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Research article

Further studies on ordinary differential equations involving the M-fractional derivative

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Abstract: In the current paper, the power series based on the M-fractional derivative is formally introduced. More peciesely, the Taylor and Maclaurin expansions are generalized for fractional-order differentiable functions in accordance with the M-fractional derivative. Some new definitions, theorems, and corollaries regarding the power series in the M sense are presented and formally proved. Several ordinary differential equations (ODEs) involving the M-fractional derivative are solved to examine the validity of the results presented in the current study.

Keywords: M-fractional derivative; power series; new definitions, theorems and corollaries; ordinary differential equations
Mathematics Subject Classification: 34A08, 35C10

1. Introduction

Fractional differential equations (FDEs) are precision tools to describe many nonlinear phenomena from porous media to other areas of scientific disciplines. Researchers have used different local and nonlocal fractional derivatives to model the phenomena around them. For example, Yang *et al.* [1]

considered an advection-dispersion equation with the conformable derivative and obtained its analytical solutions using the Fourier transform. Hosseini *et al.* [2] studied the BiswasArshed model involving the beta derivative and derived its soliton waves through the Jacobi and Kudryashov techniques. In [3], the authors steered an analytical study on a Caputo time-fractional equation using a capable analytic scheme. In another paper, Sulaiman *et al.* [4] explored the coupled Burgers system involving the Mittag-Leffler kernel through the Laplace homotopy perturbation method. Generally, the most widely used fractional derivatives that have been adopted by many authors are the conformable derivative [5–8], the beta derivative [9–12], the Caputo derivative [13–16], and the Atangana-Baleanu derivative [17–20]. For more information about the fractional derivatives, see [21–30].

The M-fractional derivative is another type of fractional derivatives that lies in the class of the local fractional derivatives (Compared to the nonlocal fractional derivatives such as the Caputo fractional derivative). This local fractional derivative is a generalization of other local fractional derivatives like the conformable fractional derivative. The M-fractional derivative was first proposed by Sousa and Oliveira in [31] that encompasses a number of ordinary derivative properties such as linearity, product rule, etc. Sousa and Oliveira [31] also developed a series of classical results from the Rolle's theorem to other theorems in the M sense. Such results led to the use of this well-behaved derivative in the studies of many researchers. In this respect, Yusuf et al. [32] gained solitons of the Ginzburg-Landau equation involving the M-fractional derivative using the generalized Bernoulli method. Özkan [33] used the simplest equation approach to derive exact solutions of Biswas-Arshed equation with the M-fractional derivatives. Tariq *et al.* [34] found optical solitons of Schrödinger-Hirota equation involving the M-fractional derivative through the Fan's method. Zafar *et al.* [35] tried to acquire optical solitons of Biswas-Arshed model with the M-fractional derivative using the sinh-Gordon method.

For $f : [0,\infty) \to \mathbb{R}$, the M-fractional derivative of f of order α is given by [31]

$$ {}_iD_M^{\alpha,\beta} f(x) = \lim_{\varepsilon\to 0} \frac{f\left(x\, {}_iE_\beta(\varepsilon x^{-\alpha})\right) - f(x)}{\varepsilon}, \tag{1.1}$$

where $x > 0$ and $\alpha \in (0,1)$. Here, ${}_iE_\beta(.)$, $\beta > 0$ is the Mittag-Leffler function [36]. If the M-fractional derivative of f of order α exists, then it is said that f is α-differentiable. Note that for the α-differentiable function f, one can define

$$ {}_iD_M^{\alpha,\beta} f(0) = \lim_{x\to 0^+} {}_iD_M^{\alpha,\beta} f(x),$$

provided that

$$\lim_{x\to 0^+} {}_iD_M^{\alpha,\beta} f(x),$$

exists.

It can be readily shown that for the α-differentiable functions, the M-fractional derivative satisfies the following features [31]:

A. ${}_iD_M^{\alpha,\beta}(af+bg) = a\left({}_iD_M^{\alpha,\beta} f\right) + b\left({}_iD_M^{\alpha,\beta} g\right),\ a,b\in\mathbb{R}.$
B. ${}_iD_M^{\alpha,\beta} x^p = \frac{p}{\Gamma(\beta+1)} x^{p-\alpha},\ p\in\mathbb{R}.$
C. If $f(x) = c$, then ${}_iD_M^{\alpha,\beta} f = 0$.
D. ${}_iD_M^{\alpha,\beta}(fg) = g\left({}_iD_M^{\alpha,\beta} f\right) + f\left({}_iD_M^{\alpha,\beta} g\right).$

E. ${}_{i}D_{M}^{\alpha,\beta}\left(\frac{f}{g}\right) = \frac{g\left({}_{i}D_{M}^{\alpha,\beta}f\right) - f\left({}_{i}D_{M}^{\alpha,\beta}g\right)}{g^2}$.

F. ${}_{i}D_{M}^{\alpha,\beta}(fog)(x) = f'\big(g(x)\big){}_{i}D_{M}^{\alpha,\beta}g(x)$, where f is differentiable at $g(x)$.

G. ${}_{i}D_{M}^{\alpha,\beta}f(x) = \frac{x^{1-\alpha}}{\Gamma(\beta+1)}\frac{df}{dx}$.

H. If $a > 0$ and $f : [a, b] \to \mathbb{R}$ is continuous and α-differentiable for some $\alpha \in (0, 1)$, then there is $c \in (a, b)$ such that

$$
{}_{i}D_{M}^{\alpha,\beta}f(c) = \alpha\left(\frac{f(b) - f(a)}{b^\alpha - a^\alpha}\right), \quad \beta > 0.
$$

Abdeljawad [37] introduced the conformable power series and applied such a representaion for a group of certain functions. The main aim of the current paper is to introduce the power series based on the M-fractional derivative and prove some new theorems and corollaries regarding it.

The next sections of the present paper are as follows: In Section 2, the power series based on the M-fractional derivative is introduced. More peciesely, the Taylor and Maclaurin expansions are generalized for fractional-order differentiable functions in accordance with the M-fractional derivative. Furthermore, some new definitions, theorems, and corollaries regarding the power series in the M sense are presented and formally proved, in this section. In Section 3, a number of ODEs with the M-fractional derivative are solved to examine the validity of the results presented. The paper totalizes the outcomes in Section 4.

2. New definitions, theorems and corollaries

In the current section, some new definitions, theorems, and corollaries regarding the power series in the M sense are presented and formally proved.

Definition 2.1. *An infinite series*

$$
a_0 + \sum_{n=1}^{\infty} a_n x^{n\alpha},
$$

is called an α-power series in x^α. Additionally, the series

$$
a_0 + \sum_{n=1}^{\infty} a_n (x^\alpha - x_0^\alpha)^n,
$$

is known as an α-power series in $x^\alpha - x_0^\alpha$ which is more general than the previous one.

Definition 2.2. *An infinite α-power series*

$$
f(x_0) + \sum_{n=1}^{\infty} \left(\frac{\Gamma(\beta+1)}{\alpha}\right)^n \frac{{}_{i}^{n}D_{M}^{\alpha,\beta}f(x_0)}{n!}(x^\alpha - x_0^\alpha)^n,
$$

is referred to as the α-Taylor expansion of the function f at x_0 provided that f is infinitely α-differentiable at x_0.

Definition 2.3. *An infinite α-power series*

$$
f(0) + \sum_{n=1}^{\infty} \left(\frac{\Gamma(\beta+1)}{\alpha}\right)^n \frac{{}_{i}^{n}D_{M}^{\alpha,\beta}f(0)}{n!} x^{n\alpha},
$$

is known as the α-Maclaurin expansion of the function f provided that f is infinitely α-differentiable at $x_0 = 0$.

Definition 2.4. *A sequence $\{f_n\}$ is called convergent uniformly to f on the set $E \subseteq \mathbb{R}$, if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in E$, $|f_n(x) - f(x)| < \varepsilon$.*

Theorem 2.5. *Assume $\{f_n\}$ converges uniformly to f on the set $E \subseteq \mathbb{R}$. Let x be a limit point of E and let $\lim_{t\to x} f_n(t) = l_n$. Then, $\{l_n\}$ converges and $\lim_{t\to x} f(t) = \lim_{n\to\infty} l_n$. Particularly*

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t).$$

For the proof, see [38, 39].

Theorem 2.6. ***(Uniform convergence and the truncated M-fractional derivative)*** *Let $0 < \alpha < 1$ and $a \geq 0$. Suppose $\{f_n\}$ is M-fractional differentiable on (a, b) such that $\{f_n(x_0)\}$ converges for some point x_0 on (a, b). If $\{{}_iD_M^{\alpha,\beta} f_n\}$ converges uniformly on (a, b), then $\{f_n\}$ converges uniformly on (a, b) to a function f and for every $x \in (a, b)$, we have*

$${}_iD_M^{\alpha,\beta} f(x) = \lim_{n\to\infty} {}_iD_M^{\alpha,\beta} f_n(x).$$

Proof. Suppose $\varepsilon > 0$ and consider $N_1 \in \mathbb{N}$ such that $m, n \geq N_1$. Now, $t \in (a, b)$ implies

$$|f_m(x_0) - f_n(x_0)| < \frac{\varepsilon}{2},$$

and

$$\left|{}_iD_M^{\alpha,\beta} f_m(t) - {}_iD_M^{\alpha,\beta} f_n(t)\right| < \frac{\alpha\varepsilon}{2(b^\alpha - a^\alpha)}. \tag{2.1}$$

If we apply the mean value theorem (H) to the function $f_m - f_n$ where $m, n \geq N_1$, from (2.1), we find

$$\begin{aligned}
|f_m(x) - f_n(x) - f_m(t) + f_n(t)| &= |(f_m(x) - f_n(x)) - (f_m(t) - f_n(t))| \\
&\leq \frac{1}{\alpha}\left|{}_iD_M^{\alpha,\beta} f_m(z) - {}_iD_M^{\alpha,\beta} f_n(z)\right| |x^\alpha - t^\alpha| \\
&\leq \frac{|x^\alpha - t^\alpha|\alpha\varepsilon}{2\alpha(b^\alpha - a^\alpha)} \\
&\leq \frac{\varepsilon}{2},
\end{aligned}$$

for every $x, t \in (a, b)$, where z is a point between x and t. Thus, for every $x \in (a, b)$ and $m, n \geq N_1$, the following

$$\begin{aligned}
|f_m(x) - f_n(x)| &= |(f_m(x) - f_n(x)) - (f_m(x_0) - f_n(x_0)) + (f_m(x_0) - f_n(x_0))| \\
&\leq |(f_m(x) - f_n(x)) - (f_m(x_0) - f_n(x_0))| + |f_m(x_0) - f_n(x_0)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

implies that $\{f_n\}$ converges uniformly on (a, b). Let $f(x) = \lim_{n\to\infty} f_n(x)$ and $x \in (a, b)$.

Let us fix a point c on (a, b). Suppose that

$$h(\lambda) = \left({}_iE_\beta\left(\lambda c^{-\alpha}\right)\right)^\alpha,$$

then

$$\begin{aligned} &h(0) = 1, \\ &\frac{dh}{d\lambda}(0) = \frac{\alpha}{\Gamma(\beta+1)} c^{-\alpha}. \end{aligned}$$

Thus, there exists a positive number γ such that

$$\left|\left(c_iE_\beta(\lambda c^{-\alpha})\right)^\alpha - c^\alpha\right| = c^\alpha\left|{}_iE_\beta(\lambda c^{-\alpha})^\alpha - 1\right| = c^\alpha\left|h(\lambda) - h(0)\right| < c^\alpha\frac{2\alpha|\lambda|}{\Gamma(\beta+1)}c^{-\alpha} = \frac{2\alpha|\lambda|}{\Gamma(\beta+1)},$$

for $0 < |\lambda| < \gamma$. Furthermore, there exists $N_2 \in \mathbb{N}$ such that $m, n \geq N_2$. Now, $x \in (a, b)$ implies

$$\left|{}_iD_M^{\alpha,\beta} f_m(x) - {}_iD_M^{\alpha,\beta} f_n(x)\right| < \frac{\Gamma(\beta+1)}{2}\varepsilon.$$

Now, for $0 < |\lambda| < \gamma$, one can define

$$\begin{aligned} g_n(\lambda) &= \frac{f_n\left(c_iE_\beta\left(\lambda c^{-\alpha}\right)\right) - f_n(c)}{\lambda}, \\ g(\lambda) &= \frac{f\left(c_iE_\beta\left(\lambda c^{-\alpha}\right)\right) - f(c)}{\lambda}. \end{aligned}$$

Since $\lim\limits_{n\to\infty} g_n(\lambda) = g(\lambda)$ and $\lim\limits_{\lambda\to 0} g_n(\lambda) = {}_iD_M^{\alpha,\beta} f_n(c)$, for $m, n \geq N_2$, we have

$$\begin{aligned} |g_m(\lambda) - g_n(\lambda)| &= \left|\frac{f_m\left(c_iE_\beta(\lambda c^{-\alpha})\right) - f_m(c)}{\lambda} - \frac{f_n\left(c_iE_\beta(\lambda c^{-\alpha})\right) - f_n(c)}{\lambda}\right| \\ &= \frac{1}{|\lambda|}\left|\left(f_m\left(c_iE_\beta(\lambda c^{-\alpha})\right) - f_n\left(c_iE_\beta(\lambda c^{-\alpha})\right)\right) - \left(f_m(c) - f_n(c)\right)\right| \\ &= \frac{1}{\alpha|\lambda|}\left|\left(c_iE_\beta(\lambda c^{-\alpha})\right)^\alpha - c^\alpha\right|\left|{}_iD_M^{\alpha,\beta}\left(f_m - f_n\right)(z)\right| \\ &< \frac{1}{\alpha|\lambda|}\frac{2\alpha|\lambda|}{\Gamma(\beta+1)}\frac{\Gamma(\beta+1)}{2}\varepsilon \\ &= \varepsilon, \end{aligned}$$

where z is a point between 0 and λ. This shows $\{g_n\}$ converges uniformly to g on $0 < |\lambda| < \gamma$. Theorem 1 implies that $\lim\limits_{\lambda\to 0} g(\lambda)$ exists and $\lim\limits_{\lambda\to 0} g(\lambda) = \lim\limits_{n\to\infty}\lim\limits_{\lambda\to 0} g_n(\lambda)$. This means that ${}_iD_M^{\alpha,\beta} f(c)$ exists and

$${}_iD_M^{\alpha,\beta} f(c) = \lim_{\lambda\to 0} g(\lambda) = \lim_{\lambda\to 0}\lim_{n\to\infty} g_n(\lambda) = \lim_{n\to\infty}\lim_{\lambda\to 0} g_n(\lambda) = \lim_{n\to\infty} {}_iD_M^{\alpha,\beta} f_n(c).$$

□

Corollary 2.7. *In Theorem 2, if for every $n \in \mathbb{N}$, f_n is differentiable in the usual context, then property (G) implies*

$$ {}_iD_M^{\alpha,\beta} f(x) = \frac{x^{1-\alpha}}{\Gamma(\beta+1)} \lim_{n\to\infty} f_n'(x), $$

for all $x \in (a, b)$.

Theorem 2.8. *Suppose that $0 < \alpha < 1$, $0 < R \leq x_0^\alpha$ and the α-power series*

$$ \sum_{n=0}^{\infty} a_n (x^\alpha - x_0^\alpha)^n, $$

converges on $I = \left(\left(x_0^\alpha - R\right)^{\frac{1}{\alpha}}, \left(x_0^\alpha + R\right)^{\frac{1}{\alpha}}\right)$, and $f(x) = \sum_{n=0}^{\infty} a_n (x^\alpha - x_0^\alpha)^n$ for $x \in I$. Then the series

$$ \sum_{n=0}^{\infty} a_n (x^\alpha - x_0^\alpha)^n, $$

converges uniformly on every closed interval of I. The function f is continuous and α-differentiable in I, and

$$ {}_iD_M^{\alpha,\beta} f(x) = \frac{\alpha}{\Gamma(\beta+1)} \sum_{n=1}^{\infty} n a_n (x^\alpha - x_0^\alpha)^{n-1}. $$

Proof. Suppose $[a, b] \subseteq I$ and p is a point in $[a, b]$ such that for every $x \in [a, b]$, $|x^\alpha - x_0^\alpha| \leq |p^\alpha - x_0^\alpha|$. Then,

$$ \left| a_n (x^\alpha - x_0^\alpha)^n \right| < \left| a_n (p^\alpha - x_0^\alpha)^n \right|, $$

for all $x \in [a, b]$. Since

$$ \sum_{n=0}^{\infty} a_n (p^\alpha - x_0^\alpha)^n, $$

converges absolutely, the Weierstrass M-test yields the uniform convergence of the series

$$ \sum_{n=0}^{\infty} a_n (x^\alpha - x_0^\alpha)^n, $$

on $[a, b]$. Since

$$ \lim_{n\to\infty} \sup \sqrt[n]{\frac{\alpha}{\Gamma(\beta+1)} n |a_n|} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}, $$

the series

$$ \sum_{n=0}^{\infty} a_n (x^\alpha - x_0^\alpha)^n $$

and

$$ \frac{\alpha}{\Gamma(\beta+1)} \sum_{n=1}^{\infty} n a_n (x^\alpha - x_0^\alpha)^{n-1} $$

have a similar interval of convergence. Accordingly

$$\frac{\alpha}{\Gamma(\beta+1)}\sum_{n=1}^{\infty} n a_n (x^\alpha - x_0^\alpha)^{n-1},$$

converges uniformly on every $[a,b] \subseteq I$. Now, if

$$s_n(x) = \sum_{k=0}^{n} a_k (x^\alpha - x_0^\alpha)^k,$$

then

$${}_iD_M^{\alpha,\beta} s_n(x) = \frac{\alpha}{\Gamma(\beta+1)}\sum_{k=1}^{n} k a_k (x^\alpha - x_0^\alpha)^{k-1}.$$

Since the sequences $\{s_n\}$ and $\{{}_iD_M^{\alpha,\beta} s_n\}$ converge uniformly on $[a,b]$, they also converge uniformly on (a,b). Therefore, Theorem 2 and its corollary imply that ${}_iD_M^{\alpha,\beta} f(x)$ exists on (a,b) and

$${}_iD_M^{\alpha,\beta} f(x) = \lim_{n\to\infty} {}_iD_M^{\alpha,\beta} s_n(x) = \frac{\alpha}{\Gamma(\beta+1)}\sum_{n=1}^{\infty} n a_n (x^\alpha - x_0^\alpha)^{n-1}.$$

But, for any x which $x \in I$, there exists a closed interval $[a,b]$ such that $x \in (a,b) \subseteq [a,b] \subseteq I$. This reveals that ${}_iD_M^{\alpha,\beta} f(x)$ exists for any $x \in I$ and

$${}_iD_M^{\alpha,\beta} f(x) = \frac{\alpha}{\Gamma(\beta+1)}\sum_{n=1}^{\infty} n a_n (x^\alpha - x_0^\alpha)^{n-1}.$$

The continuity of f is yielded from the existence of ${}_iD_M^{\alpha,\beta} f$. □

Corollary 2.9. *Under the hypotheses of Theorem 3, f has M-fractional derivatives of all orders in*

$$\left((x_0^\alpha - R)^{\frac{1}{\alpha}}, (x_0^\alpha + R)^{\frac{1}{\alpha}}\right),$$

which are given by

$${}_i^kD_M^{\alpha,\beta} f(x) = \left(\frac{\alpha}{\Gamma(\beta+1)}\right)^k \sum_{n=k}^{\infty} n(n-1)\times\cdots\times(n-k+1)\, a_n (x^\alpha - x_0^\alpha)^{n-k}.$$

In particular

$${}_i^kD_M^{\alpha,\beta} f(x_0) = \left(\frac{\alpha}{\Gamma(\beta+1)}\right)^k k! a_k.$$

Corollary 2.10. *Suppose $0 < \alpha < 1$, $R > 0$, and the α-power series*

$$\sum_{n=0}^{\infty} a_n x^{n\alpha},$$

converges on $\left(0, R^{\frac{1}{\alpha}}\right)$*, and* $f(x) = \sum_{n=0}^{\infty} a_n x^{n\alpha}$*, where* $0 < x < R^{\frac{1}{\alpha}}$*. Then, the series*

$$\sum_{n=0}^{\infty} a_n x^{n\alpha},$$

converges uniformly on every closed interval of $\left(0, R^{\frac{1}{\alpha}}\right)$*. The function* f *is continuous and* α*-differentiable in* $\left(0, R^{\frac{1}{\alpha}}\right)$*, and*

$${}_iD_M^{\alpha,\beta} f(x) = \frac{\alpha}{\Gamma(\beta+1)} \sum_{n=1}^{\infty} n a_n (x^{\alpha})^{n-1}.$$

It is easy to show that

$${}_i^k D_M^{\alpha,\beta} f(x) = \left(\frac{\alpha}{\Gamma(\beta+1)}\right)^k \sum_{n=k}^{\infty} n(n-1) \times \cdots \times (n-k+1) a_n (x^{\alpha})^{n-k}.$$

Since $\lim_{x\to 0^+} {}_i^k D_M^{\alpha,\beta} f(x)$ *exists for* $k = 0, 1, \cdots,$

$${}_i^k D_M^{\alpha,\beta} f(0) = \left(\frac{\alpha}{\Gamma(\beta+1)}\right)^k k! a_k.$$

Corollary 2.11. *If two* α*-power series*

$$\sum_{n=0}^{\infty} a_n (x^{\alpha} - x_0^{\alpha})^n, \text{ and } \sum_{n=0}^{\infty} b_n (x^{\alpha} - x_0^{\alpha})^n,$$

represent the same function in a neighborhood, then $a_n = b_n$ *for all* n*. This means that the* α*-power series expansion of a function about a given point is uniquely determined.*

3. Applications

In this section, by using the α-power series method, several linear and nonlinear ODEs with the M-fractional derivative are solved to examine the validity of the results presented in the current study.

Example 3.1. *Firstly, we deal with a problem involving the* M*-fractional derivative as [40]*

$${}_iD_M^{\alpha,\beta} y(t) = y(t) + 1, \quad \lim_{t\to 0^+} y(t) = 0, \tag{3.1}$$

where the exact solution of Eq (3.1) is

$$y(t) = \exp\left(\frac{\Gamma(\beta+1)}{\alpha} t^{\alpha}\right) - 1.$$

According to Section 2, we adopt a solution for Eq (3.1) as

$$y(t) = \sum_{n=0}^{\infty} a_n (t^{\alpha})^n. \tag{3.2}$$

By substituting Eq (3.2) into (3.1) and simplifying the resulting expression, we get

$$\frac{\alpha}{\Gamma(\beta+1)}\sum_{n=1}^{\infty} n a_n t^{(n-1)\alpha} = \sum_{n=0}^{\infty} a_n t^{n\alpha} + 1,$$

and so

$$\left(\frac{\alpha}{\Gamma(\beta+1)}a_1 - a_0 - 1\right) + \sum_{n=2}^{\infty}\left(\frac{\alpha(n+1)}{\Gamma(\beta+1)}a_{n+1} - a_n\right)t^{n\alpha} = 0.$$

By performing some simple operations, we achieve

$$\lim_{t\to 0^+} y(t) = 0 \to a_0 = 0,$$

$$\begin{aligned} &\frac{\alpha}{\Gamma(\beta+1)}a_1 - 1 = 0 \to a_1 = \frac{\Gamma(\beta+1)}{\alpha}, \\ &\frac{\alpha(n+1)}{\Gamma(\beta+1)}a_{n+1} - a_n = 0 \to a_{n+1} = \frac{\Gamma(\beta+1)}{\alpha(n+1)}a_n, \quad n = 1, 2, \cdots . \end{aligned} \tag{3.3}$$

From (3.3), it is clear that

$$\begin{aligned} a_2 &= \frac{\Gamma(\beta+1)}{2\alpha}a_1 = \frac{1}{2}\left(\frac{\Gamma(\beta+1)}{\alpha}\right)^2, \\ a_3 &= \frac{\Gamma(\beta+1)}{3\alpha}a_2 = \frac{1}{3!}\left(\frac{\Gamma(\beta+1)}{\alpha}\right)^3, \\ &\vdots \\ a_n &= \frac{1}{n!}\left(\frac{\Gamma(\beta+1)}{\alpha}\right)^n, \quad n = 2, 3, \cdots . \end{aligned}$$

By applying the above coefficients in Eq (3.2), the solution of Eq (3.1) is derived as

$$y(t) = \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{\Gamma(\beta+1)}{\alpha}\right)^n t^{n\alpha},$$

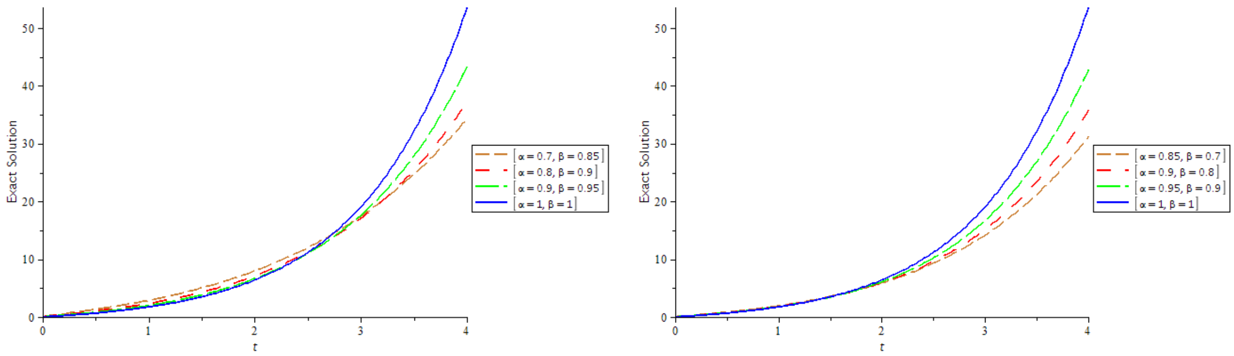
or

$$y(t) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\Gamma(\beta+1)}{\alpha}\right)^n t^{n\alpha} - 1, \quad t > 0,$$

converging to

$$y(t) = \exp\left(\frac{\Gamma(\beta+1)}{\alpha}t^{\alpha}\right) - 1.$$

The exact solution of Eq (3.1) for different sets of α and β has been plotted in Figure 1.



(a) Gold ($\alpha = 0.7, \beta = 0.85$), Red ($\alpha = 0.8, \beta = 0.9$), Green ($\alpha = 0.9, \beta = 0.95$), and Blue ($\alpha = 1, \beta = 1$), (b) Gold ($\alpha = 0.85, \beta = 0.7$), Red ($\alpha = 0.9, \beta = 0.8$), Green ($\alpha = 0.95, \beta = 0.9$), and Blue ($\alpha = 1, \beta = 1$),

Figure 1. The exact solution of Example 1 for different sets of α and β.

Example 3.2. *Secondly, we want to deal with a problem with the M-fractional derivative as*

$$ {}_iD_M^{0.5,\beta} y(t) = y(t) + t^{\frac{1}{2}}, \quad y(2) = 0, \tag{3.4} $$

where the exact solution of Eq (3.4) is

$$ y(t) = \left(2^{\frac{1}{2}} + \frac{1}{2\Gamma(\beta+1)}\right) e^{2\Gamma(\beta+1)\left(t^{\frac{1}{2}} - 2^{\frac{1}{2}}\right)} - t^{\frac{1}{2}} - \frac{1}{2\Gamma(\beta+1)}. $$

Based on Section 2, the solution of Eq (3.4) is assumed to be

$$ y(t) = \sum_{n=0}^{\infty} a_n \left(t^{\frac{1}{2}} - 2^{\frac{1}{2}}\right)^n. \tag{3.5} $$

By setting Eq (3.5) in (3.4) and simplifying the resulting expression, we find

$$ \frac{1}{2\Gamma(\beta+1)} \sum_{n=1}^{\infty} n a_n \left(t^{\frac{1}{2}} - 2^{\frac{1}{2}}\right)^{n-1} = \sum_{n=0}^{\infty} a_n \left(t^{\frac{1}{2}} - 2^{\frac{1}{2}}\right)^n + \left(t^{\frac{1}{2}} - 2^{\frac{1}{2}}\right) + 2^{\frac{1}{2}}, $$

or

$$ \begin{aligned} \left(\frac{1}{2\Gamma(\beta+1)} a_1 - a_0 - 2^{\frac{1}{2}}\right) + \left(\frac{1}{\Gamma(\beta+1)} a_2 - a_1 - 1\right)\left(t^{\frac{1}{2}} - 2^{\frac{1}{2}}\right) & \\ + \sum_{n=2}^{\infty} \left(\frac{(n+1)}{2\Gamma(\beta+1)} a_{n+1} - a_n\right)\left(t^{\frac{1}{2}} - 2^{\frac{1}{2}}\right)^n & \\ = 0. & \end{aligned} $$

Applying some simple operations, we obtain

$$ \begin{aligned} & y(2) = 0 \rightarrow a_0 = 0, \\ & \frac{1}{2\Gamma(\beta+1)} a_1 - a_0 - 2^{\frac{1}{2}} = 0 \rightarrow a_1 = 2^{\frac{3}{2}} \Gamma(\beta+1), \end{aligned} $$

$$\frac{1}{\Gamma(\beta+1)}a_2 - a_1 - 1 = 0 \rightarrow a_2 = \Gamma(\beta+1)\left(2^{\frac{3}{2}}\Gamma(\beta+1)+1\right),$$
$$\frac{(n+1)}{2\Gamma(\beta+1)}a_{n+1} - a_n = 0 \rightarrow a_{n+1} = \frac{2\Gamma(\beta+1)}{n+1}a_n, \quad n = 2, 3, \cdots. \tag{3.6}$$

From (3.6), it is found that

$$a_3 = \frac{2\Gamma(\beta+1)}{3}a_2 = \left(2^{\frac{1}{2}} + \frac{1}{2\Gamma(\beta+1)}\right)\frac{2^3\Gamma^3(\beta+1)}{3!},$$
$$a_4 = \frac{2\Gamma(\beta+1)}{4}a_3 = \left(2^{\frac{1}{2}} + \frac{1}{2\Gamma(\beta+1)}\right)\frac{2^4\Gamma^4(\beta+1)}{4!},$$
$$\vdots$$
$$a_n = \left(2^{\frac{1}{2}} + \frac{1}{2\Gamma(\beta+1)}\right)\frac{2^n\Gamma^n(\beta+1)}{n!}, \quad n = 3, 4, \cdots.$$

Inserting the above coefficients into Eq (3.5) leads to

$$y(t) = 2^{\frac{3}{2}}\Gamma(\beta+1)\left(t^{\frac{1}{2}} - 2^{\frac{1}{2}}\right) + \left(2^{\frac{1}{2}} + \frac{1}{2\Gamma(\beta+1)}\right)\sum_{n=2}^{\infty}\frac{2^n(\Gamma(\beta+1))^n}{n!}\left(t^{\frac{1}{2}} - 2^{\frac{1}{2}}\right)^n, \quad t > 0,$$

or

$$y(t) = \left(2^{\frac{1}{2}} + \frac{1}{2\Gamma(\beta+1)}\right)e^{2\Gamma(\beta+1)\left(t^{\frac{1}{2}}-2^{\frac{1}{2}}\right)} - t^{\frac{1}{2}} - \frac{1}{2\Gamma(\beta+1)}.$$

The exact solution of Eq (3.4) for different values of β has been portrayed in Figure 2.

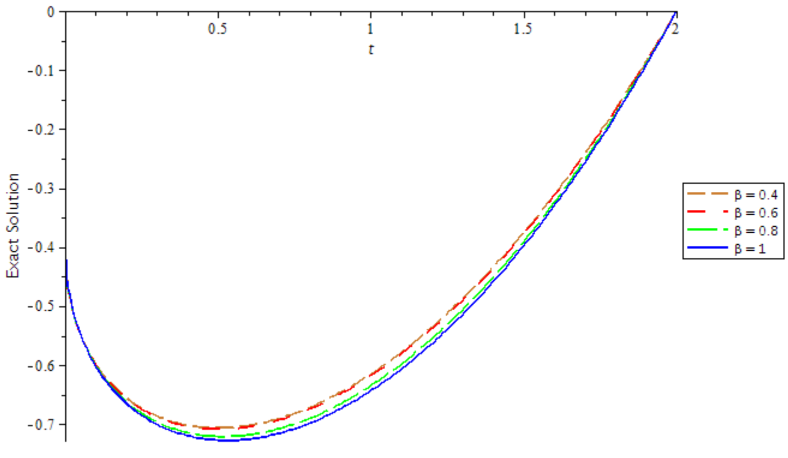


Figure 2. The exact solution of Example 2 for different values of β.

Example 3.3. *Thirdly, we deal with a problem involving the M-fractional derivative as*

$$_iD_M^{\alpha,\beta}y(t) = 1 + (y(t))^2, \quad \lim_{t\to 0^+} y(0) = 0, \tag{3.7}$$

which has the following exact solution

$$y(t) = \tan\left(\frac{\Gamma(\beta+1)}{\alpha}t^{\alpha}\right).$$

According to Section 2, we adopt a solution for Eq (3.7) as

$$y(t) = \sum_{n=0}^{\infty} a_n (t^{\alpha})^n. \tag{3.8}$$

By substituting Eq (3.8) into (3.7) and simplifying the resulting expression, we get

$$\frac{\alpha}{\Gamma(\beta+1)} \sum_{n=1}^{\infty} n a_n t^{(n-1)\alpha} = 1 + \left(\sum_{n=0}^{\infty} a_n t^{n\alpha} \right)^2,$$

or

$$\left(\frac{\alpha a_1}{\Gamma(\beta+1)} - a_0^2 - 1 \right) + \sum_{n=1}^{\infty} \left(\frac{(n+1)\alpha}{\Gamma(\beta+1)} a_{n+1} - \sum_{i=0}^{n} a_i a_{n-i} \right) t^{n\alpha} = 0.$$

By performing some simple operations, we achieve

$$\begin{aligned}
&\lim_{t \to 0^+} y(0) = 0 \to a_0 = 0, \\
&\frac{\alpha}{\Gamma(\beta+1)} a_1 - a_0^2 - 1 = 0 \to a_1 = \frac{\Gamma(\beta+1)}{\alpha}, \\
&\frac{(n+1)\alpha}{\Gamma(\beta+1)} a_{n+1} - \sum_{i=0}^{n} a_i a_{n-i} = 0 \to a_{n+1} = \frac{\Gamma(\beta+1)}{(n+1)\alpha} \sum_{i=0}^{n} a_i a_{n-i}.
\end{aligned} \tag{3.9}$$

From (3.9), it is clear that

$$\begin{aligned}
&n = 1 \to a_2 = \frac{\Gamma(\beta+1)}{2\alpha} a_0 a_1 = 0, \\
&n = 2 \to a_3 = \frac{\Gamma(\beta+1)}{3\alpha} \left(2a_0 a_2 + a_1^2 \right) = \frac{1}{3} \left(\frac{\Gamma(\beta+1)}{\alpha} \right)^3, \\
&n = 3 \to a_4 = \frac{\Gamma(\beta+1)}{4\alpha} \left(2a_0 a_3 + 2a_1 a_2 \right) = 0, \\
&n = 4 \to a_5 = \frac{\Gamma(\beta+1)}{5\alpha} \left(2a_0 a_4 + 2a_2^2 + 2a_1 a_3 \right) = \frac{2}{15} \left(\frac{\Gamma(\beta+1)}{\alpha} \right)^5, \\
&\vdots
\end{aligned}$$

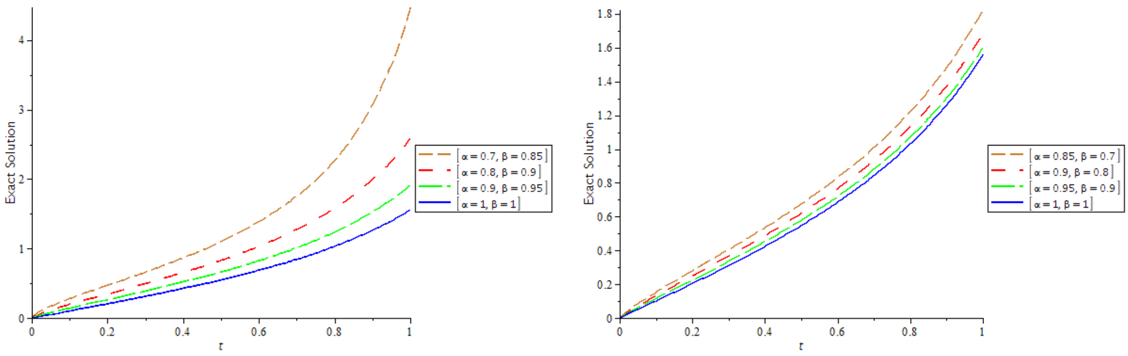
By applying the above coefficients in Eq (3.8), the solution of Eq (3.7) is derived as

$$y(t) = \frac{\Gamma(\beta+1)}{\alpha} t^{\alpha} + \frac{1}{3} \left(\frac{\Gamma(\beta+1)}{\alpha} \right)^3 t^{3\alpha} + \frac{2}{15} \left(\frac{\Gamma(\beta+1)}{\alpha} \right)^5 t^{5\alpha} + \cdots, \quad 0 < t < \left(\frac{\alpha\pi}{2\Gamma(\beta+1)} \right)^2,$$

converging to

$$y(t) = \tan \left(\frac{\Gamma(\beta+1)}{\alpha} t^{\alpha} \right).$$

The exact solution of Eq (3.7) for different sets of α and β has been plotted in Figure 3.



(a) Gold ($\alpha = 0.7, \beta = 0.85$), Red ($\alpha = 0.8, \beta = 0.9$), Green ($\alpha = 0.9, \beta = 0.95$), and Blue ($\alpha = 1, \beta = 1$),

(b) Gold ($\alpha = 0.85, \beta = 0.7$), Red ($\alpha = 0.9, \beta = 0.8$), Green ($\alpha = 0.95, \beta = 0.9$), and Blue ($\alpha = 1, \beta = 1$),

Figure 3. The exact solution of Example 3 for different sets of α and β.

Example 3.4. *In the end, we will deal with a problem with the M-fractional derivative as*

$$ {}_iD_M^{\alpha,\beta} y(t) = 1 - (y(t))^2, \quad \lim_{t\to 0^+} y(0) = 0, \tag{3.10} $$

which has the following exact solution

$$ y(t) = \tanh\left(\frac{\Gamma(\beta+1)}{\alpha} t^{\alpha}\right). $$

Based on Section 2, the solution of Eq (3.10) is supposed to be

$$ y(t) = \sum_{n=0}^{\infty} a_n (t^{\alpha})^n. \tag{3.11} $$

By setting Eq (3.11) in (3.10) and simplifying the resulting expression, we find

$$ \frac{\alpha}{\Gamma(\beta+1)} \sum_{n=1}^{\infty} n a_n t^{(n-1)\alpha} = 1 - \left(\sum_{n=0}^{\infty} a_n t^{n\alpha}\right)^2, $$

or

$$ \left(\frac{\alpha a_1}{\Gamma(\beta+1)} + a_0^2 - 1\right) + \sum_{n=1}^{\infty} \left(\frac{(n+1)\alpha}{\Gamma(\beta+1)} a_{n+1} + \sum_{i=0}^{n} a_i a_{n-i}\right) t^{n\alpha} = 0. $$

Through applying some simple operations, we obtain

$$ \begin{aligned} &\lim_{t\to 0^+} y(0) = 0 \to a_0 = 0, \\ &\frac{\alpha}{\Gamma(\beta+1)} a_1 + a_0^2 - 1 = 0 \to a_1 = \frac{\Gamma(\beta+1)}{\alpha}, \\ &\frac{(n+1)\alpha}{\Gamma(\beta+1)} a_{n+1} + \sum_{i=0}^{n} a_i a_{n-i} = 0 \to a_{n+1} = -\frac{\Gamma(\beta+1)}{(n+1)\alpha} \sum_{i=0}^{n} a_i a_{n-i}. \end{aligned} \tag{3.12} $$

From (3.12), it is found that

$$\begin{aligned}
n = 1 &\rightarrow a_2 = -\frac{\Gamma(\beta+1)}{2\alpha} a_0 a_1 = 0, \\
n = 2 &\rightarrow a_3 = -\frac{\Gamma(\beta+1)}{3\alpha}\left(2a_0a_2 + a_1^2\right) = -\frac{1}{3}\left(\frac{\Gamma(\beta+1)}{\alpha}\right)^3, \\
n = 3 &\rightarrow a_4 = -\frac{\Gamma(\beta+1)}{4\alpha}\left(2a_0a_3 + 2a_1a_2\right) = 0, \\
n = 4 &\rightarrow a_5 = -\frac{\Gamma(\beta+1)}{5\alpha}\left(2a_0a_4 + 2a_2^2 + 2a_1a_3\right) = \frac{2}{15}\left(\frac{\Gamma(\beta+1)}{\alpha}\right)^5, \\
&\vdots
\end{aligned}$$

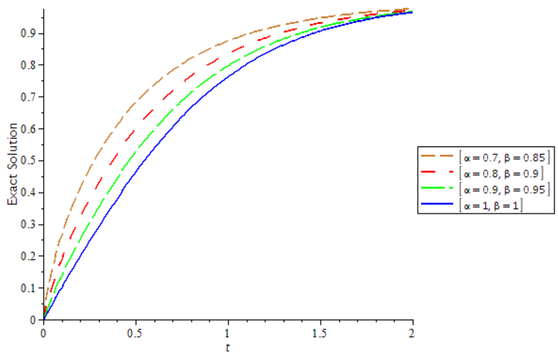
Inserting the above coefficients into Eq (3.11) leads to

$$y(t) = \frac{\Gamma(\beta+1)}{\alpha}t^{\alpha} - \frac{1}{3}\left(\frac{\Gamma(\beta+1)}{\alpha}\right)^3 t^{3\alpha} + \frac{2}{15}\left(\frac{\Gamma(\beta+1)}{\alpha}\right)^5 t^{5\alpha} - \cdots,$$

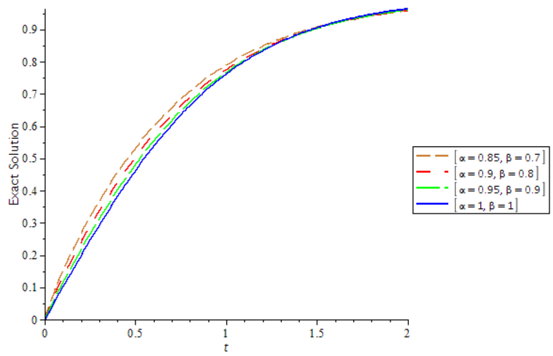
or

$$y(t) = \tanh\left(\frac{\Gamma(\beta+1)}{\alpha}t^{\alpha}\right).$$

The exact solution of Eq (3.10) for different sets of α and β has been portrayed in Figure 4.



(a) Gold ($\alpha = 0.7, \beta = 0.85$), Red ($\alpha = 0.8, \beta = 0.9$), Green ($\alpha = 0.9, \beta = 0.95$), and Blue ($\alpha = 1, \beta = 1$),



(b) Gold ($\alpha = 0.85, \beta = 0.7$), Red ($\alpha = 0.9, \beta = 0.8$), Green ($\alpha = 0.95, \beta = 0.9$), and Blue ($\alpha = 1, \beta = 1$),

Figure 4. The exact solution of Example 4 for different sets of α and β.

4. Conclusions

The key goal of the current paper was to conduct a new investigation on ordinary differential equations involving the M-fractional derivative. In this respect, first, the α-Taylor expansion and the α-Maclaurin expansion were established based on the M-fractional derivative. Then, several definitions, theorems, and corollaries regarding the power series in the M sense were given and successfully proved. To examine the effectiveness of the results provided in the present work, some ordinary differential equations involving the M-fractional derivative were solved. The Maple package as a worthwhile tool was formally adopted to deal with symbolic computations. As a possible future work, the authors will apply the power series in the M sense to solve other well-known ODEs involving the M-fractional derivative.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. S. Yang, H. Zhou, S. Zhang, L. Wang, Analytical solutions of advective-dispersive transport in porous media involving conformable derivative, *Appl. Math. Lett.*, **92** (2019) 85–92. https://doi.org/10.1016/j.aml.2019.01.004
2. K. Hosseini, M. Mirzazadeh, M. Ilie, J. F. Gómez-Aguilar, Biswas-Arshed equation with the beta time derivative: Optical solitons and other solutions, *Optik*, **217** (2020), 164801. https://doi.org/10.1016/j.ijleo.2020.164801
3. K. Hosseini, M. Ilie, M. Mirzazadeh, A. Yusuf, T. A. Sulaiman, D. Baleanu, S. Salahshour, An effective computational method to deal with a time-fractional nonlinear water wave equation in the Caputo sense, *Math. Comput. Simul.*, **187** (2021), 248–260. https://doi.org/10.1016/j.matcom.2021.02.021
4. T. A. Sulaiman, M. Yavuz, H. Bulut, H. M. Baskonus, Investigation of the fractional coupled viscous Burger's equation involving Mittag-Leffler kernel, *Physica A*, **527** (2019), 121126. https://doi.org/10.1016/j.physa.2019.121126
5. R. Khalil, M. Al-Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, *J. Comput. Appl. Math.*, **264** (2014), 65–70. https://doi.org/10.1016/j.cam.2014.01.002
6. M. Abu Hammad, R. Khalil, Abel's formula and Wronskian for conformable fractional differential equations, *Int. J. Differ. Equ. Appl.*, **13** (2014), 177–183. http://doi.org/10.12732/ijdea.v13i3.1753
7. M. S. Osman, A. Korkmaz, H. Rezazadeh, M. Mirzazadeh, M. Eslami, Q. Zhou, The unified method for conformable time fractional Schrödinger equation with perturbation terms, *Chinese J. Phys.*, **56** (2018), 2500–2506. https://doi.org/10.1016/j.cjph.2018.06.009
8. K. Hosseini, K. Sadri, M. Mirzazadeh, A. Ahmadian, Y. M. Chu, S. Salahshour, Reliable methods to look for analytical and numerical solutions of a nonlinear differential equation arising in heat transfer with the conformable derivative, *Math. Method. Appl. Sci.*, 2021. https://doi.org/10.1002/mma.7582
9. A. Atangana, D. Baleanu, A. Alsaedi, Analysis of time-fractional Hunter-Saxton equation: a model of neumatic liquid crystal, *Open Phys.*, **14** (2016), 145–149. https://doi.org/10.1515/phys-2016-0010
10. A. Atangana, R. T. Alqahtani, Modelling the spread of river blindness disease via the Caputo fractional derivative and the beta-derivative, *Entropy*, **18** (2016), 40. https://doi.org/10.3390/e18020040

11. B. Ghanbari, J. F. Gómez-Aguilar, The generalized exponential rational function method for Radhakrishnan-Kundu-Lakshmanan equation with β-conformable time derivative, *Rev. Mex. Fís.*, **65** (2019), 503–518.

12. K. Hosseini, M. Mirzazadeh, M. Ilie, J. F. Gómez-Aguilar, Soliton solutions of the Sasa-Satsuma equation in the monomode optical fibers including the beta-derivatives, *Optik*, **224** (2020), 165425. https://doi.org/10.1016/j.ijleo.2020.165425

13. M. Caputo, Linear models of dissipation whose Q is almost frequency independent-II, *Geophys. J. R. Astron. Soc.*, **13** (1967), 529–539. https://doi.org/10.1111/j.1365-246X.1967.tb02303.x

14. I. Podlubny, *Fractional differential equations*, *Math. Sci. Eng.*, **198** (1999), 41–119.

15. M. Awais, F. S. Alshammari, S. Ullah, M. Altaf Khan, S. Islam, Modeling and simulation of the novel coronavirus in Caputo derivative, *Results Phys.*, **19** (2020), 103588. https://doi.org/10.1016/j.rinp.2020.103588

16. A. Yokus, H. Durur, D. Kaya, H. Ahmad, T. A. Nofal, Numerical comparison of Caputo and conformable derivatives of time fractional Burgers-Fisher equation, *Results Phys.*, **25** (2021), 104247. https://doi.org/10.1016/j.rinp.2021.104247

17. A. Atangana, D. Baleanu, New fractional derivative with nonlocal and non-singular kernel, theory and application to heat transfer model, *Therm. Sci.*, **20** (2016), 763–769. https://doi.org10.2298/TSCI160111018A

18. D. Avci, A. Yetim, Cauchy and source problems for an advection-diffusion equation with Atangana-Baleanu derivative on the real line, *Chaos Soliton. Fract.*, **118** (2019), 361–365. https://doi.org/10.1016/j.chaos.2018.11.035

19. Z. Korpinar, M. Inc, M. Bayram, Theory and application for the system of fractional Burger equations with Mittag Leffler kernel, *Appl. Math. Comput.*, **367** (2020), 124781. https://doi.org/10.1016/j.amc.2019.124781

20. K. Hosseini, M. Ilie, M. Mirzazadeh, D. Baleanu, An analytic study on the approximate solution of a nonlinear time-fractional Cauchy reaction-diffusion equation with the Mittag-Leffler law, *Math. Method. Appl. Sci.*, **44** (2021), 6247–6258. https://doi.org/10.1002/mma.7059

21. S. G. Samko, B. Ross, Integration and differentiation to a variable fractional order, *Integr. Transf. Spec. F.*, **1** (1993), 277–300. https://doi.org/10.1080/10652469308819027

22. X. Zheng, H. Wang, An optimal-order numerical approximation to variable-order space-fractional diffusion equations on uniform or graded meshes, *SIAM J. Numer. Anal.*, **58** (2020), 330–352. https://doi.org/10.1137/19M1245621

23. X. Zheng, H. Wang, An error estimate of a numerical approximation to a hidden-memory variable-order space-time fractional diffusion equation, *SIAM J. Numer. Anal.*, **58** (2020), 2492–2514. https://doi.org/10.1137/20M132420X

24. X. Zheng, H. Wang, Optimal-order error estimates of finite element approximations to variable-order time-fractional diffusion equations without regularity assumptions of the true solutions, *IMA J. Numer. Anal.*, **41** (2021), 1522–1545.

25. X. Zheng, H. Wang, A hidden-memory variable-order time-fractional optimal control model: Analysis and approximation, *SIAM J. Control Optim.*, **59** (2021), 1851–1880. https://doi.org/10.1137/20M1344962

26. X. Zheng, H. Wang, Analysis and discretization of a variable-order fractional wave equation, *Commun. Nonlinear Sci.*, **104** (2022), 106047. https://doi.org/10.1016/j.cnsns.2021.106047
27. J. Singh, A. Ahmadian, S. Rathore, D. Kumar, D. Baleanu, M. Salimi, et al. An efficient computational approach for local fractional Poisson equation in fractal media, *Numer. Meth. Part. D. E.*, **37** (2021), 1439–1448. ttps://doi.org/10.1002/num.22589
28. M. Rahaman, S. P. Mondal, A. A. Shaikh, A. Ahmadian, N. Senu, S. Salahshour, Arbitrary-order economic production quantity model with and without deterioration: generalized point of view, *Adv. Differ. Equ.*, **2020** (2020), 16. https://doi.org/10.1186/s13662-019-2465-x
29. S. Ahmad, A. Ullah, K. Shah, S. Salahshour, A. Ahmadian, T. Ciano, Fuzzy fractional-order model of the novel coronavirus, *Adv. Differ. Equ.*, **2020** (2020), 472. https://doi.org/10.1186/s13662-020-02934-0
30. K. Shah, M. Arfan, I. Mahariq, A. Ahmadian, S. Salahshour, M. Ferrara, Fractal-fractional mathematical model addressing the situation of corona virus in Pakistan, *Results Phys.*, **19** (2020), 103560. https://doi.org/10.1016/j.rinp.2020.103560
31. J. Vanterler da C. Sousa, E. Capelas de Oliveira, A new truncated M-fractional derivative type unifying some fractional derivative types with classical properties, *Int. J. Anal. Appl.*, **16** (2018), 83–96. https://doi.org/10.28924/2291-8639-16-2018-83
32. A. Yusuf, M. Inc, D. Baleanu, Optical solitons with M-truncated and beta derivatives in nonlinear optics, *Front. Phys.*, **7** (2019), 126. https://doi.org/10.3389/fphy.2019.00126
33. Y. S. Özkan, On the exact solutions to Biswas-Arshed equation involving truncated M-fractional space-time derivative terms, *Optik*, **227** (2021), 166109. https://doi.org/10.1016/j.ijleo.2020.166109
34. K. U. Tariq, M. Younis, S. T. R. Rizvi, H. Bulut, M-truncated fractional optical solitons and other periodic wave structures with Schrödinger-Hirota equation, *Mod. Phys. Lett. B*, **34** (2020), 2050427. https://doi.org/10.1142/S0217984920504278
35. A. Zafar, A. Bekir, M. Raheel, W. Razzaq, Optical soliton solutions to Biswas-Arshed model with truncated M-fractional derivative, *Optik*, **222** (2020), 165355. https://doi.org/10.1016/j.ijleo.2020.165355
36. R. Goreno, A. A. Kilbas, F. Mainardi, S. V. Rogosin, *Mittag-Leffler functions, related topics and applications*, Berlin: Springer, 2014.
37. T. Abdeljawad, On conformable fractional calculus, *J. Comput. Appl. Math.*, **279** (2015), 57–66. https://doi.org/10.1016/j.cam.2014.10.016
38. T. M. Apostol, *Mathematical analysis*, New York: Addison-Wesley Publishing Company, 1981.
39. W. Rudin, *Principle of mathematical analysis*, New York: McGraw-Hill, 1976.
40. M. Ilie, J. Biazar, Z. Ayati, Optimal homotopy asymptotic method for first-order conformable fractional differential equations, *J. Fract. Calc. Appl.*, **10** (2019), 33–45.

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