Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(5): 9493–9507.
DOI: 10.3934/math.2022527
Received: 09 January 2022
Revised: 27 February 2022
Accepted: 01 March 2022
Published: 14 March 2022

Research article

An approach of Banach algebra in fuzzy metric spaces with an application

Saif Ur Rehman[1], **Arjamand Bano**[1], **Hassen Aydi**[2,3,4,*] **and Choonkil Park**[5,*]

[1] Institute of Numerical Sciences, Department of Mathematics, Gomal University, Dera Ismail Khan 29050, Pakistan

[2] Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, H. Sousse 4000, Tunisia

[3] China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

[4] Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa

[5] Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea

* **Correspondence:** Email: hassen.aydi@isima.rnu.tn, baak@hanyang.ac.kr.

Abstract: The purpose of this paper is to present a new concept of a Banach algebra in a fuzzy metric space (FM-space). We define an open ball, an open set and prove that every open ball in an FM-space over a Banach algebra $\mathcal{A}$ is an open set. We present some more topological properties and a Hausdorff metric on FM-spaces over $\mathcal{A}$. Moreover, we state and prove a fuzzy Banach contraction theorem on FM-spaces over a Banach algebra $\mathcal{A}$. Furthermore, we present an application of an integral equation and will prove a result dealing with the integral operators in FM-spaces over a Banach algebra.

Keywords: fixed point; Banach algebra $\mathcal{A}$; fuzzy metric space over $\mathcal{A}$; integral equation
Mathematics Subject Classification: 47H07, 47H10, 54H25

1. Introduction

In 1965, the theory of fuzzy sets was introduced by Zadeh [1]. Many authors generalized this theory in different directions for different purposes and still, it's playing a very important role in current research. By using this theory, Kramosil et al. [2] introduced the concept of a fuzzy metric space (FM-space) which performs the probabilistic metric space approach to the fuzzy setting and they proved that every metric induces an FM. In the sense of Kramosil et al. [2], Grabiec [3] proved two fixed point theorems (FP-theorems), that is, "Banach and Edelstein contraction theorems for self-mappings on complete and compact FM-spaces, respectively". Later on, the modified form of the FM-space was given by George and Vermani [4]. Gregory and Sapena [5] established some FP-theorems on

FM-spaces. After that, Rodriguez-Lopez and Romaguera [6] gave the idea of a Hausdorff metric on a given FM-space by using the concept of George and Veeramani [4] on the nonempty compact subsets. In 2011, Kiany et al. [7] established some FP and end point theorems by using set-valued fuzzy-contractive on complete FM-spaces. Recently, Shamas et al. [8, 9] established some unique FP-theorems on complete FM-spaces with integral and differential types of applications. Some more related results in the context of metric spaces and FM-spaces, can be found in (e.g., see [11–22] and the references are therein).

In 2007, Huang and Zhang [23] introduced the notion of a cone metric space which modifies the concept of metric spaces by using the Banach space instead of real numbers as a range set and proved some FP-theorems for nonlinear contractive type mappings with the normality of cone condition. Du [24] presented a note on cone metric fixed point theory and its equivalence. Later on, Cakalli et al. [25] presented the concept that any topological vector space valued cone metric space is metrizable and proved some results in topological vector space valued cone metric spaces. In 2013, Liu et al. [26] presented the new concept of a cone metric space (CM-space) over Banach algebras by the replacement of Banach algebra instead of Banach space and proved some FP-theorems by using generalized Lipschitz mappings with weaker and natural conditions of the generalized Lipschitz constant of spectral radius. In 2016, Yan et al. [27] proved the result on partially order CM-spaces over Banach algebras for FP and CFP.

The notion of a fuzzy cone metric space (FCM-space) was introduced by Oner et al. [28]. They proved some basic properties and a "fuzzy cone Banach contraction theorem" which is stated as: "A self-mapping on a complete FCM-space in which fuzzy cone contractive sequences are Cauchy has a unique FP". Later on, Oner et al. [29] defined a closed ball and pre-compact in an FCM-space, and established a Baire's theorem on a complete FCM-space. In 2017, Rehman and Li [30] extended and improved a "fuzzy cone Banach contraction theorem" and proved some generalized FP-theorems in complete FCM-spaces without the assumption of "fuzzy cone contractive sequences are Cauchy". After that, Jabeen et al. [31] proved some CFP-theorems on FCM-spaces with an application. Chen et al. [32], Priyobarta et al. [33] and Talha et al. [34, 35] proved some FP and coupled FP-results in the context of FCM-spaces with different types of applications.

In this paper, we present the new concept of an FM-space over a Banach algebra $\mathcal{A}$ and prove some basic topological properties and a Hausdorff metric on the said space. In previous literature, a mapping FM is represented by $M : U \times U \times (0, \infty) \to [0, 1]$ where U is a non-empty set. Oner et al. [28] replaced $(0, \infty)$ by cone condition and defined a mapping FCM as: $M_c : U \times U \times int(P) \to [0, 1]$, where P is a cone of a real Banach space E. Now, in this paper, we use a Banach algebra $\mathcal{A}$ instead of $(0, \infty)$ in an FM-space and present the new concept of an FM-space over a Banach algebra $\mathcal{A}$, this FM mapping over a Banach algebra $\mathcal{A}$ can be written as: $M_{\mathcal{A}} : U \times U \times \mathcal{A} \to [0, 1]$. By using this new concept, we present some basic properties and a FP-theorem for self-mappings on a G-complete FM-space over the Banach algebra $\mathcal{A}$. This new concept will play a very important role in the fixed point theory. Moreover, we present a supportive integral type application to validate our work. By using this new concept, one can prove some more topological properties and fixed point results in FM-space over a Banach algebra $\mathcal{A}$ with different types of applications.

2. Preliminaries

In this section, we shall present the basic helpful concepts related to our main results.

Let $\mathcal{A}$ represent a real Banach space in which the operation of multiplication is defined by;

(i) $u(vw) = (uv)w$,

(ii) $(u + v)w = uw + vw$ and $u(v + w) = uv + vw$,

(iii) $\beta(uv) = (\beta u)v = u(\beta v)$,

(iv) $\|uv\| \leq \|u\|\|v\|$.

for all $u, v, w \in \mathcal{A}$.

Let $e^* \in \mathcal{A}$ be the unit multiplicative identity of $\mathcal{A}$ such that $ve^* = e^*v = v$, for all $v \in \mathcal{A}$ and an element $v \in \mathcal{A}$ is said to be invertible if there is $z \in \mathcal{A}$ such that $vz = zv = e$. The inverse of v is denoted by v^{-1}. For more details, we refer the readers to [36].

Proposition 2.1. *[36] Let $\mathcal{A}$ be a Banach algebra with the identity element e^* and $v \in \mathcal{A}$. If the spectral radius $\varrho(v) < 1$, that is,*

$$\varrho(v) = \lim_{k\to+\infty} \|v^k\|^{\frac{1}{k}} = \inf_{k\geq 1} \|v^k\|^{\frac{1}{k}} < 1,$$

then $(e^* - v)$ is invertible. Therefore, we have

$$(e^* - v)^{-1} = \sum_{j=0}^{+\infty} v^j.$$

Lemma 2.2. *[37] Let u, v be any two vectors in a Banach algebra $\mathcal{A}$. If they commute with each other, then the following hold:*

(i) $\varrho(uv) \leq \varrho(u)\varrho(v)$;

(ii) $\varrho(u + v) \leq \varrho(u) + \varrho(v)$;

(iii) $|\varrho(u) - \varrho(v)| \leq \varrho(u - v)$.

Lemma 2.3. *[37] Let z be any vector in a Banach algebra $\mathcal{A}$. If $\varrho(z) \in (0, 1)$, then we have*

$$\varrho((e^* - z)^{-1}) \leq (1 - \varrho(z))^{-1}.$$

Definition 2.4. [38] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm, if

(i) $*$ is commutative, associative and is continuous,

(ii) for all $\alpha_1, \beta_1, \alpha_2, \beta_2 \in [0, 1]$, $1 * \alpha_1 = \alpha_1$, $\alpha_1 * \beta_1 \leq \alpha_2 * \beta_2$, whenever $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$.

The following are the basic three conditions of continuous t-norm (see [38]):

(i) The minimum t-norm is: $\alpha_1 * \beta_1 = \min\{\alpha_1, \beta_1\}$.

(ii) The product t-norm is: $\alpha_1 * \beta_1 = \alpha_1\beta_1$.

(iii) The Lukasiewicz t-norm is: $\alpha_1 * \beta_1 = \max\{0, \alpha_1 + \beta_1 - 1\}$.

Definition 2.5. [4] A 3-tuple $(U, M, *)$ is said to be an FM-space if U is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $U^2 \times (0, \infty)$, satisfying the conditions:

(F-1) $M(\zeta_1, \zeta_2, t) > 0$ and $M(\zeta_1, \zeta_2, t) = 1$ if and only if $\zeta_1 = \zeta_2$,

(F-2) $M(\zeta_1, \zeta_2, t) = M(\zeta_2, \zeta_1, t)$,

(F-3) $M(\zeta_1, \zeta_2, t) * M(\zeta_2, \zeta_3, s) \leq M(\zeta_1, \zeta_3, t + s)$,

(F-4) $M(\zeta_1, \zeta_2, .) : (0, \infty) \to [0, 1]$ is continuous,

for all $\zeta_1, \zeta_2, \zeta_3 \in U$ and $t, s > 0$.

Definition 2.6. [4] Let $(U, M, *)$ be an FM-space, let $\zeta \in U$ and (ζ_k) be a sequence in U. Then

(i) (ζ_k) is said to converge to ζ if for any $t > 0$ and $\rho \in (0, 1)$, there is $k_1 \in \mathbb{N}$ such that $M(\zeta_k, \zeta, t) > 1 - \rho$, for all $k \geq k_1$. We can write this $\lim_{k\to\infty} \zeta_k = \zeta$.

(ii) (ζ_k) is said to be a Cauchy sequence if for any $t > 0$ and $\rho \in (0, 1)$, there is $k_1 \in \mathbb{N}$ such that $M(\zeta_k, \zeta_m, t) > 1 - \rho$, for all $k, m \geq k_1$.

(iii) $(U, M, *)$ is complete if every Cauchy sequence is convergent in U.

Note: In the sense of Gregori et al. [5], a sequence (ζ_k) in an FM-space is said to be G-Cauchy if $\lim_{k\to\infty} M(\zeta_k, \zeta_{k+l}, t) = 1$ for $t > 0$ and an FM-space $(U, M, *)$ is called G-complete if every G-Cauchy sequence is convergent.

Definition 2.7. [6] Let $(U, M, *)$ be an FM-space. M is said to be continuous on $U^2 \times (0, \infty)$, if

$$\lim_{k\to\infty} M(\zeta_k, v_k, t_k) = M(\zeta_1, v_1, t),$$

whenever if $(\zeta_k, v_k, t_k) \in U^2 \times (0, \infty) \to (\zeta_1, v_1, t) \in U^2 \times (0, \infty)$, we have

$$\lim_{k\to\infty} M(\zeta_1, v_1, t_k) = M(\zeta_1, v_1, t), \quad \text{where } \lim_{k\to\infty} \zeta_k = \zeta_1 \text{ and } \lim_{k\to\infty} v_k = v_1.$$

Remark 2.8. [4] Let $(U, M, *)$ be an FM-space. $\mathcal{T} = \{B \subset U : \zeta_1 \in B,$ if and only if there are $t > 0$ and $\rho \in (0, 1)$ such that $\mathcal{B}(\zeta_1, \rho, t) \subset B\}$ is a topology on U.

As a further study in this paper, we introduce the new concept of an FM-space over a Banach algebra $\mathcal{A}$. We present some topological properties and a fuzzy Banach contraction theorem over the Banach algebra $\mathcal{A}$. Moreover, a supportive application of integral is given at the end to validate our work.

3. Main results

Our first definition is as follows:

Definition 3.1. A 4-tuple $(U, M_{\mathcal{A}}, \mathcal{A}, *)$ is called an FM-space over $\mathcal{A}$ if U is an arbitrary set, $\mathcal{A}$ is a Banach algebra, $*$ is a continuous t-norm and $M_{\mathcal{A}}$ is a fuzzy set on $U \times U \times \mathcal{A}$, satisfying the following conditions:

(FB-1) $M_{\mathcal{A}}(\zeta_1, \zeta_2, t) > 0$, and $M_{\mathcal{A}}(\zeta_1, \zeta_2, t) = 1$ if and only if $\zeta_1 = \zeta_2$;

(FB-2) $M_{\mathcal{A}}(\zeta_1, \zeta_2, t) = M_{\mathcal{A}}(\zeta_2, \zeta_1, t)$;

(FB-3) $M_{\mathcal{A}}(\zeta_1, \zeta_2, t) * M_{\mathcal{A}}(\zeta_2, \zeta_3, s) \leq M_{\mathcal{A}}(\zeta_1, \zeta_3, t + s)$;

(FB-4) $M_{\mathcal{A}}(\zeta_1, \zeta_2, .) : \mathcal{A} \longrightarrow [0, 1]$ is continuous

for all $\zeta_1, \zeta_2, \zeta_3 \in U$ and $s, t \in \mathcal{A}$. Then the 4-tuple $(U, M_{\mathcal{A}}, \mathcal{A}, *)$ is said to be an FM-space over $\mathcal{A}$.

Example 3.2. Let $U = \mathbb{R}$ and $*$ be a continuous t-norm, which is defined by $a * b = ab$, for all $a, b \in [0, 1]$ and $M_{\mathcal{A}} : U \times U \times \mathcal{A} \to [0, 1]$ be defined by

$$M_{\mathcal{A}}(\zeta_1, \zeta_2, t) = \left(e^{\|\zeta_1 - \zeta_2\| t^{-1}}\right)^{-1},$$

for all $\zeta_1, \zeta_2 \in U$ and $t \in \mathcal{A}$, $M_{\mathcal{A}}$ is an FM-space over $\mathcal{A}$.
Proof.

(FB-1) For all $\zeta_1, \zeta_2 \in U$ and $t \in \mathcal{A}$, $M_{\mathcal{A}}(\zeta_1, \zeta_2, t) > 0$, where $\zeta_1 \neq \zeta_2$. Assume that $\zeta_2 = \zeta_1$. Then $\|\zeta_1 - \zeta_2\| = 0$ and hence we get

$$\left(e^{\|\zeta_1 - \zeta_2\| t^{-1}}\right)^{-1} = 1.$$

So $M_{\mathcal{A}}(\zeta_1, \zeta_2, t) = 1$. Conversely, suppose that $M_{\mathcal{A}}(\zeta_1, \zeta_2, t) = 1$. Then $\left(e^{\|\zeta_1 - \zeta_2\| t^{-1}}\right)^{-1} = 1$, and so $e^{\|\zeta_1 - \zeta_2\| t^{-1}} = e^0$. Due to the same base, we compare the power on both sides and multiplying by t, we have that $\|\zeta_1 - \zeta_2\| t^{-1} t = 0$, and so $\|\zeta_1\| = \|\zeta_2\|$. Hence, it is proved that $M_{\mathcal{A}}(\zeta_1, \zeta_2, t) = 1$ if and only if $\zeta_1 = \zeta_2$.

(FB-2) If $\|\zeta_1 - \zeta_2\| = \|\zeta_2 - \zeta_1\|$, for all $\zeta_1, \zeta_2 \in U$ and $t \in \mathcal{A}$, then

$$\left(e^{\|\zeta_1 - \zeta_2\| t^{-1}}\right)^{-1} = \left(e^{\|\zeta_2 - \zeta_1\| t^{-1}}\right)^{-1}.$$

This implies that

$$M_{\mathcal{A}}(\zeta_1, \zeta_2, t) = M_{\mathcal{A}}(\zeta_2, \zeta_1, t).$$

(FB-3) We will prove that $M_{\mathcal{A}}(\zeta_1, \zeta_2, s + t) \geq M_{\mathcal{A}}(\zeta_1, \zeta_3, s) * M_{\mathcal{A}}(\zeta_3, \zeta_2, t)$, for all $\zeta_1, \zeta_2, \zeta_3 \in U$ and $s, t \in \mathcal{A}$. Since

$$\|\zeta_1 - \zeta_2\| \leq \|\zeta_1 - \zeta_3\| \left(s^{-1}(s + t)\right) + \|\zeta_3 - \zeta_2\| \left(t^{-1}(s + t)\right),$$

$$\|\zeta_1 - \zeta_2\|(s + t)^{-1} \leq \|\zeta_1 - \zeta_3\| s^{-1} + \|\zeta_3 - \zeta_2\| t^{-1}.$$

Thus we have

$$e^{\|\zeta_1 - \zeta_2\|(s+t)^{-1}} \leq e^{\|\zeta_1 - \zeta_3\| s^{-1}}\ e^{\|\zeta_3 - \zeta_2\| t^{-1}}.$$

Since e^{ζ} is an increasing function for $0 < \zeta \in U$, we have

$$\left(e^{\|\zeta_1 - \zeta_2\|(s+t)^{-1}}\right)^{-1} \geq \left(e^{\|\zeta_1 - \zeta_3\| s^{-1}}\right)^{-1} * \left(e^{\|\zeta_3 - \zeta_2\| t^{-1}}\right)^{-1}.$$

Hence, it is proved that $M_{\mathcal{A}}(\zeta_1, \zeta_2, s + t) \geq M_{\mathcal{A}}(\zeta_2, \zeta_3, s) * M_{\mathcal{A}}(\zeta_3, \zeta_2, t)$.

(FB-4) Let us take a sequence (t_k) in $\mathcal{A}$ such that $t_k \to t \in \mathcal{A}$. That is, $\lim_k \|t_k - t\| = 0$.

Without loss of the generality, fix $\zeta_1, \zeta_2 \in U$. Since the function e^{ζ_1} is continuous on $\mathbb{R}$, we have

$$e^{\|\zeta_1 - \zeta_2\| t_k^{-1}} \to e^{\|\zeta_1 - \zeta_2\| t^{-1}}, \quad \text{as } t_k \to t,$$

with respect to the usual metric. Therefore, $M_{\mathcal{A}}(\zeta_1, \zeta_2, .) : \mathcal{A} \to [0, 1]$ is continuous.

Hence, it is proved that the four-tuple $(U, M_{\mathcal{A}}, \mathcal{A}, *)$ is an FM-space over $\mathcal{A}$.

Proposition 3.3. *Let $(U_1, M_{\mathcal{A}_1}, \mathcal{A}, *)$ and $(U_2, M_{\mathcal{A}_2}, \mathcal{A}, *)$ be two FM-spaces over $\mathcal{A}$. For $(\zeta_1, \zeta_2), (v_1, v_2) \in U_1 \times U_2$ and $t \in \mathcal{A}$ such that*

$$M_{\mathcal{A}}((\zeta_1, \zeta_2), (v_1, v_2), t) = M_{\mathcal{A}_1}(\zeta_1, v_1, t) * M_{\mathcal{A}_2}(\zeta_2, u_2, t),$$

$M_{\mathcal{A}}$ is an FM-space over $\mathcal{A}$ on $U_1 \times U_2$.

Proof.

(FB-1) Since $M_{\mathcal{A}_1}(\zeta_1, v_1, t) > 0$ and $M_{\mathcal{A}_2}(\zeta_2, v_2, t) > 0$,

$$M_{\mathcal{A}_1}(\zeta_1, v_1, t) * M_{\mathcal{A}_2}(\zeta_2, v_2, t) > 0 \quad \text{for } t \in \mathcal{A}.$$

Therefore, we have that

$$M_{\mathcal{A}}((\zeta_1, \zeta_2), (v_1, v_2), t) > 0 \quad \text{for } t \in \mathcal{A}.$$

Further, we suppose that for $t \in \mathcal{A}$, $(\zeta_1, v_1, t) = (\zeta_2, v_2, t)$. This implies that $\zeta_1 = v_1$ and $\zeta_2 = v_2$, for $t \in \mathcal{A}$. Hence, we get

$$M_{\mathcal{A}_1}(\zeta_1, v_1, t) = 1 \quad \text{and} \quad M_{\mathcal{A}_2}(\zeta_2, v_2, t) = 1.$$

It follows that,

$$M_{\mathcal{A}}((\zeta_1, \zeta_2), (v_1, v_2), t) = 1.$$

Conversely, suppose that $M_{\mathcal{A}}((\zeta_1, \zeta_2), (v_1, v_2), t) = 1$, for $t \in \mathcal{A}$. Then

$$M_{\mathcal{A}_1}(\zeta_1, v_1, t) * M_{\mathcal{A}_2}(\zeta_2, v_2, t) = 1.$$

So

$$0 < M_{\mathcal{A}_1}(\zeta_1, v_1, t) \leq 1 \quad \text{and} \quad 0 < M_{\mathcal{A}_2}(\zeta_2, v_2, t) \leq 1.$$

It follows that

$$M_{\mathcal{A}_1}(\zeta_1, v_1, t) = 1 \quad \text{and} \quad M_{\mathcal{A}_2}(\zeta_2, v_2, t) = 1.$$

Thus, we get that $\zeta_1 = \zeta_2$ and $v_1 = v_2$. Therefore, $(\zeta_1, \zeta_2, t) = (v_1, v_2, t)$ for $t \in \mathcal{A}$.

(FB-2) We will prove that $M_{\mathcal{A}}((\zeta_1, \zeta_2), (v_1, v_2), t) = M_{\mathcal{A}}((v_1, v_2), (\zeta_1, \zeta_2), t)$, for $t \in \mathcal{A}$. Now, we observe that

$$M_{\mathcal{A}_1}(\zeta_1, v_1, t) = M_{\mathcal{A}_1}(v_1, \zeta_1, t) \text{ and } M_{\mathcal{A}_2}(\zeta_2, v_2, t) = M_{\mathcal{A}_2}(v_2, \zeta_2, t).$$

It follows that for all $(\zeta_1, \zeta_2), (v_1, v_2) \in U_1 \times U_2$ and $t \in \mathcal{A}$,

$$M_{\mathcal{A}}((\zeta_1, \zeta_2), (v_1, v_2), t) = M_{\mathcal{A}}((v_1, v_2), (\zeta_1, \zeta_2), t).$$

(FB-3) Since $(U_1, M_{\mathcal{A}_1}, \mathcal{A}, *)$ and $(U_2, M_{\mathcal{A}_2}, \mathcal{A}, *)$ are FM-spaces over $\mathcal{A}$, we have that

$$M_{\mathcal{A}_1}(\zeta_1, v_1, t+s) \geq M_{\mathcal{A}_1}(\zeta_1, w_1, t) * M_{\mathcal{A}_1}(w_1, v_1, s),$$

and

$$M_{\mathcal{A}_2}(\zeta_2, v_2, t+s) \geq M_{\mathcal{A}_2}(\zeta_2, w_2, t) * M_{\mathcal{A}_2}(w_2, v_2, s),$$

for all $(\zeta_1, \zeta_2), (v_1, v_2), (w_1, w_2) \in U_1 \times U_2$ and for $s, t \in \mathcal{A}$. Therefore, we have

$$\begin{aligned} M_{\mathcal{A}}((\zeta_1, \zeta_2), (v_1, v_2), t+s) &= M_{\mathcal{A}_1}(\zeta_1, v_1, t+s) * M_{\mathcal{A}_2}(\zeta_2, v_2, t+s) \\ &\geq M_{\mathcal{A}_1}(v_1, w_1, t) * M_{\mathcal{A}_1}(w_1, v_1, s) * M_{\mathcal{A}_2}(\zeta_2, w_2, t) * M_{\mathcal{A}_2}(w_2, v_2, s) \\ &= M_{\mathcal{A}_1}(\zeta_1, w_1, t) * M_{\mathcal{A}_2}(\zeta_2, w_2, t) * M_{\mathcal{A}_1}(w_1, v_1, s) * M_{\mathcal{A}_2}(w_2, v_2, s) \\ &= M_{\mathcal{A}}((\zeta_1, \zeta_2), (w_1, w_2), t) * M_{\mathcal{A}}((w_1, w_2), (v_1, v_2), s). \end{aligned}$$

(FB-4) Note that $M_{\mathcal{A}_1}(\zeta_1, v_1, t)$ and $M_{\mathcal{A}_2}(\zeta_2, v_2, t)$ are continuous with resprct to $t \in \mathcal{A}$ and $*$ is also continuous. Therefore, it follows that

$$M_{\mathcal{A}}((\zeta_1, \zeta_2), (v_1, v_2), t) = M_{\mathcal{A}_1}(\zeta_1, v_1, t) * M_{\mathcal{A}_2}(\zeta_2, v_2, t)$$

is also continuous at $t \in \mathcal{A}$.

Definition 3.4. Let $(U, M_{\mathcal{A}}, \mathcal{A}, *)$ be an FM-space over $\mathcal{A}$. For every $t > 0$, in $\mathcal{A}$, the open ball $\mathcal{B}(\zeta_0, \rho, t)$ with center at $\zeta_0 \in U$ and radius $\rho \in (0, 1)$ is given as

$$\mathcal{B}(\zeta_0, \rho, t) = \{\zeta \in U : M_{\mathcal{A}}(\zeta_0, \zeta, t) > 1 - \rho\}.$$

Definition 3.5. A subset V of $(U, M_{\mathcal{A}}, \mathcal{A}, *)$ is said be open if for any given point $v \in V$, there are $\rho \in (0, 1)$ and $t \in \mathcal{A}$ such that

$$\mathcal{B}(v, \rho, t) \subseteq V.$$

Theorem 3.6. *Every open ball in $(U, M_{\mathcal{A}}, \mathcal{A}, *)$ is an open set.*

Proof. Suppose that $\mathcal{B}(\zeta_0, \rho, t)$ is an open ball in a $(U, M_{\mathcal{A}}, \mathcal{A}, *)$. Then $\zeta \in \mathcal{B}(\zeta_0, \zeta, t)$ implies that $M_{\mathcal{A}}(\zeta_0, \zeta, t) > 1 - \rho$. Let us take $t^* \in (0, t)$ in $\mathcal{A}$ such that

$$M_{\mathcal{A}}(\zeta_0, \zeta, t^*) > 1 - \rho.$$

Let $r^* = M_{\mathcal{A}}(\zeta_0, \zeta, t^*) > 1 - \rho \Rightarrow \rho^* > 1 - \rho$. Then we can choose a point $s \in (0, 1)$ such that

$$\rho^* > 1 - s > 1 - \rho.$$

Now, for given ρ^* and s such that $\rho^* > 1 - s$, we take another point $\rho' \in (0, 1)$ such that $\rho^* * \rho' \geq 1 - s$. Now, by considering the ball $\mathcal{B}(\zeta, 1 - \rho', t - t^*)$, we claim that

$$\mathcal{B}(\zeta, 1 - \rho', t - t^*) \subset \mathcal{B}(\zeta_0, \rho, t).$$

Again, we choose a point $\zeta_1 \in \mathcal{B}(\zeta, 1 - \rho', t - t^*)$ such that $M_{\mathcal{A}}(\zeta, \zeta_1, t - t^*) > \rho'$. Then we have

$$M_{\mathcal{A}}(\zeta_0, \zeta_1, t) \geq M_{\mathcal{A}}(\zeta_0, \zeta, t^*) * M_{\mathcal{A}}(\zeta, \zeta_1, t - t^*) \geq \rho^* * \rho' \geq 1 - s > 1 - \rho.$$

Therefore, we get that $\zeta_1 \in \mathcal{B}(\zeta_0, \rho, t)$ and hence

$$\mathcal{B}(\zeta, 1 - \rho', t - t^*) \subset \mathcal{B}(\zeta_0, \rho, t).$$

Thus, the proof is completed.

Theorem 3.7. *Let $(U, M_{\mathcal{A}}, \mathcal{A}, *)$ be an FM-space over $\mathcal{A}$. If we define*

$$\mathcal{T}_{\mathcal{A}} = \{A \subset U : \zeta_1 \in A \text{ if and only if there exist } \rho \in (0,1) \text{ and } t \in \mathcal{A} \text{ such that } \mathcal{B}(\zeta_1, \rho, t) \subset A\},$$

then $\mathcal{T}_{\mathcal{A}}$ is a topology on U.

Proof.

i) If $\zeta_1 \in \emptyset$, then $\emptyset = \mathcal{B}(\zeta_1, \rho, t) \subseteq \emptyset$. Hence, $\emptyset \in \mathcal{T}_{\mathcal{A}}$. So for any $\zeta_1 \in U$, $0 < \rho < 1$ and $t \in \mathcal{A}$ such that $\mathcal{B}(\zeta_1, \rho, t) \subseteq U$, we have $U \in \mathcal{T}_{\mathcal{A}}$.

ii) Let $A_k \in \mathcal{T}_{\mathcal{A}}$ for every $k \in I$ and $\zeta_1 \in \bigcup_{k \in I} \subset A_k$. Then there exists $k_0 \in I$ such that $\zeta_1 \in A_{k_0}$. So there exist $t \in \mathcal{A}$ and $\rho \in (0,1)$ such that $\mathcal{B}(\zeta_1, \rho, t) \subset A_{k_0}$. Since $A_{k_0} \subset \bigcup_{k \in I}$, $\mathcal{B}(\zeta_1, \rho, t) \subset \bigcup_{k \in I} A_k$. Thus, $\bigcup_{k \in I} A_k \in \mathcal{T}_{\mathcal{A}}$.

iii) Suppose that $A, B \in \mathcal{T}_{\mathcal{A}}$ and $\zeta_1 \in A \cap B$. Then, $\zeta_1 \in A$ and $\zeta_1 \in B$ and so there are $t_1, t_2 \in \mathcal{A}$ and $0 < \rho_1, \rho_2 < 1$ such that $\mathcal{B}(\zeta_1, \rho_1, t_1) \subset A$ and $\mathcal{B}(\zeta_1, \rho_2, t_2) \subset B$. By choosing $t \in \mathcal{A}$ such that $t < (t_1, t_2)$ and taking $\rho = \min\{\rho_1, \rho_2\}$, we have

$$\mathcal{B}(\zeta_1, \rho, t) \subset \mathcal{B}(\zeta_1, \rho_1, t_1) \cap \mathcal{B}(\zeta_1, \rho_2, t_2) \subset A \cap B.$$

Thus, $A \cap B \in \mathcal{T}_{\mathcal{A}}$. Hence, it is proved that $\mathcal{T}_{\mathcal{A}}$ is a topology on U.

Theorem 3.8. *Every FM-space over $\mathcal{A}$ is Hausdorff.*

Proof. Let ζ_1, ζ_2 be any two distinct points of U. Then from the definition of an FM over $\mathcal{A}$, $0 < M_{\mathcal{A}}(\zeta_1, \zeta_2, t) < 1$ for $t \in \mathcal{A}$. We can say that $M_{\mathcal{A}}(\zeta_1, \zeta_2, t) = \rho$ for some $\rho \in (0,1)$. For each $\rho_0 \in (\rho, 1)$ there exists $\rho_1 \in (0,1)$ such that $\rho_1 * \rho_1 > \rho_0$. Now, we consider the sets $\mathcal{B}(\zeta_1, 1 - \rho_1, t/2)$ and $\mathcal{B}(\zeta_2, 1 - \rho_1, t/2)$. Then we have

$$\mathcal{B}\left(\zeta_1, 1 - \rho_1, \frac{t}{2}\right) \cap \mathcal{B}\left(\zeta_2, 1 - \rho_1, \frac{t}{2}\right) = \emptyset,$$

for $t \in \mathcal{A}$. Suppose that

$$\mathcal{B}\left(\zeta_1, 1 - \rho_1, \frac{t}{2}\right) \cap \mathcal{B}\left(\zeta_2, 1 - \rho_1, \frac{t}{2}\right) \neq \emptyset,$$

for $t \in \mathcal{A}$. Then there exists $y \in \mathcal{B}\left(u_1, 1 - \rho_1, \frac{t}{2}\right) \cap \mathcal{B}\left(\zeta_2, 1 - \rho_1, \frac{t}{2}\right)$. Therefore, we have $M_{\mathcal{A}}(\zeta_1, y, t/2) > 1 - (1 - \rho_1) = \rho_1$ and $M_{\mathcal{A}}(\zeta_2, y, t/2) > 1 - (1 - \rho_1) = \rho_1$ for $t \in \mathcal{A}$. From Definition 3.1 (fmB3),

$$\rho = M_{\mathcal{A}}(\zeta_1, \zeta_2, t) \geq M_{\mathcal{A}}\left(\zeta_1, 1 - \rho_1, \frac{t}{2}\right) \cap M_{\mathcal{A}}\left(\zeta_2, 1 - \rho_1, \frac{t}{2}\right) \neq \emptyset,$$

for $t \in \mathcal{A}$. Then $\rho_1 * \rho_1 < \rho$, which implies that $\rho < \rho_0 < \rho$. This is a contradiction.
Thus $\mathcal{B}\left(\zeta_1, 1 - \rho_1, \frac{t}{2}\right) \cap \mathcal{B}\left(\zeta_2, 1 - \rho_1, \frac{t}{2}\right) = \emptyset$ for $t \in \mathcal{A}$.

Definition 3.9. Let $(U, M_{\mathcal{A}}, \mathcal{A}, *)$ be an FM-space over $\mathcal{A}$. A subset B of U is called F-bounded over $\mathcal{A}$ if there are $t \in \mathcal{A}$ and $\rho \in (0,1)$, such that $M_{\mathcal{A}}(\zeta_1, \zeta_2, t) > 1 - \rho$ for all $\zeta_1, \zeta_2 \in B$.

Theorem 3.10. *Let $(U, M_{\mathcal{A}}, \mathcal{A}, *)$ be an FM-space over $\mathcal{A}$. Then every compact set in U is closed and F-bounded over $\mathcal{A}$.*

Proof. Let B be a compact subset of U, $t \in \mathcal{A}$ and $\rho \in (0,1)$. Since $\{\mathcal{B}(\zeta,\rho,t) : \zeta \in B\}$ is an open cover of B, there are $\zeta_1, \zeta_2, ..., \zeta_j \in B$ such that

$$B \subset \cup_{k=1}^{j} \mathcal{B}(\zeta_j, \rho, t) \quad \text{for } t \in \mathcal{A}.$$

For any $\zeta, v \in B$, there exist $1 \leq k, l \leq j$ such that $\zeta \in \mathcal{B}(\zeta_k, \rho, t)$ and $v \in \mathcal{B}(\zeta_k, \rho, t)$. Then we can write

$$M_{\mathcal{A}}(\zeta, \zeta_k, t) > 1 - \rho \quad \text{and} \quad M_{\mathcal{A}}(\zeta, \zeta_l, t) > 1 - \rho,$$

for $t \in \mathcal{A}$. Let us take $\gamma = \min\{M_{\mathcal{A}}(\zeta_k, \zeta_l, t) : 1 \leq k, l \leq j\}$. Then we have

$$\begin{aligned} M_{\mathcal{A}}(\zeta, v, 3t) &\geq M_{\mathcal{A}}(\zeta, \zeta_k, t) * M_{\mathcal{A}}(\zeta_k, \zeta_l, t) * M_{\mathcal{A}}(\zeta_l, v, t) \\ &\geq (1-\rho) * \gamma * (1-\rho). \end{aligned}$$

Let $t^* = 3t$ and choose a point $\rho^* \in (0,1)$ such that $(1-\rho) * \gamma * (1-\rho) > 1 - \rho^*$. Then, for any $\zeta, v \in B$, we have that $M_{\mathcal{A}}(\zeta, v, t^*) > 1 - \rho^*$, and hence B is F-bounded over $\mathcal{A}$. On the other hand, from Theorem 3.8, every FM-space over a Banach algebra $\mathcal{A}$ is Hausdorff and every compact subset of a Hausdorff space is closed, that is, B is closed.

Definition 3.11. Let $(U, M_{\mathcal{A}}, \mathcal{A}, *)$ be an FM-space over $\mathcal{A}$ and (ζ_k) be a sequence in U. Then, (ζ_k) is said to converge to a point $\zeta \in U$, if for any $t \in \mathcal{A}$ and $\rho \in (0,1)$, there is $k_0 \in \mathbb{N}$ such that $M_{\mathcal{A}}(\zeta_k, \zeta, t) > 1 - \rho$, for all $k \geq k_0$. We can write this as,

$$\lim_{k\to\infty} \zeta_k = \zeta \quad \text{or} \quad \zeta_k \to \zeta, \quad \text{as } k \to \infty.$$

Theorem 3.12. *Let $(U, M_{\mathcal{A}}, \mathcal{A}, *)$ be an FM-space over $\mathcal{A}$. Then a sequence (ζ_k) in U converges to a point $\zeta \in U$ if and only if $M_{\mathcal{A}}(\zeta_k, \zeta, t) \to 1$, as $k \to \infty$, for each $t \in \mathcal{A}$.*

Proof. Assume that a sequence (ζ_k) converges to a point $\zeta \in U$. Then for each $t \in \mathcal{A}$ and $\rho \in (0,1)$, there is $k_0 \in \mathbb{N}$ such that $M_{\mathcal{A}}(\zeta_k, \zeta, t) > 1 - \rho$, for all $k \geq k_0$. Hence,

$$M_{\mathcal{A}}(\zeta_k, \zeta, t) \to 1, \quad \text{as } k \to \infty.$$

Conversely, suppose that $M_{\mathcal{A}}(\zeta_k, \zeta, t) \to 1$ as $k \to \infty$. Then, for each $t \in \mathcal{A}$ and $\rho \in (0,1)$, there is $k_0 \in \mathbb{N}$ such that $\rho > 1 - M_{\mathcal{A}}(\zeta_k, \zeta, t)$, for all $k \geq k_0$, which implies that $M_{\mathcal{A}}(\zeta_k, \zeta, t) > 1 - \rho$. Hence, $\zeta_k \to \zeta \in U$ as $k \to \infty$.

Definition 3.13. Let $(U, M_{\mathcal{A}}, \mathcal{A}, *)$ be an FM-space over $\mathcal{A}$ and (ζ_k) be a sequence in U. Then

(1) (ζ_k) is said to be a Cauchy sequence if for any $t \in \mathcal{A}$ and $\rho \in (0,1)$, there is $k_0 \in \mathbb{N}$ such that $M_{\mathcal{A}}(\zeta_k, \zeta_m, t) > 1 - \rho$, for all $k, m \geq k_0$;

(2) $(U, M_{\mathcal{A}}, \mathcal{A}, *)$ is complete if every Cauchy sequence is convergent in U.

Definition 3.14. A sequence (ζ_k) in $(U, M_{\mathcal{A}}, \mathcal{A}, *)$ is called $G_{\mathcal{A}}$-Cauchy sequence if $\lim_{k\to\infty} M_{\mathcal{A}}(\zeta_k, \zeta_{k+l}, t) = 1$, for all $k, m \geq k_0$ and $t \in \mathcal{A}$. $(U, M_{\mathcal{A}}, \mathcal{A}, *)$ is $G_{\mathcal{A}}$-complete if every $G_{\mathcal{A}}$-Cauchy sequence is convergent in U.

Lemma 3.15. *Let $(U, M_{\mathcal{A}}, \mathcal{A}, *)$ be an FM-space over $\mathcal{A}$. Then $M_{\mathcal{A}}(\zeta_1, \zeta_2, .) : \mathcal{A} \to [0, 1]$ is non-decreasing for all $\zeta_1, \zeta_2 \in U$.*

Proof. On the contrary, suppose that $M_{\mathcal{A}}(\zeta_1, \zeta_2, t) > M_{\mathcal{A}}(\zeta_1, \zeta_2, s)$, for $s, t \in \mathcal{A}$ with $s > t$. Then

$$M_{\mathcal{A}}(\zeta_1, \zeta_2, t) * M_{\mathcal{A}}(\zeta_2, \zeta_2, s - t) \leq M_{\mathcal{A}}(\zeta_1, \zeta_2, s) < M_{\mathcal{A}}(\zeta_1, \zeta_2, t) \quad \text{for } s, t \in \mathcal{A}.$$

By Definition 3.1 (2), i.e., $M_{\mathcal{A}}(\zeta_2, \zeta_2, s - t) = 1$, we get that

$$M_{\mathcal{A}}(\zeta_1, \zeta_2, t) < M_{\mathcal{A}}(\zeta_1, \zeta_2, s) < M_{\mathcal{A}}(\zeta_1, \zeta_2, t) \quad \text{for } s, t \in \mathcal{A},$$

which is a contradiction.

Theorem 3.16. *Let $(U, M_{\mathcal{A}}, \mathcal{A}, *)$ be a $G_{\mathcal{A}}$-complete FM-space over $\mathcal{A}$. Let a mapping $T : U \to U$ be a mapping satisfying*

$$M_{\mathcal{A}}(T\zeta, Tv, \alpha t) \geq M_{\mathcal{A}}(\zeta, v, t), \tag{3.1}$$

for all $\zeta, v \in U$, $\alpha \in (0, 1)$ and $t \in \mathcal{A}$. Then T has a unique fixe point (FP).

Proof. Fix $\zeta_0 \in U$ and construct an iterative sequence in U such that $\zeta_{k+1} = T\zeta_k$ with $k \geq 0$, for $t \in \mathcal{A}$. Then, by (3.1), for $t \in \mathcal{A}$,

$$M_{\mathcal{A}}(\zeta_k, \zeta_{k+1}, \alpha t) = M_{\mathcal{A}}(T\zeta_{k-1}, T\zeta_k, \alpha t) \geq M_{\mathcal{A}}(\zeta_{k-1}, \zeta_k, t).$$

Then we get

$$M_{\mathcal{A}}(\zeta_k, \zeta_{k+1}, t) \geq M_{\mathcal{A}}\left(\zeta_{k-1}, \zeta_k, \frac{t}{\alpha}\right) \quad \text{for } t \in \mathcal{A}. \tag{3.2}$$

Similarly,

$$\begin{aligned} M_{\mathcal{A}}(\zeta_{k-1}, \zeta_k, t) &= M_{\mathcal{A}}\left(\zeta_{k-1}, \zeta_k, \alpha\left(\frac{t}{\alpha}\right)\right) \\ &= M_{\mathcal{A}}\left(T\zeta_{k-2}, T\zeta_{k-1}, \alpha\left(\frac{t}{\alpha}\right)\right) \geq M_{\mathcal{A}}\left(\zeta_{k-2}, \zeta_{k-1}, \frac{t}{\alpha}\right) \quad \text{for } t \in \mathcal{A}. \end{aligned}$$

This implies that

$$M_{\mathcal{A}}\left(\zeta_{k-1}, \zeta_k, \frac{t}{\alpha}\right) \geq M_{\mathcal{A}}\left(\zeta_{k-2}, \zeta_{k-1}, \frac{t}{\alpha^2}\right) \quad \text{for } t \in \mathcal{A}. \tag{3.3}$$

Now, from (3.2), (3.3), and by induction, for $t \in \mathcal{A}$, we have

$$\begin{aligned} M_{\mathcal{A}}(\zeta_k, \zeta_{k+1}, t) &\geq M_{\mathcal{A}}\left(\zeta_{k-1}, \zeta_k, \frac{t}{\alpha}\right) \geq M_{\mathcal{A}}\left(\zeta_{k-2}, \zeta_{k-1}, \frac{t}{\alpha^2}\right) \\ &\geq \cdots \geq M_{\mathcal{A}}\left(\zeta_0, \zeta_1, \frac{t}{\alpha^k}\right) \to 1, \quad \text{as } k \to \infty. \end{aligned}$$

Hence we get that

$$\lim_{k\to\infty} M_{\mathcal{A}}(\zeta_k, \zeta_{k+1}, t) = 1 \quad \text{for } t \in \mathcal{A}. \tag{3.4}$$

Now, for any positive integer ℓ and for $t \in \mathcal{A}$, from (3.4), we have

$$\begin{aligned}
M_{\mathcal{A}}(\zeta_k, \zeta_{k+\ell}, t) &\geq M_{\mathcal{A}}\left(\zeta_k, \zeta_{k+1}, \frac{t}{\ell}\right) * M_{\mathcal{A}}\left(\zeta_{k+1}, \zeta_{k+2}, \frac{t}{\ell}\right) * \cdots * M_{\mathcal{A}}\left(\zeta_{k+\ell-1}, \zeta_{k+\ell}, \frac{t}{\ell}\right) \\
&\geq M_{\mathcal{A}}\left(\zeta_0, \zeta_1, \frac{t}{\ell\alpha^k}\right) * M_{\mathcal{A}}\left(\zeta_0, \zeta_1, \frac{t}{\ell\alpha^{k+1}}\right) * \cdots * M_{\mathcal{A}}\left(\zeta_0, \zeta_1, \frac{t}{\ell\alpha^{k+\ell-1}}\right) \\
&\to 1 * 1 * \cdots * 1 = 1, \quad \text{as } k \to \infty.
\end{aligned}$$

Hence it is proved that (ζ_k) is a $G_{\mathcal{A}}$-Cauchy sequence. Since $(U, M_{\mathcal{A}}, \mathcal{A}, *)$ is $G_{\mathcal{A}}$-complete, there is $z \in U$ such that

$$\lim_{k\to\infty} M_{\mathcal{A}}(\zeta_k, z, t) = 1 \quad \text{for } t \in \mathcal{A}. \tag{3.5}$$

Now, from (3.1), (3.2) and (3.5), we have for all $t \in \mathcal{A}$,

$$\begin{aligned}
M_{\mathcal{A}}(z, Tz, t) &\geq M_{\mathcal{A}}\left(z, \zeta_{k+1}, \frac{t}{2}\right) * M_{\mathcal{A}}\left(T\zeta_k, Tz, \frac{t}{2}\right) \\
&\geq M_{\mathcal{A}}\left(z, \zeta_{k+1}, \frac{t}{2}\right) * M_{\mathcal{A}}\left(\zeta_k, z, \frac{t}{2\alpha}\right) \\
&\to 1 * 1* = 1, \quad \text{as } k \to \infty.
\end{aligned}$$

Hence we get that $M_{\mathcal{A}}(z, Tz, t) = 1$, i.e., $Tz = z$.

For uniqueness, assume that w is another FP of the mapping T in U. Then, from (3.1) and (3.2), we have for all $t \in \mathcal{A}$,

$$\begin{aligned}
1 \geq M_{\mathcal{A}}(z, w, t) = M_{\mathcal{A}}(Tz, Tw, t) &\geq M_{\mathcal{A}}\left(z, w, \frac{t}{\alpha}\right) = M_{\mathcal{A}}\left(Tz, Tw, \frac{t}{\alpha}\right) \\
&\geq M_{\mathcal{A}}\left(z, w, \frac{t}{\alpha^2}\right) \geq \cdots \geq M_{\mathcal{A}}\left(z, w, \frac{t}{\alpha^k}\right) \to 1, \quad \text{as } k \to \infty.
\end{aligned}$$

Hence $M_{\mathcal{A}}(z, w, t) = 1 \;\Rightarrow\; z = w$.

4. Application

This section deals with an application of a nonlinear integral equation (NIE) to support our work. Let $U = C([0,\rho], \mathbb{R})$ be the space of $\mathbb{R}$-valued continuous functions on $[0,\rho]$, where $0 < \rho \in \mathbb{R}$. The NIE is

$$\zeta(\kappa) = \int_0^\kappa P(\kappa, s, \zeta(s))ds, \quad \text{for all } \zeta \in U, \tag{4.1}$$

where $\kappa, s \in [0,\rho]$ and $P : [0,\rho] \times [0,\rho] \times \mathbb{R} \to \mathbb{R}$. The induced metric $m : U \times U \to \mathbb{R}$ is defined by

$$m(\zeta, \nu) = \sup_{\kappa\in[0,\rho]} |\zeta(\kappa) - \nu(\kappa)| = \|\zeta - \nu\|, \quad \text{where } \zeta, \nu \in C([0,\rho], \mathbb{R}) = U.$$

The binary operation $*$ is defined as $\delta * \varrho = \delta\varrho$ for all $\delta, \varrho \in [0,\rho]$. An FM over $\mathcal{A}$ $M_{\mathcal{A}} : U \times U \times \mathcal{A} \to [0,1]$ is defined by

$$M_{\mathcal{A}}(\zeta, \nu, t) = \left(e^{\|\zeta-\nu\|t^{-1}}\right)^{-1}, \tag{4.2}$$

for all $\zeta, \nu \in U$ and $t \in \mathcal{A}$. Then easily it can be verified that $(U, M_{\mathcal{A}}, \mathcal{A}, *)$ is a complete FM space over a Banach algebra $\mathcal{A}$.

Theorem 4.1. *Assume that there exist $A_\zeta, A_\nu \in U$ such that*

$$A_\zeta(\kappa) = \int_0^\kappa P(\kappa, s, \zeta(s))ds \quad \text{and} \quad A_\nu(\kappa) = \int_0^\kappa P(\kappa, s, \nu(s))ds, \tag{4.3}$$

where $\kappa \in [0, \rho]$. If there is $\beta \in (0, 1)$ such that

$$||A_\zeta - A_\nu) \leq \beta||\zeta - \nu||, \tag{4.4}$$

for all $\zeta, \nu \in U$. Then the NIE (4.1) *has a unique solution in U.*

Proof. We define an integral operator $T : U \to U$ by

$$T(\zeta) = A_\zeta + \hbar, \quad T(\nu) = A_\nu + \hbar,$$

for all $\zeta, \nu, \hbar \in U$. Now, from (4.2) and (4.4), for $t \in \mathcal{A}$, we have

$$\begin{aligned} M_{\mathcal{A}}(T\zeta, T\nu, \alpha t) &= \left(e^{||T\zeta - T\nu||(\alpha t)^{-1}}\right)^{-1} \\ &= \left(e^{||A_\zeta - A_\nu||\alpha^{-1}t^{-1}}\right)^{-1} \\ &\geq \left(e^{\beta||\zeta - \nu||\alpha^{-1}t^{-1}}\right)^{-1} \\ &= \left(e^{||\zeta - \nu||\beta\alpha^{-1}t^{-1}}\right)^{-1} \\ &\geq \left(e^{||\zeta - \nu||t^{-1}}\right)^{-1}, \quad \text{where } \beta\alpha^{-1} \leq 1 \text{ and } \alpha, \beta \in (0, 1) \\ &= M_{\mathcal{A}}(\zeta, \nu, t). \end{aligned}$$

This implies that

$$M_{\mathcal{A}}(T\zeta, T\nu, \alpha t) \geq M_{\mathcal{A}}(\zeta, \nu, t),$$

for all $\zeta, \nu \in U$ and $t \in \mathcal{A}$. This means that the inequality (3.1) of Theorem 3.16 is satisfied. Thus, the integral operator T satisfies all the conditions of Theorem 3.16 and T has a unique FP, i.e., (4.1) has a unique solution in U.

5. Conclusions

In this paper, we discussed a new concept of an FM-space over a Banach algebra $\mathcal{A}$, i.e., four-tuple $(U, M_{\mathcal{A}}, \mathcal{A}, *)$. We proved some basic properties and a Hausdorff metric in the FM-space over $\mathcal{A}$. We proved a "Banach contraction principle for fixed point" on the FM-space over $\mathcal{A}$. Moreover, we presented an application of a nonlinear integral equation. One can use this concept to present some more properties, FP and CFP theorems for different contractive type mappings in FM-space over $\mathcal{A}$ with different types of integral operators.

Conflict of interest

The authors declare that they have no competing interests.

References

1. L. A. Zadeh, Fuzzy sets, *Inf. Control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
2. O. Kramosil, J. Michalek, Fuzzy metric and statistical metric spaces, *Kybernetika*, **11** (1975), 336–344.
3. M. Grabiec, Fixed point in fuzzy metric spaces, *Fuzzy Set. Syst.*, **27** (1988), 385–389. https://doi.org/10.1016/0165-0114(88)90064-4
4. A. George, P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Set. Syst.*, **64** (1994), 395–399. https://doi.org/10.1016/0165-0114(94)90162-7
5. V. Gregori, A. Sapena, On fixed point theorems in fuzzy metric spaces, *Fuzzy Set. Syst.*, **125** (2002), 245–252. https://doi.org/10.1016/S0165-0114(00)00088-9
6. J. Rodriguez-Lopez, S. Romaguera, The Haudorff fuzzy metric on compact sets, *Fuzzy Set. Syst.*, **147** (2004), 273–283. https://doi.org/10.1016/j.fss.2003.09.007
7. F. Kiani, A. Amini-Haradi, Fixed point and endpoint theorems for set-valued fuzzy contraction mapps in fuzzy metric spaces, *Fixed Point Theory Appl.*, **2011** (2011), 94. https://doi.org/10.1186/1687-1812-2011-94
8. I. Shamas, S. U. Rehman, H. Aydi, T. Mahmood, E. Ameer, Unique fixed-point results in fuzzy metric spaces with an application Fredholm integral equations, *J. Funct. Space.*, **2021** (2021), 4429173. https://doi.org/10.1155/2021/4429173
9. I. Shamas, S. U. Rehman, N. Jan, A. Gumaei, M. Al-Rakhami, A new approach to fuzzy differential equations using weakly-compatible self-mappings in fuzzy metric spaces, *J. Funct. Space.*, **2021** (2021), 6123154. https://doi.org/10.1155/2021/6123154
10. S. U. Rehman, R. Chinram, C. Boonpok, Rational type fuzzy-contraction results in fuzzy metric spaces with an application, *J. Math.*, **2021** (2021), 6644491. https://doi.org/10.1155/2021/6644491
11. M. A. Erceg, Metric spaces in fuzzy set theorey, *J. Math. Anal. Appl.*, **69** (1979), 205–230. https://doi.org/10.1016/0022-247X(79)90189-6
12. P. Debnath, N. Konwar, S. Radenović, *Metric fixed point theory: Applications in science, engineering and behavioural sciences,* Springer Nature, Singapore, 2021. https://doi.org/10.1007/978-981-16-4896-0
13. A. George, P. Veeramani, On some results of analysis for fuzzy metric spaces, *Fuzzy Set. Syst.*, **90** (1997), 365–368. https://doi.org/10.1016/S0165-0114(96)00207-2
14. O. Hadzic, E. Pap, Fixed point theorem for multivalued mappings in probabilistic metric spaces and an applications in fuzzy metric spaces, *Fuzzy Set. Syst.*, **127** (2002), 333–344. https://doi.org/10.1016/S0165-0114(01)00144-0
15. D. Rakić, A. Mukheimer, T. Došenović, Z. D. Mitrović, S. Radenović, On some new fixed point results in fuzzy *b*-metric spaces, *J. Inequal. Appl.*, **2020** (2020), 99. https://doi.org/10.1186/s13660-020-02371-3.
16. Z. Sadeghi, S. M. Vaezpour, C. Park, R. Saadati, C. Vetro, Set-valued mappings in partially ordered fuzzy metric spaces, *J. Inequal. Appl.*, **157** (2014). https://doi.org/10.1186/1029-242X-2014-157

17. S. Sedghi, N. Shobkolaei, T. Došenović, S. Radenović, Suzuki-type of common fixed point theorems in fuzzy metric spaces, *Math. Slovaca*, **68** (2018), 1–12. https://doi.org/10.1515/ms-2017-0115

18. P. Debnath, Lacunary ideal convergence in intuitionistic fuzzy normed linear spaces, *Comput. Math. Appl.*, **63** (2012), 708–715. https://doi.org/10.1016/j.camwa.2011.11.034

19. N. Konwar, P. Debnath, Some new contractive conditions and related fixed point theorems in intuitionistic fuzzy n-Banach spaces, *J. Intell. Fuzzy Syst.*, **34** (2018), 361–372. https://doi.org/10.3233/JIFS-171372

20. N. Konwar, B. Davvaz, P. Debnath, Approximation of new bounded operators in intuitionistic fuzzy n-Banach spaces, *J. Intell. Fuzzy Syst.*, **35** (2018), 6301–6312. https://doi.org/10.3233/JIFS-181094

21. N. Konwar, P. Debnath, Continuity and Banach contraction principle in intuitionistic fuzzy n-normed linear spaces, *J. Intell. Fuzzy Syst.*, **33** (2017), 2363–2373. https://doi.org/10.3233/JIFS-17500

22. N. Konwar, B. Davvaz, P. Debnath, Results on generalized intuitionistic fuzzy hypergroupoids, *J. Intell. Fuzzy Syst.*, **36** (2019), 2571–2580. https://doi.org/10.3233/JIFS-181522

23. L. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, **332** (2007), 1468–1476. https://doi.org/10.1016/j.jmaa.2005.03.087

24. W. S. Du, A note on cone metric fixed point theory and its equivalence, *Nonlinear Anal.*, **72** (2010), 2259–2261. https://doi.org/10.1016/j.na.2009.10.026

25. H. Cakalli, A. Sonmez, C. Genc, On an equivalence of topological vector space valued cone metric spaces and metric spaces, *Appl. Math. Lett.*, **25** (2012), 429–433. https://doi.org/10.1016/j.aml.2011.09.029

26. H. Liu, S. Xu, Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz map, *Fixed Point Theory Appl.*, **2013** (2013), 320. https://doi.org/10.1186/1687-1812-2013-320

27. Q. Yan, J. Yin, T. Wang, Fixed point and common fixed point theorems on ordered cone metric spaces over Banach algebras, *J. Nonlinear Sci. Appl.*, **9** (2016), 1581–1589. https://doi.org/10.22436/jnsa.009.04.15

28. T. Oner, M. B. Kandemir, B. Tanay, Fuzzy cone metric spaces, *J. Nonlinear Sci. Appl.*, **8** (2015), 610–616. https://doi.org/10.22436/jnsa.008.05.13

29. T. Oner, Some topological properties of fuzzy cone metric spaces, *J. Nonlinear Sci. Appl.*, **9** (2016), 799–805. https://doi.org/10.22436/jnsa.009.03.08

30. S. U. Rehman, H. X. Li, Fixed point theorems in fuzzy cone metric spaces, *J. Nonlinear Sci. Appl.*, **10** (2017), 5763–5769. https://doi.org/10.22436/jnsa.010.11.14

31. S. Jabeen, S. U. Rman, Z. Zheng, W. Wei, Weakly compatible and Quasi-contraction results in fuzzy cone metric spaces with application to the Urysohn type integral equations, *Adv. Differ. Equ.*, **2020** (2020), 280. https://doi.org/10.1186/s13662-020-02743-5

32. G. X. Chen, S. Jabeen, S. U. Rehman, A. M. Khalil, F. Abbas, A. Kanwal, Coupled fixed point analysis in fuzzy cone metric spaces with application to nonlinear integral equations, *Adv. Differ. Equ.*, **2020** (2020), 671. https://doi.org/10.1186/s13662-020-03132-8

33. N. Priyobarta, Y. Rohen, B. B. Upadhyay, Some fixed point results in fuzzy cone metric spaces, *Int. J. Pure Appl. Math.*, **109** (2016), 573–582. https://doi.org/10.12732/ijpam.v109i3.7

34. M. T. Waheed, S. U. Rehman, N. Jan, A. Gumaei, M. Al-Rakhami, Some new coupled fixed-point findings depending on another function in fuzzy cone metric spaces with application, *Math. Probl. Eng.*, **2021** (2021), 4144966. https://doi.org/10.1155/2021/4144966

35. M. T. Waheed, S. U. Rehman, N. Jan, A. Gumaei, M. Al-Rakhami, An approach of Lebesgue integral in fuzzy cone metric spaces via unique coupled fixed point theorems, *J. Funct. Space.*, **2021** (2021), 8766367. https://doi.org/10.1155/2021/8766367

36. W. Rudin, *Functional analysis*, 2 Eds., McGraw-Hill, New York, 1991.

37. S. Xu, S. Radenović, Fixed point theorems of generalized Lipschitz maps on cone metric spaces over Banach algebras without the assumtion of normality, *Fixed Point Theory Appl.*, **2014** (2014), 102. https://doi.org/10.1186/1687-1812-2014-102

38. B. Schweizer, A. Sklar, Statistical metric spaces, *Pac. J. Math.*, **10** (1960), 314–334. https://doi.org/10.2140/pjm.1960.10.313