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Research article

Study of multivalued fixed point problems for generalized contractions in double controlled dislocated quasi metric type spaces

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Abstract: The main purpose of this research is to establish a new generalized ξ^*-Kannan type double controlled contraction on a sequence and obtain fixed point results for a pair of multivalued mappings in left K-sequentially complete double controlled dislocated quasi metric type spaces. New results in different setting of generalized metric spaces and ordered spaces and also new results for graphic contractions can be obtained as corollaries of our results. An example is presented to show the novelty of results. In this paper, we unify and extend some recent results in the existing literature.

Keywords: common fixed point; left K-sequentially complete double controlled dislocated quasi metric type space; generalized ξ^*-Kannan type double controlled contraction; partial order; multivalued mapping
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1. Introduction and preliminaries

Fixed point theory is an important branch of functional analysis and its applications are used in various fields of pure and applied mathematics. Nadler [18] was the first author who showed the contraction principle for multivalued mappings in a complete metric space. This proof basically uses the concept of Hausdorff's distance between sets. Nadler's result proved as a source of inspiration and a large number of researchers have started their research works in this field. Multivalued mappings are generalizitions of single valued mappings. Fixed point results for multivalued mappings have a lot of applications in engineering, economics, Nash equilibria and game theory [7, 10, 11, 16]. Due to its important applications in various subjects, many authors have showed interesting results for multivalued mappings, which can be seen in [8, 26, 27, 31, 33].

By removing one and a half restriction from out of three restriction of a metric space, we get dislocated quasi metric space [33]. Complete dislocated quasi metric space is a generalization of 0-complete and complete quasi-partial metric space [3, 12, 15]. Dislocated quasi metric also generalizes dislocated metric and partial metric. Fixed point results established by various researchers in dislocated quasi metric space can be seen in [4, 6, 30, 37].

For the solution of matrix equations, Ran and Reurings [21] showed a significant result with an order. Lateral, Nieto et al. [20] extended the result in [21] governed a solution for ODE with periodic boundary conditions for ordered mappings. In [2] Abdeljawad et al., proved a significant result for best proximity points for cyclical contraction mappings. Altun et al. [5] renovated the approach to common fixed point of mappings, satisfying a generalized contraction with new order condition in a complete ordered metric space. For more results with order see [9, 13, 19].

On the other hand Kamran et al. [14] introduced a new concept of generalized b-metric spaces, named as extended b-metric spaces see also [29]. They replaced the parameter $b \geq 1$ in the triangle inequality by the control function $\theta : X \times X \to [1, \infty)$. Recently, Mlaiki et al. [17] conceptualized the triangle inequality in b-metric spaces using a different styled controlled function and introduced controlled metric type spaces. After this, Abdeljawad et al. [1] auditioned the concept of controlled metric type spaces by introducing two control functions $\alpha(w, g)$ and $\mu(w, g)$ and set up double controlled metric type spaces. Recently Shoaib et al. [36] introduced the notion of double controlled metric type spaces which is a generalization of [1] and proved fixed point results for multivalued mappings. Furthermore, some recent useful results on this setting can be seen in [34, 35]. The study recollects rudimentary concepts worth important for it to employ the definition in order to prove upcoming some new generalized results.

Definition 1.1. [33] Let ϑ is a nonempty set. Then, $\xi_q : \vartheta \times \vartheta \to [0, \infty)$ is called a dislocated quasi metric (or simply ξ_q-metric) if the following conditions hold for any $l, g, z \in \vartheta$:
(i) If $\xi_q(l, g) = \xi_q(g, l) = 0$, then $l = g$;
(ii) $\xi_q(l, g) \leq \xi_q(l, z) + \xi_q(z, g)$.
The pair (ϑ, ξ_q) is called a dislocated quasi metric space.

It is clear that if $\xi_q(l, g) = \xi_q(g, l) = 0$, then from (i), $l = g$. But if $l = g$, then $\xi_q(l, g)$ may not be 0. It is observed that if $\xi_q(l, g) = \xi_q(g, l)$ for all $l, g \in \vartheta$, then (ϑ, ξ_q) becomes a dislocated metric space (metric-like space).

Example 1.2. [33] Let $\vartheta = R^+ \cup \{0\}$ and $\xi_q(l, g) = l + \max\{l, g\}$ for any $l, g \in \vartheta$. Then, (ϑ, ξ_q) is a dislocated quasi metric space.

Definition 1.3. [1] Given non-comparable functions $\alpha, \mu : \vartheta \times \vartheta \to [1, \infty)$. If $q : \vartheta \times \vartheta \to [0, \infty)$ satisfies:
(q1) $q(l, g) = 0$ if and only if $l = g$,
(q2) $q(l, g) = q(g, l)$,
(q3) $q(l, g) \leq \alpha(l, z)q(l, z) + \mu(z, g)q(z, g)$, for all $l, z, g \in \vartheta$. Then, q is called double controlled metric

type with the functions α, μ and the pair (ϑ, q) is called double controlled metric type space with the functions α, μ.

Theorem 1.4. [1] Let (ϑ, q) be a complete double controlled metric type space with the functions $\alpha, \mu : \vartheta \times \vartheta \to [1, \infty)$ and let $T : \vartheta \to \vartheta$ be a given mapping. Suppose that the following conditions are satisfied:

There exists $t \in (0, 1)$ such that

$$q(T(l), T(g) \leq t\,(q(l, g)), \text{ for all } l, g \in \vartheta.$$

For $\upsilon_0 \in \vartheta$, choose $\upsilon_{\acute{n}} = T^{\acute{n}}\upsilon_0$. Assume that

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\alpha\,(\upsilon_{i+1}, \upsilon_{i+2})}{\alpha\,(\upsilon_i, \upsilon_{i+1})}\mu\,(\upsilon_{i+1}, \upsilon_m) < \frac{1}{t}.$$

In addition, for every $\upsilon \in \vartheta$, we have

$$\lim_{\acute{n}\to\infty} \alpha\,(\upsilon, \upsilon_{\acute{n}}) \text{ and } \lim_{\acute{n}\to\infty} \mu\,(\upsilon_{\acute{n}}, \upsilon) \text{ exist and are finite.}$$

Then T has a unique fixed point $\upsilon^* \in \vartheta$.

Definition 1.5. [36] Given non-comparable functions $\alpha, \mu : \vartheta\times\vartheta\to [1, \infty)$. If $\xi_q : \vartheta\times\vartheta\to [0, \infty)$ satisfies:
$(\xi_q 1)$ $\xi_q(l, g) = \xi_q(g, l) = 0$, then $l = g$,
$(\xi_q 2)$ $\xi_q(l, g) \leq \alpha(l, z)q(l, z) + \mu(z, g)q(z, g)$,
for all $l, z, g \in\vartheta$. Then, ξ_q is called a double controlled dislocated quasi metric type space with the functions α and μ and (ϑ, ξ_q) is called a double controlled dislocated quasi metric type space. If $\mu(z, g) = \alpha(z, g)$ then (ϑ, ξ_q) is called a controlled quasi metric type space.

Remark 1.6. Any dislocated quasi metric space or double controlled metric type space is double controlled dislocated quasi metric type space but, the converse is not true in general. Also, a controlled dislocated quasi metric type space is double controlled quasi metric type space. The converse is not true in general (see Examples (1.7 and 2.4)).

Example 1.7. Let $\vartheta = \{0, 1, 2, 3\}$. Define $\xi_q : \vartheta \times \vartheta \to [0,\infty)$ by $\xi_q(0, 1) = 0$, $\xi_q(0, 2) = 1$, $\xi_q(0, 3) = \frac{1}{4}$, $\xi_q(1, 0) = \frac{1}{2}$, $\xi_q\,(1, 2) = 2$, $\xi_q(1, 3) = \frac{1}{3}$, $\xi_q(2, 0) = \frac{1}{2}$, $\xi_q(2, 1) = 1$, $\xi_q(2, 3) = \frac{1}{3}$, $\xi_q(3, 0) = \frac{3}{2}$, $\xi_q(3, 1) = 2$, $\xi_q(3, 2) = \frac{1}{4}$, $\xi_q(0, 0) = \frac{1}{2}$, $\xi_q(1, 1) = 0$, $\xi_q(2, 2) = 2$, $\xi_q(3, 3) = 0$. Define $\alpha, \mu : \vartheta \times \vartheta \to [1,\infty)$ as $\alpha(0, 1) = \alpha\,(1, 2) = \alpha(2, 1) = \alpha(0, 2) = 1$, $\alpha(2, 0) = \alpha(3, 2) = 2$, $\alpha(3, 1) = \alpha(1, 0) = \alpha(3, 0) = \alpha(0, 3) = \frac{4}{3}$, $\alpha(1, 3) = \alpha(2, 3) = 3$, $\alpha(0, 0) = \alpha(1, 1) = \alpha(2, 2) = \alpha(3, 3) = 1$,

$\mu(1, 2) = \mu\,(2, 1) = \frac{3}{2}$, $\mu(2, 0) = 2$, $\mu(3, 0) = \mu(0, 3) = \mu(1, 0) = \mu(0, 1) = \mu(1, 3) = \mu(3, 1) = \mu(0, 0) = \mu(1, 1) = \mu(2, 2) = \mu(3, 3) = 1$, $\mu(3, 2) = 4$, $\mu(2, 3) = 1$, $\mu(0, 2) = 2$. It is obvious that ξ_q is double controlled dislocated quasi metric type for all $l, g, z \in X$. It is clear that ξ_q is not double controlled metric type space. Also, it is not controlled dislocated quasi metric type. Indeed,

$$\xi_q(1, 2) = 2 > \frac{3}{2} = \alpha\,(1, 3)\,\xi_q(1, 3) + \alpha(3, 2)\xi_q(3, 2).$$

Definition 1.8. [36] Let (ϑ, ξ_q) be a double controlled dislocated quasi metric type space.
(i) A sequence $\{l_n\}$ in (ϑ, ξ_q) is called left K-Cauchy if for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\xi_q(l_m, l_n) < \varepsilon$, $\forall\ n > m \geq n_0$.
(ii) A sequence $\{l_n\}$ is double controlled dislocated quasi-converges to l if $\lim_{n\to\infty} \xi_q(l_n, l) = \lim_{n\to\infty} \xi_q(l, l_n) = 0$ or for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that for all $n > n_0$, $\xi_q(l, l_n) < \varepsilon$ and $\xi_q(l_n, l) < \varepsilon$. In this case l is called a ξ_q-limit of $\{l_n\}$.
(iii) (ϑ, ξ_q) is called left K-sequentially complete if every left K-Cauchy sequence in $\left(\vartheta, \xi_q\right)$ convergent to a point $l \in \vartheta$ such that $\xi_q(l, l) = 0$.

Definition 1.9. [34] Let (ϑ, ξ_q) be a double controlled dislocated quasi metric type space. Let K be a nonempty subset of ϑ and let $l \in \vartheta$. An element $g_0 \in K$ is called a best approximation in K if

$$\begin{aligned} \xi_q(l, K) &= \xi_q(l, g_0), \text{ where } \xi_q(l, K) = \inf_{g\in K} \xi_q(l, g) \\ \text{and } \xi_q(K, l) &= \xi_q(g_0, l), \text{ where } \xi_q(K, l) = \inf_{g\in K} \xi_q(g, l). \end{aligned}$$

If each $l \in \vartheta$ has at least one best approximation in K, then K is called a proximinal set. We denote the set of all proximinal subsets of ϑ by $P(\vartheta)$.

Definition 1.10. [34] The function $H_{\xi_q} : P(\vartheta) \times P(\vartheta) \to [0, \infty)$, defined by

$$H_{\xi_q}(A, B) = \max \left\{ \sup_{a\in A} \xi_q(a, B),\ \ \sup_{b\in B} \xi_q(A, b) \right\}$$

is called double controlled dislocated quasi Hausdorff metric type on $P(\vartheta)$. Also $(P(\vartheta), H_{\xi_q})$ is known as double controlled dislocated quasi Hausdorff metric type space.

Lemma 1.11. *[35] Let (ϑ, ξ_q) be a double controlled dislocated quasi metric type space. Let $(P(\vartheta), H_{\xi_q})$ be a double controlled dislocated quasi Hausdorff metric type space on $P(\vartheta)$. Then, for all $A, B \in P(\vartheta)$ and for each $a \in A$, there exists $b_a \in B$, such that $H_{\xi_q}(A, B) \geq \xi_q(a, b_a)$ and $H_{\xi_q}(B, A) \geq \xi_q(b_a, a)$.*

Lemma 1.12. *[35] Let (ϑ, ξ_q) be a double controlled dislocated quasi metric type space. For $A, B \in P(\vartheta)$ and $a, b, z \in \vartheta$, then*

$$\begin{aligned} \rho_q(a, B) &\leq \alpha(a, z) q(a, z) + \mu(z, B) q(z, B), \\ \rho_q(A, b) &\leq \alpha(A, z) q(A, z) + \mu(z, b) q(z, b), \end{aligned}$$

where

$$\begin{aligned} \mu(z, B) &= \inf \{\mu(z, a),\ a \in B\}, \\ \alpha(A, z) &= \inf \{\alpha(b, z),\ b \in A\}. \end{aligned}$$

2. Results and discussion

Let (ϑ, ξ_q) be a double controlled dislocated quasi metric type space, $g_0 \in \vartheta$ and $T : \vartheta \to P(\vartheta)$ be a multifunction on ϑ. Let $g_1 \in Tg_0$ be an element such that $\xi_q(g_0, Tg_0) = \xi_q(g_0, g_1)$, $\xi_q(Tg_0, g_0) =$

$\xi_q(g_1, g_0)$. Let $g_2 \in Tg_1$ be such that $\xi_q(g_1, Tg_1) = \xi_q(g_1, g_2)$, $\xi_q(Tg_1, g_1) = \xi_q(g_2, g_1)$. Let $g_3 \in Tg_2$ be such that $\xi_q(g_2, Tg_2) = \xi_q(g_2, g_3)$ and so on. Thus, we construct a sequence g_n of points in ϑ such that $g_{2n+1} \in Tg_{2n}$ and $g_{2n+2} \in Tg_{2n+1}$, with $\xi_q(g_{2n}, Tg_{2n}) = \xi_q(g_{2n}, g_{2n+1})$, $\xi_q(Tg_{2n}, g_{2n}) = \xi_q(g_{2n+1}, g_{2n})$, and $\xi_q(g_{2n+1}, Tg_{2n+1}) = \xi_q(g_{2n+1}, g_{2n+2})$, $\xi_q(Tg_{2n+1}, g_{2n+1}) = \xi_q(g_{2n+2}, g_{2n+1})$, where $n = 0, 1, 2, \cdots$. We denote this iterative sequence by $\{\vartheta T(g_n)\}$. We say that $\{\vartheta T(g_n)\}$ is a sequence in ϑ generated by g_0 under double controlled dislocated quasi metric ξ_q. If ξ_q is dislocated quasi b-metric, then we say that $\{\vartheta T(g_n)\}$ is a sequence in ϑ generated by g_0 under dislocated quasi b-metric ξ_q. We can define $\{\vartheta T(g_n)\}$ in other metrics in a similar way. Let $M \subseteq \vartheta$, define $\xi^*(w, M) = \inf\{\xi(w, a),\ a \in M\}$ and $\xi^*(M, g) = \inf\{\xi(b, g),\ b \in M\}$. Let us introduce the following definition:

Definition 2.1. Let ϑ be a nonempty set and $\xi : \vartheta \times \vartheta \to [0, +\infty)$ be a mapping such that $\xi(w, g) \geq 1$ and $\xi(g, w) \geq 1$ imply $w = g$. Let $\Omega, T : \vartheta \to P(\vartheta)$ be a multivalued mappings and $\{\vartheta T(g_n)\}$ is a sequence in ϑ generated by g_0 under double controlled dislocated quasi metric ξ_q, then Ω, T are said to be ξ^*-ξ_q multivalued mappings, if for each $w \in \{\vartheta T(g_n)\}$, then we have

$$\text{(a) } \xi^*(w, \Omega w) \geq 1 \text{ implies } \xi^*(\Omega g, g) \geq 1,$$

$$\text{(b) } \xi^*(\Omega w, w) \geq 1 \text{ implies } \xi^*(g, \Omega g) \geq 1,$$

where $\xi_q(w, Tw) = \xi_q(w, g)$ and $\xi_q(Tw, w) = \xi_q(g, w)$.

Definition 2.2. Let (ϑ, ξ_q) be a complete double controlled dislocated quasi metric type space and Ω, T be ξ^*-ξ_q multivalued mappings. Then the pair (Ω, T) is called ξ^* Kannan type double controlled contraction, if for every two consecutive points w, g belonging to the range of an iterative sequence $\{\vartheta T(g_n)\}$ with $\xi^*(\Omega g, g) \geq 1$, $\xi^*(w, \Omega w) \geq 1$ or $\xi^*(\Omega w, w) \geq 1$, $\xi^*(g, \Omega g) \geq 1$ and $\xi_q(w, g) > 0$, we have

$$H_{\xi_q}(Tw, Tg) \leq t(\xi_q(w, Tw) + \xi_q(g, Tg)), \tag{2.1}$$

whenever, $t \in [0, \frac{1}{2})$ and for $g_0 \in \{\vartheta T(g_n)\}$,

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\alpha(g_{i+1}, g_{i+2})}{\alpha(g_i, g_{i+1})} \mu(g_{i+1}, g_m) < \frac{1 - t}{t}. \tag{2.2}$$

Theorem 2.3. *Let (ϑ, ξ_q) be a left K-sequentially complete double controlled dislocated quasi metric type space. Let a pair (Ω, T) be a ξ^* Kannan type double controlled contraction. Assume that:*
(i) The set $G(\Omega) = \{w : \xi^*(w, \Omega w) \geq 1\}$ is closed and contained g_0.
(ii) For every $g \in \{\vartheta T(g_n)\}$, we have

$$\lim_{n \to \infty} \alpha(g, g_n) < \frac{1}{t} \text{ and } \lim_{n \to \infty} \mu(g_n, g) < \frac{1}{t}. \tag{2.3}$$

Then $\{\vartheta T(g_n)\} \to u \in \vartheta$. Also, if (2.1) holds for each $w, g \in \{u\}$, then Ω and T have a common fixed point u in ϑ and $\xi_q(u, u) = 0$.

Proof. Since g_0 is an arbitrary element of $G(\Omega)$, from condition (i) $\xi^*(g_0, \Omega g_0) \geq 1$. Let $\{\vartheta T(g_n)\}$ be the iterative sequence in ϑ generated by a point $g_0 \in \vartheta$.

Since $\xi^*(g_0, \Omega g_0) \geq 1$, $\xi_q(g_0, Tg_0) = \xi_q(g_0, g_1)$ and $\xi_q(Tg_0, g_0) = \xi_q(g_1, g_0)$. As (Ω, T) is ξ^* multivalued mapping, so $\xi^*(\Omega g_1, g_1) \geq 1$. Now, $\xi^*(\Omega g_1, g_1) \geq 1$, $\xi_q(g_1, Tg_1) = \xi_q(g_1, g_2)$ and

$\xi_q(Tg_1, g_1) = \xi_q(g_2, g_1)$ imply that $\xi^*(g_2, \Omega g_2) \geq 1$. By induction we deduce that $\xi^*\left(g_{2p}, \Omega g_{2p}\right) \geq 1$ and $\xi^*\left(\Omega g_{2p+1}, g_{2p+1}\right) \geq 1$, for all $p = 0, 1, 2, \cdots$. Now, by Lemma 1.11, we have

$$\xi_q(g_{2p}, g_{2p+1}) \leq H_{\xi q}(Tg_{2p-1}, Tg_{2p}). \tag{2.4}$$

Since $g_{2p}, g_{2p-1} \in \{\vartheta T(g_n)\}$, $\xi^*\left(g_{2p}, \Omega g_{2p}\right) \geq 1$ and $\xi^*\left(\Omega g_{2p-1}, g_{2p-1}\right) \geq 1$, by the condition (2.1), we get

$$\begin{aligned}
\xi_q(g_{2p}, g_{2p+1}) &\leq t(\xi_q(g_{2p-1}, Tg_{2p-1}) + \xi_q(g_{2p}, Tg_{2p}) \\
&\leq t\left(\xi_q(g_{2p-1}, g_{2p}) + \xi_q(g_{2p}, g_{2p+1})\right)) \leq \frac{t}{1-t}\left(\xi_q(g_{2p-1}, g_{2p})\right) \\
&= \mu\left(\xi_q(g_{2p-1}, g_{2p})\right), \text{ where } \mu = \frac{t}{1-t}.
\end{aligned} \tag{2.5}$$

Now, by Lemma 1.11, we have

$$\xi_q(g_{2p-1}, g_{2p}) \leq H_{\xi q}(Tg_{2p-2}, Tg_{2p-1}).$$

Since $g_{2p-2}, g_{2p-1} \in \{\vartheta T(g_n)\}$, $\xi^*\left(g_{2p-2}, \Omega g_{2p-2}\right) \geq 1$ and $\xi^*\left(\Omega g_{2p-1}, g_{2p-1}\right) \geq 1$, by the condition (2.1), we have

$$\begin{aligned}
\xi_q(g_{2p-1}, g_{2p}) &\leq t\left(\xi_q(g_{2p-2}, Tg_{2p-2}) + \xi_q(g_{2p-1}, Tg_{2p-1})\right) \\
&\leq t\left(\xi_q(g_{2p-2}, g_{2p-1}) + \xi_q(g_{2p-1}, g_{2p})\right) \\
&\leq \frac{t}{1-t}\left(\xi_q(g_{2p-2}, g_{2p-1})\right) \leq \mu\left(\xi_q(g_{2p-2}, g_{2p-1})\right).
\end{aligned} \tag{2.6}$$

Using (2.6) in (2.5), we have

$$\xi_q(g_{2p}, g_{2p+1}) \leq \mu^2\left(\xi_q(g_{2p-2}, g_{2p-1})\right). \tag{2.7}$$

Now, by (2.4) we have

$$\xi_q(g_{2p-2}, g_{2p-1}) \leq H_{\xi q}(Tg_{2p-3}, Tg_{2p-2}).$$

Since $g_{2p-2}, g_{2p-1} \in \{\vartheta T(g_n)\}$, $\xi^*\left(\Omega g_{2p-1}, g_{2p-1}\right) \geq 1$ and $\xi^*\left(g_{2p-2}, \Omega g_{2p-2}\right) \geq 1$, by the condition (2.1), we get

$$\begin{aligned}
\xi_q(g_{2p-2}, g_{2p-1}) &\leq t\left(\xi_q(g_{2p-3}, g_{2p-2}) + \xi_q(g_{2p-2}, g_{2p-1})\right) \\
&\leq \mu^3\left(\xi_q(g_{2p-3}, g_{2p-2})\right).
\end{aligned} \tag{2.8}$$

From (2.7) and (2.8), we have

$$\mu^2(\xi_q(g_{2p-2}, g_{2p-1})) \leq \mu^3(\xi_q(g_{2p-3}, g_{2p-2})). \tag{2.9}$$

Using (2.9) in (2.5), we have

$$\xi_q(g_{2p}, g_{2p+1})\} \leq \mu^3(\xi_q(g_{2p-3}, g_{2p-2})) \text{ for all } p \in \mathbb{N}.$$

Similarly, we can obtain

$$\xi_q(g_{2p-1}, g_{2p})\} \leq \mu^{2p-1}(\xi_q(g_0, g_1)) \text{ for all } p \in \mathbb{N}.$$

Continuing in this way, we get

$$\xi_q(g_{2p}, g_{2p+1})\} \leq \mu^{2p}(\xi_q(g_0, g_1)). \tag{2.10}$$

Now, we can write (2.10) as

$$\xi_q(g_n, g_{n+1}) \leq \mu^{n}(\xi_q(g_0, g_1)). \tag{2.11}$$

Now, to prove that $\{g_n\}$ is a Cauchy sequence, for all natural numbers $n < m$, we have

$$\begin{aligned}
\xi_q(g_n, g_m) &\leq \alpha(g_n, g_{n+1})q(g_n, g_{n+1}) + \mu(g_{n+1}, g_m)q(g_{n+1}, g_m) \\
&\leq \alpha(g_n, g_{n+1})q(g_n, g_{n+1}) + \mu(g_{n+1}, g_m)\alpha(g_{n+1}, g_{n+2})q(g_{n+1}, g_{n+2}) \\
&\quad +\mu(g_{n+1}, g_m)\mu(g_{n+2}, g_m)q(g_{n+2}, g_m) \\
&\leq \alpha(g_n, g_{n+1})q(g_n, g_{n+1}) + \mu(g_{n+1}, g_m)\alpha(g_{n+1}, g_{n+2})q(g_{n+1}, g_{n+2}) \\
&\quad +\mu(g_{n+1}, g_m)\mu(g_{n+2}, g_m)\alpha(g_{n+2}, g_{n+3})q(g_{n+2}, g_{n+3}) \\
&\quad +\mu(g_{n+1}, g_m)\mu(g_{n+2}, g_m)\mu(g_{n+3}, g_m)q(g_{n+3}, g_m) \leq \ldots \\
&\leq \alpha(g_n, g_{n+1})q(g_n, g_{n+1}) + \\
&\quad \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\mu\left(g_j, g_m\right)\right)\alpha\left(g_i, g_{i+1}\right)q(g_i, g_{i+1}) \\
&\quad + \prod_{k=n+1}^{m-1}\mu\left(g_k, g_m\right)q(g_{m-1}, g_m) \\
&\leq \alpha(g_n, g_{n+1})q(g_n, g_{n+1}) + \\
&\quad \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\mu\left(g_j, g_m\right)\right)\alpha\left(g_i, g_{i+1}\right)\left(\frac{t}{1-t}\right)^{i}q(g_0, g_1) \\
&\quad + \prod_{i=n+1}^{m-1}\mu\left(g_i, g_m\right)\alpha\left(g_{m-1}, g_m\right)\left(\frac{t}{1-t}\right)^{m-1}q(g_0, g_1) \\
&= \alpha(g_n, g_{n+1})\left(\frac{t}{1-t}\right)^{n}q(g_0, g_1) + \\
&\quad \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\mu\left(g_j, g_m\right)\right)\alpha\left(g_i, g_{i+1}\right)\left(\frac{t}{1-t}\right)^{i}q(g_0, g_1) \\
&\leq \alpha(g_n, g_{n+1})\left(\frac{t}{1-t}\right)^{n}q(g_0, g_1) + \\
&\quad \sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\mu\left(g_j, g_m\right)\right)\alpha\left(g_i, g_{i+1}\right)\left(\frac{t}{1-t}\right)^{m-1}q(g_0, g_1).
\end{aligned}$$

We used $\alpha(w, g) \geq 1$. Let

$$\Omega_p = \sum_{i=0}^{p}\left(\prod_{j=0}^{i}\mu\left(g_j, g_m\right)\right)\alpha\left(g_i, g_{i+1}\right)\left(\frac{t}{1-t}\right)^{i}.$$

Hence, we have

$$\xi_q(g_n, g_m) \leq q(g_0, g_1)\left[\left(\frac{t}{1-t}\right)^n \alpha(g_n, g_{n+1}) + \Omega_{m-1} - \Omega_n\right]. \tag{2.12}$$

The ratio test together with (2.2) implies that the limit of the real number sequence $\{\Omega_n\}$ exists, and so $\{\Omega_n\}$ is left Cauchy. Indeed, the ratio test is applied to the term $a_i = \left(\prod_{j=0}^{i} \mu\left(g_j, g_m\right)\right) \alpha\left(g_i, g_{i+1}\right)$. Letting $n, m \to \infty$ in (2.12), we get

$$\lim_{n,m\to\infty} \xi_q(g_n, g_m) = 0. \tag{2.13}$$

So the sequence $\{\vartheta T(g_n)\}$ is a left Cauchy. Since (ϑ, ξ_q) is left K-sequentially double controlled complete dislocated quasi metric type space, $\{\vartheta T(g_n)\} \to u$, that is,

$$\lim_{n\to\infty} \xi_q(g_n, u) = \lim_{n\to\infty} \xi_q(u, g_n) = 0. \tag{2.14}$$

Since $G(\Omega)$ is a closed set, $G(\Omega)$ is left K-sequentially complete. Since $\{g_{2p}\}$ is a subsequence of $\{\vartheta T(g_n)\}$ contained in $G(\Omega)$, $\{g_{2p}\} \to u$. Completeness of $G(\Omega)$ implies $u \in G(\Omega)$, that is,

$$\xi^*\left(u, \Omega u\right) \geq 1. \tag{2.15}$$

Now,

$$\xi_q(u, u) \leq \alpha(u, u_n)\xi_q(u, g_n) + \mu\left(u_n, u\right)\xi_q(g_n, u).$$

This implies $\xi_q(u, u) = 0$ as $n \to \infty$. Now, we show that u is a common fixed point. We claim that $q_b\left(u, Tu\right) = 0$. On contrary suppose $\xi_q(u, Tu) > 0$. Now by Lemma 1.11, we have

$$\xi_q(g_{2n+2}, Tu) \leq H_{\xi_q}(Tg_{2n+1}, Tu).$$

Since, $\xi^*\left(\Omega g_{2n}, g_{2n}\right) \geq 1$ and $\xi^*\left(u, \Omega u\right) \geq 1$, by (2.1), we get

$$\xi_q(g_{2n+1}, Tu) \leq t\left[\xi_q(g_{2n}, g_{2n+1}) + \xi_q(u, Tu)\right]. \tag{2.16}$$

Taking the lim as $n \to \infty$ on both sides of (2.16), we get

$$\lim_{n\to\infty}\xi_q(g_{2n+1}, Tu) \leq \lim_{n\to\infty} t\left[\xi_q(g_{2n}, g_{2n+1}) + \xi_q(u, Tu))\right],$$

$$\lim_{n\to\infty}\xi_q(g_{2n+1}, Tu) \leq t(\xi_q(u, Tu)). \tag{2.17}$$

Now by Lemma 1.12, we have

$$\xi_q(u, Tu) \leq \alpha(u, g_{2n+1})\xi_q(u, g_{2n+1}) + \mu(g_{2n+1}, Tu)\xi_q(g_{2n+1}, Tu).$$

Taking the lim as $n \to \infty$ and using (2.3) and (2.14), we get

$$\xi_q(u, Tu) \leq \mu(g_{2n+1}, Tu)\xi_q(g_{2n+1}, Tu).$$

By (2.3) and (2.17), we get

$$\xi_q(u, Tu) < \xi_q(u, Tu).$$

It is a contraction. Therefore

$$\xi_q(u, Tu) = 0. \tag{2.18}$$

Thus $u \in Tu$. Now, suppose $\xi_q(Tu, u) > 0$. By Lemma 1.11, we have

$$\xi_q(Tu, g_{2n-1}) \leq H_{\xi_q}(Tu, Tg_{2n-2}).$$

Since, $\xi^*(\Omega g_{2n}, g_{2n}) \geq 1$ and $\xi^*(u, \Omega u) \geq 1$, by (2.1), we get

$$\xi_q(Tu, g_{2n-1}) \leq t\left[\xi_q(u, Tu)) + \xi_q(g_{2n-1}, g_{2n})\right].$$

Taking the lim as $n \to \infty$ on both sides of the above inequality, we get

$$\lim_{n\to\infty} \xi_q(Tu, g_{2n-1}) \leq t(\xi_q(u, Tu)).$$

Now by Lemma 1.12, we have

$$\xi_q(Tu, g_{2n}) \leq \alpha(Tu, g_{2n-1})\xi_q(Tu, g_{2n-1}) + \mu(g_{2n-1}, g_{2n})\xi_q(g_{2n-1}, g_{2n})).$$

Taking the lim as $n \to \infty$ and inequality (2.3) and (2.14), we get

$$\xi_q(Tu, u) < \xi_q(u, Tu) = 0, \text{ by (2.18)}.$$

It is a contradiction. Hence $u \in Tu$. As $\xi^*(u, \Omega u) \geq 1$ and $\xi_q(u, Tu) = \xi_q(Tu, u) = 0$, Definition 2.1 implies

$$\xi^*(\Omega u, u) \geq 1. \tag{2.19}$$

From (2.15) and (2.19) $\xi^*(u, \Omega u) \geq 1$, $\xi^*(\Omega u, u) \geq 1$. This implies $\xi(u, g) \geq 1$, $\xi(g, u) \geq 1$, for all $g \in \Omega u$. Thus $u = g$. Hence, u is a common fixed point for Ω and T. □

Example 2.4. Let $\vartheta = [0, 4] \cap \mathbb{Q}^+$. Define the function $\xi_q : \vartheta \times \vartheta \to [0, +\infty)$ by $\xi_q(w, g) = (w + 2g)^2$ if $w \neq g$ and $\xi_q(w, g) = 0$, if $w = g$. Then (ϑ, ξ_q) is a complete double controlled dislocated quasi metric type space with

$$\alpha(w, g) = \begin{cases} 2, \text{ if } w, g \geq 1 \\ \frac{w+2}{2}, \text{ otherwise} \end{cases}, \ \mu(w, g) = \begin{cases} 1 \text{ if } w, g \geq 1 \\ \frac{g+2}{2}, \text{ otherwise} \end{cases}$$

Define the mappings $\Omega, T : \vartheta \to P(\vartheta)$ as follows:

$$T(g) = \begin{cases} \left[\frac{g}{8}, \frac{g}{4}\right] \cap \mathbb{Q}^+, \text{ for all } g \in \{0, 1, \frac{1}{8}, \frac{1}{64}, \frac{1}{512}, \frac{1}{4096}, \cdots\} \\ \left[g + 2, \ 2\,(g + 1)\right], \qquad \text{otherwise.} \end{cases}$$

$$\Omega(g) = \begin{cases} \{\frac{1}{8}g\} \cap \mathbb{Q}^+, \text{ for all } g \in \{0, 1, \frac{1}{8}, \frac{1}{64}, \frac{1}{512}, \frac{1}{4096}, \cdots\} \\ \left[g + 1, g + 3\right], \qquad \text{otherwise} \end{cases}.$$

Let

$$A = \{r : \beta^*(w, \Omega w) \geq 1\} = \left\{0, 1, \frac{1}{64}, \frac{1}{4096}, ...\right\},$$

$$B = \{g : \beta^* (\Omega g, g) \geq 1\} = \left\{0, \frac{1}{8}, \frac{1}{512}, ...\right\},$$

$$\xi_q (w, g) = \begin{cases} 1, \text{ if } w \in A, \ g \in \mathcal{B} \\ \frac{1}{4}, \text{ otherwise.} \end{cases}$$

Then ξ_q is not a controlled dislocated quasi metric type space for the function α. Indeed,

$$\xi_q(1, 3) = 49 > 37.5 = \alpha (1, 0)\, \xi_q(1, 0) + \alpha(0, 3)\xi_q(0, 3).$$

Now, $\xi_q (w_0, \Omega w_0) = \xi_q (1, \Omega 1) = \xi_q \left(1, \frac{1}{8}\right) = \left(1 + \frac{2}{8}\right)^2 = \left(\frac{5}{4}\right)^2$. We define the sequence $\{\vartheta T(g_n)\} = \{1, \frac{1}{8}, \frac{1}{64}, \frac{1}{512}, \frac{1}{4096}, \cdots\}$ in ϑ generated by $g_0 = 1$. Let $w_0 = 1$. Then we have

$$\begin{aligned} G(\Omega) &= \{w : \xi^* (w, \Omega w) \geq 1 \text{ and } w \in \{\vartheta T (g_n)\}\} \\ &= \left\{0, 1, \frac{1}{64}, \frac{1}{4096}, ...\right\}. \end{aligned}$$

So (i) is satisfied.
Take $\frac{1}{64} \in \{\vartheta T (g_n)\}$. Then we have

$$\xi_q(\frac{1}{64}, T\frac{1}{64}) = \xi_q(\frac{1}{64}, \frac{1}{512}).$$

Also,

$$\xi_q(T\frac{1}{64}, \frac{1}{64}) = \xi_q(\frac{1}{512}, \frac{1}{64}).$$

Note that $\xi^* (w, \Omega w) \geq 1$, for all $w \in A$ implies $\xi^* (\Omega g, g) \geq 1$, for all $g \in B$. Also, $\xi^* (\Omega w, w) \geq 1$, for all $w \in B$ implies $\xi^* (g, \Omega g) \geq 1$, for all $g \in A$. So the pair (Ω, T) is ξ^*-ξ_q multivalued mapping on $\{\vartheta T (g_n)\}$.
Now, for all $w, g \in \vartheta \cap \{\vartheta T w_n\}$ with $\xi^* (\Omega g, g) \geq 1$, $\xi^* (w, \Omega w) \geq 1$ and $t = \frac{2}{5}$, we have the following cases:
In the case: $w < g$. Let $w = \frac{1}{64}$, $g = \frac{1}{8}$. Then we have

$$\begin{aligned} H_{\xi_q}(Tw, Tg) &= H_{\xi_q}\left(\left[\frac{1}{512}, \frac{1}{256}\right], \left[\frac{1}{64}, \frac{1}{32}\right]\right) \\ &= \max\left\{\xi_q\left(\frac{1}{256}, \frac{1}{512}\right), \xi_q\left(\frac{1}{512}, \frac{1}{32}\right)\right\} \\ &= \max\left\{\left(\frac{1}{256} + \frac{2}{512}\right)^2, \left(\frac{1}{512} + \frac{2}{32}\right)^2\right\} \end{aligned}$$

$$H_{\xi_q}(Tw, Tg) = \left(\frac{1}{512} + \frac{2}{32}\right)^2 = 0.00415.$$

Now, we have

$$
\begin{aligned}
t(\xi_q(w,Tw)+\xi_q(g,Tg)) &= \frac{2}{5}\left[\xi_q\left(\frac{1}{64},\left[\frac{1}{512},\frac{1}{256}\right]\right)+\xi_q\left(\frac{1}{8},\left[\frac{1}{64},\frac{1}{32}\right]\right)\right]\\
&= \frac{2}{5}\left[\left(\frac{1}{64}+\frac{2}{512}\right)^2+\left(\frac{1}{8}+\frac{2}{64}\right)^2\right]\\
&= \frac{2}{5}\left[\left(\frac{10}{512}\right)^2+\left(\frac{10}{64}\right)^2\right]=0.00998.
\end{aligned}
$$

In the case $w>g$. Take $w=1$, $g=\frac{1}{8}$. Then we have

$$
\begin{aligned}
H_{\xi_q}(Tw,Tg) &= H_{\xi_q}\left(\left[\frac{1}{8},\frac{1}{4}\right],\left[\frac{1}{64},\frac{1}{32}\right]\right)\\
&= \max\left\{\xi_q\left(\frac{1}{8},\frac{1}{64}\right),\xi_q\left(\frac{1}{8},\frac{1}{32}\right)\right\}\\
&= \max\left\{\left(\frac{1}{8}+\frac{2}{64}\right)^2,\left(\frac{1}{8}+\frac{2}{32}\right)^2\right\}
\end{aligned}
$$

$$
H_{\xi_q}(Tw,Tg)=\left(\frac{1}{8}+\frac{2}{32}\right)^2.
$$

Now,

$$
\begin{aligned}
t(\xi_q(w,Tw)+\xi_q(g,Tg)) &= \frac{2}{5}\left[\left(1+\frac{2}{8}\right)^2+\left(\frac{1}{8}+\frac{2}{64}\right)^2\right]\\
&= \frac{2}{5}\left[\left(\frac{5}{4}\right)^2+\left(\frac{5}{32}\right)^2\right]=0.6347.
\end{aligned}
$$

In the case: $w=0$, $g>0$. Let $w=0$, $g=\frac{1}{8}$. Then we have

$$
\begin{aligned}
H_{\xi_q}(Tw,Tg) &= H_{\xi_q}\left([0,0],\left[\frac{1}{64},\frac{1}{32}\right]\right),\\
&= \max\left\{\xi_q\left(0,\frac{1}{64}\right),\xi_q\left(0,\frac{1}{32}\right)\right\},\\
&= \max\left\{\left(\frac{2}{64}\right)^2,\left(\frac{2}{32}\right)^2\right\}=\left(\frac{2}{32}\right)^2.
\end{aligned}
$$

Now,

$$
\begin{aligned}
t(\xi_q(w,Tw)+\xi_q(g,Tg)) &= \frac{2}{5}\left[\xi_q(0,0)+\xi_q\left(\frac{1}{8},\left[\frac{1}{64},\frac{1}{32}\right]\right)\right],\\
&= \left(\frac{1}{8}+\frac{2}{64}\right)^2=\left(\frac{10}{64}\right)^2=.0097.
\end{aligned}
$$

In the case vi: Take $w = \frac{1}{8}$, and $g = 0$. Then we have

$$\begin{aligned} H_{\xi_q}(Tw, Tg) &= H_{\xi_q}\left([\frac{1}{64}, \frac{1}{32}], [0,0]\right) \\ &= \max\left\{\xi_q\left(\frac{1}{32}, 0\right), \xi_q\left(\frac{1}{64}, 0\right)\right\} \\ &= \max\left\{\left(\frac{1}{32}\right)^2, \left(\frac{1}{64}\right)^2\right\} = \left(\frac{1}{32}\right)^2. \end{aligned}$$

Now,

$$\begin{aligned} t(\xi_q(w, Tw) + \xi_q(g, Tg)) &= \frac{2}{5}\left[\xi_q\left(\frac{1}{8}, [\frac{1}{64}, \frac{1}{32}]\right) + \xi_q(0,0)\right] \\ &= \frac{2}{5} \times \left(\frac{5}{32}\right)^2 = 0.009. \end{aligned}$$

And the case $w = 0$ and $g = 0$ is trivially true. So all cases are satisfied. Let $g_0 = 1$. Then we have $g_1 = Tg_0 = \frac{1}{8}$, $g_2 = Tg_1 = \frac{1}{64}$, $g_3 = Tg_2 = \frac{1}{512}, \ldots$.

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\alpha(g_{i+1}, g_{i+2})}{\alpha(g_i, g_{i+1})} \mu(g_{i+1}, g_m) = 0.71 < \frac{1-t}{t} = \frac{3}{2}.$$

That is, the pair (Ω, T) is ξ^* Kannan type double controlled contraction. Finally, for every $g \in \{\vartheta T(g_n)\}$, we have

$$\lim_{n \to \infty} \alpha(g, 0) < \frac{5}{2} \text{ and } \lim_{n \to \infty} \mu(0, g) < \frac{5}{2}.$$

Hence all the hypothesis of Theorem 2.3 are satisfied and 0 is a common fixed point of Ω and T.

The next Theorem 2.6 is a special case of our main Theorem 2.3 if we use in Theorem 2.6 $\alpha(w, g) = \mu((w, g) = b$ for all $w, g \in \vartheta$ then we get the result on dislocated quasi-b metric type space instead double controlled dislocated quasi type metric spaces.

Definition 2.5. Let (ϑ, ξ_q) be a left K-sequentially complete dislocated quasi b-metric type space and (Ω, T) be a ξ^*-ξ_q multivalued mapping under dislocated quasi b-metric ξ_q, with $b > 1$ then the pair (Ω, T) is called ξ^* Kannan type b-contraction, if for every two consecutive points w, g belonging to the range of an iterative sequence $\{\vartheta T(g_n)\}$ with $\xi^*(\Omega g, g) \geq 1$, $\xi^*(w, \Omega w) \geq 1$ or $\xi^*(\Omega w, w) \geq 1$, $\xi^*(g, \Omega g) \geq 1$ and $\xi_q(w,g) > 0$, we have

$$H_{\xi_q}(Tw, Tg) \leq t(\xi_q(w, Tw) + \xi_q(g, Tg)), \tag{2.20}$$

whenever, $t \in [0, \frac{1}{2})$ and $b < \frac{1-t}{t}$.

Theorem 2.6. *Let (ϑ, ξ_q) be a left K-sequentially complete dislocated quasi b-metric space and a pair (Ω, T) be a ξ^* Kannan type b-contraction. Assume that:*
The set $G(\Omega) = \{w : \xi^*(w, \Omega w) \geq 1\}$ is closed and contained in g_0. Then $\{\vartheta T(g_n)\} \to u \in \vartheta$. Also, if (2.20) holds for each $w, g \in \{u\}$, then Ω and T have a common fixed point u in ϑ and $\xi_q(u, u) = 0$.

Remark 2.7. In Example 2.4, $\xi_q(w,g) = (w+2g)^2$ is dislocated quasi b-metric with $b \geq 2$, but we can not apply Theorem 2.6 because the pair (Ω, T) is not ξ^* Kannan type b-contraction, indeed $b \nleq \frac{3}{2} = \frac{1-t}{t}$.

Definition 2.8. Let (ϑ, d) be a complete metric space and (Ω, T) be a ξ^*- d multivalued mapping under metric d, then the pair (Ω, T) is called ξ^* Kannan type contraction, if for every two consecutive points w, g belonging to the range of an iterative sequence $\{\vartheta T(g_n)\}$ with $\xi^*(\Omega g, g) \geq 1$, $\xi^*(w, \Omega w) \geq 1$ and $d(w,g) > 0$, we have

$$H(Tw, Tg) \leq t(d(w, Tw) + d(g, Tg)), \tag{2.21}$$

whenever, $t \in [0, \frac{1}{2})$.

Theorem 2.9. *Let (ϑ, d) be a complete metric space. Let a pair (Ω, T) be a ξ^* Kannan type contraction. Assume that the set $G(\Omega) = \{w : \xi^*(w, \Omega w) \geq 1\}$ is closed and contained g_0. Then $\{\vartheta T(g_n)\} \to u \in \vartheta$. Also, if (2.21) holds for each $w, g \in \{u\}$, then Ω and T have a common fixed point u in ϑ.*

Motivated by the result Samet et al. [28] we have obtained the upcoming results for order. Further results which generalized partial order can be seen in ([22–25]). Recall that if X is a nonempty set, $\preceq$ is a partial order on X, then $(X, \preceq)$ is called non empty partially ordered set. Let $a \in X$ and $B \subseteq X$. We say that $a \preceq B$ whenever for all $b \in B$, we have $a \preceq b$.

Definition 2.10. Let $(\vartheta, \preceq)$ is a non empty partially ordered set. Let $\Omega, T : \vartheta \to P(\vartheta)$ be the multivalued mappings and $\{\vartheta T(g_n)\}$ is a sequence in ϑ generated by g_0 under metric d, then (Ω, T) is said to be $\preceq$-d multivalued mapping, if $w \in \{\vartheta T(g_n)\}$, we have

$$\text{(a) } w \preceq \Omega w \text{ implies } \Omega g \succeq g,$$

$$\text{(b) } \Omega w \succeq w \text{ implies } g \preceq \Omega g,$$

where $d(w, Tw) = d(w, g)$.

Definition 2.11. Let $(\vartheta, \preceq, d)$ is an ordered complete metric space and (Ω, T) be a $\preceq$-d multivalued mapping. Let $\alpha : \vartheta \times \vartheta \to [0, \infty)$ be a function, then the pair (Ω, T) is called $\preceq$ Kannan type contraction, if for every two consecutive points w, g belonging to the range of an iterative sequence $\{\vartheta T(g_n)\}$ with $\Omega g \succeq g$, $w \preceq \Omega w$ or $\Omega w \succeq w$, $g \preceq \Omega g$ and $d(w,g) > 0$, we have

$$H(Tw, Tg) \leq t(d(w, Tw) + d(g, Tg)), \tag{2.22}$$

whenever, $t \in [0, \frac{1}{2})$.

Theorem 2.12. *Let (ϑ, d) is a complete metric space. Let a pair (Ω, T) be a $\preceq$ Kannan type contraction. Let $\alpha : \vartheta \times \vartheta \to [0, \infty)$ be a function, and assume that the set $G(\Omega) = \{w : w \preceq \Omega w\}$ is closed and contained g_0. Then $\{\vartheta T(g_n)\} \to u \in \vartheta$. Also, if (2.22) holds for each $w, g \in \{u\}$, then Ω and T have a common fixed point u in ϑ.*

Proof. Let $\alpha : \vartheta \times \vartheta \to [0, +\infty)$ be a mapping such that

$$\alpha(\ell, \jmath) = \begin{cases} 1 \text{ if } \ell \preceq \jmath \text{ or } \jmath \preceq \ell \\ 0 \quad \text{otherwise} \end{cases}.$$

As $(\Omega\ T)$ is $\preceq$-d multivalued mapping so $d(w, Tw) = d(w, g)$ then $w \preceq \Omega w$ implies $\Omega g \succeq g$ or $w \preceq b$ for all $b \in \Omega w$ implies $e \succeq g$ for all $e \in \Omega g$ or $\alpha(w, b) = 1\ \forall\ b \in \Omega w$ implies $\alpha(e, g) = 1\ \forall\ e \in \Omega g$ or $\inf\{\alpha(w, b) : b \in \Omega w\} = 1$ implies $\inf\{\alpha(e, g) : e \in \Omega g\} = 1$ or

$$\alpha^*(w, \Omega w) \geq 1 \text{ implies } \alpha^*(g, \Omega g) \geq 1\} \qquad (a)$$

Similarly case (b) $\Omega w \succeq w$ implies $g \preceq \Omega g$ gives $\alpha^*(\Omega w, w) \geq 1$ implies $\alpha^*(g, \Omega g) \geq 1$. So $(\Omega\ T)$ is a α^*-d multivalued mapping. As $(\Omega\ T)$ is $\preceq$ Kannan type contraction. By using definition of α we can easily prove that $(\Omega\ T)$ is α^* Kannan type contraction. The $G(\Omega) = \{w : w \preceq \Omega w\}$ is closed and contained g_0 implies $G(\Omega) = \{w : \alpha^*(w, \Omega w) \geq 1\}$ is closed and contained g_0. Then, by Theorem 2.9, we have $\{\vartheta T(g_n)\}$ is a sequence in ϑ and $\{\vartheta T(g_n)\} \to u \in \vartheta$. Also, (2.22) holds for $w, g \in \{u\}$ implies (2.21) holds for $w, g \in \{u\}$. Hence, by Theorem 2.9 Ω and T have a common fixed point u in ϑ. □

3. Conclusions

In this research, we have achieved sufficient conditions to prove the existence of common fixed point for a pair of multivalued mappings satisfying a new generalized Kannan type double controlled contraction with a sequence in left K-sequentially complete double controlled dislocated quasi type metric spaces. An example is given to show the variety of our results. Moreover, we investigated our results in a better framework of double controlled dislocated quasi-metric spaces. New results in dislocated quasi b-metric spaces, controlled quasi metric spaces, dislocated quasi metric spaces and metric spaces can be obtained as corollaries of our results.

Conflict of interest

The authors declare that they have no competing interests.

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