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Research article

Interpolative Ćirić-Reich-Rus-type best proximity point results with applications

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Abstract: In this paper, we introduce the notion of ω-interpolative Ćirić-Reich-Rus-type proximal contraction. We obtain some best proximity point results for these mappings using the concept of ω-admissibility in complete metric spaces. Some best proximity results are extended to partial ordered metric spaces and graphical metric spaces. Several new definitions are presented by considering the special cases of aforementioned results. The application of these results in fixed point theory is also discussed. The acquired results extend ω-interpolative Ćirić-Reich-Rus-type contraction for obtaining fixed points.

Keywords: complete metric space; ordered metric space; graph theory; interpolative proximal contraction
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1. Introduction and preliminaries

The theory of fixed point is a study of different conditions on contractive self-mappings having a fixed point. While considering self-mapping, the solution x^* of an operative equation $Jx = x$ is called the fixed point of the mapping J. Generally, the researchers used the sequences of Picard iteration which then become Cauchy sequences and their limits are fixed points of the operator J.

In 1922, Banach opened a flourishing area of investigation in metrical fixed point theory by stating the classical Banach contraction principle [5], which asserts that every contraction in a complete metric space possesses a unique fixed point. This result captivated and inspired numerous researchers around the world. During the last century, they generalized and improved this key principle by using several techniques. This would include generalizing, modifying and extending it (refer to [12, 15, 16, 21, 29, 33, 37]). The applications of fixed point theory plays a vital role in various fields of sciences. It provides a technique for solving a variety of applied problems in many branches of mathematics (see [9, 19, 25, 36]). Many fixed point theorems were established for contractive type maps in partially ordered metric spaces and their applications to ordinary differential equations (for details, see [2]). The applications of fixed point theory is versatile as it is immensely used in areas other than mathematics, for instance in digital programming to develop algorithms (see [35]), in dynamic coding (for details, see [34]), in economics to identify an equilibrium point, in biology for the cell replicative study. However, all these results were generated by considering self-mappings. Rhoades [30] compared and contrasted distinct types of contractive mappings, which includes Kannan [22], Chatterjea [10], Reich [32], Ćirić [11], Hardy and Rogers [17]. It is observed that while using self-mappings we can find fixed points of the underlying mapping but it come more interesting when self-mappings are replaced with nonself mappings on different and disjoint subsets of same metric space, then the fixed point the nonself will not exist. In such cases, it comes more interesting to find the approximate fixed points of the underlying non-self mappings.

While considering non-self maps, say $J : L \to N$, where L, N denote nonvoid subsets of a metric space (Y, d), where d is the distance defined on Y, then J might not have a fixed point. In that case, the key interest is to identify any element x, which would be closest to Jx. This point is called the best proximity point. The best proximity point theorems are used to set forth sufficient conditions to assure the existence of an optimum solution to the query of minimizing the distance between x and Jx (see [13]), mathematically

$$\min_{x \in L} d(x, Jx). \tag{1.1}$$

Since $d(x, Jx) \geq d(L, N)$ for every x, where $d(L, N)$ be a distance of set L to set N, so any element x that may satisfy the equation

$$d(x, Jx) = d(L, N),$$

is then the best proximity point (that is, an optimal solution) of minimization problem (1.1), where

$$d(L, N) = \inf\{d(x, y),\ \text{for all } x \in L \text{ and } y \in N\}.$$

Note that when $L \cap N$ is not empty then the best proximity point reduces to a fixed point, for more details, see [4, 6, 7, 14, 18, 20].

One more interesting generalization of Banach's contraction is a F-contraction. Especially, in 2014 Cosentino and Vetro introduced F-contraction of Roger-Hardy type and proved a fixed point theorem in the setting of complete metric space. In 2018, the best proximity point theorem for Roger-Hardy type generalized F-contraction in complete metric spaces with the way of an optimal approximate solution using nonexpansive mappings (see [26]). In 2017, a notion of Suzuki $\alpha - F$-proximal multi-valued mappings in metric space was defined along with some examples, for more detail, see [28]. Same year, Geraghty contraction was studied in detail and the result of Geraghty was of particular interest due to its connection to real world problems in partially ordered complete metric space. Further, the best coincidence point and fixed point theorems were obtained for α-Geraghty contractions in the setting of complete metric spaces in [27]. These authors presented the presence and convergence of best proximity point using different contractive mappings which generalizes the Banach contraction principle.

From now and onward, consider L and N as two nonempty subsets of a metric space Y, along with the following essential notions to be used here and ahead:

$$\begin{aligned} d(L,N) &= \inf\{d(x,y) : x \in L \text{ and } y \in N\} \text{ (distance between sets } L \text{ and } N\text{)},\\ L_0 &= \{x \in L : d(x,y) = d(L,N) \text{ for some } y \in N\},\\ N_0 &= \{y \in N : d(x,y) = d(L,N) \text{ for some } x \in L\},\\ d^*(x,y) &= d(x,y) - d(L,N). \end{aligned}$$

Definition 1.1. *A set N is called approximately compact with respect to L, if for every sequence $\{y_n\}$ in N there is a convergent subsequence for some x in L, provided $d(x,y_n) \to d(x,N)$ holds.*

It is understood that every compact set is approximately compact, and every set with respect to itself is approximately compact. Moreover, L_0 and N_0 are nonvoid subsets, if L is compact and N is approximately compact with respect to L, as $L \cap N$ is contained in both L_0 and N_0.

In 1968, Kannan [22] generalized Banach contraction principle and made the world familiar with Kannan fixed point theorem. More knowledge concerning this renowned theorem can be gained from [33]. In 2018, Karapinar [23] made fusion of Kannan fixed point theorem and interpolative theory. The basic intention behind this combination was to maximize the rate of convergence to obtain fixed point.

The main result presented in [23] states as follow.

Theorem 1.1. *[23] A self-mapping J on a complete metric space Y, admits a unique fixed point such that*

$$d(Jx,Jy) \le \lambda (d(x,Jx))^{\gamma} (d(y,Jy))^{1-\gamma}, \tag{1.2}$$

where $\lambda \in [0,1)$, $\gamma \in (0,1)$, and $x, y \in Y$ with $x \neq Jx$.

In [24], it was pointed out that the obtained fixed point from inequality (1.2) may not be unique. Therefore, the Kannan fixed point theorem was restated (for details, see [1,3]). The following was the revised form of it.

Theorem 1.2. *[24] In the framework of a complete metric space Y, a self-mapping J admits a unique fixed point in Y, if there exist constants $\lambda \in [0,1)$ and $\gamma \in (0,1)$ such that*

$$d(Jx,Jy) \le \lambda (d(x,Jx))^{\gamma} (d(y,Jy))^{1-\gamma},$$

for all $x, y \in Y \backslash Fix(J)$, where $Fix(J) = \{a \in Y : Ja = a\}$ (set of fixed points)·

Next, the Ćirić-Reich-Rus theorem generalized both Ćirić type and Reich type contractions and as a result, enhanced Banach and Kannan type fixed point theorems (for further details, see [11,31,32]).

Theorem 1.3. *A self-mapping J on a complete metric space Y such that*

$$d(Jx,Jy) \le \lambda (d(x,y) + d(x,Jx) + d(y,Jy)),$$

for every x, y belong to Y, where $\lambda \in [0, \frac{1}{3})$, admits a unique fixed point.

Definition 1.2. *[8] Let Ψ represent the set of all nondecreasing mappings $\psi : [0,\infty) \to [0,\infty)$ such that*

$$\sum_{n=1}^{\infty} \psi^n(t) < \infty, \quad \text{for each } t > 0,$$

further, for every $\psi \in \Psi$, we observe that $\psi(0) = 0$ and $\psi(t) < t$ for each $t > 0$.

The following is a new definition of ω-interpolative Ćirić-Reich-Rus-type contraction that was introduced in [3]. Our main result is inspired by it.

Definition 1.3. *[3] Consider Y as a metric space, then $J : Y \to Y$ is said to be an ω-interpolative Ćirić-Reich-Rus-type contractive mapping, such that*

$$\omega(x,y)\, d(Jx, Jy) \leq \psi\left((d(x,y))^{\gamma} (d(x,Jx))^{\beta} (d(y,Jy))^{1-\gamma-\beta}\right), \tag{1.3}$$

for all $x, y \in Y \backslash Fix(J)$, provided $\psi \in \Psi$, $\omega : Y \times Y \to [0,\infty)$ and positive real numbers γ, β satisfying $\gamma + \beta < 1$.

The concept of ω-interpolative Kannan-type contractions was introduced in [3] and is defined in following definition.

Definition 1.4. *[3] Let Y be a metric space, then $J : Y \to Y$ is called an ω-interpolative Kannan-type contraction, such that*

$$\omega(x,y) d(Jx, Jy) \leq \psi([d(x,Jx)]^{\beta}[d(y,Jy)]^{1-\beta}), \tag{1.4}$$

for all $x, y \in Y \backslash Fix(J)$, provided $\psi \in \Psi$, $\omega : Y \times Y \to [0,\infty)$ and $\beta \in (0,1)$.

Definition 1.5. *[7] An element x^* in L is said to be a best proximity point of non-self mapping $J : L \to N$ if*

$$d(x^*, Jx^*) = d(L,N).$$

Definition 1.6. *[20] Let $J : L \to N$ and $\omega : L \times L \to [0,\infty)$, then J is known as ω-proximal admissible, if*

$$\left.\begin{array}{l} \omega(x_1,x_2) \geq 1 \\ d(u_1, Jx_1) = d(L,N) \\ d(u_2, Jx_2) = d(L,N) \end{array}\right\} \implies \omega(u_1,u_2) \geq 1,$$

for all x_1, x_2, u_1 and $u_2 \in L$.

Definition 1.7. *[37] Let $J : Y \to Y$ and $\omega : L \times L \to [0,\infty)$, then J is said to be a ω-admissible mapping, if*

$$\omega(x_1,x_2) \geq 1 \implies \omega(Jx_1, Jx_2) \geq 1, \tag{1.5}$$

for all $x_1, x_2 \in Y$.

Remark 1.1. *If J is a self-mapping, then every ω-proximal admissible becomes ω-admissible mapping.*

The concept of the weak P-property was first introduced in [14] and is defined as:

Definition 1.8. *[14] Consider (L,N) as a pair of nonempty subsets of a (Y,d) with $L_0 \neq \emptyset$. Then (L,N) is said to have the weak P-property if and only if for any $x_1, x_2 \in L_0$ and $y_1, y_2 \in N_0$,*

$$\left.\begin{array}{l} d(x_1,y_1) = d(L,N) \\ d(x_2,y_2) = d(L,N) \end{array}\right\} \implies d(x_1,x_2) \leq d(y_1,y_2).$$

Remark 1.2. *It is imperative to note that all sets that satisfies the P-property also abides by the weak P-property.*

In this manuscript, we extend the concept of ω-interpolative Ćirić-Reich-Rus-type contractions for obtaining fixed point [3] to ω-interpolative Ćirić-Reich-Rus-type proximal contraction for best proximity points.

2. Main result

In this section, we defined ω-interpolative Ćirić-Reich-Rus-type proximal contraction and related best proximity point result followed by supporting example.

Definition 2.1. *Let L and N be two nonvoid subsets of (Y, d). A mapping $J : L \to N$ is said to be ω-interpolative Ćirić-Reich-Rus-type proximal contraction, if there exist $\psi \in \Psi$, $\omega : L \times L \to [0, \infty)$ and positive real numbers γ, β satisfying $\gamma + \beta < 1$ such that*

$$\omega(x, y)d(Jx, Jy) \leq \psi\left((d(x, y))^{\gamma} (d^{*}(x, Jx))^{\beta} (d^{*}(y, Jy))^{1-\gamma-\beta}\right), \tag{2.1}$$

for all $x, y \in L \backslash B_{est}(J)$ with $\omega(x, y) \geq 1$, where $B_{est}(J) = \{x \in L\colon d(x, Jx) = d(L, N)\}$ and $d^{*}(x, Jx) = d(x, Jx) - d(L, N)$.

Theorem 2.1. *Let (Y, d) be a complete metric space and (L, N) be a pair of nonvoid closed subsets of Y such that L_0 is nonempty. Let $J : L \to N$ be a continuous mapping, satisfying*

- (i) *$J(L_0) \subseteq N_0$ and (L, N) abide by the weak P-property;*
- (ii) *J is ω-proximal admissible;*
- (iii) *There exist $x_0, x_1 \in L_0$ such that $d(x_1, Jx_0) = d(L, N)$ and $\omega(x_0, x_1) \geq 1$;*
- (iv) *J is ω-interpolative Ćirić-Reich-Rus-type proximal contraction.*

Then J has a unique best proximity point in L.

Proof. Owing to (iii), there exist $x_0, x_1 \in L_0$ such that

$$d(x_1, Jx_0) = d(L, N) \text{ and } \omega(x_0, x_1) \geq 1. \tag{2.2}$$

Since $J(L_0) \subseteq N_0$, it is observed that Jx_1 is an element of $J(L_0)$ which is also contained in N_0, consequently there is some element x_2 in L_0 such that

$$d(x_2, Jx_1) = d(L, N). \tag{2.3}$$

Then, by Eqs (2.2) and (2.3) and using the definition of ω-proximal admissibility, we have

$$\begin{aligned} \omega(x_0, x_1) &\geq 1, \\ d(x_1, Jx_0) &= d(L, N), \\ d(x_2, Jx_1) &= d(L, N), \end{aligned}$$

which implies that $\omega(x_1, x_2) \geq 1$. Thus,

$$d(x_2, Jx_1) = d(L, N) \text{ and } \omega(x_1, x_2) \geq 1.$$

On similar lines, for Jx_2 which belongs to N_0, there exists x_3 in L_0 such that

$$d(x_3, Jx_2) = d(L, N).$$

Since J is ω-proximal admissible, we conclude that $\omega(x_2, x_3) \geq 1$. Thus, we obtain

$$d(x_3, Jx_2) = d(L, N) \text{ and } \omega(x_2, x_3) \geq 1.$$

On similar lines, using induction, we construct a sequence $\{x_n\}$ in L_0 such that

$$d(x_{n+1}, Jx_n) = d(L, N) \text{ and } \omega(x_n, x_{n+1}) \geq 1, \tag{2.4}$$

for every $n \in \mathbb{N} \cup \{0\}$. If there is n_0 so that $x_{n_0} = x_{n_0+1}$, then

$$d(x_{n_0}, Jx_{n_0}) = d(x_{n_0+1}, Jx_{n_0}) = d(L, N).$$

That is, x_{n_0} is a best proximity point of J. Assuming $x_n \neq x_{n+1}$ for each n. Then, by using (iv) and (2.4), we get

$$\begin{aligned} d(Jx_n, Jx_{n+1}) &\leq \psi\left((d(x_n, x_{n+1}))^{\gamma} (d^*(x_n, Jx_n))^{\beta} (d^*(x_{n+1}, Jx_{n+1}))^{1-\gamma-\beta}\right) \\ &\leq \psi\Big((d(x_n, x_{n+1}))^{\gamma} (d(x_n, x_{n+1}) + d(x_{n+1}, Jx_n) - d(L, N))^{\beta} \\ &\quad (d(x_{n+1}, x_{n+2}) + d(x_{n+2}, Jx_{n+1}) - d(L, N))^{1-\gamma-\beta}\Big) \\ &\leq \psi\left((d(x_n, x_{n+1}))^{\gamma}(d(x_n, x_{n+1}))^{\beta}(d(x_{n+1}, x_{n+2}))^{1-\gamma-\beta}\right). \end{aligned} \tag{2.5}$$

Since (L, N) satisfies the weak P-property, we deduce that $d(x_{n+1}, x_{n+2}) \leq d(Jx_n, Jx_{n+1})$. Thus, from (2.5), we have

$$d(x_{n+1}, x_{n+2}) \leq \psi\left((d(x_n, x_{n+1}))^{\gamma+\beta}(d(x_{n+1}, x_{n+2}))^{1-(\gamma+\beta)}\right). \tag{2.6}$$

Considering $\psi(t) < t$ for every $t > 0$,

$$d(x_{n+1}, x_{n+2}) < (d(x_n, x_{n+1}))^{\gamma+\beta}(d(x_{n+1}, x_{n+2}))^{1-(\gamma+\beta)},$$

implies that

$$(d(x_{n+1}, x_{n+2}))^{\gamma+\beta} < (d(x_n, x_{n+1}))^{\gamma+\beta}.$$

And so,

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}), \text{ for all } n \geq 0. \tag{2.7}$$

So, it is observed that the sequence $\{d(x_n, x_{n+1})\}$ is a decreasing sequence. Consequently, there exists a real number $l \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = l$. From inequality (2.7), we can write

$$\begin{aligned} (d(x_n, x_{n+1}))^{\gamma+\beta}(d(x_{n+1}, x_{n+2}))^{1-(\gamma+\beta)} &\leq (d(x_n, x_{n+1}))^{\gamma+\beta}(d(x_n, x_{n+1}))^{1-(\gamma+\beta)} \\ &= d(x_n, x_{n+1}). \end{aligned} \tag{2.8}$$

From (2.6) and the nondecreasing property of ψ, we infer

$$d(x_{n+1}, x_{n+2}) \leq \psi\left((d(x_n, x_{n+1}))^{\gamma+\beta} (d(x_{n+1}, x_{n+2}))^{1-(\gamma+\beta)}\right) \leq \psi(d(x_n, x_{n+1})).$$

By repeating this argument, we deduce that

$$d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1})) \leq \psi^2(d(x_{n-1}, x_n)) \leq \cdots \leq \psi^{n+1}(d(x_0, x_1)). \quad (2.9)$$

Fix $\varepsilon > 0$ and let $n(\varepsilon) \in \mathbb{N}$ such that $\sum_{n \geq n(\varepsilon)} \psi^n(d(x_0, x_1)) < \varepsilon$. Let $n, m \in \mathbb{N}$ with $m > n > n(\varepsilon)$, using the triangular inequality, we obtain

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \psi^i(d(x_0, x_1)) \leq \sum_{n \geq n(\varepsilon)} \psi^n(d(x_0, x_1)) < \varepsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence in L. Since L is a closed subset of the complete metric space (Y, d), then there exists $x \in L$ so that

$$\lim_{n\to\infty} d(x_n, x) = 0. \quad (2.10)$$

Since J is continuous, one writes

$$\lim_{n\to\infty} d(Jx_n, Jx) = 0. \quad (2.11)$$

Combining (2.4), (2.10) and (2.11), we get

$$d(L, N) = \lim_{n\to\infty} d(x_{n+1}, Jx_n) = d(x, Jx).$$

Consequently, x is a best proximity point of J.

Now, to prove uniqueness of the best proximity point of mapping J, on contrary, suppose that $y \in L_0$ is another best proximity point (different from x) of the mapping J such that

$$\begin{aligned} \omega(x, y) &\geq 1, \\ d(x, Jx) &= d(L, N), \\ d(y, Jy) &= d(L, N). \end{aligned}$$

Since the pair of subsets (L, N) satisfies the weak property, then we have $d(x, y) \leq d(Jx, Jy)$, and mapping J is ω-interpolative Ćirić-Reich-Rus-type proximal contraction, then we have

$$\begin{aligned} d(x, y) &\leq d(Jx, Jy) \leq \omega(x, y) d(Jx, Jy) \\ &\leq \psi\left((d(x, y))^\gamma (d^*(x, Jx))^\beta (d^*(y, Jy))^{1-\gamma-\beta}\right), \\ &= \psi\left((d(x, y))^\gamma (d(x, Jx) - d(L, N))^\beta (d(y, Jy)) - d(L, N))^{1-\gamma-\beta}\right) \\ &= 0 \end{aligned}$$

which is a contradiction, hence best proximity point of the mapping J is unique. □

Coming up result can be given by incorporating the following property in place of continuity of the mapping J:

(K) Say the sequence $\{x_n\}$ in L exists in a way that $\omega(x_n, x_{n+1}) \geq 1$ for each n also $x_n \to x \in L$ as $n \to \infty$, hence $\omega(x_n, x) \geq 1$ for every n.

Theorem 2.2. *Let (Y, d) be a complete metric space and (L, N) be a pair of nonvoid closed subsets of Y such that L_0 is nonempty. Consider $J : L \to N$ be a non-self mapping such that the conditions (i)–(iv) in Theorem 2.1 and the property (K) are satisfied. Then J admits unique best proximity point in L.*

Proof. Pursuing the proof of Theorem 2.1, there exists a Cauchy sequence $\{x_n\} \subset L$ satisfying

$$d(x_{n+1}, Jx_n) = d(L, N) \text{ and } \omega(x_n, x_{n+1}) \geq 1,$$

for every $n \in \mathbb{N} \cup \{0\}$. Also, there exists $x \in L$ such that $x_n \to x$ as $n \to \infty$. Thus, from (K), we infer that $\omega(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. As $J(L_0) \subseteq N_0$, there is $y \in L_0$ so that

$$d(y, Jx) = d(L, N).$$

Thus, we get

$$\begin{aligned} \omega(x_n, x) &\geq 1, \\ d(x_{n+1}, Jx_n) &= d(L, N), \\ d(y, Jx) &= d(L, N). \end{aligned}$$

Considering (iv), we have

$$\begin{aligned} d(Jx_n, Jx) &\leq \psi\left((d(x_n, x))^{\gamma} (d^*(x_n, Jx_n))^{\beta} (d^*(x, Jx))^{1-\gamma-\beta}\right) \\ &\leq \psi\Big((d(x_n, x))^{\gamma} (d(x_n, x_{n+1}) + d(x_{n+1}, Jx_n) - d(L, N))^{\beta} \\ &\quad (d(x, Jx))^{1-\gamma-\beta}\Big) \\ &\leq \psi((d(x_n, x))^{\gamma} (d(x_n, x_{n+1}))^{\beta} (d(x, Jx))^{1-\gamma-\beta}). \end{aligned} \tag{2.12}$$

Since (L, N) satisfies the weak P-property, we deduce that $d(x_{n+1}, y) \leq d(Jx_n, Jx)$. From (2.12), we have

$$d(x_{n+1}, y) \leq \psi((d(x_n, x))^{\gamma} (d(x_n, x_{n+1}))^{\beta} (d(x, Jx))^{1-\gamma-\beta}).$$

Taking limit as $n \to \infty$, we get $\lim_{n\to\infty} d(x_{n+1}, y) = 0$. By the uniqueness of limit, we obtain $y = x$. Therefore, $d(x, Jx) = d(L, N)$. The uniqueness of the best proximity point followed on the same lines as proved in Theorem 2.1. □

Theorem 2.3. *Let (Y, d) be a complete metric space and (L, N) be a pair of nonvoid closed subsets of Y such that L_0 is nonvoid. Let $J : L \to N$ be a non-self mapping such that*

$$\omega(x, y)\, d(Jx, Jy) \leq \psi\left((d(x, y))^{\gamma} (d^*(x, Jx))^{\beta} (d^*(y, Jy))^{1-\gamma-\beta}\right), \tag{2.13}$$

for all $x, y \in L \backslash B_{est}(J)$, where $\psi \in \Psi$, $\omega : L \times L \to [0, \infty)$ and $\gamma, \beta > 0$ satisfying $\gamma + \beta < 1$. Assume that the following assertions hold:

(i) *$J(L_0) \subseteq N_0$ and (L, N) abides by the weak P-property;*
(ii) *J is ω-proximal admissible;*
(iii) *There exist $x_0, x_1 \in L_0$ such that $d(x_1, Jx_0) = d(L, N)$ and $\omega(x_0, x_1) \geq 1$;*
(iv) *Either J is continuous or property (K) holds.*

Then J admits unique best proximity point in L.

Proof. Let $x, y \in L\backslash B_{est}(J)$ such that $\omega(x,y) \geq 1$. By (2.13), we have

$$d(Jx, Jy) \leq \omega(x,y)\, d(Jx, Jy) \leq \psi\left((d(x,y))^{\gamma} (d^{*}(x, Jx))^{\beta} (d^{*}(y, Jy))^{1-\gamma-\beta}\right)$$

and hence

$$d(Jx, Jy) \leq \psi\left((d(x,y))^{\gamma} (d^{*}(x, Jx))^{\beta} (d^{*}(y, Jy))^{1-\gamma-\beta}\right),$$

for all $x, y \in L\backslash B_{est}(J)$ such that $\omega(x,y) \geq 1$. This implies that the inequality (2.1) holds. Thus, the rest of proof follows from Theorem 2.1 (resp. Theorem 2.2). □

By taking $\psi(t) = \lambda t$ (where $\lambda \in [0,1)$) in Theorem 2.1 (or Theorem 2.2), we have the following result.

Corollary 2.1. *Let (Y, d) be a complete metric space and (L, N) be a pair of nonvoid closed subsets of Y such that L_0 is nonvoid. Let $J : L \to N$ be a non-self mapping such that*

$$d(Jx, Jy) \leq \lambda\left((d(x,y))^{\gamma} (d^{*}(x, Jx))^{\beta} (d^{*}(y, Jy))^{1-\gamma-\beta}\right),$$

for all $x, y \in L\backslash B_{est}(J)$ with $\omega(x,y) \geq 1$, where $\lambda \in [0,1)$, $\omega : L \times L \to [0,\infty)$ and $\gamma, \beta > 0$ satisfying $\gamma + \beta < 1$. Assume that the following assertions hold:

(1) *$J(L_0) \subseteq N_0$ and (L, N) abides by the weak P-property;*
(2) *J is ω-proximal admissible;*
(3) *There exist $x_0, x_1 \in L_0$ such that $d(x_1, Jx_0) = d(L, N)$ and $\omega(x_0, x_1) \geq 1$;*
(4) *Either J is continuous or property (K) holds.*

Then J admits unique best proximity point in L.

Example 2.1. *Suppose $Y = \mathbb{R}^2$ to be a metric space defined by $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$, for all $(x_1, x_2) = x$, $(y_1, y_2) = y \in \mathbb{R}^2$. Consider L and N be two nonempty subsets of Y given as*

$$L = \{(-1,1), (-1,0), (-1,-1), (-1,-2)\}$$

and

$$N = \{(-4,1), (-4,0), (-4,-1), (-4,-2)\}.$$

Simple calculation yields $d(L, N) = 3$ and also the pair (L, N) admits the P-property (by Remark 1.2 then it also satisfies the weak P-property). It is clear that $L = L_0$ and $N = N_0$. Define a mapping $J : L \to N$ as

$$J(a) = \begin{cases} (-4,0), & \text{if } a \in \{(-1,1), (-1,0)\}, \\ (-4,1), & \text{if } a \in \{(-1,-1), (-1,-2)\}, \end{cases}$$

clearly $J(a) \in N_0$ for all $a \in L_0$. Also define $\omega : Y \times Y \to [0,\infty)$ as

$$\omega(x,y) = \exp^{\frac{1}{d(x,y)+6}}.$$

Let

$$\psi(t) = \frac{9t}{10}. \tag{2.14}$$

We have to show that J is ω-interpolative Ćirić-Reich-Rus-type proximal contraction on L. Let $\gamma = 0.2$ *and* $\beta = 0.5$ *for all the coming up cases.*

Case 1. *If* $u \in \{(-1,1)\}$, $r \in \{(-1,-1)\}$, $x, y \in \{(-1,0),(-1,-2)\}$ *are the subsets of L, then*

$$d(Ju, Jr) = d((-4,0),(-4,1)) = 1$$

and

$$\omega(u,r) = \exp^{\frac{1}{d(u,r)+6}} = \exp^{\frac{1}{2+6}} = 1.13,$$

therefore,

$$\omega(u,r)\, d(Ju, Jr) = 1.13. \tag{2.15}$$

Using (2.15) in the inequality (2.13),

$$\begin{aligned} 1.13 &= \omega(u,r)\, d(Ju, Jr) & (2.16)\\ &\leq \psi\left((d(u,r))^{\gamma} (d(u,Ju) - d(L,N))^{\beta} (d(r,Jr) - d(L,N))^{1-\gamma-\beta}\right)\\ &= \psi((d((-1,1),(-1,-1)))^{\gamma} (d((-1,1),(-4,0)) - 3)^{\beta} (d((-1,-1),(-4,1)) - 3)^{1-\gamma-\beta})\\ &= \psi\left(2^{1-\beta}\right). \end{aligned}$$

By using the value of β and apply the function ψ, then above inequality can be written as

$$1.13 \leq \frac{9}{10}\left(2^{1-0.5}\right) \leq 1.27,$$

thereby inequality (2.13) is satisfied.

Case 2. *If* $u \in \{(-1,1)\}$, $r \in \{(-1,-2)\}$, $x, y \in \{(-1,0),(-1,-1)\}$ *are the subsets of L, then*

$$d(Ju, Jr) = d((-4,0),(-4,1)) = 1$$

and

$$\omega(u,r) = \exp^{\frac{1}{d(u,r)+6}} = \exp^{\frac{1}{3+6}} = 1.12,$$

hence,

$$\omega(u,r)\, d(Ju, Jr) = 1.12. \tag{2.17}$$

Substituting (2.17) in inequality (2.13),

$$\begin{aligned} 1.12 &\leq \psi((d((-1,1),(-1,-2)))^{\gamma} (d((-1,1),(-4,0)) - 3)^{\beta} (d((-1,-2),(-4,1)) - 3)^{1-\gamma-\beta}\\ &\leq \psi(3^{1-\beta+\gamma}) = 1.39. \end{aligned}$$

Considering the value of β and γ, inequality (2.13) holds.

Case 3. *If* $u \in \{(-1,0)\}$, $r \in \{(-1,-1)\}$, $x, y \in \{(-1,-2),(-1,1)\}$ *are the subsets of L, then*

$$d(Ju, Jr) = d((-4,0),(-4,1)) = 1$$

and

$$\omega(u,r) = \exp^{\frac{1}{d(u,r)+6}} = \exp^{\frac{1}{1+6}} = 1.15,$$

hence,

$$\omega(u,r)\,d(Ju,Jr) = 1.15.$$

Therefore, the inequality (2.13) becomes

$$1.15 \le \psi((3)^{\beta}(2)^{1-\gamma-\beta}) = 1.92.$$

Considering the values of γ and β, then above inequality holds. For this case also the inequality (2.13) holds true.
Case 4. *If $u \in \{(-1,0)\}$, $r \in \{(-1,-2)\}$, $x, y \in \{(-1,-1), (-1,1)\}$ are the subsets of L, then*

$$d(Ju,Jr) = d((-4,0),(-4,1)) = 1$$

and

$$\omega(u,r) = \exp^{\frac{1}{d(u,r)+6}} = \exp^{\frac{1}{2+6}} = 1.13,$$

hence,

$$\omega(u,r)\,d(Ju,Jr) = 1.13.$$

Therefore, (2.13) becomes

$$1.13 \le \psi((3)^{\beta}) = 1.55.$$

So inequality (2.13) is satisfied.

Now we will collectively discuss remaining cases in which inequality (2.13) naturally holds, if $u, r = \{(-1,1), (-1,0)\}$ respectively or $u, r = \{(-1,-1), (-1,-2)\}$ respectively, since in such situations it is observed that

$$\omega(u,r)\,d(Ju,Jr) = 0.$$

Hence it can be concluded that for all possible cases inequality (2.13) is satisfied, abiding by all the conditions of Theorem 2.3 and using simple calculation, we can observe that $(-1,0)$ is the unique best proximity point of the mapping J.

3. Some results

In this segment, we are concerned with best proximity point deductions for ordered interpolative Ćirić-Reich-Rus-type proximal contractions on a metric space endowed with a partial ordering/graph, with the aid of results presented in the preceding section. Define

$$\Delta = \{x, y \in L \text{ such that } x \preceq y \text{ or } y \preceq x\}$$

and

$$\omega : L \times L \to [0,\infty), \quad \text{where } \omega(x,y) = \begin{cases} 1, & \text{if } (x,y) \in \Delta, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3.1. *[6] Let Y be a nonvoid set. Then $(Y, d, \preceq)$ is called a partially ordered metric space if the following conditions are satisfied:*

(1) *d is a metric on Y;*
(2) *$\preceq$ is a partial order on Y.*

Definition 3.2. *[6] Consider $(Y, d, \preceq)$ as a partially ordered metric space and (L, N) as a pair of nonvoid subsets of Y. A mapping $J : L \to N$ is called proximally order-preserving, if*

$$\left.\begin{array}{l} y_1 \preceq y_2 \\ d(x_1, Jy_1) = d(L, N) \\ d(x_2, Jy_2) = d(L, N) \end{array}\right\} \implies x_1 \preceq x_2,$$

for all $x_1, x_2, y_1, y_2 \in L$.

Definition 3.3. *Consider L and N are nonvoid subsets of partially ordered metric space $(Y, d, \preceq)$. A mapping $J : L \to N$ is said to be ordered interpolative Ćirić-Reich-Rus-type proximal contraction, if there exist $\psi \in \Psi$ and positive real numbers γ, β satisfying $\gamma + \beta < 1$ such that*

$$d(Jx, Jy) \leq \psi\left((d(x, y))^{\gamma} (d^{*}(x, Jx))^{\beta} (d^{*}(y, Jy))^{1-\gamma-\beta}\right),$$

for all $x, y \in L \backslash B_{est}(J)$ with $(x, y) \in \Delta$, where $d^{*}(x, Jx) = d(x, Jx) - d(L, N)$.

($\preceq_H$) Say the sequence $\{x_n\}$ in L exists in a way that $x_n \preceq x_{n+1}$ for every n also $x_n \to x \in L$ as $n \to \infty$, hence $x_n \preceq x$ for every n.

Then the following result is a direct consequence of Theorem 2.1 (resp. Theorem 2.2).

Theorem 3.1. *Consider $(Y, d, \preceq)$ as a complete partially ordered metric space and (L, N) as a pair of nonvoid closed subsets of Y such that L_0 is nonvoid. Let $J : L \to N$ be a non-self mapping such that*

(1) *$J(L_0) \subseteq N_0$ and (L, N) abides by the weak P-property;*
(2) *J is proximally order-preserving;*
(3) *There exist $x_0, x_1 \in L_0$ such that $d(x_1, Jx_0) = d(L, N)$ and $(x_0, x_1) \in \Delta$;*
(4) *J is ordered interpolative Ćirić-Reich-Rus-type proximal contraction;*
(5) *Either J is continuous or property ($\preceq_H$) holds.*

Then J admits unique best proximity point in L.

Let Y be a nonvoid set and ∇ designates the diagonal of Cartesian product $Y \times Y$ and $G = (V(G), E(G))$ be a directed graph without parallel edges in a way that the vertices set $V(G)$ coincides with Y and $\nabla \subset E(G)$, where $E(G)$ is the set of the edges of the graph, which contains all loops, like that $\nabla \subseteq Y \times Y$. Apprehend that, a graph G is connected if there is a path between any two vertices and it is weakly connected if G is connected, where G is an undirected form of the graph G in which direction of edges have not any role. In a graph G, by antipole the direction of edges we obtain the graph G^{-1}, whose set of edges and set of vertices are given by

$$E(G^{-1}) = \{(x_1, x_2) \in Y \times Y : (x_1, x_2) \in E(G)\} \text{ and } V(G) = V(G^{-1}). \tag{3.1}$$

In the presence of this manner, we get

$$E(G) = E(G) \cup E(G^{-1}). \tag{3.2}$$

Define

$$\omega : L \times L \to [0, \infty), \text{ where } \quad \omega(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3.4. *Consider (Y, d) be a metric space endowed with a graph G and (L, N) be a pair of nonvoid subsets of Y. A mapping $J : L \to N$ is called proximally G-preserving, if for all $x_1, x_2, y_1, y_2 \in L$,*

$$\left.\begin{array}{l}(y_1, y_2) \in E(G)\\ d(x_1, Jy_1) = d(L, N)\\ d(x_2, Jy_2) = d(L, N)\end{array}\right\} \Longrightarrow (x_1, x_2) \in E(G).$$

Definition 3.5. *Consider (Y, d) be a metric space endowed with a graph G and (L, N) be a pair of nonvoid subsets of Y. A mapping $J : L \to N$ is said to be G-interpolative Ćirić-Reich-Rus-type proximal contraction, if there exist $\psi \in \Psi$ and positive real numbers γ, β satisfying $\gamma + \beta < 1$ such that*

$$d(Jx, Jy) \leq \psi\left((d(x, y))^{\gamma} (d^{*}(x, Jx))^{\beta} (d^{*}(y, Jy))^{1-\gamma-\beta}\right),$$

for all $x, y \in L \backslash B_{est}(J)$ with $(x, y) \in E(G)$ where $d^{*}(x, Jx) = d(x, Jx) - d(L, N)$.

(G_H) Say the sequence $\{x_n\}$ in L exists in a way that $(x_n, x_{n+1}) \in E(G)$ for every n also $x_n \to x \in L$ as $n \to \infty$, hence $(x_n, x) \in E(G)$ for each n.

Therefore, the next result is a direct consequence of Theorem 2.1 (resp. Theorem 2.2).

Theorem 3.2. *Consider (Y, d) be a complete metric space endowed with a graph G and (L, N) be a pair of nonvoid closed subsets of Y such that L_0 is nonempty. Moreover, $J : L \to N$ be a non-self mapping such that*

(1) *$J(L_0) \subseteq N_0$ and (L, N) satisfies the weak P-property;*
(2) *J is proximally G-preserving;*
(3) *There exist $x_0, x_1 \in L_0$ such that $d(x_1, Jx_0) = d(L, N)$ and $(x_0, x_1) \in E(G)$;*
(4) *J is G-interpolative Ćirić-Reich-Rus-type proximal contraction;*
(5) *Either J is continuous or property (G_H) holds.*

Then J admits unique best proximity point in L.

4. Applications to fixed point theory

This section contains related results to fixed point theory for ω-interpolative Ćirić-Reich-Rus-type contractions.

If $L = N = Y$, then the following contractive condition can be defined.

Since $d(x, Jx) = d(L, N) = 0$ for self mappings, meaning $x = Jx$, whereby best proximity point reduces to fixed point. In this context, ω-interpolative Ćirić-Reich-Rus-type contraction also reduces to fixed point problem.

Definition 4.1. *Let (Y, d) be a metric space. A mapping $J : Y \to Y$ is said to be ω-interpolative Ćirić-Reich-Rus-type contraction, if there exist $\psi \in \Psi$, $\omega : Y \times Y \to [0, \infty)$ and positive real numbers γ, β satisfying $\gamma + \beta < 1$ such that*

$$d(Jx, Jy) \leq \psi\left((d(x, y))^{\gamma} (d(x, Jx))^{\beta} (d(y, Jy))^{1-\gamma-\beta}\right),$$

for all $x, y \in Y \backslash Fix(J)$ with $\omega(x, y) \geq 1$.

(F) Say the sequence $\{x_n\}$ in Y exists such as $\omega(x_n, x_{n+1}) \geq 1$ for each n and $x_n \to x \in Y$ as $n \to \infty$, hence $\omega(x_n, x) \geq 1$ for every n.

The upcoming result is a consequence of Theorem 2.1 (resp. Theorem 2.2).

Theorem 4.1. *Let (Y, d) be a complete metric space and $J : Y \to Y$ be a self mapping such that*

(1) *J is ω-admissible;*
(2) *There exists $x_0 \in Y$ such that $\omega(x_0, Jx_0) \geq 1$;*
(3) *J is ω-interpolative Ćirić-Reich-Rus-type contraction;*
(4) *Either J is continuous or property (F) holds.*

Then J has unique fixed point in Y.

Definition 4.2. *Consider $(Y, d, \preceq)$ as a partially ordered metric space. A mapping $J : Y \to Y$ is called an ordered-interpolative Ćirić-Reich-Rus-type contraction, for there exist $\psi \in \Psi$ and positive real numbers γ, β satisfying $\gamma + \beta < 1$ in a way that*

$$d(Jx, Jy) \leq \psi\left((d(x, y))^{\gamma} (d(x, Jx))^{\beta} (d(y, Jy))^{1-\gamma-\beta}\right),$$

for all $x, y \in Y \backslash Fix(J)$ with $(x, y) \in \Delta$.

$(\preceq_F)$ Say the sequence $\{x_n\}$ in Y exists in a way that $x_n \preceq x_{n+1}$ for each n and $x_n \to x \in Y$ as $n \to \infty$, then $x_n \preceq x$ for all n.

The following result is a direct consequence of Theorem 3.1.

Theorem 4.2. *Let $(Y, d, \preceq)$ be a complete partially ordered metric space and $J : Y \to Y$ be defined as*

(1) *J is nondecreasing;*
(2) *There exists $x_0 \in Y$ such that $x_0 \preceq Jx_0$;*
(3) *J is ordered-interpolative Ćirić-Reich-Rus-type contraction;*
(4) *Either J is continuous or property $(\preceq_F)$ holds.*

Then J admits unique fixed point in Y.

Definition 4.3. *Consider (Y, d) a metric space empowered by a graph G. A mapping $J : Y \to Y$ is said to be G-interpolative Ćirić-Reich-Rus-type contraction, if there exist $\psi \in \Psi$ and positive real numbers γ, β satisfying $\gamma + \beta < 1$ such that*

$$d(Jx, Jy) \leq \psi\left((d(x, y))^{\gamma} (d(x, Jx))^{\beta} (d(y, Jy))^{1-\gamma-\beta}\right),$$

for all $x, y \in Y \backslash Fix(J)$ with $(x, y) \in E(G)$.

Definition 4.4. *Consider (Y, d) be a metric space empowered by a graph G. A mapping $J : Y \to Y$ is called G-preserving, if for all $x_1, x_2 \in Y$,*

$$(x_1, x_2) \in E(G) \Longrightarrow (Jx_1, Jx_2) \in E(G).$$

(G_F) Say the sequence $\{x_n\}$ in Y exists in a way that $(x_n, x_{n+1}) \in E(G)$ for each n and $x_n \to x \in Y$ as $n \to \infty$, then $(x_n, x) \in E(G)$ for every n.

The following result is a direct consequence of Theorem 3.2.

Theorem 4.3. *Let (Y, d) be a complete metric space empowered with a graph G and $J : Y \to Y$ be a self mapping such that*

(1) *J is G-preserving;*
(2) *There exists $x_0 \in Y$ such that $(x_0, Jx_0) \in E(G)$;*
(3) *J is G-interpolative Ćirić-Reich-Rus-type contraction;*
(4) *Either J is continuous or property (G_F) holds.*

Then J contains unique fixed point in Y.

5. Conclusions

This article covers various results related to existence of a best proximity point for ω-interpolative Ćirić-Reich-Rus-type proximal contraction, where the mapping J is imposed with necessary conditions for it to be contractive. For supporting the results, we present an example. Special cases are discussed with necessary proves, which show that the transformation of ω-interpolative Ćirić-Reich-Rus-type proximal contraction into ordered-interpolative Ćirić-Reich-Rus-type proximal contraction in partial ordered and G-interpolative Ćirić-Reich-Rus-type proximal contraction in a complete metric space endowed with a graph G. As an application, some results related to fixed point for the aforementioned contractive map are provided.

Conflicts of interest

The authors declare that they have no competing interests.

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