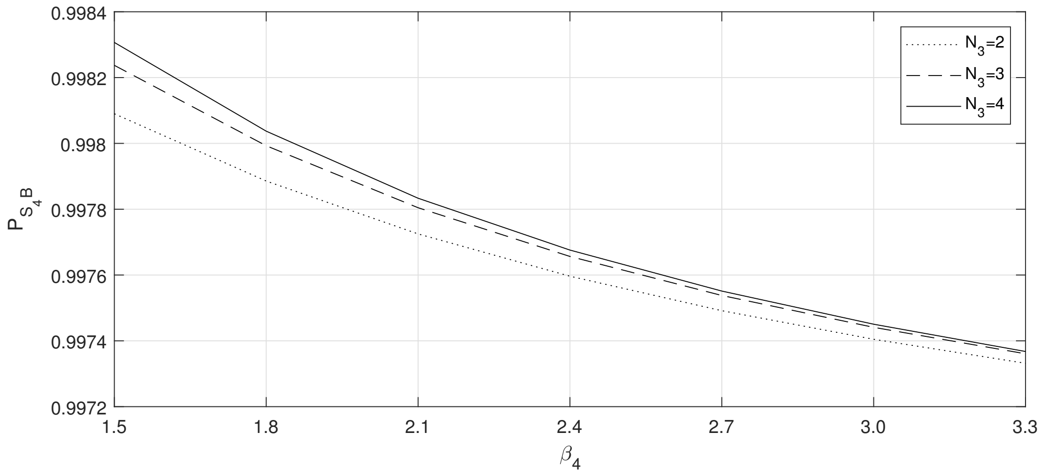

Figure 14. Probability that Server-4 is busy on β_4 vs. N_3.

5.2. Interpretation of Expected Total Cost Value of the System

Example 5. *For the purposes of this example, the expected total cost of the considered seven-dimensional stochastic process is to be explored, along with α_1, α_2, α_3, α_4, β_1, β_2, β_3, β_4, and β_5 as shown in Table 3.*

1. *Of course, when dealing with this type of business in the real world, we all experience some degree of ambiguity regarding the typical total cost.*
2. *This example will be extremely beneficial to all businesses in order to eliminate such ambiguity; however, despite the fact that the system consists of five heterogeneous servers, the average service time of each server is inversely related to predicted total cost (i.e., for each service rate of β_1 and β_i, where $i \in S_3^5$ grows, the expected total cost reduces) but for β_2 it increases because Server-2 performs a purchasing job—this will cause an increase in total cost.*
3. *As predicted, when we observe the mean arrival rate of all Class-i customers, where $i \in S_1^4$, the projected total cost is directly proportional to the number of α_i customers, where $i \in S_1^4$.*
4. *Furthermore, the cost value analysis provides the predicted rise in the expected total cost value, which is presented in Table 4. Many readers will be inspired by this example to conduct further investigation into this type of topic in the future. Providing a satisfactory service to all types of consumers under one QIS with a variety of dedicated servers is a difficult undertaking. The total cost incurred by our study, on the other hand, will provide valuable information to many readers and business people.*
5. *This is the most significant and necessary conversation in this proposed paradigm, and it should not be skipped.*

Table 3. Interpretation of parameters on E_{TCV}.

				α_1	**0.4**							
				α_2	**0.02**				**0.04**			
				α_3	**0.01**		**0.03**		**0.01**		**0.03**	
				α_4	**0.7**	**0.9**	**0.7**	**0.9**	**0.7**	**0.9**	**0.7**	**0.9**
β_1	β_2	β_3	β_4	β_5								
3.5	2.3	2.4	1.5	1.8	14.3814	14.4014	14.6691	14.7118	13.7803	13.7960	14.7620	14.7867
				2	14.3742	14.3939	14.6546	14.6938	13.7749	13.7903	14.7531	14.7771
			1.8	1.8	14.3813	14.4013	14.6673	14.7099	13.7803	13.7960	14.7611	14.7859
				2	14.3741	14.3938	14.6527	14.6920	13.7749	13.7902	14.7522	14.7762
		2.7	1.5	1.8	14.3724	14.3924	14.6621	14.7047	13.7715	13.7872	14.7535	14.7782
				2	14.3652	14.3849	14.6476	14.6867	13.7661	13.7815	14.7447	14.7686
			1.8	1.8	14.3723	14.3923	14.6602	14.7028	13.7715	13.7872	14.7527	14.7774
				2	14.3651	14.3848	14.6457	14.6849	13.7661	13.7814	14.7438	14.7677
	2.5	2.4	1.5	1.8	14.3844	14.4044	14.6683	14.7110	13.7833	13.7990	14.7626	14.7873
				2	14.3772	14.3969	14.6538	14.6930	13.7779	13.7933	14.7537	14.7777
			1.8	1.8	14.3843	14.4043	14.6664	14.7091	13.7832	13.7990	14.7617	14.7865
				2	14.3771	14.3968	14.6519	14.6911	13.7779	13.7932	14.7528	14.7768
		2.7	1.5	1.8	14.3754	14.3953	14.6613	14.7039	13.7745	13.7902	14.7541	14.7788
				2	14.3682	14.3878	14.6468	14.6859	13.7691	13.7844	14.7453	14.7692
			1.8	1.8	14.3753	14.3952	14.6594	14.7020	13.7744	13.7902	14.7533	14.7780
				2	14.3681	14.3877	14.6449	14.6840	13.7691	13.7844	14.7444	14.7683
3.5	2.3	2.4	1.5	1.8	14.3650	14.3849	14.6503	14.6929	13.7638	13.7796	14.7419	14.7666
				2	14.3578	14.3774	14.6358	14.6750	13.7584	13.7738	14.7330	14.7570
			1.8	1.8	14.3649	14.3848	14.6484	14.6911	13.7638	13.7795	14.7410	14.7657
				2	14.3576	14.3773	14.6339	14.6731	13.7584	13.7738	14.7321	14.7561
		2.7	1.5	1.8	14.3560	14.3759	14.6432	14.6858	13.7550	13.7708	14.7334	14.7581
				2	14.3488	14.3684	14.6287	14.6679	13.7496	13.7650	14.7245	14.7485
			1.8	1.8	14.3558	14.3758	14.6413	14.6839	13.7550	13.7707	14.7325	14.7572
				2	14.3486	14.3683	14.6268	14.6660	13.7496	13.7650	14.7236	14.7476
	2.5	2.4	1.5	1.8	14.3679	14.3879	14.6494	14.6921	13.7668	13.7825	14.7425	14.7672
				2	14.3607	14.3804	14.6349	14.6741	13.7614	13.7768	14.7336	14.7576
			1.8	1.8	14.3678	14.3878	14.6475	14.6902	13.7668	13.7825	14.7416	14.7663
				2	14.3606	14.3803	14.6330	14.6722	13.7614	13.7768	14.7327	14.7567
		2.7	1.5	1.8	14.3589	14.3789	14.6424	14.6850	13.7580	13.7737	14.7340	14.7587
				2	14.3517	14.3714	14.6279	14.6670	13.7526	13.7680	14.7251	14.7491
			1.8	1.8	14.3588	14.3788	14.6405	14.6831	13.7580	13.7737	14.7331	14.7578
				2	14.3516	14.3713	14.6260	14.6652	13.7526	13.7679	14.7242	14.7482

Table 3. *Cont.*

				α_1	0.6							
				α_2	0.02				0.04			
				α_3	0.01		0.03		0.01		0.03	
				α_4	0.7	0.9	0.7	0.9	0.7	0.9	0.7	0.9
β_1	β_2	β_3	β_4	β_5								
3.5	2.3	2.4	1.5	1.8	15.4725	15.4925	15.7625	15.8051	14.8714	14.8871	15.8565	15.8812
				2	15.4653	15.4850	15.7479	15.7871	14.8660	14.8813	15.8476	15.8716
			1.8	1.8	15.4724	15.4924	15.7606	15.8032	14.8713	14.8871	15.8556	15.8804
				2	15.4652	15.4848	15.7460	15.7853	14.8659	14.8813	15.8467	15.8707
		2.7	1.5	1.8	15.4635	15.4834	15.7554	15.7980	14.8626	14.8783	15.8480	15.8727
				2	15.4563	15.4759	15.7409	15.7800	14.8572	14.8725	15.8391	15.8631
			1.8	1.8	15.4634	15.4833	15.7535	15.7961	14.8625	14.8783	15.8471	15.8719
				2	15.4562	15.4758	15.7390	15.7782	14.8571	14.8725	15.8382	15.8622
	2.5	2.4	1.5	1.8	15.4755	15.4954	15.7616	15.8043	14.8743	14.8901	15.8571	15.8818
				2	15.4683	15.4879	15.7471	15.7863	14.8690	14.8843	15.8482	15.8722
			1.8	1.8	15.4753	15.4953	15.7597	15.8024	14.8743	14.8900	15.8562	15.8810
				2	15.4681	15.4878	15.7452	15.7844	14.8689	14.8843	15.8473	15.8713
		2.7	1.5	1.8	15.4664	15.4864	15.7546	15.7972	14.8655	14.8813	15.8486	15.8733
				2	15.4592	15.4789	15.7401	15.7792	14.8602	14.8755	15.8397	15.8637
			1.8	1.8	15.4663	15.4863	15.7527	15.7953	14.8655	14.8812	15.8477	15.8725
				2	15.4591	15.4788	15.7382	15.7773	14.8601	14.8755	15.8389	15.8628
3.5	2.3	2.4	1.5	1.8	15.3930	15.4130	15.6799	15.7226	14.7919	14.8076	15.7724	15.7971
				2	15.3858	15.4055	15.6654	15.7046	14.7865	14.8019	15.7635	15.7875
			1.8	1.8	15.3929	15.4129	15.6780	15.7207	14.7918	14.8076	15.7715	15.7963
				2	15.3857	15.4054	15.6635	15.7027	14.7864	14.8018	15.7626	15.7866
		2.7	1.5	1.8	15.3840	15.4040	15.6729	15.7155	14.7831	14.7988	15.7639	15.7886
				2	15.3768	15.3964	15.6584	15.6975	14.7777	14.7930	15.7550	15.7790
			1.8	1.8	15.3839	15.4038	15.6710	15.7136	14.7830	14.7988	15.7630	15.7877
				2	15.3767	15.3963	15.6565	15.6956	14.7776	14.7930	15.7541	15.7781
	2.5	2.4	1.5	1.8	15.3960	15.4160	15.6791	15.7218	14.7949	14.8106	15.7730	15.7977
				2	15.3888	15.4085	15.6646	15.7038	14.7895	14.8048	15.7641	15.7881
			1.8	1.8	15.3959	15.4159	15.6772	15.7199	14.7948	14.8106	15.7721	15.7969
				2	15.3887	15.4083	15.6627	15.7019	14.7894	14.8048	15.7632	15.7872
		2.7	1.5	1.8	15.3870	15.4069	15.6721	15.7146	14.7861	14.8018	15.7645	15.7892
				2	15.3798	15.3994	15.6576	15.6967	14.7807	14.7960	15.7556	15.7796
			1.8	1.8	15.3869	15.4068	15.6702	15.7128	14.7860	14.8017	15.7636	15.7883
				2	15.3797	15.3993	15.6557	15.6948	14.7806	14.7960	15.7547	15.7787

Table 4. Interpretation of cost values on E_{TCV}.

				HC_{FP}	0.2				0.4			
				HC_{RP}	0.2		0.3		0.2		0.3	
				HC_{RFP}	0.1	0.3	0.1	0.3	0.1	0.3	0.1	0.3
WC_{Q1}	WC_{Q2}	WC_{Q3}	WC_{Q4}	SC_{FP}								
5	4	5	5	5	12.942	13.138	13.037	13.233	12.942	13.138	13.037	13.233
				10	13.828	14.025	13.923	14.120	13.828	14.025	13.923	14.120
			7	5	13.528	13.725	13.623	13.820	13.528	13.725	13.623	13.820
				10	14.415	14.612	14.510	14.707	14.415	14.612	14.510	14.707
		6	5	5	13.894	14.090	13.989	14.185	13.894	14.090	13.989	14.185
				10	14.780	14.977	14.875	15.072	14.780	14.977	14.875	15.072
			7	5	14.481	14.677	14.576	14.772	14.481	14.677	14.576	14.772
				10	15.367	15.564	15.462	15.659	15.367	15.564	15.462	15.659
	6	5	5	5	14.965	15.161	15.060	15.256	14.965	15.161	15.060	15.256
				10	15.852	16.048	15.947	16.143	15.852	16.048	15.947	16.143
			7	5	15.552	15.748	15.647	15.843	15.552	15.748	15.647	15.843
				10	16.438	16.635	16.533	16.730	16.438	16.635	16.533	16.730
		6	5	5	15.917	16.114	16.012	16.209	15.917	16.114	16.012	16.209
				10	16.804	17.000	16.899	17.095	16.804	17.000	16.899	17.095
			7	5	16.504	16.701	16.599	16.796	16.504	16.701	16.599	16.796
				10	17.391	17.587	17.486	17.682	17.391	17.587	17.486	17.682
7	4	5	5	5	13.280	13.477	13.375	13.572	13.280	13.477	13.375	13.572
				10	14.167	14.363	14.262	14.458	14.167	14.363	14.262	14.458
			7	5	13.867	14.063	13.962	14.159	13.867	14.063	13.962	14.159
				10	14.754	14.950	14.849	15.045	14.754	14.950	14.849	15.045
		6	5	5	14.232	14.429	14.327	14.524	14.232	14.429	14.327	14.524
				10	15.119	15.315	15.214	15.410	15.119	15.315	15.214	15.410
			7	5	14.819	15.016	14.914	15.111	14.819	15.016	14.914	15.111
				10	15.706	15.902	15.801	15.997	15.706	15.902	15.801	15.997
	6	5	5	5	15.303	15.500	15.398	15.595	15.303	15.500	15.398	15.595
				10	16.190	16.387	16.285	16.482	16.190	16.387	16.285	16.482
			7	5	15.890	16.087	15.985	16.182	15.890	16.087	15.985	16.182
				10	16.777	16.974	16.872	17.069	16.777	16.974	16.872	17.069
		6	5	5	16.256	16.452	16.351	16.547	16.256	16.452	16.351	16.547
				10	17.142	17.339	17.237	17.434	17.142	17.339	17.237	17.434
			7	5	16.842	17.039	16.938	17.134	16.843	17.039	16.938	17.134
				10	17.729	17.926	17.824	18.021	17.729	17.926	17.824	18.021

Table 4. *Cont.*

				HC_{FP}	**0.2**				**0.4**			
				HC_{RP}	**0.2**		**0.3**		**0.2**		**0.3**	
				HC_{RFP}	**0.1**	**0.3**	**0.1**	**0.3**	**0.1**	**0.3**	**0.1**	**0.3**
LC_{Q1}	LC_{Q2}	LC_{Q3}	LC_{Q4}	SC_{FP}								
0.5	0.2	0.2	0.1	5	13.953	14.150	14.048	14.245	13.953	14.150	14.048	14.245
				10	14.840	15.036	14.935	15.131	14.840	15.036	14.935	15.131
			0.3	5	13.971	14.167	14.066	14.262	13.971	14.167	14.066	14.262
				10	14.858	15.054	14.953	15.149	14.858	15.054	14.953	15.149
		0.4	0.1	5	14.032	14.228	14.127	14.323	14.032	14.228	14.127	14.323
				10	14.918	15.115	15.013	15.210	14.918	15.115	15.013	15.210
			0.3	5	14.049	14.246	14.144	14.341	14.049	14.246	14.144	14.341
				10	14.936	15.133	15.031	15.228	14.936	15.133	15.031	15.228
	0.4	0.2	0.1	5	13.986	14.183	14.081	14.278	13.986	14.183	14.081	14.278
				10	14.873	15.069	14.968	15.164	14.873	15.069	14.968	15.164
			0.3	5	14.004	14.200	14.099	14.295	14.004	14.200	14.099	14.295
				10	14.890	15.087	14.985	15.182	14.890	15.087	14.985	15.182
		0.4	0.1	5	14.065	14.261	14.160	14.356	14.065	14.261	14.160	14.356
				10	14.951	15.148	15.046	15.243	14.951	15.148	15.046	15.243
			0.3	5	14.082	14.279	14.177	14.374	14.082	14.279	14.177	14.374
				10	14.969	15.166	15.064	15.261	14.969	15.166	15.064	15.261
0.4	0.2	0.2	0.1	5	14.163	14.360	14.258	14.455	14.163	14.360	14.258	14.455
				10	15.050	15.246	15.145	15.341	15.050	15.246	15.145	15.341
			0.3	5	14.181	14.377	14.276	14.472	14.181	14.377	14.276	14.472
				10	15.068	15.264	15.163	15.359	15.068	15.264	15.163	15.359
		0.4	0.1	5	14.242	14.438	14.337	14.533	14.242	14.438	14.337	14.533
				10	15.128	15.325	15.223	15.420	15.128	15.325	15.223	15.420
			0.3	5	14.259	14.456	14.354	14.551	14.259	14.456	14.354	14.551
				10	15.146	15.343	15.241	15.438	15.146	15.343	15.241	15.438
	0.7	0.2	0.1	5	14.196	14.393	14.291	14.488	14.196	14.393	14.291	14.488
				10	15.083	15.279	15.178	15.374	15.083	15.279	15.178	15.374
			0.3	5	14.214	14.410	14.309	14.505	14.214	14.410	14.309	14.505
				10	15.100	15.297	15.195	15.392	15.100	15.297	15.195	15.392
		0.4	0.1	5	14.275	14.471	14.370	14.566	14.275	14.471	14.370	14.566
				10	15.161	15.358	15.256	15.453	15.161	15.358	15.256	15.453
			0.3	5	14.292	14.489	14.387	14.584	14.292	14.489	14.387	14.584
				10	15.179	15.376	15.274	15.471	15.179	15.376	15.274	15.471

6. Conclusions

Markovian queuing environments can be arranged using a model created using the research presented in this paper. According to information gleaned from the queuing inventory literature, there are currently no publications examining the combination of FP, RFP, and RP with multi-type servers and queues. This work is an attempt to fill a void in the inventory literature that has been identified. Four classes of customers are welcomed to use the inventory system,

1. To purchase an FP.
2. To sell their OP and buy a new FP.
3. To buy used things (second-hand shops are commonplace in many countries).
4. Require a repair service of their defective product.

These customers benefit from the proposed MQIS, which provides a multi-type service facility to them. In addition, each of the three items included in the system must be acquired based on the customer's satisfaction with the product's features and total cost of ownership. These customer-focused services are only offered by a limited number of companies. The multi-type service facility provided by the system is considered a customer-oriented service.

- Assuming a customer-oriented service model, this system's performance is in line with real-world inventory businesses.
- The notion is theoretically described as a seven-dimensional stochastic process and its full analysis is carried out by the NMAM.
- Using LRA, the minimal non-negative solution of the matrix quadratic equation is found for the proposed MQIS.
- The system's performance metrics can be calculated when the stationary probability vector has been computed.
- The discussed model comes in under budget. From the detailed interpretation of the numerical discussion provided for each queue, one can observe that the overall expected inflow of a customer in the system (the sum of the expected number of customers in each queue) is raised.

6.1. Insights and Limitations

As a result, the reader can gain new insights into the FP and RFP service processes under the assumption of probability.

1. In this MQIS, the probability q_{r_1} is expected to play an important role. In addition to increasing the number of customers in Queue-1 and Queue-3, it also raises the amount of RPs and RFPs.
2. The probability q_{r_3} must be smaller than q_{r_1} and q_{r_2} because it represents the loss of customers in Queue-2.
3. Despite the fact that customers have been lost in all of Queue-i, where $i = 1, 3, 4$, the loss of customers in Queue-2 will have a significant impact on the system's overall costs and profits.
4. Customer dissatisfaction cannot be prevented, but it can be managed through the use of real-world examples. This study shows that the probability p_{f_2}, q_{r_3}, and r_2 should always be kept at reduced values while also never going to zero. In the event that they are considered to be zero, it conflicts with reality.
5. According to the proposed model, these probabilities could be reduced by readers or business people. In the meantime, they will need a creative strategy or design to meet the needs of all kinds of customers. These days, just a handful of businesses, such as online retailers Amazon and Flipkart, make an attempt to appeal to customers of various socioeconomic backgrounds. So, if a customer can have all of their needs met in a single place, they are less likely to look elsewhere.
6. The purchase of a new product when returning the old is increasing because every month there is new software and upgrades are introduced in many mobile phones, laptops, fridges, air-conditioning companies, etc. These upgrades stimulate the customer to buy a new product. Even though their previously purchased product will not expire soon, they are interested in buying the new one if a company will give such an opportunity (the sale of a new product when buying the customer's old product).
7. This model explores a circular economy that will bring business opportunities to the business people.

6.2. Future Directions

A busy-period analysis of each server and waiting time distribution of each queue work is under process. Discussion of this topic in a Markovian arrival process setting may be possible later on. The purchasing option will be given to a customer to choose FP or RFP. The repair work on the defective product will be performed using a phase-type distribution.

Author Contributions: Conceptualization, K.J.; Data curation, K.J.; Formal analysis, K.J.; S.S. (S. Selvakumar); Funding acquisition, G.P.J., J.R. and W.C.; Investigation, S.S. (S. Selvakumar), S.S. (S. Saravanan), and N.A.; Methodology, S.S. (S. Saravanan) and N.A.; Project administration, J.R. and W.C.; Resources, J.R.; Software, S.S. (S. Selvakumar) and S.A.; Supervision, G.P.J., J.R. and W.C.; Validation, W.C.; Visualization, N.A. and S.A.; Writing—original draft, K.J.; Writing—review and editing, G.P.J. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: Anbazhagan and Amutha would like to thank RUSA Phase 2.0 (F 24-51/2014-U), DST-FIST (SR/FIST/MS-I/2018/17), DST-PURSE 2nd Phase programme (SR/PURSE Phase 2/38), Govt. of India.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

QIS	Queuing-Inventory System
FP	Fresh Product
OP	Old Product
NP	New Product
RFP	Refurbished Product
RP	Returned Product
$MQIS$	Markovian Queuing-Inventory System
TaC	Terms and Conditions
TOM	Technical Operation Manual
$CTMC$	Continuous Time Markov Chain
LRA	Logarithmic Reduction Algorithm
TCV	Total Cost Value
$NMAM$	Neuts Matrix Analytic Method
Notations	
$\mathbb{Z}^+$	The set of all non-negative integers
S_l^m	$\{l, l+1, l+2, \cdots, m\}, \ \ l, m \in \mathbb{Z}^+$
$\mathbf{0}$	Zero matrix of an appropriate order
I	Identity matrix of an appropriate order
I_r	Identity matrix of order r
$\mathbf{e}$	Column matrix containing all ones of an appropriate order
δ_{ij}	$\begin{cases} 1, & \text{if } j = i, \\ 0, & \text{otherwise} \end{cases}$
$\bar{\delta}_{ij}$	$1 - \delta_{ij}$
$J_1(t)$	Number of customer in Queue-1 at any time
$J_2(t)$	Number of available FP at any time
$J_3(t)$	Number of customer in Queue-2 at any time
$J_4(t)$	Number of RP exists at any time

$J_5(t)$	Number of customer in Queue-3 at any time
$J_6(t)$	Number of available RFP at any time
$J_7(t)$	Number of customer in Queue-4 at any time
K	$\{(j_1, j_2, j_3, j_4, j_5, j_5, j_7) : \ j_1 \in \mathbb{Z}^+, j_2 \in S_1^{P_1}, j_3 \in S_0^{N_1}, j_4 \in S_0^{P_2}, j_5 \in S_0^{N_2},$ $j_6 \in S_0^{P_3}, j_7 \in S_0^{N_3}\}$
γ_0	$P_1(N_1+1)(P_2+1)(N_2+1)(P_3+1)(N_3+1)$
γ_1	$(N_1+1)(P_2+1)(N_2+1)(P_3+1)(N_3+1)$
γ_2	$(P_2+1)(N_2+1)(P_3+1)(N_3+1)$
γ_3	$(N_2+1)(P_3+1)(N_3+1)$
γ_4	$(P_3+1)(N_3+1)$
γ_5	(N_3+1)
HC_{FP}	Holding cost of per FP per unit time
HC_{RP}	Holding cost of per RP per unit time
HC_{RFP}	Holding cost of per RFP per unit time
SC_{FP}	Set up cost of per order of FP per unit time
WC_{Q1}	Waiting cost of per customer in Queue-1 per unit time
WC_{Q2}	Waiting cost of per customer in Queue-2 per unit time
WC_{Q3}	Waiting cost of per customer in Queue-3 per unit time
WC_{Q4}	Waiting cost of per customer in Queue-4 per unit time
LC_{Q1}	Lost cost of per customer in Queue-1 per unit time
LC_{Q2}	Lost cost of per customer in Queue-2 per unit time
LC_{Q3}	Lost cost of per customer in Queue-3 per unit time
LC_{Q4}	Lost cost of per customer in Queue-4 per unit time

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