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Research article

On solution of Fredholm integral equations via fuzzy b-metric spaces using triangular property

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Abstract: In the present paper, we prove common fixed point without continuity by using triangular property on fuzzy b-metric space. Our results generalize and expand some of the literature's well-known results. We also explore some of the application of our key results to Fredholm integral equation.

Keywords: fuzzy metric spaces; fuzzy b-metric space; common fixed point
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1. Introduction and preliminaries

In 1960, Schweizer and Sklar [1] introduced the notion of continuous triangular norm. After that in 1965, Zadeh [2] introduces the theory of fuzzy sets. Using the concept of fuzziness, in 1975, Kramosil and Michalek [3] defined the fuzzy metric space with the help of continuous t-norm. The fuzzy approach to the distance follows from the idea that it is not necessary that there always exist a real number to define the distance between any two points which we have to approximate or to find, but it is a fuzzy notion. In 1994, George and Veeramani [6] modified the definition of fuzzy metric spaces. Grabeic [4] extend the well known fixed point theorem of Banach to fuzzy metric spaces in the sense of Karamosil and Michalek [3]. After that, Gregori and Sapena [5] extended the fuzzy banach contraction theorem to fuzzy metric space in the sense George and Veeramani's [6]. In 2012, Sedghi and Shobe [7], proved common fixed point theorem in b-fuzzy metric space. In 2020, Abbas et al. [8], proved fixed point theorems in fuzzy b-metric spaces. Recently, Shamas et al. [9] proved fixed point results without continuity by using triangular property in fuzzy metric spaces. In this paper, we prove fixed point theorems without continuity by using triangular property on fuzzy b-metric space.

Now, we present some basic definitions and lemma as follows:

Definition 1.1. [7] A 3-tuple $(\Upsilon, \Psi_\varrho, *)$ is called a fuzzy b-metric space(FBM space) if Υ is an arbitrary (non-empty) set, $*$ is a continuous δ-norm and Ψ_ϱ is a fuzzy set on $\Upsilon^2 \times (0, \infty)$, satisfying the following conditions for each $\aleph, \varpi, \xi \in \Upsilon$, $\mathfrak{r}, \mathfrak{g} > 0$ and a given real number $\mathfrak{s} \geq 1$,
(FBM1) $\Psi_\varrho(\aleph, \varpi, \mathfrak{r}) > 0$,
(FBM2) $\Psi_\varrho(\aleph, \varpi, \mathfrak{r}) = 1$ iff $\aleph = \varpi$,
(FBM3) $\Psi_\varrho(\aleph, \varpi, \mathfrak{r}) = \Psi_\varrho(\varpi, \aleph, \mathfrak{r})$,
(FBM4) $\Psi_\varrho(\aleph, \varpi, \mathfrak{s}(\mathfrak{r} + \mathfrak{g})) \geq \Psi_\varrho(\aleph, \xi, \mathfrak{r}) * \Psi_\varrho(\xi, \varpi, \mathfrak{g})$,
(FBM5) $\Psi_\varrho(\aleph, \varpi, .) : (0, \infty) \to [0, 1]$ is continuous.

The function Ψ_ϱ is called a fuzzy b-metric.

Definition 1.2. [7] Let $(\Upsilon, \Psi_\varrho, *)$ be a FBM space.

(D1) A sequence $\{\aleph_\mathfrak{j}\}$ converges to $\aleph \in \Upsilon$ if $\lim_{\mathfrak{j}\to\infty} \Psi_\varrho(\aleph_\mathfrak{j}, \aleph, \mathfrak{r}) = 1$ for all $\mathfrak{r} > 0$ and denoted as $\aleph_\mathfrak{j} \to \aleph$.

(D2) If
$$\lim_{\mathfrak{j},\mathfrak{k}\to\infty} \Psi_\varrho(\aleph_\mathfrak{j}, \aleph_\mathfrak{k}, \mathfrak{r}) = 1$$
for all sufficiently large $\mathfrak{j}, \mathfrak{k}$ and for any $\mathfrak{r} > 0$ then $\aleph_\mathfrak{j}$ is called a Cauchy sequence in Υ.

(D3) If every Cauchy sequence is convergent in Υ then Υ is called a complete FBM space.

Definition 1.3. Let $(\Upsilon, \Psi_\varrho, *)$ be a FBM space with $\mathfrak{s} \geq 1$. The FBM space Ψ_ϱ is triangular if
$$\frac{1}{\Psi_\varrho(\aleph, \varpi, \mathfrak{r})} - 1 \leq \mathfrak{s}\left(\frac{1}{\Psi_\varrho(\aleph, \xi, \mathfrak{r})} - 1 + \frac{1}{\Psi_\varrho(\xi, \varpi, \mathfrak{r})} - 1\right)$$

Lemma 1.4. *A fuzzy b-metric space Ψ_ϱ is triangular.*

Proof. Let $\Psi_\varrho : \Upsilon^2 \times (0, \infty) \to [0, 1]$ be a fuzzy b-metric (FBM) defined by
$$\Psi_\varrho(\aleph, \varpi, \mathfrak{r}) = \frac{\mathfrak{r}}{\mathfrak{r} + |\aleph - \varpi|^2}, \text{ for all } \aleph, \varpi \in \Upsilon, \mathfrak{r} > 0.$$

We need the following inequality which we can easily obtained from results proved in [11, 12],
$$|\aleph - \xi||\xi - \varpi| \leq \left(\frac{|\aleph - \xi| + |\xi - \varpi|}{2}\right)^2 \leq \frac{|\aleph - \xi|^2 + |\xi - \varpi|^2}{2}. \tag{1.1}$$

Now,

$$\begin{aligned}
\frac{1}{\Psi_\varrho(\aleph,\varpi,\mathfrak{r})}-1&=\frac{|\aleph-\varpi|^2}{\mathfrak{r}}=\frac{|\aleph-\xi+\xi-\varpi|^2}{\mathfrak{r}}\\
&\leq\frac{|\aleph-\xi|^2+|\xi-\varpi|^2+2|\aleph-\xi||\xi-\varpi|}{\mathfrak{r}}\\
&\leq\frac{|\aleph-\xi|^2+|\xi-\varpi|^2+|\aleph-\xi|^2+|\xi-\varpi|^2}{\mathfrak{r}},\ \textit{by using}\ (1.1)\\
&=2\Big(\frac{|\aleph-\xi|^2}{\mathfrak{r}}+\frac{|\xi-\varpi|^2}{\mathfrak{r}}\Big)\\
&\leq\mathfrak{s}\Big(\frac{|\aleph-\xi|^2}{\mathfrak{r}}+\frac{|\xi-\varpi|^2}{\mathfrak{r}}\Big)\\
&=\mathfrak{s}\Big(\frac{1}{\Psi_\varrho(\aleph,\xi,\mathfrak{r})}-1+\frac{1}{\Psi_\varrho(\xi,\varpi,\mathfrak{r})}-1\Big),
\end{aligned}$$

which implies that

$$\frac{1}{\Psi_\varrho(\aleph,\varpi,\mathfrak{r})}-1\leq\mathfrak{s}\Big(\frac{1}{\Psi_\varrho(\aleph,\xi,\mathfrak{r})}-1+\frac{1}{\Psi_\varrho(\xi,\varpi,\mathfrak{r})}-1\Big),\ \text{for}\ \mathfrak{r}>0.$$

Hence, FBM with $\mathfrak{s}\geq 2$, Ψ_ϱ is triangular. □

Motivated by Shamas et al. [9], we prove fixed point theorems without continuity by using triangular property on FBM space with an application. This idea is new and we hope that this research article will open new horizon for interested researcher in this field. Now, we give our main results in the following section. Next, we establish the application of our results in the next section for better understanding of our main result.

2. Main results

In this section, we prove common fixed point theorems to FBM spaces without continuity by using triangular property.

Theorem 2.1. *Let $(\Upsilon,\Psi_\varrho,*)$ be a complete FBM space with $\mathfrak{s}\geq 1$ in which Ψ_ϱ is triangular. Let $\Gamma,\Omega\colon\Upsilon\to\Upsilon$ be a pair of self-mappings such that*

$$\begin{aligned}
\frac{1}{\Psi_\varrho(\Gamma\aleph,\Omega\varpi,\mathfrak{r})}-1\leq{}&\alpha_1\Big(\frac{1}{\Psi_\varrho(\aleph,\varpi,\mathfrak{r})}-1\Big)\\
&+\alpha_2\Big(\frac{\Psi_\varrho(\aleph,\varpi,\mathfrak{r})}{\Psi_\varrho(\aleph,\Omega\varpi,2\mathfrak{s}\mathfrak{r})*\Psi_\varrho(\varpi,\Gamma\aleph,2\mathfrak{s}\mathfrak{r})}-1\Big)\\
&+\alpha_3\Big(\frac{\Psi_\varrho(\aleph,\Gamma\aleph,\mathfrak{r})*\Psi_\varrho(\varpi,\Omega\varpi,\mathfrak{r})}{\Psi_\varrho(\aleph,\varpi,\mathfrak{r})*\Psi_\varrho(\aleph,\Omega\varpi,2\mathfrak{s}\mathfrak{r})*\Psi_\varrho(\varpi,\Gamma\aleph,2\mathfrak{s}\mathfrak{r})}-1\Big)\\
&+\alpha_4\Big(\frac{1}{\Psi_\varrho(\aleph,\Gamma\aleph,\mathfrak{r})}-1+\frac{1}{\Psi_\varrho(\varpi,\Omega\varpi,\mathfrak{r})}-1\Big),
\end{aligned}\tag{2.1}$$

for all $\aleph,\varpi\in\Upsilon,\mathfrak{r}>0,\alpha_1\in(0,1)$ and $\alpha_2,\alpha_3,\alpha_4\geq 0$ with $\alpha_1+\alpha_2+\alpha_3+2\alpha_4<\frac{1}{\mathfrak{s}}$. Then, Γ and Ω have a common fixed point in Υ.

Proof. Fix $\aleph_0 \in \Upsilon$ and construct a sequence of points in Υ such that

$$\begin{aligned}\aleph_{2j+1} &= \Gamma\aleph_{2j},\\ \aleph_{2j+2} &= \Omega\aleph_{2j+1},\end{aligned}$$

$j \geq 0$. Then,

$$\begin{aligned}\frac{1}{\Psi_\varrho(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}-1 &= \frac{1}{\Psi_\varrho(\Gamma\aleph_{2j},\Omega\aleph_{2j+1},\mathfrak{r})}-1\\
&\leq \alpha_1\Big(\frac{1}{\Psi_\varrho(\aleph_{2j},\aleph_{2j+1},\mathfrak{r})}-1\Big)\\
&+\alpha_2\Big(\frac{\Psi_\varrho(\aleph_{2j},\aleph_{2j+1},\mathfrak{r})}{\Psi_\varrho(\aleph_{2j},\Omega\aleph_{2j+1},2\mathfrak{s}\mathfrak{r})*\Psi_\varrho(\aleph_{2j+1},\Gamma\aleph_{2j},2\mathfrak{s}\mathfrak{r})}-1\Big)\\
&+\alpha_3\Big(\frac{\Psi_\varrho(\aleph_{2j},\Gamma\aleph_{2j},\mathfrak{r})*\Psi_\varrho(\aleph_{2j+1},\Omega\aleph_{2j+1},\mathfrak{r})}{\Psi_\varrho(\aleph_{2j},\aleph_{2j+1},\mathfrak{r})*\Psi_\varrho(\aleph_{2j},\Omega\aleph_{2j+1},2\mathfrak{s}\mathfrak{r})*\Psi_\varrho(\aleph_{2j+1},\Gamma\aleph_{2j},2\mathfrak{s}\mathfrak{r})}-1\Big)\\
&+\alpha_4\Big(\frac{1}{\Psi_\varrho(\aleph_{2j},\Gamma\aleph_{2j},\mathfrak{r})}-1+\frac{1}{\Psi_\varrho(\aleph_{2j+1},\Omega\aleph_{2j+1},\mathfrak{r})}-1\Big)\\
&= \alpha_1\Big(\frac{1}{\Psi_\varrho(\aleph_{2j},\aleph_{2j+1},\mathfrak{r})}-1\Big)\\
&+\alpha_2\Big(\frac{\Psi_\varrho(\aleph_{2j},\aleph_{2j+1},\mathfrak{r})}{\Psi_\varrho(\aleph_{2j},\aleph_{2j+2},2\mathfrak{s}\mathfrak{r})*\Psi_\varrho(\aleph_{2j+1},\aleph_{2j+1},2\mathfrak{s}\mathfrak{r})}-1\Big)\\
&+\alpha_3\Big(\frac{\Psi_\varrho(\aleph_{2j},\aleph_{2j+1},\mathfrak{r})*\Psi_\varrho(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}{\Psi_\varrho(\aleph_{2j},\aleph_{2j+1},\mathfrak{r})*\Psi_\varrho(\aleph_{2j},\aleph_{2j+2},2\mathfrak{s}\mathfrak{r})*\Psi_\varrho(\aleph_{2j+1},\aleph_{2j+1},2\mathfrak{s}\mathfrak{r})}-1\Big)\\
&+\alpha_4\Big(\frac{1}{\Psi_\varrho(\aleph_{2j},\aleph_{2j+1},\mathfrak{r})}-1+\frac{1}{\Psi_\varrho(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}-1\Big)\\
&= \alpha_1\Big(\frac{1}{\Psi_\varrho(\aleph_{2j},\aleph_{2j+1},\mathfrak{r})}-1\Big)+\alpha_2\Big(\frac{\Psi_\varrho(\aleph_{2j},\aleph_{2j+1},\mathfrak{r})}{\Psi_\varrho(\aleph_{2j},\aleph_{2j+2},2\mathfrak{s}\mathfrak{r})}-1\Big)\\
&+\alpha_3\Big(\frac{\Psi_\varrho(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}{\Psi_\varrho(\aleph_{2j},\aleph_{2j+2},2\mathfrak{s}\mathfrak{r})}-1\Big)\\
&+\alpha_4\Big(\frac{1}{\Psi_\varrho(\aleph_{2j},\aleph_{2j+1},\mathfrak{r})}-1+\frac{1}{\Psi_\varrho(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}-1\Big).\end{aligned}$$

Since, $\Psi_\varrho(\aleph_{2j},\aleph_{2j+2},2\mathfrak{s}\mathfrak{r}) \geq \Psi_\varrho(\aleph_{2j},\aleph_{2j+1},\mathfrak{r})*\Psi_\varrho(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})$, for $\mathfrak{r}>0$, we have further

$$\begin{aligned}\frac{1}{\Psi_\varrho(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}-1 &\leq \alpha_1\Big(\frac{1}{\Psi_\varrho(\aleph_{2j},\aleph_{2j+1},\mathfrak{r})}-1\Big)\\
&+\alpha_2\Big(\frac{\Psi_\varrho(\aleph_{2j},\aleph_{2j+1},\mathfrak{r})}{\Psi_\varrho(\aleph_{2j},\aleph_{2j+1},\mathfrak{r})*\Psi_\varrho(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}-1\Big)\\
&+\alpha_3\Big(\frac{\Psi_\varrho(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}{\Psi_\varrho(\aleph_{2j},\aleph_{2j+1},\mathfrak{r})*\Psi_\varrho(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}-1\Big)\\
&+\alpha_4\Big(\frac{1}{\Psi_\varrho(\aleph_{2j},\aleph_{2j+1},\mathfrak{r})}-1+\frac{1}{\Psi_\varrho(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}-1\Big).\end{aligned}$$

After simplification, we get that

$$\frac{1}{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}-1\leq\gamma\Big(\frac{1}{\Psi_{\varrho}(\aleph_{2j},\aleph_{2j+1},\mathfrak{r})}-1\Big),\tag{2.2}$$

for $\mathfrak{r}>0$, where $\gamma=(\alpha_1+\alpha_3+\alpha_4)/(1-\alpha_2-\alpha_4)<1$, since $\alpha_2,\alpha_3,\alpha_4\geq 0$ with $\alpha_1+\alpha_2+\alpha_3+2\alpha_4<\frac{1}{\mathfrak{s}}$. Similarly,

$$\begin{aligned}
\frac{1}{\Psi_{\varrho}(\aleph_{2j+2},\aleph_{2j+3},\mathfrak{r})}-1&=\frac{1}{\Psi_{\varrho}(\Gamma\aleph_{2j+1},\Omega\aleph_{2j+2},\mathfrak{r})}-1\\
&\leq\alpha_1\Big(\frac{1}{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}-1\Big)\\
&+\alpha_2\Big(\frac{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}{\Psi_{\varrho}(\aleph_{2j+1},\Omega\aleph_{2j+2},2\mathfrak{s}\mathfrak{r})*\Psi_{\varrho}(\aleph_{2j+2},\Gamma\aleph_{2j+1},2\mathfrak{s}\mathfrak{r})}-1\Big)\\
&+\alpha_3\Big(\frac{\Psi_{\varrho}(\aleph_{2j+1},\Gamma\aleph_{2j+1},\mathfrak{r})*\Psi_{\varrho}(\aleph_{2j+2},\Omega\aleph_{2j+2},\mathfrak{r})}{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})*\Psi_{\varrho}(\aleph_{2j+1},\Omega\aleph_{2j+2},2\mathfrak{s}\mathfrak{r})*\Psi_{\varrho}(\aleph_{2j+2},\Gamma\aleph_{2j+1},2\mathfrak{s}\mathfrak{r})}-1\Big)\\
&+\alpha_4\Big(\frac{1}{\Psi_{\varrho}(\aleph_{2j+1},\Gamma\aleph_{2j+1},\mathfrak{r})}-1+\frac{1}{\Psi_{\varrho}(\aleph_{2j+2},\Omega\aleph_{2j+2},\mathfrak{r})}-1\Big)\\
&=\alpha_1\Big(\frac{1}{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}-1\Big)\\
&+\alpha_2\Big(\frac{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+3},2\mathfrak{s}\mathfrak{r})*\Psi_{\varrho}(\aleph_{2j+2},\aleph_{2j+2},2\mathfrak{s}\mathfrak{r})}-1\Big)\\
&+\alpha_3\Big(\frac{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})*\Psi_{\varrho}(\aleph_{2j+2},\aleph_{2j+3},\mathfrak{r})}{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})*\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+3},2\mathfrak{s}\mathfrak{r})*\Psi_{\varrho}(\aleph_{2j+2},\aleph_{2j+2},2\mathfrak{s}\mathfrak{r})}-1\Big)\\
&+\alpha_4\Big(\frac{1}{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}-1+\frac{1}{\Psi_{\varrho}(\aleph_{2j+2},\aleph_{2j+3},\mathfrak{r})}-1\Big)\\
&=\alpha_1\Big(\frac{1}{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}-1\Big)+\alpha_2\Big(\frac{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+3},2\mathfrak{s}\mathfrak{r})}-1\Big)\\
&+\alpha_3\Big(\frac{\Psi_{\varrho}(\aleph_{2j+2},\aleph_{2j+3},\mathfrak{r})}{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+3},2\mathfrak{s}\mathfrak{r})}-1\Big)\\
&+\alpha_4\Big(\frac{1}{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}-1+\frac{1}{\Psi_{\varrho}(\aleph_{2j+2},\aleph_{2j+3},\mathfrak{r})}-1\Big).
\end{aligned}$$

Since, $\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+3},2\mathfrak{r})\geq\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})*\Psi_{\varrho}(\aleph_{2j+2},\aleph_{2j+3},\mathfrak{r})$, for $\mathfrak{r}>0$, we have

$$\begin{aligned}
\frac{1}{\Psi_{\varrho}(\aleph_{2j+2},\aleph_{2j+3},\mathfrak{r})}-1\leq\alpha_1&\Big(\frac{1}{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}-1\Big)\\
&+\alpha_2\Big(\frac{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})*\Psi_{\varrho}(\aleph_{2j+2},\aleph_{2j+3},\mathfrak{r})}-1\Big)\\
&+\alpha_3\Big(\frac{\Psi_{\varrho}(\aleph_{2j+2},\aleph_{2j+3},\mathfrak{r})}{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})*\Psi_{\varrho}(\aleph_{2j+2},\aleph_{2j+3},\mathfrak{r})}-1\Big)\\
&+\alpha_4\Big(\frac{1}{\Psi_{\varrho}(\aleph_{2j+1},\aleph_{2j+2},\mathfrak{r})}-1+\frac{1}{\Psi_{\varrho}(\aleph_{2j+2},\aleph_{2j+3},\mathfrak{r})}-1\Big).
\end{aligned}$$

After simplification, we have

$$\frac{1}{\Psi_{\varrho}(\aleph_{2\mathfrak{j}+2}, \aleph_{2\mathfrak{j}+3}, \mathfrak{r})} - 1 \leq \gamma\Big(\frac{1}{\Psi_{\varrho}(\aleph_{2\mathfrak{j}+1}, \aleph_{2\mathfrak{j}+2}, \mathfrak{r})} - 1\Big), \tag{2.3}$$

for $\mathfrak{r} > 0$, where $\gamma = (\alpha_1 + \alpha_3 + \alpha_4)/(1 - \alpha_2 - \alpha_4) < 1$, since $\alpha_2, \alpha_3, \alpha_4 \geq 0$ with $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 < \frac{1}{\mathfrak{s}}$. Now, from (2.2) and (2.3) and by induction, we have

$$\begin{aligned}
\frac{1}{\Psi_{\varrho}(\aleph_{2\mathfrak{j}+2}, \aleph_{2\mathfrak{j}+3}, \mathfrak{r})} - 1 &\leq \gamma\Big(\frac{1}{\Psi_{\varrho}(\aleph_{2\mathfrak{j}+1}, \aleph_{2\mathfrak{j}+2}, \mathfrak{r})} - 1\Big) \\
&\leq \gamma^2\Big(\frac{1}{\Psi_{\varrho}(\aleph_{2\mathfrak{j}}, \aleph_{2\mathfrak{j}+1}, \mathfrak{r})} - 1\Big) \\
&\vdots \\
&\leq \gamma^{2\mathfrak{j}+2}\Big(\frac{1}{\Psi_{\varrho}(\aleph_0, \aleph_1, \mathfrak{r})} - 1\Big).
\end{aligned}$$

Therefore,

$$\lim_{\mathfrak{j}\to\infty} \Psi_{\varrho}(\aleph_{2\mathfrak{j}+1}, \aleph_{2\mathfrak{j}+2}, \mathfrak{r}) = 1, \tag{2.4}$$

for $\mathfrak{r} > 0$. Note that Ψ_{ϱ} is triangular, then, for all $\mathfrak{k} > \mathfrak{j} \geq \mathfrak{j}_0$,

$$\begin{aligned}
\frac{1}{\Psi_{\varrho}(\aleph_{\mathfrak{j}}, \aleph_{\mathfrak{k}}, \mathfrak{r})} - 1 &\leq \mathfrak{s}\Big(\frac{1}{\Psi_{\varrho}(\aleph_{\mathfrak{j}}, \aleph_{\mathfrak{j}+1}, \mathfrak{r})} - 1 + \frac{1}{\Psi_{\varrho}(\aleph_{\mathfrak{j}+1}, \aleph_{\mathfrak{k}}, \mathfrak{r})} - 1\Big) \\
&\leq \mathfrak{s}\Big(\frac{1}{\Psi_{\varrho}(\aleph_{\mathfrak{j}}, \aleph_{\mathfrak{j}+1}, \mathfrak{r})} - 1\Big) \\
&+ \mathfrak{s}^2\Big(\frac{1}{\Psi_{\varrho}(\aleph_{\mathfrak{j}+1}, \aleph_{\mathfrak{j}+2}, \mathfrak{r})} - 1\Big) \\
&+ \mathfrak{s}^3\Big(\frac{1}{\Psi_{\varrho}(\aleph_{\mathfrak{j}+2}, \aleph_{\mathfrak{j}+3}, \mathfrak{r})} - 1\Big) \\
&\vdots \\
&+ \mathfrak{s}^{\mathfrak{k}-\mathfrak{j}-1}\Big(\frac{1}{\Psi_{\varrho}(\aleph_{\mathfrak{k}-1}, \aleph_{\mathfrak{k}}, \mathfrak{r})} - 1\Big) \\
&\leq (\mathfrak{s}\gamma^{\mathfrak{j}} + \mathfrak{s}^2\gamma^{\mathfrak{j}+1} + ... + \mathfrak{s}^{\mathfrak{k}-\mathfrak{j}-1}\gamma^{k-1})\Big(\frac{1}{\Psi_{\varrho}(\aleph_0, \aleph_1, \mathfrak{r})} - 1\Big) \\
&\leq \frac{\mathfrak{s}\gamma^{\mathfrak{j}}}{1 - \mathfrak{s}\gamma}\Big(\frac{1}{\Psi_{\varrho}(\aleph_0, \aleph_1, \mathfrak{r})} - 1\Big) \to 0, \quad as \quad \mathfrak{j} \to \infty.
\end{aligned}$$

Therefore, $(\aleph_{\mathfrak{j}})$ is a Cauchy sequence in Υ. Since Υ is a complete, there is $\varpi_1 \in \Upsilon$ such that

$$\lim_{\mathfrak{j}\to\infty} \Psi_{\varrho}(\aleph_{2\mathfrak{j}+1}, \varpi_1, \mathfrak{r}) = 1, \tag{2.5}$$

for $\mathfrak{r} > 0$. Next, we show that $\Omega\varpi_1 = \varpi_1$. Since Ψ_{ϱ} is triangular,

$$\frac{1}{\Psi_{\varrho}(\varpi_1, \Omega\varpi_1, \mathfrak{r})} - 1 \leq \mathfrak{s}\Big(\frac{1}{\Psi_{\varrho}(\varpi_1, \aleph_{2\mathfrak{j}+1}, \mathfrak{r})} - 1 + \frac{1}{\Psi_{\varrho}(\aleph_{2\mathfrak{j}+1}, \Omega\varpi_1, \mathfrak{r})} - 1\Big), \tag{2.6}$$

for $\mathfrak{r} > 0$. By (2.1), (2.4) and (2.5), for $\mathfrak{r} > 0$,

$$\begin{aligned}\frac{1}{\Psi_\varrho(\aleph_{2\mathfrak{j}+1}, \Omega\varpi_1, \mathfrak{r})} - 1 &= \frac{1}{\Psi_\varrho(\Gamma\aleph_{2\mathfrak{j}}, \Omega\varpi_1, \mathfrak{r})} - 1\\ &\leq \alpha_1\Big(\frac{1}{\Psi_\varrho(\aleph_{2\mathfrak{j}}, \varpi_1, \mathfrak{r})} - 1\Big)\\ &+ \alpha_2\Big(\frac{\Psi_\varrho(\aleph_{2\mathfrak{j}}, \varpi_1, \mathfrak{r})}{\Psi_\varrho(\aleph_{2\mathfrak{j}}, \Omega\varpi_1, 2\mathfrak{s}\mathfrak{r}) * \Psi_\varrho(\varpi_1, \Gamma\aleph_{2\mathfrak{j}}, 2\mathfrak{s}\mathfrak{r})} - 1\Big)\\ &+ \alpha_3\Big(\frac{\Psi_\varrho(\aleph_{2\mathfrak{j}}, \Gamma\aleph_{2\mathfrak{j}}, \mathfrak{r}) * \Psi_\varrho(\varpi_1, \Omega\varpi_1, \mathfrak{r})}{\Psi_\varrho(\aleph_{2\mathfrak{j}}, \varpi_1, \mathfrak{r}) * \Psi_\varrho(\aleph_{2\mathfrak{j}}, \Omega\varpi_1, 2\mathfrak{s}\mathfrak{r}) * \Psi_\varrho(\varpi_1, \Gamma\aleph_{2\mathfrak{j}}, 2\mathfrak{s}\mathfrak{r})} - 1\Big)\\ &+ \alpha_4\Big(\frac{1}{\Psi_\varrho(\aleph_{2\mathfrak{j}}, \Gamma\aleph_{2\mathfrak{j}}, \mathfrak{r})} - 1 + \frac{1}{\Psi_\varrho(\varpi_1, \Omega\varpi_1, \mathfrak{r})} - 1\Big)\\ &= \alpha_1\Big(\frac{1}{\Psi_\varrho(\aleph_{2\mathfrak{j}}, \varpi_1, \mathfrak{r})} - 1\Big)\\ &+ \alpha_2\Big(\frac{\Psi_\varrho(\aleph_{2\mathfrak{j}}, \varpi_1, \mathfrak{r})}{\Psi_\varrho(\aleph_{2\mathfrak{j}}, \Omega\varpi_1, 2\mathfrak{s}\mathfrak{r}) * \Psi_\varrho(\varpi_1, \aleph_{2\mathfrak{j}+1}, 2\mathfrak{s}\mathfrak{r})} - 1\Big)\\ &+ \alpha_3\Big(\frac{\Psi_\varrho(\aleph_{2\mathfrak{j}}, \aleph_{2\mathfrak{j}+1}, \mathfrak{r}) * \Psi_\varrho(\varpi_1, \Omega\varpi_1, \mathfrak{r})}{\Psi_\varrho(\aleph_{2\mathfrak{j}}, \varpi_1, \mathfrak{r}) * \Psi_\varrho(\aleph_{2\mathfrak{j}}, \Omega\varpi_1, 2\mathfrak{s}\mathfrak{r}) * \Psi_\varrho(\varpi_1, \aleph_{2\mathfrak{j}+1}, 2\mathfrak{s}\mathfrak{r})} - 1\Big)\\ &+ \alpha_4\Big(\frac{1}{\Psi_\varrho(\aleph_{2\mathfrak{j}}, \aleph_{2\mathfrak{j}+1}, \mathfrak{r})} - 1 + \frac{1}{\Psi_\varrho(\varpi_1, \Omega\varpi_1, \mathfrak{r})} - 1\Big).\end{aligned}$$

Since

$$\Psi_\varrho(\aleph_{2\mathfrak{j}}, \Gamma\varpi_1, 2\mathfrak{s}\mathfrak{r}) \geq \Psi_\varrho(\aleph_{2\mathfrak{j}}, \varpi_1, \mathfrak{r}) * \Psi_\varrho(\varpi_1, \Gamma\varpi_1, \mathfrak{r}),$$

for $\mathfrak{r} > 0$. Then,

$$\begin{aligned}\frac{1}{\Psi_\varrho(\aleph_{2\mathfrak{j}+1}, \Omega\varpi_1, \mathfrak{r})} - 1 &\leq \alpha_1\Big(\frac{1}{\Psi_\varrho(\aleph_{2\mathfrak{j}}, \varpi_1, \mathfrak{r})} - 1\Big)\\ &+ \alpha_2\Big(\frac{\Psi_\varrho(\aleph_{2\mathfrak{j}}, \varpi_1, \mathfrak{r})}{\Psi_\varrho(\aleph_{2\mathfrak{j}}, \varpi_1, \mathfrak{r}) * \Psi_\varrho(\varpi_1, \Omega\varpi_1, \mathfrak{r}) * \Psi_\varrho(\varpi_1, \aleph_{2\mathfrak{j}+1}, 2\mathfrak{s}\mathfrak{r})} - 1\Big)\\ &+ \alpha_3\Big(\frac{\Psi_\varrho(\aleph_{2\mathfrak{j}}, \aleph_{2\mathfrak{j}+1}, \mathfrak{r}) * \Psi_\varrho(\varpi_1, \Omega\varpi_1, \mathfrak{r})}{\Psi_\varrho(\aleph_{2\mathfrak{j}}, \varpi_1, \mathfrak{r}) * \Psi_\varrho(\aleph_{2\mathfrak{j}}, \varpi_1, \mathfrak{r}) * \Psi_\varrho(\varpi_1, \Omega\varpi_1, \mathfrak{r}) * \Psi_\varrho(\varpi_1, \aleph_{2\mathfrak{j}+1}, 2\mathfrak{s}\mathfrak{r})} - 1\Big)\\ &+ \alpha_4\Big(\frac{1}{\Psi_\varrho(\aleph_{2\mathfrak{j}}, \aleph_{2\mathfrak{j}+1}, \mathfrak{r})} - 1 + \frac{1}{\Psi_\varrho(\varpi_1, \Omega\varpi_1, \mathfrak{r})} - 1\Big)\\ &\to (\alpha_2 + \alpha_4)\Big(\frac{1}{\Psi_\varrho(\varpi_1, \Omega\varpi_1, \mathfrak{r})} - 1\Big), \text{ as } \mathfrak{j} \to \infty.\end{aligned}$$

Then,

$$\limsup_{\mathfrak{j}\to\infty}\Big(\frac{1}{\Psi_\varrho(\aleph_{2\mathfrak{j}+1}, \Omega\varpi_1, \mathfrak{r})} - 1\Big) \leq (\alpha_2 + \alpha_4)\Big(\frac{1}{\Psi_\varrho(\varpi_1, \Omega\varpi_1, \mathfrak{r})} - 1\Big),$$

for $\mathfrak{r} > 0$. By (2.5) and (2.6), we get

$$\frac{1}{\Psi_\varrho(\varpi_1, \Omega\varpi_1, \mathfrak{r})} - 1 \leq \mathfrak{s}(\alpha_2 + \alpha_4)\Big(\frac{1}{\Psi_\varrho(\varpi_1, \Omega\varpi_1, \mathfrak{r})} - 1\Big).$$

Note that $(\alpha_2 + \alpha_4) < \frac{1}{\mathfrak{s}}$ because $\alpha_2, \alpha_3, \alpha_4 \geq 0$ with $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 < \frac{1}{\mathfrak{s}}$. Then,

$$\Psi_\varrho(\varpi_1, \Omega\varpi_1, \mathfrak{r}) = 1.$$

Therefore, $\Omega\varpi_1 = \varpi_1$. Similarly, we can show that $\Gamma\varpi_1 = \varpi_1$ because Ψ_ϱ is triangular. Therefore,

$$\frac{1}{\Psi_\varrho(\varpi_1, \Gamma\varpi_1, \mathfrak{r})} - 1 \leq \mathfrak{s}\Big(\frac{1}{\Psi_\varrho(\varpi_1, \aleph_{2\mathrm{j}+2}, \mathfrak{r})} - 1 + \frac{1}{\Psi_\varrho(\aleph_{2\mathrm{j}+2}, \Gamma\varpi_1, \mathfrak{r})} - 1\Big), \tag{2.7}$$

for $\mathfrak{r} > 0$. By (2.1), (2.4) and (2.5), for $\mathfrak{r} > 0$,

$$\begin{aligned}\frac{1}{\Psi_\varrho(\aleph_{2\mathrm{j}+2}, \Gamma\varpi_1, \mathfrak{r})} - 1 &= \frac{1}{\Psi_\varrho(\Gamma\varpi_1, \Omega\aleph_{2\mathrm{j}+1}, \mathfrak{r})} - 1\\
&\leq \alpha_1\Big(\frac{1}{\Psi_\varrho(\varpi_1, \aleph_{2\mathrm{j}+1}, \mathfrak{r})} - 1\Big)\\
&\quad + \alpha_2\Big(\frac{\Psi_\varrho(\varpi_1, \aleph_{2\mathrm{j}+1}, \mathfrak{r})}{\Psi_\varrho(\varpi_1, \Omega\aleph_{2\mathrm{j}+1}, 2\mathfrak{s}\mathfrak{r}) * \Psi_\varrho(\aleph_{2\mathrm{j}+1}, \Gamma\varpi_1, 2\mathfrak{s}\mathfrak{r})} - 1\Big)\\
&\quad + \alpha_3\Big(\frac{\Psi_\varrho(\varpi_1, \Gamma\varpi_1, \mathfrak{r}) * \Psi_\varrho(\aleph_{2\mathrm{j}+1}, \Omega\aleph_{2\mathrm{j}+1}, \mathfrak{r})}{\Psi_\varrho(\varpi_1, \aleph_{2\mathrm{j}+1}, \mathfrak{r}) * \Psi_\varrho(\varpi_1, \Omega\aleph_{2\mathrm{j}+1}, 2\mathfrak{s}\mathfrak{r}) * \Psi_\varrho(\aleph_{2\mathrm{j}+1}, \Gamma\varpi_1, 2\mathfrak{s}\mathfrak{r})} - 1\Big)\\
&\quad + \alpha_4\Big(\frac{1}{\Psi_\varrho(\varpi_1, \Gamma\varpi_1, \mathfrak{r})} - 1 + \frac{1}{\Psi_\varrho(\aleph_{2\mathrm{j}+1}, \Omega\aleph_{2\mathrm{j}+1}, \mathfrak{r})} - 1\Big)\\
&= \alpha_1\Big(\frac{1}{\Psi_\varrho(\varpi_1, \aleph_{2\mathrm{j}+1}, \mathfrak{r})} - 1\Big)\\
&\quad + \alpha_2\Big(\frac{\Psi_\varrho(\varpi_1, \aleph_{2\mathrm{j}+1}, \mathfrak{r})}{\Psi_\varrho(\varpi_1, \aleph_{2\mathrm{j}+2}, 2\mathfrak{s}\mathfrak{r}) * \Psi_\varrho(\aleph_{2\mathrm{j}+1}, \Gamma\varpi_1, 2\mathfrak{s}\mathfrak{r})} - 1\Big)\\
&\quad + \alpha_3\Big(\frac{\Psi_\varrho(\varpi_1, \Gamma\varpi_1, \mathfrak{r}) * \Psi_\varrho(\aleph_{2\mathrm{j}+1}, \aleph_{2\mathrm{j}+2}, \mathfrak{r})}{\Psi_\varrho(\varpi_1, \aleph_{2\mathrm{j}+1}, \mathfrak{r}) * \Psi_\varrho(\varpi_1, \aleph_{2\mathrm{j}+2}, 2\mathfrak{s}\mathfrak{r}) * \Psi_\varrho(\aleph_{2\mathrm{j}+1}, \Gamma\varpi_1, 2\mathfrak{s}\mathfrak{r})} - 1\Big)\\
&\quad + \alpha_4\Big(\frac{1}{\Psi_\varrho(\varpi_1, \Gamma\varpi_1, \mathfrak{r})} - 1 + \frac{1}{\Psi_\varrho(\aleph_{2\mathrm{j}+1}, \aleph_{2\mathrm{j}+2}, \mathfrak{r})} - 1\Big).\end{aligned}$$

Since

$$\Psi_\varrho(\aleph_{2\mathrm{j}+1}, \Gamma\varpi_1, 2\mathfrak{s}\mathfrak{r}) \geq \Psi_\varrho(\aleph_{2\mathrm{j}+1}, \varpi_1, \mathfrak{r}) * \Psi_\varrho(\varpi_1, \Gamma\varpi_1, \mathfrak{r}),$$

for $\mathfrak{r} > 0$. Then,

$$\begin{aligned}&\frac{1}{\Psi_\varrho(\aleph_{2\mathrm{j}+1}, \Gamma\varpi_1, \mathfrak{r})} - 1\\
&\leq \alpha_1\Big(\frac{1}{\Psi_\varrho(\varpi_1, \aleph_{2\mathrm{j}+1}, \mathfrak{r})} - 1\Big)\\
&\quad + \alpha_2\Big(\frac{\Psi_\varrho(\varpi_1, \aleph_{2\mathrm{j}+1}, \mathfrak{r})}{\Psi_\varrho(\varpi_1, \aleph_{2\mathrm{j}+2}, 2\mathfrak{s}\mathfrak{r}) * \Psi_\varrho(\aleph_{2\mathrm{j}+1}, \varpi_1, \mathfrak{r}) * \Psi_\varrho(\varpi_1, \Gamma\varpi_1, \mathfrak{r})} - 1\Big)\\
&\quad + \alpha_3\Big(\frac{\Psi_\varrho(\varpi_1, \Gamma\varpi_1, \mathfrak{r}) * \Psi_\varrho(\aleph_{2\mathrm{j}+1}, \aleph_{2\mathrm{j}+2}, \mathfrak{r})}{\Psi_\varrho(\varpi_1, \aleph_{2\mathrm{j}+1}, \mathfrak{r}) * \Psi_\varrho(\varpi_1, \aleph_{2\mathrm{j}+2}, 2\mathfrak{s}\mathfrak{r}) * \Psi_\varrho(\aleph_{2\mathrm{j}+1}, \varpi_1, \mathfrak{r}) * \Psi_\varrho(\varpi_1, \Gamma\varpi_1, \mathfrak{r})} - 1\Big)\end{aligned}$$

$$+\alpha_4\Big(\frac{1}{\Psi_\varrho(\varpi_1,\Gamma\varpi_1,\mathfrak{r})}-1+\frac{1}{\Psi_\varrho(\aleph_{2\mathrm{j}+1},\aleph_{2\mathrm{j}+2},\mathfrak{r})}-1\Big)$$
$$\to(\alpha_2+\alpha_4)\Big(\frac{1}{\Psi_\varrho(\varpi_1,\Gamma\varpi_1,\mathfrak{r})}-1\Big),\ \text{as } \mathrm{j}\to\infty.$$

Therefore,

$$\limsup_{\mathrm{j}\to\infty}\Big(\frac{1}{\Psi_\varrho(\aleph_{2\mathrm{j}+1},\Gamma\varpi_1,\mathfrak{r})}-1\Big)\le(\alpha_2+\alpha_4)\Big(\frac{1}{\Psi_\varrho(\varpi_1,\Gamma\varpi_1,\mathfrak{r})}-1\Big),$$

for $\mathfrak{r}>0$. By (2.7) and (2.5), we get

$$\frac{1}{\Psi_\varrho(\varpi_1,\Gamma\varpi_1,\mathfrak{r})}-1\le\mathfrak{s}(\alpha_2+\alpha_4)\Big(\frac{1}{\Psi_\varrho(\varpi_1,\Gamma\varpi_1,\mathfrak{r})}-1\Big),$$

for $\mathfrak{r}>0$. Note that $\alpha_2+\alpha_4<\frac{1}{\mathfrak{s}}$, since $\alpha_1+\alpha_2+\alpha_3+2\alpha_4<\frac{1}{\mathfrak{s}}$. Then,

$$\Psi_\varrho(\varpi_1,\Gamma\varpi_1,\mathfrak{r})=1.$$

Therefore, $\Gamma\varpi_1=\varpi_1$. Hence, ϖ_1 is a common fixed point of Γ and Ω. □

Example 2.2. Let $\Upsilon=[0,\infty)$, $*$ be a continuous δ-norm and $\Psi_\varrho:\Upsilon^2\times(0,\infty)\to[0,1]$ by

$$\Psi_\varrho(\aleph,\varpi,\mathfrak{r})=\frac{\mathfrak{r}}{\mathfrak{r}+|\aleph-\varpi|^2},\ \text{for all } \aleph,\varpi\in\Upsilon,\mathfrak{r}>0.$$

Then, easily one can verify that Ψ_ϱ is triangular and $(\Psi_\varrho,*)$ is a complete FBM space with $\mathfrak{s}\ge 1$. Define $\Gamma,\Omega:\Upsilon\to\Upsilon$ by

$$\Gamma(\aleph)=\begin{cases}\frac{\aleph}{2}, & \text{if } \aleph\in[0,1),\\ \frac{\aleph-1}{2}, & \text{if } \aleph\in[1,\infty).\end{cases}$$

and

$$\Omega(\aleph)=\begin{cases}\frac{\aleph}{3}, & \text{if } \aleph\in[0,1),\\ \frac{\aleph-1}{3}, & \text{if } \aleph\in[1,\infty).\end{cases}$$

Let $\aleph,\varpi\in[0,1)$ and $\mathfrak{r}>0$, then

$$\begin{aligned}\frac{1}{\Psi_\varrho(\Gamma(\aleph),\Omega(\varpi),\mathfrak{r})}-1&=\frac{|\Gamma(\aleph)-\Omega(\varpi)|^2}{\mathfrak{r}}\\&=\frac{|3\aleph-2\varpi|^2}{36\mathfrak{r}}\\&\le\frac{|3\aleph-2\varpi|^2}{36\mathfrak{r}}\\&\le\frac{|\aleph-\varpi|^2}{4\mathfrak{r}}\\&=\frac{1}{4}\Big(\frac{1}{\Psi_\varrho(\aleph,\varpi,\mathfrak{r})}-1\Big)\\&\le\frac{1}{4}\Big(\frac{1}{\Psi_\varrho(\aleph,\varpi,\mathfrak{r})}-1\Big)\end{aligned}$$

$$+\frac{1}{3}\Big(\frac{\Psi_\varrho(\aleph,\varpi,\mathfrak{r})}{\Psi_\varrho(\aleph,\Omega\varpi,2\mathfrak{s}\mathfrak{r})*\Psi_\varrho(\varpi,\Gamma\aleph,2\mathfrak{s}\mathfrak{r})}-1\Big)$$
$$+\frac{1}{5}\Big(\frac{\Psi_\varrho(\aleph,\Gamma\aleph,\mathfrak{r})*\Psi_\varrho(\varpi,\Omega\varpi,\mathfrak{r})}{\Psi_\varrho(\aleph,\varpi,\mathfrak{r})*\Psi_\varrho(\aleph,\Omega\varpi,2\mathfrak{s}\mathfrak{r})*\Psi_\varrho(\varpi,\Gamma\aleph,2\mathfrak{s}\mathfrak{r})}-1\Big)$$
$$+\frac{1}{6}\Big(\frac{1}{\Psi_\varrho(\aleph,\Gamma\aleph,\mathfrak{r})}-1+\frac{1}{\Psi_\varrho(\varpi,\Omega\varpi,\mathfrak{r})}-1\Big).$$

Let $\aleph\in[0,1)$, $\varpi\in[1,\infty)$ and $\mathfrak{r}>0$, then

$$\begin{aligned}\frac{1}{\Psi_\varrho(\Gamma(\aleph),\Omega(\varpi),\mathfrak{r})}-1&=\frac{|\Gamma(\aleph)-\Omega(\varpi)|^2}{\mathfrak{r}}\\&=\frac{|3\aleph-2\varpi+2|^2}{36\mathfrak{r}}\\&\leq\frac{|\aleph-\varpi|^2}{4\mathfrak{r}}\\&=\frac{1}{4}\Big(\frac{1}{\Psi_\varrho(\aleph,\varpi,\mathfrak{r})}-1\Big)\\&\leq\frac{1}{4}\Big(\frac{1}{\Psi_\varrho(\aleph,\varpi,\mathfrak{r})}-1\Big)\\&+\frac{1}{3}\Big(\frac{\Psi_\varrho(\aleph,\varpi,\mathfrak{r})}{\Psi_\varrho(\aleph,\Omega\varpi,2\mathfrak{s}\mathfrak{r})*\Psi_\varrho(\varpi,\Gamma\aleph,2\mathfrak{s}\mathfrak{r})}-1\Big)\\&+\frac{1}{5}\Big(\frac{\Psi_\varrho(\aleph,\Gamma\aleph,\mathfrak{r})*\Psi_\varrho(\varpi,\Omega\varpi,\mathfrak{r})}{\Psi_\varrho(\aleph,\varpi,\mathfrak{r})*\Psi_\varrho(\aleph,\Omega\varpi,2\mathfrak{s}\mathfrak{r})*\Psi_\varrho(\varpi,\Gamma\aleph,2\mathfrak{s}\mathfrak{r})}-1\Big)\\&+\frac{1}{6}\Big(\frac{1}{\Psi_\varrho(\aleph,\Gamma\aleph,\mathfrak{r})}-1+\frac{1}{\Psi_\varrho(\varpi,\Omega\varpi,\mathfrak{r})}-1\Big).\end{aligned}$$

Let $\varpi\in[0,1)$, $\aleph\in[1,\infty)$ and $\mathfrak{r}>0$, then

$$\begin{aligned}\frac{1}{\Psi_\varrho(\Gamma(\aleph),\Omega(\varpi),\mathfrak{r})}-1&=\frac{|\Gamma(\aleph)-\Omega(\varpi)|^2}{\mathfrak{r}}\\&=\frac{|3\aleph-2\varpi-3|^2}{36\mathfrak{r}}\\&\leq\frac{|\aleph-\varpi|^2}{4\mathfrak{r}}\\&=\frac{1}{4}\Big(\frac{1}{\Psi_\varrho(\aleph,\varpi,\mathfrak{r})}-1\Big)\\&\leq\frac{1}{4}\Big(\frac{1}{\Psi_\varrho(\aleph,\varpi,\mathfrak{r})}-1\Big)\\&+\frac{1}{3}\Big(\frac{\Psi_\varrho(\aleph,\varpi,\mathfrak{r})}{\Psi_\varrho(\aleph,\Omega\varpi,2\mathfrak{s}\mathfrak{r})*\Psi_\varrho(\varpi,\Gamma\aleph,2\mathfrak{s}\mathfrak{r})}-1\Big)\\&+\frac{1}{5}\Big(\frac{\Psi_\varrho(\aleph,\Gamma\aleph,\mathfrak{r})*\Psi_\varrho(\varpi,\Omega\varpi,\mathfrak{r})}{\Psi_\varrho(\aleph,\varpi,\mathfrak{r})*\Psi_\varrho(\aleph,\Omega\varpi,2\mathfrak{s}\mathfrak{r})*\Psi_\varrho(\varpi,\Gamma\aleph,2\mathfrak{s}\mathfrak{r})}-1\Big)\\&+\frac{1}{6}\Big(\frac{1}{\Psi_\varrho(\aleph,\Gamma\aleph,\mathfrak{r})}-1+\frac{1}{\Psi_\varrho(\varpi,\Omega\varpi,\mathfrak{r})}-1\Big).\end{aligned}$$

Let $\aleph, \varpi \in [1, \infty)$ and $\mathfrak{r} > 0$, then

$$\begin{aligned}
&\frac{1}{\Psi_\varrho(\Gamma(\aleph), \Omega(\varpi), \mathfrak{r})} - 1 \\
&= \frac{|\Gamma(\aleph) - \Omega(\varpi)|^2}{\mathfrak{r}} \\
&= \frac{|3\aleph - 3 - 2\varpi + 2|^2}{36\mathfrak{r}} \\
&= \frac{|3\aleph - 2\varpi - 1|^2}{36\mathfrak{r}} \\
&\leq \frac{|\aleph - \varpi|^2}{4\mathfrak{r}} \\
&= \frac{1}{4}\Big(\frac{1}{\Psi_\varrho(\aleph, \varpi, \mathfrak{r})} - 1\Big) \\
&\leq \frac{1}{4}\Big(\frac{1}{\Psi_\varrho(\aleph, \varpi, \mathfrak{r})} - 1\Big) \\
&+ \frac{1}{3}\Big(\frac{\Psi_\varrho(\aleph, \varpi, \mathfrak{r})}{\Psi_\varrho(\aleph, \Omega\varpi, 2\mathfrak{s}\mathfrak{r}) * \Psi_\varrho(\varpi, \Gamma\aleph, 2\mathfrak{s}\mathfrak{r})} - 1\Big) \\
&+ \frac{1}{5}\Big(\frac{\Psi_\varrho(\aleph, \Gamma\aleph, \mathfrak{r}) * \Psi_\varrho(\varpi, \Omega\varpi, \mathfrak{r})}{\Psi_\varrho(\aleph, \varpi, \mathfrak{r}) * \Psi_\varrho(\aleph, \Omega\varpi, 2\mathfrak{s}\mathfrak{r}) * \Psi_\varrho(\varpi, \Gamma\aleph, 2\mathfrak{s}\mathfrak{r})} - 1\Big) \\
&+ \frac{1}{6}\Big(\frac{1}{\Psi_\varrho(\aleph, \Gamma\aleph, \mathfrak{r})} - 1 + \frac{1}{\Psi_\varrho(\varpi, \Omega\varpi, \mathfrak{r})} - 1\Big).
\end{aligned}$$

Therefore, all the conditions of Theorem 2.1 are fulfilled. Hence Γ and Ω have a common fixed point.

Corollary 2.3. *Let $(\Upsilon, \Psi_\varrho, *)$ be a complete FBM space with $\mathfrak{s} \geq 1$ in which Ψ_ϱ is triangular. Let $\Gamma, \Omega\colon \Upsilon \to \Upsilon$ be a pair of self-mappings such that*

$$\begin{aligned}
\frac{1}{\Psi_\varrho(\Gamma\aleph, \Omega\varpi, \mathfrak{r})} - 1 \leq{}& \alpha_1\Big(\frac{1}{\Psi_\varrho(\aleph, \varpi, \mathfrak{r})} - 1\Big) \\
&+ \alpha_2\Big(\frac{\Psi_\varrho(\aleph, \varpi, \mathfrak{r})}{\Psi_\varrho(\aleph, \Omega\varpi, 2\mathfrak{s}\mathfrak{r}) * \Psi_\varrho(\varpi, \Gamma\aleph, 2\mathfrak{s}\mathfrak{r})} - 1\Big) \\
&+ \alpha_4\Big(\frac{1}{\Psi_\varrho(\aleph, \Gamma\aleph, \mathfrak{r})} - 1 + \frac{1}{\Psi_\varrho(\varpi, \Omega\varpi, \mathfrak{r})} - 1\Big),
\end{aligned} \tag{2.8}$$

for all $\aleph, \varpi \in \Upsilon, \mathfrak{r} > 0, \alpha_1 \in (0, 1)$ and $\alpha_2, \alpha_4 \geq 0$ with $\alpha_1 + \alpha_2 + 2\alpha_4 < \frac{1}{\mathfrak{s}}$. Then, Γ and Ω have a unique common fixed point in Υ.

Proof. It follows from the proof of Theorem 2.1 that ϖ_1 is a common fixed point of Γ and Ω in Υ such that $\Gamma\varpi_1 = \Omega\varpi_1 = \varpi_1$. For uniqueness, let $\aleph_1$ be another common fixed point of Γ and Ω in Υ such that $\Gamma\aleph_1 = \Omega\aleph_1 = \aleph_1$. Then,

$$\begin{aligned}
\frac{1}{\Psi_\varrho(\aleph_1, \varpi_1, \mathfrak{r})} - 1 &= \frac{1}{\Psi_\varrho(\Gamma\aleph_1, \Omega\varpi_1, \mathfrak{r})} - 1 \\
&\leq \alpha_1\Big(\frac{1}{\Psi_\varrho(\aleph_1, \varpi_1, \mathfrak{r})} - 1\Big)
\end{aligned}$$

$$+\alpha_2\Big(\frac{\Psi_\varrho(\aleph_1,\varpi_1,\mathfrak{r})}{\Psi_\varrho(\aleph_1,\Omega\varpi_1,2\mathfrak{s}\mathfrak{r})*\Psi_\varrho(\varpi_1,\Gamma\aleph_1,2\mathfrak{s}\mathfrak{r})}-1\Big)$$
$$+\alpha_4\Big(\frac{1}{\Psi_\varrho(\aleph_1,\Gamma\aleph_1,\mathfrak{r})}-1+\frac{1}{\Psi_\varrho(\varpi_1,\Omega\varpi_1,\mathfrak{r})}-1\Big).$$

Since,

$$\begin{aligned}\Psi_\varrho(\varpi_1,\Gamma\aleph_1,2\mathfrak{s}\mathfrak{r}) &\geq \Psi_\varrho(\varpi_1,\aleph_1,\mathfrak{r})*\Psi_\varrho(\aleph_1,\Gamma\aleph_1,\mathfrak{r})\\ &= \Psi_\varrho(\varpi_1,\aleph_1,\mathfrak{r})*1\\ &= \Psi_\varrho(\varpi_1,\aleph_1,\mathfrak{r})\end{aligned}$$

and

$$\begin{aligned}\Psi_\varrho(\aleph_1,\Omega\varpi_1,2\mathfrak{s}\mathfrak{r}) &\geq \Psi_\varrho(\aleph_1,\varpi_1,\mathfrak{r})*\Psi_\varrho(\varpi_1,\Omega\varpi_1,\mathfrak{r})\\ &= \Psi_\varrho(\aleph_1,\varpi_1,\mathfrak{r})*1\\ &= \Psi_\varrho(\aleph_1,\varpi_1,\mathfrak{r}),\end{aligned}$$

for $\mathfrak{r}>0$. Consequently,

$$\begin{aligned}\frac{1}{\Psi_\varrho(\aleph_1,\varpi_1,\mathfrak{r})}-1 &\leq \alpha_1\Big(\frac{1}{\Psi_\varrho(\aleph_1,\varpi_1,\mathfrak{r})}-1\Big)\\ &\quad+\alpha_2\Big(\frac{\Psi_\varrho(\aleph_1,\varpi_1,\mathfrak{r})}{\Psi_\varrho(\aleph_1,\varpi_1,\mathfrak{r})*\Psi_\varrho(\varpi_1,\aleph_1,\mathfrak{r})}-1\Big)\\ &\quad+\alpha_4\Big(\frac{1}{\Psi_\varrho(\aleph_1,\aleph_1,\mathfrak{r})}-1+\frac{1}{\Psi_\varrho(\varpi_1,\varpi_1,\mathfrak{r})}-1\Big)\\ &= (\alpha_1+\alpha_2)\Big(\frac{1}{\Psi_\varrho(\aleph_1,\varpi_1,\mathfrak{r})}-1\Big)\\ &= (\alpha_1+\alpha_2)\Big(\frac{1}{\Psi_\varrho(\Gamma\aleph_1,\Omega\varpi_1,\mathfrak{r})}-1\Big)\\ &\leq (\alpha_1+\alpha_2)^2\Big(\frac{1}{\Psi_\varrho(\aleph_1,\varpi_1,\mathfrak{r})}-1\Big)\\ &\ \ \vdots\\ &\leq (\alpha_1+\alpha_2)^{\mathfrak{j}}\Big(\frac{1}{\Psi_\varrho(\aleph_1,\varpi_1,\mathfrak{r})}-1\Big)\\ &\to 0,\ \text{ as } \mathfrak{j}\to\infty.\end{aligned}$$

Since, $\alpha_1+\alpha_2<1$, therefore $\Psi_\varrho(\aleph_1,\varpi_1,\mathfrak{r})=1$. Hence $\aleph_1=\varpi_1$, for $\mathfrak{r}>0$. □

Example 2.4. Let $\Upsilon=[0,\infty)$, $*$ be a continuous δ-norm and $\Psi_\varrho:\Upsilon^2\times(0,\infty)\to[0,1]$ by

$$\Psi_\varrho(\aleph,\varpi,\mathfrak{r})=\frac{\mathfrak{r}}{\mathfrak{r}+|\aleph-\varpi|^2},\ \text{ for all } \aleph,\varpi\in\Upsilon,\mathfrak{r}>0.$$

Then, easily one can verify that $\varPsi_{\varrho}$ is triangular and $(\varPsi_{\varrho}, *)$ is a complete FBM space with $\mathfrak{s} \geq 1$. Define $\varGamma, \varOmega : \varUpsilon \to \varUpsilon$ by

$$\varGamma(\aleph) = \begin{cases} \frac{\aleph}{2}, & \text{if } \aleph \in [0,1), \\ \frac{\aleph}{2} - 1, & \text{if } \aleph \in [1,\infty), \end{cases}$$

and

$$\varOmega(\aleph) = \begin{cases} \frac{\aleph}{3}, & \text{if } \aleph \in [0,1), \\ \frac{\aleph}{3} - 1, & \text{if } \aleph \in [1,\infty). \end{cases}$$

Let $\aleph, \varpi \in [0,1)$ and $\mathfrak{r} > 0$, then

$$\begin{aligned}
&\frac{1}{\varPsi_{\varrho}(\varGamma(\aleph), \varOmega(\varpi), \mathfrak{r})} - 1 \\
&= \frac{|\varGamma(\aleph) - \varOmega(\varpi)|^2}{\mathfrak{r}} \\
&= \frac{|3\aleph - 2\varpi|^2}{36\mathfrak{r}} \\
&\leq \frac{|\aleph - \varpi|^2}{2\mathfrak{r}} \\
&= \frac{1}{2}\Big(\frac{1}{\varPsi_{\varrho}(\aleph, \varpi, \mathfrak{r})} - 1\Big) \\
&\leq \frac{1}{2}\Big(\frac{1}{\varPsi_{\varrho}(\aleph, \varpi, \mathfrak{r})} - 1\Big) \\
&+ \frac{1}{3}\Big(\frac{\varPsi_{\varrho}(\aleph, \varpi, \mathfrak{r})}{\varPsi_{\varrho}(\aleph, \varOmega\varpi, 2\mathfrak{s}\mathfrak{r}) * \varPsi_{\varrho}(\varpi, \varGamma\aleph, 2\mathfrak{s}\mathfrak{r})} - 1\Big) \\
&+ \frac{1}{5}\Big(\frac{\varPsi_{\varrho}(\aleph, \varGamma\aleph, \mathfrak{r}) * \varPsi_{\varrho}(\varpi, \varOmega\varpi, \mathfrak{r})}{\varPsi_{\varrho}(\aleph, \varpi, \mathfrak{r}) * \varPsi_{\varrho}(\aleph, \varOmega\varpi, 2\mathfrak{s}\mathfrak{r}) * \varPsi_{\varrho}(\varpi, \varGamma\aleph, 2\mathfrak{s}\mathfrak{r})} - 1\Big) \\
&+ \frac{1}{6}\Big(\frac{1}{\varPsi_{\varrho}(\aleph, \varGamma\aleph, \mathfrak{r})} - 1 + \frac{1}{\varPsi_{\varrho}(\varpi, \varOmega\varpi, \mathfrak{r})} - 1\Big).
\end{aligned}$$

Let $\aleph \in [0,1)$, $\varpi \in [1,\infty)$ and $\mathfrak{r} > 0$, then

$$\begin{aligned}
\frac{1}{\varPsi_{\varrho}(\varGamma(\aleph), \varOmega(\varpi), \mathfrak{r})} - 1 &= \frac{|\varGamma(\aleph) - \varOmega(\varpi)|^2}{\mathfrak{r}} \\
&= \frac{|3\aleph - 2\varpi + 6|^2}{36\mathfrak{r}} \\
&\leq \frac{|\aleph - \varpi|^2}{2\mathfrak{r}} \\
&= \frac{1}{2}\Big(\frac{1}{\varPsi_{\varrho}(\aleph, \varpi, \mathfrak{r})} - 1\Big) \\
&\leq \frac{1}{2}\Big(\frac{1}{\varPsi_{\varrho}(\aleph, \varpi, \mathfrak{r})} - 1\Big) \\
&+ \frac{1}{3}\Big(\frac{\varPsi_{\varrho}(\aleph, \varpi, \mathfrak{r})}{\varPsi_{\varrho}(\aleph, \varOmega\varpi, 2\mathfrak{s}\mathfrak{r}) * \varPsi_{\varrho}(\varpi, \varGamma\aleph, 2\mathfrak{s}\mathfrak{r})} - 1\Big)
\end{aligned}$$

$$+\frac{1}{5}\Big(\frac{\Psi_{\varrho}(\aleph,\Gamma\aleph,\mathfrak{r})*\Psi_{\varrho}(\varpi,\Omega\varpi,\mathfrak{r})}{\Psi_{\varrho}(\aleph,\varpi,\mathfrak{r})*\Psi_{\varrho}(\aleph,\Omega\varpi,2\mathfrak{s}\mathfrak{r})*\Psi_{\varrho}(\varpi,\Gamma\aleph,2\mathfrak{s}\mathfrak{r})}-1\Big)$$
$$+\frac{1}{6}\Big(\frac{1}{\Psi_{\varrho}(\aleph,\Gamma\aleph,\mathfrak{r})}-1+\frac{1}{\Psi_{\varrho}(\varpi,\Omega\varpi,\mathfrak{r})}-1\Big).$$

Let $\varpi\in[0,1)$, $\aleph\in[1,\infty)$ and $\mathfrak{r}>0$, then

$$\begin{aligned}\frac{1}{\Psi_{\varrho}(\Gamma(\aleph),\Omega(\varpi),\mathfrak{r})}-1&=\frac{|\Gamma(\aleph)-\Omega(\varpi)|^2}{\mathfrak{r}}\\&=\frac{|3\aleph-2\varpi-6|^2}{36\mathfrak{r}}\\&\le\frac{|\aleph-\varpi|^2}{2\mathfrak{r}}\\&=\frac{1}{2}\Big(\frac{1}{\Psi_{\varrho}(\aleph,\varpi,\mathfrak{r})}-1\Big)\\&\le\frac{1}{2}\Big(\frac{1}{\Psi_{\varrho}(\aleph,\varpi,\mathfrak{r})}-1\Big)\\&+\frac{1}{3}\Big(\frac{\Psi_{\varrho}(\aleph,\varpi,\mathfrak{r})}{\Psi_{\varrho}(\aleph,\Omega\varpi,2\mathfrak{s}\mathfrak{r})*\Psi_{\varrho}(\varpi,\Gamma\aleph,2\mathfrak{s}\mathfrak{r})}-1\Big)\\&+\frac{1}{5}\Big(\frac{\Psi_{\varrho}(\aleph,\Gamma\aleph,\mathfrak{r})*\Psi_{\varrho}(\varpi,\Omega\varpi,\mathfrak{r})}{\Psi_{\varrho}(\aleph,\varpi,\mathfrak{r})*\Psi_{\varrho}(\aleph,\Omega\varpi,2\mathfrak{s}\mathfrak{r})*\Psi_{\varrho}(\varpi,\Gamma\aleph,2\mathfrak{s}\mathfrak{r})}-1\Big)\\&+\frac{1}{6}\Big(\frac{1}{\Psi_{\varrho}(\aleph,\Gamma\aleph,\mathfrak{r})}-1+\frac{1}{\Psi_{\varrho}(\varpi,\Omega\varpi,\mathfrak{r})}-1\Big).\end{aligned}$$

Let $\aleph,\varpi\in[1,\infty)$ and $\mathfrak{r}>0$, then

$$\begin{aligned}\frac{1}{\Psi_{\varrho}(\Gamma(\aleph),\Omega(\varpi),\mathfrak{r})}-1&=\frac{|\Gamma(\aleph)-\Omega(\varpi)|^2}{\mathfrak{r}}\\&=\frac{|3\aleph-2\varpi|^2}{36\mathfrak{r}}\\&\le\frac{|\aleph-\varpi|^2}{2\mathfrak{r}}\\&=\frac{1}{2}\Big(\frac{1}{\Psi_{\varrho}(\aleph,\varpi,\mathfrak{r})}-1\Big)\\&\le\frac{1}{2}\Big(\frac{1}{\Psi_{\varrho}(\aleph,\varpi,\mathfrak{r})}-1\Big)\\&+\frac{1}{3}\Big(\frac{\Psi_{\varrho}(\aleph,\varpi,\mathfrak{r})}{\Psi_{\varrho}(\aleph,\Omega\varpi,2\mathfrak{s}\mathfrak{r})*\Psi_{\varrho}(\varpi,\Gamma\aleph,2\mathfrak{s}\mathfrak{r})}-1\Big)\\&+\frac{1}{5}\Big(\frac{\Psi_{\varrho}(\aleph,\Gamma\aleph,\mathfrak{r})*\Psi_{\varrho}(\varpi,\Omega\varpi,\mathfrak{r})}{\Psi_{\varrho}(\aleph,\varpi,\mathfrak{r})*\Psi_{\varrho}(\aleph,\Omega\varpi,2\mathfrak{s}\mathfrak{r})*\Psi_{\varrho}(\varpi,\Gamma\aleph,2\mathfrak{s}\mathfrak{r})}-1\Big)\\&+\frac{1}{6}\Big(\frac{1}{\Psi_{\varrho}(\aleph,\Gamma\aleph,\mathfrak{r})}-1+\frac{1}{\Psi_{\varrho}(\varpi,\Omega\varpi,\mathfrak{r})}-1\Big).\end{aligned}$$

Based on Theorem 2.1, it is concluded that ϖ is a common fixed point and by Corollary 2.3 follows that it is unique.

Theorem 2.5. *Let $(\Upsilon, \Psi_\varrho, *)$ be a complete FBM space with $\mathfrak{s} \geq 1$ in which Ψ_ϱ is triangular. Let $\Gamma, \Omega\colon \Upsilon \to \Upsilon$ be a pair of self-mappings such that*

$$\begin{aligned}\frac{1}{\Psi_\varrho(\Gamma\aleph, \Omega\varpi, \mathfrak{r})} - 1 \leq\ & \alpha_1\Big(\frac{1}{\Psi_\varrho(\aleph, \varpi, \mathfrak{r})} - 1\Big)\\ & + \alpha_2\Big(\frac{\Psi_\varrho(\aleph, \varpi, \mathfrak{r})}{\Psi_\varrho(\aleph, \Omega\varpi, 2\mathfrak{s}\mathfrak{r}) * \Psi_\varrho(\varpi, \Gamma\aleph, 2\mathfrak{s}\mathfrak{r})} - 1\Big)\\ & + \alpha_3\Big(\frac{\Psi_\varrho(\aleph, \Gamma\aleph, \mathfrak{r}) * \Psi_\varrho(\varpi, \Omega\varpi, \mathfrak{r})}{\Psi_\varrho(\aleph, \varpi, \mathfrak{r}) * \Psi_\varrho(\aleph, \Omega\varpi, 2\mathfrak{s}\mathfrak{r}) * \Psi_\varrho(\varpi, \Gamma\aleph, 2\mathfrak{s}\mathfrak{r})} - 1\Big)\\ & + \alpha_4\Big(\frac{1}{\Psi_\varrho(\varpi, \Gamma\aleph, \mathfrak{r})} - 1 + \frac{1}{\Psi_\varrho(\aleph, \Omega\varpi, \mathfrak{r})} - 1\Big), \end{aligned} \tag{2.9}$$

for all $\aleph, \varpi \in \Upsilon, \mathfrak{r} > 0, \alpha_1 \in (0, 1)$ and $\alpha_2, \alpha_3, \alpha_4 \geq 0$ with $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 < \frac{1}{\mathfrak{s}}$. Then, Γ and Ω have a common fixed point in Υ.

Proof. The proof is similar as the proof of Theorem 2.1. □

Corollary 2.6. *Let $(\Upsilon, \Psi_\varrho, *)$ be a complete FBM space with $\mathfrak{s} \geq 1$ in which Ψ_ϱ is triangular. Let $\Gamma, \Omega\colon \Upsilon \to \Upsilon$ be a pair of self-mappings such that*

$$\begin{aligned}\frac{1}{\Psi_\varrho(\Gamma\aleph, \Omega\varpi, \mathfrak{r})} - 1 \leq\ & \alpha_1\Big(\frac{1}{\Psi_\varrho(\aleph, \varpi, \mathfrak{r})} - 1\Big)\\ & + \alpha_2\Big(\frac{\Psi_\varrho(\aleph, \varpi, \mathfrak{r})}{\Psi_\varrho(\aleph, \Omega\varpi, 2\mathfrak{s}\mathfrak{r}) * \Psi_\varrho(\varpi, \Gamma\aleph, 2\mathfrak{s}\mathfrak{r})} - 1\Big)\\ & + \alpha_4\Big(\frac{1}{\Psi_\varrho(\varpi, \Gamma\aleph, \mathfrak{r})} - 1 + \frac{1}{\Psi_\varrho(\aleph, \Omega\varpi, \mathfrak{r})} - 1\Big), \end{aligned} \tag{2.10}$$

for all $\aleph, \varpi \in \Upsilon, \mathfrak{r} > 0, \alpha_1 \in (0, 1)$ and $\alpha_2, \alpha_4 \geq 0$ with $\alpha_1 + \alpha_2 + 2\alpha_4 < \frac{1}{\mathfrak{s}}$. Then, Γ and Ω have a unique common fixed point in Υ.

Proof. The proof is similar as the proof of Corollary 2.3. □

3. Applications

In this section, we study the existence and unique solution to an Fredholm integral equations as an application of Theorem 2.3. Let $\Upsilon = C([0, \eta], \mathbb{R})$ be the space of all real-valued continuous functions on the interval $[0, \eta]$, where $0 < \eta \in \mathbb{R}$. The Fredholm integral equations are

$$\begin{aligned}\aleph(\tau) &= \int_0^\eta \mathcal{K}_1(\tau, \mathfrak{h}, \aleph(\mathfrak{h}))d\mathfrak{h},\\ \aleph(\tau) &= \int_0^\eta \mathcal{K}_2(\tau, \mathfrak{h}, \aleph(\mathfrak{h}))d\mathfrak{h}.\end{aligned} \tag{3.1}$$

The binary operation $*$ is defined by $\mathfrak{p} * \mathfrak{q} = \mathfrak{p}\mathfrak{q}$, $\forall\ \mathfrak{p}, \mathfrak{q} \in [0, \eta]$. The standard fuzzy metric Ψ_ϱ : $\Upsilon \times \Upsilon \times (0, \infty) \to [0, 1]$ defined by

$$\Psi_\varrho(\aleph, \varpi, \mathfrak{r}) = \frac{\mathfrak{r}}{\mathfrak{r} + |\aleph - \varpi|^2}, \text{ for all } \aleph, \varpi \in \Upsilon \text{ and } \mathfrak{r} > 0.$$

Then, one can easily verify that Ψ_ϱ is triangular and $(\Upsilon, \Psi_\varrho, *)$ is a complete FBM space with $\mathfrak{s} \geq 1$.

Theorem 3.1. *Assume that*

(T1) there is a continuous function $\theta : [0,\eta] \times [0,\eta] \to [0,\infty)$ *and* $\alpha_1 \in (0,1)$ *such that*

$$|\mathcal{K}_1(\tau,\mathfrak{h},\aleph(\mathfrak{h})) - \mathcal{K}_2(\tau,\mathfrak{h},\varpi(\mathfrak{h}))| \leq \sqrt{\alpha_1\theta(\tau,\mathfrak{h})}|\aleph(\mathfrak{h}) - \varpi(\mathfrak{h})|.$$

(T2) $\int_0^\eta \theta(\tau,\mathfrak{h})d\mathfrak{h} \leq 1$.

Then, the Eq (3.1) *have a unique common solution in* Υ.

Proof. Define the mappings $\Gamma, \Omega : \Upsilon \to \Upsilon$ by

$$\Gamma(\aleph(\tau)) = \int_0^\eta \mathcal{K}_1(\tau,\mathfrak{h},\aleph(\mathfrak{h}))d\mathfrak{h},$$
$$\Omega(\aleph(\tau)) = \int_0^\eta \mathcal{K}_2(\tau,\mathfrak{h},\aleph(\mathfrak{h}))d\mathfrak{h}.$$

Notice that

$$\begin{aligned}
\frac{1}{\Psi_\varrho(\Gamma\aleph(\tau),\Omega\varpi(\tau)),\mathfrak{r})} - 1 &= \frac{|\aleph(\tau)-\varpi(\tau)|^2}{\mathfrak{r}} \\
&= \frac{\left(\int_0^\eta |\mathcal{K}_1(\tau,\mathfrak{h},\aleph(\mathfrak{h})) - \mathcal{K}_2(\tau,\mathfrak{h},\varpi(\mathfrak{h}))|d\mathfrak{h}\right)^2}{\mathfrak{r}} \\
&\leq \frac{\int_0^\eta |\mathcal{K}_1(\tau,\mathfrak{h},\aleph(\mathfrak{h})) - \mathcal{K}_2(\tau,\mathfrak{h},\varpi(\mathfrak{h}))|^2 d\mathfrak{h}}{\mathfrak{r}} \\
&\leq \frac{\int_0^\eta \alpha_1\theta(\tau,\mathfrak{h})|\aleph(\mathfrak{h}) - \varpi(\mathfrak{h})|^2 d\mathfrak{h}}{\mathfrak{r}} \\
&\leq \alpha_1\frac{|\aleph(\mathfrak{h}) - \varpi(\mathfrak{h})|^2}{\mathfrak{r}} \\
&= \alpha_1\Big(\frac{1}{\Psi_\varrho(\aleph(\tau),\varpi(\tau)),\mathfrak{r})} - 1\Big).
\end{aligned}$$

Therefore, all the conditions of Corollary 2.3 are fulfilled with $\alpha_1 \in (0,1)$ and $\alpha_2 = \alpha_4 = 0$. Hence Γ and Ω have a unique common fixed point in Υ. □

4. Conclusions and future work

In this paper, we introduced triangular property, proved common fixed point theorems on FBM spaces. Recently, Huang et al. [10] proved fixed point theorems on extended b-metric spaces. It is an interesting open problem to study the fuzzy extended b-metric spaces instead of fuzzy b-metric spaces and obtain common fixed point results on fuzzy extended b-metric spaces.

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Conflict of interest

The authors declare no conflicts of interest.

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