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Research article

Some geometric properties of multivalent functions associated with a new generalized q-Mittag-Leffler function

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Abstract: In this article, a new generalized q-Mittag-Leffler function is introduced and investigated. Motivated by the newly defined function and using the concept of differential subordination, a new subclass of multivalent functions is introduced. Some geometric properties of them are obtained. Furthermore, the radii for the aforementioned subclass associated with a generalized Srivastava-Attiya integral operator are also studied.

Keywords: generalized Mittag-Leffler function; multivalent function; Hadamard product; convex linear combination; generalized Srivastava-Attiya operator
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1. Introduction and preliminaries

In 1903, Mittag-Leffler [22] provided the function $E_\sigma(z)$ defined by

$$E_\sigma(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\sigma j + 1)}, (\sigma, z \in \mathbb{C}, \mathcal{R}(\sigma) > 0),$$

where Γ is the gamma function and $\mathcal{R}$ means the real part.

Wiman [34] introduced the following generalized Mittag-Leffler function

$$E_{\sigma,\mu}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\sigma j + \mu)}, (\sigma, \mu, z \in \mathbb{C}, [\mathcal{R}(\sigma), \mathcal{R}(\mu)] > 0).$$

Prabhakar [25] introduced the following function $E^{\rho}_{\sigma,\mu}(z)$ in the form

$$E^{\rho}_{\sigma,\mu}(z) = \sum_{j=0}^{\infty} \frac{(\rho)_j}{\Gamma(\mu+\sigma j)}.\frac{z^j}{j!}, \quad (\sigma,\mu,\rho,z \in \mathbb{C}, [\mathcal{R}(\sigma),\mathcal{R}(\mu),\mathcal{R}(\rho)] > 0).$$

Later, Shukla and Prajapati [27] (see also [32]) defined another generalized Mittag-Leffler function

$$E^{\rho,k}_{\sigma,\mu}(z) = \sum_{j=0}^{\infty} \frac{(\rho)_{kj}}{\Gamma(\mu+\sigma j)}\frac{z^j}{j!}, (\sigma,\mu,\rho,z \in \mathbb{C}, [\mathcal{R}(\sigma),\mathcal{R}(\mu),\mathcal{R}(\rho)] > 0)$$

where $k \in (0,1) \bigcup \mathbb{N}$ and $(\rho)_{kj} = \frac{\Gamma(\rho+kj)}{\Gamma(\rho)}$ is the generalized Pochhammer symbol defined as

$$k^{kj} \prod_{m=1}^{k} \left(\frac{\rho+m-1}{k}\right)_j \text{ if } k \in \mathbb{N}.$$

Bansal and Prajapat [5] and Srivastava and Bansal [31] investigated geometric properties of the Mittag-Leffler function $E_{\sigma,\mu}(z)$, including starlikeness, convexity, and close-to-convexity (see [1, 4, 6, 8, 12, 13, 17, 28, 29]). In reality, the generalized Mittag-Leffler function $E_{\sigma,\mu}(z)$ and its extensions are still widely used in geometric function theory and in a variety of applications (see, for details, [2, 3, 7, 16, 24]).

Let $\mathcal{S}(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j, \tag{1.1}$$

where f is holomorphic and multivalent in the open unit disk $\mathcal{O} = \{z : |z| < 1\}$.

Let f and $\mathcal{F}$ be two functions in $\mathcal{S}(p)$. Then *the convolution (or Hadamard product)*, denoted by $f * \mathcal{F}$, is defined as

$$(f * \mathcal{F})(z) = z^p + \sum_{j=p+1}^{\infty} a_j d_j z^j = (\mathcal{F} * f)(z),$$

where $f(z)$ is in (1.1) and $\mathcal{F}(z) = z^p + \sum_{j=p+1}^{\infty} d_j z^j$.

Let $f(z)$ and $h(z)$ be two analytic functions defined in $\mathcal{O}$. The function $f(z)$ is called subordinate to $h(z)$, or $h(z)$ is superordinate to $f(z)$, denoted by $f(z) \prec h(z)$ and $h(z) \prec f(z)$, respectively, if there is a Schwarz function φ with $\varphi(z) = 0, |\varphi(z)| < 1$ and $f(z) = h(\varphi(z))$. If the function h is univalent in $\mathcal{O}$, then the following equivalence is true if

$$f(z) \prec h(z) \quad (z \in \mathcal{O}) \Leftrightarrow f(0) = h(0) \text{ and } f(\mathcal{O}) \subset h(\mathcal{O}).$$

Definition 1.1 ([18]). Let $0 < q < 1$. Then $[j]_q!$ denotes the q-factorial, which is defined as follows:

$$[j]_q! = \begin{cases} [j]_q[j-1]_q \dots [2]_q[1]_q, & j = 1, 2, 3, \dots \\ 1, & j = 0 \end{cases}$$

where $[j]_q = \frac{1-q^j}{1-q} = 1 + \sum_{m=1}^{j-1} q^m$ and $[0]_q = 0$.

Definition 1.2 ([18])**.** The q-generalized Pochhammer symbol $[\rho]_{j,q}$, $\rho \in \mathbb{C}$, is given as

$$[\rho]_{j,q} = [\rho]_q[\rho + 1]_q[\rho + 2]_q \ldots [\rho + j - 1]_q,$$

and the q-Gamma function is defined as

$$\Gamma_q(\rho + 1) = [\rho]_q\Gamma_q(\rho) \text{ and } \Gamma_q(1) = 1.$$

It follows that $\Gamma_q(j + 1) = [j]_q!$.

Lately, many results have been given for some related special functions such as the Wright function [3] and multivalent functions (see [10, 23, 26]).

Here, we propose a q-extension of specific extensions of the Mittag-Leffler function, motivated by the success of Mittag-Leffler function applications in physics, biology, engineering, and applied sciences. We generalize the Mittag-Leffler function given by Shukla and Prajapati [27] and obtain a new generalized q-Mittag-Leffler function.

Now, we present a new generalized q-Mittag-Leffler function as follows

$$\mathcal{E}^{\rho}_{\sigma,\mu}(q; z) = z + \sum_{j=2}^{\infty} \frac{(\rho)_{kj}}{\Gamma_q(\mu + \sigma j)} \frac{z^j}{j!}. \tag{1.2}$$

It is obvious that, when $q \to 1^-$, the resulting function is the generalized Mittag-Leffler function, which is given by Shukla and Prajapati [27].

Corresponding to the function $\mathcal{E}^{\rho}_{\sigma,\mu}(q; z)$ in (1.2), we establish the following generalized q-Mittag-Leffler function $\mathcal{E}^{\rho}_{\sigma,\mu}(p, q; z)$ in multivalent functions $\mathcal{S}(p)$, as given below

$$\mathcal{E}^{\rho}_{\sigma,\mu}(p, q; z) = z^p + \sum_{j=p+1}^{\infty} \frac{(\rho)_{k(j-p)}}{\Gamma_q(\mu + \sigma(j - p))} \frac{z^j}{(j - p)!}. \tag{1.3}$$

Again, using the new function (1.3), we define the following function:

$$\mathcal{G}^{\rho}_{\sigma,\mu}(p, q; z) := z^p\Gamma_q(\mu)\,\mathcal{E}^{\rho}_{\sigma,\mu}(p, q; z) = z^p + \sum_{j=p+1}^{\infty} \frac{\Gamma_q(\mu)\,(\rho)_{k(j-p)}}{\Gamma_q(\mu + \sigma(j - p))} \frac{z^j}{(j - p)!}. \tag{1.4}$$

Definition 1.3. For $f \in \mathcal{S}(p)$, we define the new linear operator $\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z) : \mathcal{S}(p) \to \mathcal{S}(p)$ by

$$\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z) = \mathcal{G}^{\rho}_{\sigma,\mu}(p, q; z) * f(z) = z^p + \sum_{j=p+1}^{\infty} \chi_j a_j z^j, \tag{1.5}$$

where $\chi_j = \frac{\Gamma_q(\mu)(\rho)_{kj}}{\Gamma_q(\mu+\sigma j)j!}$.

We now define a subclass $Q^{\mu,\rho;k}_{\sigma;q}(\mathcal{M}, \mathcal{N}; \tau, p)$ of the family $\mathcal{S}(p)$ using the multivalent linear operator in (1.5) and the subordination concept.

Definition 1.4. Let $\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z)$ be an operator in (1.5). A function $f(z) \in \mathcal{S}(p)$ is said to be in the class $Q^{\mu,\rho;k}_{\sigma;q}(\mathcal{M}, \mathcal{N}; \tau, p)$ if satisfies the following subordination condition

$$\frac{1}{p - \tau}\left(\frac{z\left(\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z)\right)'}{\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z)} - \tau\right) \prec \frac{1 + \mathcal{M}z}{1 + \mathcal{N}z}, \quad (z \in \mathcal{O}) \tag{1.6}$$

or equivalently

$$\frac{z\left(\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z)\right)'}{\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z)} \prec \frac{p+(p\mathcal{N}+(\mathcal{M}-\mathcal{N})(p-\tau))z}{1+\mathcal{N}z}, \quad (z\in O)$$

and

$$\left|\frac{\frac{z\left(\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z)\right)'}{\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z)}-p}{\mathcal{N}\frac{z\left(\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z)\right)'}{\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z)}-[p\mathcal{N}+(\mathcal{M}-\mathcal{N})(p-\tau)]}\right|<1, \tag{1.7}$$

where $-1\leq\mathcal{M}<\mathcal{N}\leq 1,\ 0\leq\tau<p$, and $p\in\mathbb{N}$.

Remark 1.1. Some well-known special classes of the class $Q^{\mu,\rho;k}_{\sigma;q}(\mathcal{M},\mathcal{N};\tau,p)$ can be obtained by choosing the values of the parameters $\varsigma,\mu,\rho;\tau,k,p,q,\ \mathcal{M}$, and $\mathcal{N}$.

(1) $Q^{0,0,1}_{0,1}(\mathcal{M},\mathcal{N};\tau,p)=S^*_p(\mathcal{M},\mathcal{N};\tau,p)$ was provided by Aouf [2].

(2) $Q^{0,0,1}_{0,1}(\mathcal{M},\mathcal{N};0,p)=S^*_p(\mathcal{M},\mathcal{N};p)$ was provided by Goel and Sohi [16].

In this work, we introduce a new subclass of multivalent functions $Q^{\mu,\rho;k}_{\sigma;q}(\mathcal{M},\mathcal{N};\tau,p)$ defined by the new linear operator $\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z)$. And we study some geometric properties for the class $Q^{\mu,\rho;k}_{\sigma;q}(\mathcal{M},\mathcal{N};\tau,p)$ such as the coefficient estimates, convexity and convex linear combination. Finally, the radius theorems associated with the generalized Srivastava-Attiya integral operator will be investigated.

2. Main results

The first theorem in this section presents the necessary and sufficient condition for the function $f(z)$ in (1.1) belong to the class $Q^{\mu,\rho;k}_{\sigma;q}(\mathcal{M},\mathcal{N};\tau,p)$.

Theorem 2.1. *A function $f(z)$ is in the class $Q^{\mu,\rho;k}_{\sigma;q}(\mathcal{M},\mathcal{N};\tau,p)$ if and only if*

$$\sum_{j=p+1}^{\infty}((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau))\chi_j\left|a_j\right|\leq(\mathcal{M}-\mathcal{N})(p-\tau), \tag{2.1}$$

where $1\leq\mathcal{M}<\mathcal{N}\leq 1,\ 0\leq\tau<p$, *and* $p\in\mathbb{N}$.

Proof. Assume that the condition (2.1) is true. Then by (1.7), we have

$$\begin{aligned}
&\left|z\left(\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z)\right)'-p\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z)\right|-\left|\mathcal{N}z\left(\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z)\right)'-[(\mathcal{M}-\mathcal{N})(p-\tau)+p\mathcal{N}]\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z)\right|\\
&=\left|\sum_{j=p+1}^{\infty}(j-p)\chi_j a_j z^j\right|-\left|(\mathcal{M}-\mathcal{N})(p-\tau)z^j-\sum_{j=p+1}^{\infty}[\mathcal{N}j-((\mathcal{M}-\mathcal{N})(p-\tau)+p\mathcal{N})]\chi_j a_j z^j\right|\\
&\leq-(\mathcal{M}-\mathcal{N})(p-\tau)+\sum_{j=p+1}^{\infty}[(1+\mathcal{N})(j-p)+((\mathcal{M}-\mathcal{N})(p-\tau))]\chi_j\left|a_j\right|\\
&\leq 0.
\end{aligned}$$

By maximum modulus theorem [11], we get $f(z)\in Q^{\mu,\rho;k}_{\sigma;q}(\mathcal{M},\mathcal{N};\tau,p)$.

Conversely, suppose that $f(z) \in Q^{\mu,\rho;k}_{\sigma;q}(\mathcal{M},\mathcal{N};\tau,p)$. Then

$$\begin{aligned}&\left|\frac{\frac{z\left(\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z)\right)'}{\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z)}-p}{\mathcal{N}\frac{z\left(\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z)\right)'}{\mathcal{A}^{\mu,\rho;k}_{\sigma;p,q}f(z)}-[p\mathcal{N}+(\mathcal{M}-\mathcal{N})(p-\tau)]}\right|\\ =&\left|\frac{\sum_{j=p+1}^{\infty}(j-p)\chi_j a_j z^j}{(\mathcal{M}-\mathcal{N})(p-\tau)z^j-\sum_{j=p+1}^{\infty}[\mathcal{N}j-((\mathcal{M}-\mathcal{N})(p-\tau)+p\mathcal{N})]\chi_j a_j z^j}\right|\\ <&\ 1.\end{aligned}$$

Since $\mathcal{R}(z) \leq |z|$, we get

$$\mathcal{R}\{\frac{\sum_{j=p+1}^{\infty}(j-p)\chi_j a_j z^j}{(\mathcal{M}-\mathcal{N})(p-\tau)z^j-\sum_{j=p+1}^{\infty}[\mathcal{N}j-((\mathcal{M}-\mathcal{N})(p-\tau)+p\mathcal{N})]\chi_j a_j z^j}\}<1.$$

Taking $z \to 1^-$, we have

$$\sum_{j=p+1}^{\infty}((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau))\chi_j\left|a_j\right| \leq (\mathcal{M}-\mathcal{N})(p-\tau).$$

This completes the proof. □

Theorem 2.2. *Let f_1 and f_2 be analytic functions in the class $Q^{\mu,\rho;k}_{\sigma;q}(\mathcal{M},\mathcal{N};\tau,p)$. Then $f_1 * f_2 \in Q^{\mu,\rho;k}_{\sigma;q}(\mathcal{M},\mathcal{N};\tau,p)$, where*

$$\tau_1 = p-\frac{(1-p)(1+\mathcal{N})(\mathcal{M}-\mathcal{N})(p-\tau)^2\chi_1}{\left[((1+\mathcal{N})(1-p)+(\mathcal{M}-\mathcal{N})(p-\tau_1))\chi_1\right]^2-(\mathcal{M}-\mathcal{N})^2(p-\tau)^2\chi_1}, \tag{2.2}$$

where $\chi_1 = \frac{\Gamma_q(\mu)(\rho)_k}{\Gamma_q(\mu+\varsigma)}$.

Proof. We will show that τ_1 is the largest satisfying

$$\sum_{j=p+1}^{\infty}\frac{((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau_1))\chi_j}{(\mathcal{M}-\mathcal{N})(p-\tau_1)}a_{j,1}a_{j,2} \leq 1. \tag{2.3}$$

Since $f_1, f_2 \in Q^{\mu,\rho;k}_{\sigma;q}(\mathcal{M},\mathcal{N};\tau,p)$, by the condition (2.1) and the Cauchy-Schwarz inequality, we get

$$\sum_{j=p+1}^{\infty}\frac{((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau))\chi_j}{(\mathcal{M}-\mathcal{N})(p-\tau)}\sqrt{a_{j,1}a_{j,2}} \leq 1. \tag{2.4}$$

From (2.3) and (2.4), we observe that

$$\sqrt{a_{j,1}a_{j,2}} \leq \frac{\left[((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau))\chi_j\right](p-\tau_1)}{\left[((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau_1))\chi_j\right](p-\tau)}.$$

From (2.4), it is necessary to prove

$$\frac{(\mathcal{M}-\mathcal{N})(p-\tau)}{((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau))\chi_j} \leq \frac{\left[((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau))\chi_j\right](p-\tau_1)}{\left[((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau_1))\chi_j\right](p-\tau)}. \quad (2.5)$$

Furthermore, from the inequality (2.5) it follows that

$$\tau_1 \leq p - \frac{(j-p)(1+\mathcal{N})(\mathcal{M}-\mathcal{N})(p-\tau)^2\chi_j}{\left[((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau_1))\chi_j\right]^2-(\mathcal{M}-\mathcal{N})^2(p-\tau)^2\chi_j}.$$

Now, set

$$E(j) = p - \frac{(j-p)(1+\mathcal{N})(\mathcal{M}-\mathcal{N})(p-\tau)^2\chi_j}{\left[((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau_1))\chi_j\right]^2-(\mathcal{M}-\mathcal{N})^2(p-\tau)^2\chi_j}.$$

We observe that the function $E(j)$ is increasing for $j \in \mathbb{N}$. Putting $j = 1$, we have

$$\tau_1 = E(1) = p - \frac{(1-p)(1+\mathcal{N})(\mathcal{M}-\mathcal{N})(p-\tau)^2\chi_1}{\left[((1+\mathcal{N})(1-p)+(\mathcal{M}-\mathcal{N})(p-\tau_1))\chi_1\right]^2-(\mathcal{M}-\mathcal{N})^2(p-\tau)^2\chi_1}.$$

This completes the proof. □

Theorem 2.3. *Let f_1 and f_2 be analytic functions in the class $Q_{\sigma;q}^{\mu,\rho;k}(\mathcal{M},\mathcal{N};\tau,p)$ of forms given in (1.1) with $a_{j,1}$ and $a_{j,2}$, respectively. Then*

$$w(z) = z^p + \sum_{j=p+1}^{\infty}(a_{j,1}^2+a_{j,2}^2)z^j \in Q_{\sigma;q}^{\mu,\rho;k}(\mathcal{M},\mathcal{N};\tau,p),$$

where

$$\eta = p - \frac{(1-p)(1+\mathcal{N})(\mathcal{M}-\mathcal{N})(p-\tau)^2\chi_1}{\left[((1+\mathcal{N})(1-p)+(\mathcal{M}-\mathcal{N})(p-\tau_1))\chi_1\right]^2-(\mathcal{M}-\mathcal{N})^2(p-\tau)^2\chi_1}.$$

Proof. By Theorem 2.1, we have

$$\sum_{j=p+1}^{\infty}\left[\frac{((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau))\chi_j}{(\mathcal{M}-\mathcal{N})(p-\tau)}\right]^2 a_{j,s}^2$$
$$\leq \sum_{j=p+1}^{\infty}\left[\frac{((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau))\chi_j}{(\mathcal{M}-\mathcal{N})(p-\tau)}a_{j,s}\right]^2 \leq 1,\ (s=1,2).$$

From the above inequality, we obtain

$$\sum_{j=p+1}^{\infty}\frac{1}{2}\left[\frac{((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau))\chi_j}{(\mathcal{M}-\mathcal{N})(p-\tau)}\right]^2(a_{j,1}^2+a_{j,2}^2) \leq 1.$$

Therefore, the largest η can be obtained such that

$$\frac{((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau))\chi_j}{(\mathcal{M}-\mathcal{N})(p-\tau)} \leq \frac{1}{2}\left[\frac{((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau))\chi_j}{(\mathcal{M}-\mathcal{N})(p-\tau)}\right]^2.$$

That is,

$$\eta \leq p - \frac{2(j-p)(1+\mathcal{N})(\mathcal{M}-\mathcal{N})(p-\tau)^2\chi_1}{\left[((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau_1))\chi_1\right]^2 - 2(\mathcal{M}-\mathcal{N})^2(p-\tau)^2\chi_1}.$$

Now, set

$$E(j) = p - \frac{2(j-p)(1+\mathcal{N})(\mathcal{M}-\mathcal{N})(p-\tau)^2\chi_1}{\left[((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau_1))\chi_1\right]^2 - 2(\mathcal{M}-\mathcal{N})^2(p-\tau)^2\chi_1}.$$

We observe that the function $E(j)$ is increasing for $j \in \mathbb{N}$. Putting $j = 1$, we have

$$\eta = E(1) = p - \frac{2(1-p)(1+\mathcal{N})(\mathcal{M}-\mathcal{N})(p-\tau)^2\chi_1}{\left[((1+\mathcal{N})(1-p)+(\mathcal{M}-\mathcal{N})(p-\tau_1))\chi_1\right]^2 - 2(\mathcal{M}-\mathcal{N})^2(p-\tau)^2\chi_1}.$$

This completes the proof. □

Theorem 2.4. *Let $f_1, f_2 \in \mathcal{Q}^{\mu,\rho;k}_{\sigma;q}(\mathcal{M},\mathcal{N};\tau,p)$. Then for $\gamma \in [0,1]$, the function $F(z) = (1-\gamma) f_1 + \gamma f_2$ belongs to the class $\mathcal{Q}^{\mu,\rho;k}_{\sigma;q}(\mathcal{M},\mathcal{N};\tau,p)$.*

Proof. Since the functions f_1 and f_2 belong to the class $\mathcal{Q}^{\mu,\rho;k}_{\sigma;q}(\mathcal{M},\mathcal{N};\tau,p)$,

$$F(z) = (1-\gamma) f_1 + \gamma f_2 = z^p + \sum_{j=p+1}^{\infty} \eta_j z^j,$$

where $\eta_j = (1-\gamma) a_{j,1} + \gamma a_{j,2}$.

By (2.1), we observe that

$$\begin{aligned}
&\sum_{j=p+1}^{\infty} ((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau))\chi_j[(1-\gamma) a_{j,1} + \gamma a_{j,2}] \\
&= (1-\gamma)\sum_{j=p+1}^{\infty} ((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau))\chi_j a_{j,1} \\
&+\gamma \sum_{j=p+1}^{\infty} ((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau))\chi_j a_{j,2} \\
&\leq (1-\gamma)(\mathcal{M}-\mathcal{N})(p-\tau) + \gamma(\mathcal{M}-\mathcal{N})(p-\tau).
\end{aligned}$$

Hence $F(z) \in \mathcal{Q}^{\mu,\rho;k}_{\sigma;q}(\mathcal{M},\mathcal{N};\tau,p)$. □

Theorem 2.5. *Let $f_s(z) = z^p + \sum_{j=p+1}^{\infty} a_{j,s} z^j$ be in the class $\mathcal{Q}^{\mu,\rho;k}_{\sigma;q}(\mathcal{M},\mathcal{N};\tau,p)$ for $s = 1,2,\ldots,m$. Then the function $\mathcal{P}(z) = \sum_{s=1}^{m} \aleph_s f_s$, where $\sum_{s=1}^{m} \aleph_s = 1$, is also in the class $\mathcal{Q}^{\mu,\rho;k}_{\sigma;q}(\mathcal{M},\mathcal{N};\tau,p)$.*

Proof. By Theorem 2.1, we have

$$\sum_{j=p+1}^{\infty} \frac{((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau))\chi_j}{(\mathcal{M}-\mathcal{N})(p-\tau)} a_{j,s} \leq 1.$$

Since

$$\mathcal{P}(z) = \sum_{s=1}^{m} \aleph_s f_s = \sum_{s=1}^{m} \aleph_s (z^p + \sum_{j=p+1}^{\infty} a_{j,s} z^j) = z^p + \sum_{j=p+1}^{\infty} (\sum_{s=1}^{m} \aleph_s a_{j,s}) z^j ,$$

$$\sum_{j=p+1}^{\infty} \frac{((1+\mathcal{N})(j-p) + (\mathcal{M}-\mathcal{N})(p-\tau))\chi_j}{(\mathcal{M}-\mathcal{N})(p-\tau)} \sum_{s=1}^{m} \aleph_s a_{j,s} \leq 1.$$

Thus $\mathcal{P}(z) \in Q_{\sigma;q}^{\mu,\rho;k}(\mathcal{M},\mathcal{N};\tau,p)$. □

3. Radius property of the class $Q_{\sigma;q}^{\mu,\rho;k}(\mathcal{M},\mathcal{N};\tau,p)$ with differintegral operator

In this section, we investigate radii of multivalent starlikeness, multivalent convexity, and multivalent close-to-convex for the function $f(z)$ in the class $Q_{\sigma;q}^{\mu,\rho;k}(\mathcal{M},\mathcal{N};\tau,p)$ with the generalized integral operator of Srivastava-Attiya.

Jung et al. [19] introduced an integral operator with one parameter as follows:

$$\mathcal{I}^{\delta}(f)(z) := \frac{2^{\delta}}{z\Gamma(\delta)} \int_0^z \left(\log\left(\frac{z}{v}\right)\right)^{\delta-1} f(v)\, dv = z + \sum_{j=2}^{\infty} \left(\frac{2}{j+1}\right)^{\delta} a_j z^j \quad (\delta > 0; f \in \mathcal{S}).$$

In 2007, Srivastava and Attiya [30] investigated a new integral operator, which is called Srivastava-Attiya operator, given by

$$\mathcal{J}_{u,m} f(z) = z + \sum_{j=1}^{\infty} \left(\frac{1+u}{j+u}\right)^{\delta} a_j z^j.$$

Many studies are concerned with the study of the operator of Srivastava-Attiya (see [9, 14, 15, 20]).

Mishra and Gochhayat [21] (also [33]) provided a fractional differintegral operator $\mathcal{J}_{u,p}^{m} f(z) : \mathcal{S}(p) \to \mathcal{S}(p)$ which is called a generalized of Srivastava-Attiya integral operator, defined by

$$\mathcal{J}_{u,p}^{m} f(z) = z^p + \sum_{j=p+1}^{\infty} \left(\frac{p+u}{j+u}\right)^{\delta} a_j z^j. \tag{3.1}$$

Theorem 3.1. *If $f(z) \in Q_{\sigma;q}^{\mu,\rho;k}(\mathcal{M},\mathcal{N};\tau,p)$ and $0 \leq \tau < p$, then $\mathcal{J}_{u,p}^{m} f(z)$ in* (3.1) *is multivalent starlike of order τ in $|z| \leq \mathrm{r}_1$, where*

$$\mathrm{r}_1 = \inf_{j \geq p+1} \left\{ \frac{((1+\mathcal{N})(j-p) + (\mathcal{M}-\mathcal{N})(p-\tau))\chi_j (j+u)^{\delta}}{(\mathcal{M}-\mathcal{N})(j-2p+\tau)(p+u)^{\delta}} \right\}. \tag{3.2}$$

Proof. According to the definition of a starlike function in [28], we have

$$\left| \frac{z\left(\mathcal{J}_{u,p}^{m} f(z)\right)'}{\mathcal{J}_{u,p}^{m} f(z)} - p \right| \leq p - \tau, \tag{3.3}$$

$$\left| \frac{z\left(J_{u,p}^{m} f(z)\right)'}{\mathcal{J}_{u,p}^{m} f(z)} - p \right| = \left| \frac{\sum_{j=p+1}^{\infty} (j-p) \left(\frac{p+u}{j+u}\right)^{\delta} a_j z^j}{\sum_{j=p+1}^{\infty} \left(\frac{p+u}{j+u}\right)^{\delta} a_j z^j} \right| \leq \frac{\sum_{j=p+1}^{\infty} (j-p) \left(\frac{p+u}{j+u}\right)^{\delta} a_j |z|^j}{\sum_{j=p+1}^{\infty} \left(\frac{p+u}{j+u}\right)^{\delta} a_j |z|^j}.$$

By (3.2), we have

$$\sum_{j=p+1}^{\infty} \frac{(j-2p+\tau)(p+u)^{\delta} a_j |z|^j}{(p-\tau)(j+u)^{\delta}} \leq 1.$$

By (2.1) in Theorem 2.1, it is clear that

$$\frac{(j-2p+\tau)(p+u)^{\delta}}{(p-\tau)(j+u)^{\delta}} |z|^j \leq \frac{((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau))\chi_j}{(\mathcal{M}-\mathcal{N})(p-\tau)}.$$

Therefore,

$$|z| \leq \left\{ \frac{((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau))\chi_j (j+u)^{\delta}}{(\mathcal{M}-\mathcal{N})(j-2p+\tau)(p+u)^{\delta}} \right\}^{\frac{1}{j}}.$$

This completes the proof. □

Theorem 3.2. *If* $f(z) \in Q_{\sigma;q}^{\mu,\rho;k}(\mathcal{M},\mathcal{N};\tau,p)$ *and* $0 \leq \tau < p$, *then* $\mathcal{J}_{u,p}^{m} f(z)$ *in* (3.1) *is multivalent convex of order* τ *in* $|z| \leq \mathrm{r}_2$, *where*

$$\mathrm{r}_2 = \inf_{j \geq p+1} \left\{ \frac{((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau))\chi_j p (j+u)^{\delta}}{(\mathcal{M}-\mathcal{N})[j(j-2p+\tau)](p+u)^{\delta}} \right\}. \tag{3.4}$$

Proof. To verify (3.4), it is necessary to prove

$$\left| \left(1 + \frac{z\left(\mathcal{J}_{u,p}^{m} f(z)\right)''}{\left(\mathcal{J}_{u,p}^{m} f(z)\right)'} \right) - p \right| \leq p - \tau,$$

but the result is obtained by repeating the steps in Theorem 3.1. □

Corollary 3.1. *If* $f(z) \in Q_{\sigma;q}^{\mu,\rho;k}(\mathcal{M},\mathcal{N};\tau,p)$ *and* $0 \leq \tau < p$, *then* $\mathcal{J}_{u,p}^{m} f(z)$ *in* (3.1) *is multivalent close-to-convex of order* τ *in* $|z| \leq \mathrm{r}_3$, *where*

$$\mathrm{r}_3 = \inf_{j \geq 1} \left\{ \frac{((1+\mathcal{N})(j-p)+(\mathcal{M}-\mathcal{N})(p-\tau))\chi_j (j+u)^{\delta}}{(\mathcal{M}-\mathcal{N}) j (p+u)^{\delta}} \right\}. \tag{3.5}$$

4. Conclusions

In this work, we established and investigated a new generalized Mittag-Leffler function, which is a generalization of q-Mittag-Leffler function defined by Shukla and Prajapati [27]. Also, we studied some of the geometric properties of a certain subclass of multivalent functions. In addition, we introduced radius theorem using a generalized Srivastava-Attiya integral operator. Since the Mittag-Leffler function is of importance, it is related to a wide range of problems in mathematical physics, engineering, and the applied sciences. The results obtained in this article may have many other applications in special functions.

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Conflict of interest

The authors declare that they have no competing interests concerning the publication of this article.

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