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Diverse approaches to search for solitary wave solutions of the fractional modified Camassa–Holm equation

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ABSTRACT

In this study, an integrable dispersive modified Camassa–Holm equation is considered with the essence of fractional beta derivative. The aforesaid equation is a shallow water equation and a bi-Hamiltonian having an associated isospectral problem of second order. Three diverse techniques namely the extended Jacobi's elliptic function expansion, the new version of Kudryashov and the Exp_a function methods are enforced. A variety of complex solitary wave solutions including, Jacobi's elliptic function solutions, bright and dark solitons and many other analytical solutions are developed. The obtained results are explicated graphically depending upon the physical and fractional parameters. These results may also be used to illuminate the significance of applied methods to many other related non-linear physical phenomena.

Introduction

Nonlinear partial differential equations (NLPDEs) have much importance because of their use in applied sciences. NLPDEs always represent some nonlinear physical phenomena in different physical or applied fields such as water wave theory, quantum physics, optics, fluid mechanics, applied chemistry and many others [1–5]. Nonlinear Schrödinger equations are some important model equations that represent the physical phenomenons in the structures of mathematical NLPDEs [6–10]. Furthermore, fractional differential equations (FDEs) are the generalized forms of FDEs [11,12]. The most important task is to find their approximate and analytical solutions through some reliable techniques. Among different kinds of solutions, the exact solitary wave solutions of any FDEs have much significant role to understand the corresponding physical. Many many analytical schemes have been employed to secure such wave solutions for nonlinear FDEs [13–17].

Moreover, Biswas and Alqahtani have discussed the Semi-inverse Variational method for two types of solutions to the PGI equation [18]. Also, the variational principle method has been explored for periodic type wave solutions to the KMN equation in $(2+1)$-dimension [19]. Moreover, the Riccati equation method has been employed to secure several optical solitons in the field of optics [20]. The modified extended tanh expansion technique has been used to discuss the non-linearity of Biswas and Arshed model in [21].

Here, we consider the simplified modified Camassa–Holm equation from the family of significant equations called the modified α-equations discussed by Wazwaz [22]:

$$q_t - q_{xxt} + (\alpha+1)q^2 q_{xx} - \alpha q_x q_{xx} - q_{xxx} = 0. \qquad \alpha > 0. \tag{1}$$

By taking $\alpha = 2$ in the above Eq. (1), then the Eq. (1) takes the form

$$q_t - q_{xxt} + 3q^2 q_{xx} - 2q_x q_{xx} - q_{xxx} = 0. \tag{2}$$

This form is known as modified Camassa–Holm equation. Further simplified form of the Eq. (2) is given as

$$q_t + 2\delta q_x - q_{xxt} + \gamma q^2 q_x = 0. \qquad \delta \in \Re, \quad \gamma > 0. \tag{3}$$

Here δ and γ are the non-zero parameters and this form is called the simplified modified Camassa–Holm equation [23].

Different techniques have been utilized to find the different kinds of wave solutions of the aforementioned model. For example, the Generalized (G'/G)-Expansion method was applied to determine the exact

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solitons for the aforesaid model in [23]. The solitary wave solutions for the same model have been reported by applying the Exp-function method [24]. Distinct solutions of this model have been found by applying Riccati–Bernoulli sub-ODE method [25]. Moreover, solitary wave solutions have also been determined for the simplified MCH-equation in [26].

The extended Jacobi elliptic expansion function method (JEEFM), the Kudryashov method (KM) and the Exp_a function method have been exercised to explore many prolific mathematical models. For example, different soliton solutions have been gained for highly dispersive non-linear Schrödinger equation with CQS nonlinearities by using the extended JEEFM [27]. Optical soliton solutions of highly dispersive non-linear Schrödinger's equation with Kerr law nonlinearity has been produced by applying extended JEEFM [28]. Diverse traveling wave solutions have been constructed by using the extended JEEFM in [29]. With the help of KM, some new types of solutions of the GEW-Burgers models have been reported [30]. The PGI equation was explored for soliton solutions via KM by Hosseini et al. [31]. Similarly, these techniques have also been applied to solve the other nonlinear Schrödinger equations in [32].

In fractional calculus, the $\beta-$ derivative is known as the generalization of classical derivative [33]. We recall the definition and some of its characteristics as

Definition. Consider a defined function $g(\theta)$ ∀ non-negative θ. Consequently, beta-time fractional derivative of g of order β is as follow:

$$D^{\beta}(g(\theta)) = \frac{d^{\beta}g(\theta)}{d\theta^{\beta}} = \lim_{\epsilon\to 0}\frac{g(\theta+\epsilon(\theta+\frac{1}{\Gamma(\beta)})^{1-\beta})-g(\theta)}{\epsilon}, \quad \beta\in(0,1].$$

Few expedient characteristics of the beta-time fractional derivative have been given in [34–36].

Since Zulfiqar and Ahmad [24] obtained rational Exp-function type exact solutions of the simplified MCH-Equation. The hyperbolic and trigonometric function solutions have been reported for the said model by Alam and Akbar in [23]. But our task is to search for some new analytical solutions, for the aforesaid equation possessing space–time fractional derivatives, in the form of Jacobi's elliptic, rational hyperbolic and rational exponential functions. The extended JEFEM, the KM and the Exp_a function method are employed for the first time to complete this task.

Description of the model and the schemes of solutions

Consider the simplified MCH-equation given in the Eq. (3) with beta space–time fractional derivative:

$$\frac{\partial^{\beta}q}{\partial t^{\beta}} + 2\delta\,\frac{\partial^{\beta}q}{\partial x^{\beta}} - \frac{\partial^{3\beta}q}{\partial x^{2\beta}\partial t^{\beta}} + \gamma\, q^2\frac{\partial^{\beta}q}{\partial x^{\beta}} = 0. \tag{4}$$

Now suppose the given traveling wave transformations:

$$q(x,\ t) = Y(\eta), \quad where \quad \eta = \frac{\omega}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta} \tag{5}$$

here ω and λ are the non-zero constants. By inserting the above transformations into the Eq. (4), we yield the following nonlinear ODE

$$(\lambda+2\delta\omega)Y - \lambda\omega^2 Y'' + \frac{\gamma\omega}{3}Y^3 = 0. \tag{6}$$

Description of the extended JEEFM

Let us consider the following PDE to explain the main steps for the extended JEEFM [27].

$$G(j_t, j^2 j_t, j_x, j_{tt}, j_{xx}, j_{xt}, \ldots) = 0 \tag{7}$$

with $g = g(x,t)$. Let μ represent the soliton speed and assume the given wave transformation.

$$j(x,t) = Y(\eta), \quad \eta = x - \mu t \tag{8}$$

Inserting the Eq. (8) into Eq. (7), we obtain the below non-linear ordinary differential equation:

$$F(Y, Y^2Y', Y'', Y''', \ldots) = 0. \tag{9}$$

The above Eq. (9) has the below shape of solution by utilizing the extended JEEFM:

$$Y(\eta) = \sum_{k=-M}^{N}\alpha_k J^k(\eta) \tag{10}$$

here $M,\ N,\ \alpha_k(k = -M,\ldots,N)$ are undetermined to be found later while J shows the Jacobi's elliptic (JE) function, *i.e.* , $J = J(\eta) = sn\eta = sn(\eta,m)$ or $cn(\eta,n)$ or $dn(\eta,n)$ where $0 < n < 1$ is the amplitude of JE functions. The highest derivative and nonlinear terms in Eq. (9) will produce the values of M and N due to the homogeneous balance principle. After that, Eq. (10) into the Eq. (9), we may have a system of algebraic equations in terms of $\alpha_k(k = -M,\ldots,N)$. Now by using symbolic software Mathematica, one can solve the above system of equations for α_k. By inserting the obtained values into Eq. (10) then the general form of JE function solutions of Eq. (7) can be secured.

Moreover, $n \to 1$, JE functions will be transformed into hyperbolic functions as:

$\mathrm{sn}(\eta,n) \to \tanh(\eta)$, $\mathrm{cn}(\eta,n) \to \mathrm{sech}(\eta)$ and $\mathrm{dn}(\eta,n) \to \mathrm{sech}(\eta)$.

Description of Kudryashov method (KM)

The procedure of KM [31] is explained abiding by:

Step 1: Assume Eqs. (7), (8) & (9).

Step 2: Consider the solutions of the Eq, (9) is of the type:

$$Y(\eta) = \sum_{k=0}^{m}\alpha_k\phi^k(\eta). \tag{11}$$

with $\alpha_k(k = 0,1,2,3,\ldots,m)$ are unknowns and $\alpha_j \neq 0$. The positive integer m will be determined by homogeneous balance method.

The function $\phi(\eta)$ accomplish the below auxiliary equation:

$$(\phi'(\eta))^2 = \phi^2(\eta)(1-\pi\phi^2(\eta)). \tag{12}$$

with

$$\phi(\eta) = \frac{4a}{(4a^2-\pi)\sinh(\eta) + (4a^2+\pi)\cosh(\eta)}, \tag{13}$$

$\pi = 4ab$, a and b are constants. Step 3: By putting the Eq. (11) in Eq. (9) along with Eq. (12) and adding up the same order terms in $\phi(\eta)$ coefficients. Setting each coefficient equals to zero to get over a system of algebraic expressions having α_k, μ and other parameters. One can use an appropriate symbolic software for the solutions of said system.

Step 4: Using the unknowns obtained in Step 3 along with the solution Eq. (13), one will acquire the solutions of Eq. (7).

Representation of Exp_a function method

Consider Eqs. (7), (8) and (9), let us assume a solution of the Eq. (9) is of the below type [37]:

$$Y(\tau) = \frac{\alpha_0 + \alpha_1 R^{\tau} + \ldots + \alpha_m R^{m\tau}}{\beta_0 + \beta_1 R^{\tau} + \cdots + \beta_m R^{m\tau}}, \quad \alpha \neq 0, 1, \tag{14}$$

here $\alpha_k(0 \le k \le m)$ and $\beta_k(0 \le k \le m)$ are unknown parameters and to be found later. The positive integer m in Eq. (9) is produced with the help of homogeneous balance principle. Using Eq. (14) in Eq. (9), yields

$$\wp(R^{\tau}) = \ell_0 + \ell_1 D^{\tau} + \cdots + \ell_t D^{t\tau} = 0. \tag{15}$$

Putting ℓ_k $(0 \le k \le t)$ into Eq. (9) equals to zero, a set of algebraic equations is secured given as.

$$\ell_k = 0, \qquad where \quad k = 0, \dots, t. \tag{16}$$

On solving the above set of equations, one can get the non-trival solutions of Eq. (7).

Solutions with the extended JEEFM

Balancing the terms Y'' and Y^3 using homogeneous principle on Eq. (6) implies $M = N = 1$. So the Eq. (10) reduces

$$Y(\eta) = \alpha_{-1} J^{-1}(\eta) + \alpha_0 + \alpha_1 J(\eta). \tag{17}$$

Case 1: If $J = J(\eta) = sn(\eta, m)$, Eq. (17) becomes:

$$Y(\eta) = \alpha_{-1} sn^{-1}(\eta, m) + \alpha_0 + \alpha_1 sn(\eta, m). \tag{18}$$

By putting Eq. (18) into the Eq. (16), yields the successive solution sets:

Set 1:

$$\left\{\alpha_{-1} = 0,\ \alpha_0 = 0, \alpha_1 = \mp \frac{2\sqrt{3}\sqrt{\delta}\sqrt{m}\omega}{\sqrt{-\gamma\left((m+1)\omega^2+1\right)}}, \lambda = -\frac{2\delta\omega}{(m+1)\omega^2+1}\right\}. \tag{19}$$

By using Eqs. (19) and (18) into Eq. (5), we get

$$q(x,t) = \mp \frac{2\sqrt{3}\sqrt{\delta}\sqrt{m}\omega}{\sqrt{-\gamma\left((m+1)\omega^2+1\right)}} \operatorname{sn}(\frac{\omega}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta - \frac{2\delta\omega}{\beta((m+1)\omega^2+1)}(t + \frac{1}{\Gamma(\beta)})^\beta). \tag{20}$$

Set 2:

$$\{\alpha_{-1} = \mp 2i\omega \sqrt{\frac{3\delta}{\gamma\left(\left(m - 6\sqrt{m}+1\right)\omega^2+1\right)}}, \alpha_0 = 0,$$
$$\alpha_1 = \pm 2i\omega \sqrt{\frac{3\delta m}{\gamma\left(\left(m - 6\sqrt{m}+1\right)\omega^2+1\right)}},$$
$$\lambda = -\frac{2\delta\omega}{\left(m - 6\sqrt{m}+1\right)\omega^2+1}\}. \tag{21}$$

By inserting Eqs. (21) and (18) into Eq. (5), we get

$$q(x,t) = 2i\omega \sqrt{\frac{3\delta}{\gamma((m-6\sqrt{m}+1)\omega^2+1)}} (\mp \operatorname{sn}^{-1}(\frac{\omega}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta - \frac{2\delta\omega}{\beta((m-6\sqrt{m}+1)\omega^2+1)}(t+\frac{1}{\Gamma(\beta)})^\beta) \pm \sqrt{m}\operatorname{sn}(\frac{\omega}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta - \frac{2\delta\omega}{\beta((m-6\sqrt{m}+1)\omega^2+1)}(t+\frac{1}{\Gamma(\beta)})^\beta)). \tag{22}$$

Set 3:

$$\{\alpha_{-1} = -2i\omega \sqrt{\frac{3\delta}{\gamma((m+6\sqrt{m}+1)\omega^2+1)}}, \alpha_0 = 0,$$
$$\alpha_1 = -2i\omega \sqrt{\frac{3\delta m}{\gamma((m+6\sqrt{m}+1)\omega^2+1)}},$$
$$\lambda = -\frac{2\delta\omega}{(m+6\sqrt{m}+1)\omega^2+1}\}. \tag{23}$$

By using Eqs. (23) and (18) into Eq. (5), we obtain

$$q(x,t) = -2i\omega \sqrt{\frac{3\delta}{\gamma((m+6\sqrt{m}+1)\omega^2+1)}} (\operatorname{sn}^{-1}(\frac{\omega}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta - \frac{2\delta\omega}{\beta((m+6\sqrt{m}+1)\omega^2+1)}(t+\frac{1}{\Gamma(\beta)})^\beta) + \sqrt{m}\operatorname{sn}(\frac{\omega}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta - \frac{2\delta\omega}{\beta((m+6\sqrt{m}+1)\omega^2+1)}(t+\frac{1}{\Gamma(\beta)})^\beta)). \tag{24}$$

Set 4:

$$\{\alpha_{-1} = 2i\omega \sqrt{\frac{3\delta}{\gamma\left(\left(m + 6\sqrt{m}+1\right)\omega^2+1\right)}}, \alpha_0 = 0,$$
$$\alpha_1 = 2i\omega \sqrt{\frac{3\delta m}{\gamma\left(\left(m + 6\sqrt{m}+1\right)\omega^2+1\right)}}, \tag{25}$$
$$\lambda = -\frac{2\delta\omega}{\left(m + 6\sqrt{m}+1\right)\omega^2+1}\}. \tag{26}$$

By using Eqs. (25) and (18) into Eq. (5), we secure

$$q(x,t) = 2i\omega \sqrt{\frac{3\delta}{\gamma((m+6\sqrt{m}+1)\omega^2+1)}} (\operatorname{sn}^{-1}(\frac{\omega}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta - \frac{2\delta\omega}{\beta((m+6\sqrt{m}+1)\omega^2+1)}(t+\frac{1}{\Gamma(\beta)})^\beta) + \sqrt{m}\operatorname{sn}(\frac{\omega}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta - \frac{2\delta\omega}{\beta((m+6\sqrt{m}+1)\omega^2+1)}(t+\frac{1}{\Gamma(\beta)})^\beta)). \tag{27}$$

Set 5:

$$\left\{\alpha_{-1} = \mp \frac{2\sqrt{3}\sqrt{\delta}\omega}{\sqrt{-\gamma\left((m+1)\omega^2+1\right)}}, \alpha_0 = 0, \alpha_1 = 0, \lambda = -\frac{2\delta\omega}{(m+1)\omega^2+1}\right\}. \tag{28}$$

By using Eqs. (28) and (18) into Eq. (5), we get

$$q(x,\ t) = \mp \frac{2\sqrt{3}\sqrt{\delta}\omega}{\sqrt{-\gamma\left((m+1)\omega^2+1\right)}} sn^{-1}(\frac{\omega}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta - \frac{2\delta\omega}{\beta((m+1)\omega^2+1)}(t + \frac{1}{\Gamma(\beta)})^\beta). \tag{29}$$

Soliton solutions

When $m \to 1$ then Eqs. (20), (22), (24), (27) and (29) produce the dark and singular soliton solutions:

$$q(x,\ t) = \mp \frac{2\sqrt{3}\sqrt{\delta}\omega}{\sqrt{-\gamma\left(2\omega^2+1\right)}} \tanh(\frac{\omega}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta - \frac{2\delta\omega}{\beta(2\omega^2+1)}(t + \frac{1}{\Gamma(\beta)})^\beta). \tag{30}$$

$$q(x,t) = 2i\omega \sqrt{\frac{3\delta}{\gamma(-4\omega^2+1)}} (\mp \coth(\frac{\omega}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta - \frac{2\delta\omega}{\beta(-4\omega^2+1)}(t + \frac{1}{\Gamma(\beta)})^\beta) \pm \tanh(\frac{\omega}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta - \frac{2\delta\omega}{\beta(-4\omega^2+1)}(t + \frac{1}{\Gamma(\beta)})^\beta)). \tag{31}$$

$$q(x,t) = -2i\omega \sqrt{\frac{3\delta}{\gamma(8\omega^2+1)}} (\coth(\frac{\omega}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta - \frac{2\delta\omega}{\beta(8\omega^2+1)}(t + \frac{1}{\Gamma(\beta)})^\beta) + \tanh(\frac{\omega}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta - \frac{2\delta\omega}{\beta(8\omega^2+1)}(t + \frac{1}{\Gamma(\beta)})^\beta)). \tag{32}$$

$$q(x,t) = 2i\omega \sqrt{\frac{3\delta}{\gamma(8\omega^2+1)}} (\coth(\frac{\omega}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta - \frac{2\delta\omega}{\beta(8\omega^2+1)}(t + \frac{1}{\Gamma(\beta)})^\beta) + \tanh(\frac{\omega}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta - \frac{2\delta\omega}{\beta(8\omega^2+1)}(t + \frac{1}{\Gamma(\beta)})^\beta)). \tag{33}$$

$$q(x,t) = \mp \frac{2\sqrt{3}\sqrt{\delta}\omega}{\sqrt{-\gamma\left(2\omega^2+1\right)}} \coth(\frac{\omega}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta - \frac{2\delta\omega}{\beta(2\omega^2+1)}(t + \frac{1}{\Gamma(\beta)})^\beta). \tag{34}$$

Since Eqs. (20), (22), (24), (27) and (29) produce the dark soliton (30), singular soliton (34) and combined dark-singular soliton (31). The $3D$

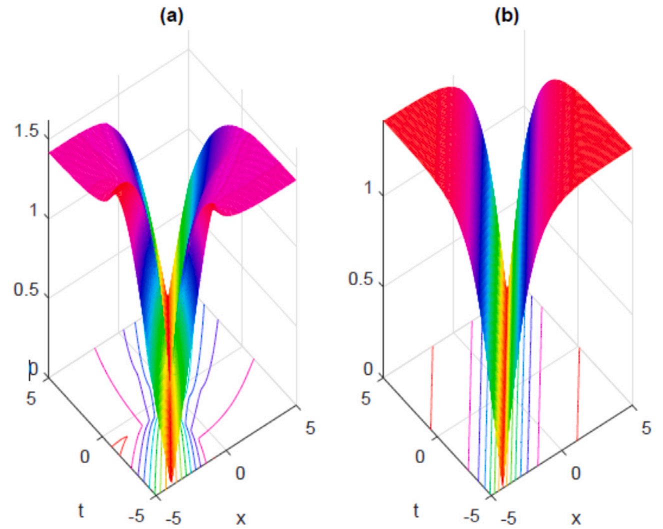

Fig. 1. 3*D* wave simulations of Eq. (30) are presented in (a) and (b) for fractional parameter $\beta = 0.75, 1.0$ and $\omega = 0.5$.

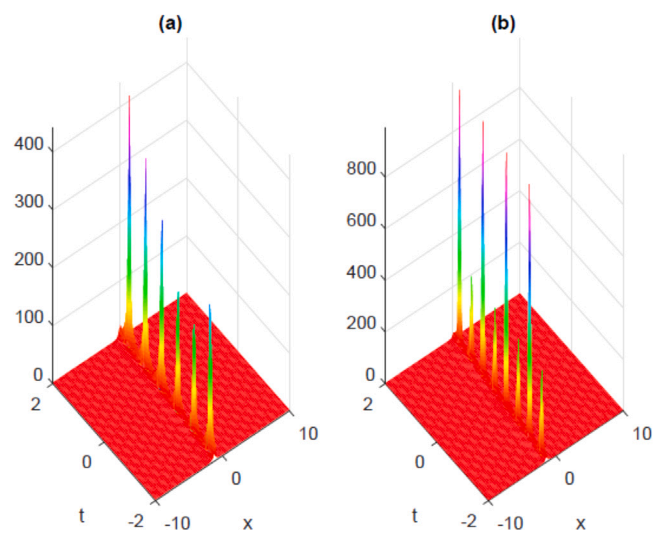

Fig. 2. 3*D* wave simulations of Eq. (34) are presented in (a) and (b) for fractional parameter $\beta = 0.75, 1.0$ and $\omega = 1$.

graphs, by taking $\gamma = 0.6$ and $\delta = 0.6$, are displayed in Figs. 1–3 for aforesaid solutions to visualized their dynamics.

Case 2: If $Y = Y(\xi) = cn(\xi, m)$, then Eq. (17) becomes:

$$Y(\eta) = \alpha_{-1} cn^{-1}(\eta, m) + \alpha_0 + \alpha_1 cn(\eta, m) \tag{35}$$

By putting Eq. (35) in Eq. (16), we have the successive solution sets:

Set 1:

$$\left\{\alpha_{-1} = \mp \frac{2i\sqrt{3}\sqrt{\delta}\sqrt{m-1}\omega}{\sqrt{\gamma\left((2m-1)\omega^2-1\right)}}, \alpha_0 = 0, \alpha_1 = 0, \lambda = -\frac{2\delta\omega}{-2m\omega^2+\omega^2+1}\right\}. \tag{36}$$

By using Eqs. (35) and (36) into Eq. (5), we get

$$q(x,t) = \mp \frac{2i\sqrt{3}\sqrt{\delta}\sqrt{m-1}\omega}{\sqrt{\gamma\left((2m-1)\omega^2-1\right)}}\ \mathrm{cn}^{-1}\left(\frac{\omega}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \frac{2\delta\omega}{\beta(-2m\omega^2+\omega^2+1)}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right). \tag{37}$$

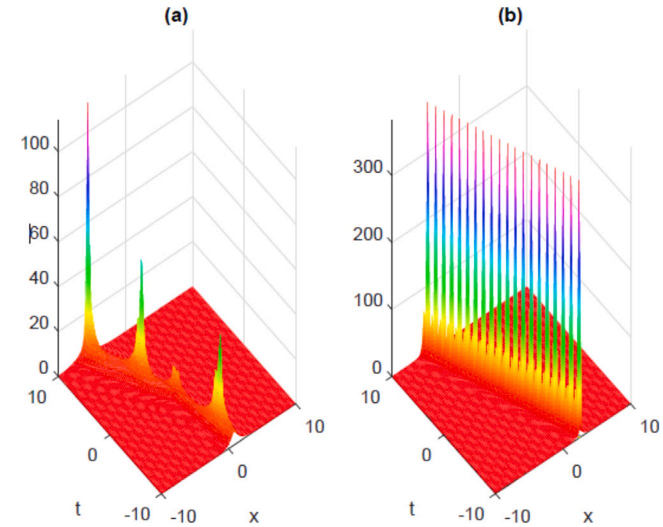

Fig. 3. 3*D* wave simulations of Eq. (31) are presented in (a) and (b) for fractional parameter $\beta = 0.75, 1.0$ and $\omega = 1$.

Set 2:

$$\left\{\alpha_{-1} = 0, \alpha_0 = 0, \alpha_1 = \mp \frac{2\sqrt{3}\sqrt{\delta}\sqrt{m}\omega}{\sqrt{\gamma\left((1-2m)\omega^2+1\right)}}, \lambda = -\frac{2\delta\omega}{-2m\omega^2+\omega^2+1}\right\}. \tag{38}$$

By using Eqs. (38) and (35) into the Eq. (5), we have

$$q(x,t) = \mp \frac{2\sqrt{3}\sqrt{\delta}\sqrt{m}\omega}{\sqrt{\gamma\left((1-2m)\omega^2+1\right)}} \mathrm{cn}\left(\frac{\omega}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \frac{2\delta\omega}{\beta(-2m\omega^2+\omega^2+1)}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right). \tag{39}$$

Set 3:

$$\begin{aligned}\{\alpha_{-1} &= \mp 2\omega\sqrt{\frac{3\delta(m-1)\left(6\sqrt{(m-1)m\omega^4}+(2m-1)\omega^2-1\right)}{\gamma\left(\left(32m^2-32m-1\right)\omega^4+(4m-2)\omega^2-1\right)}}, \alpha_0 = 0,\\ \alpha_1 &= \pm 2\omega\sqrt{\frac{3\delta m\left(6\sqrt{(m-1)m\omega^4}+(2m-1)\omega^2-1\right)}{\gamma\left(\left(32m^2-32m-1\right)\omega^4+(4m-2)\omega^2-1\right)}},\\ \lambda &= -\frac{2\delta\omega\left(6\sqrt{(m-1)m\omega^4}+(2m-1)\omega^2-1\right)}{\left(32m^2-32m-1\right)\omega^4+(4m-2)\omega^2-1}\}.\end{aligned} \tag{40}$$

By using Eqs. (40) and (35) into Eq. (5), we obtain

$$\begin{aligned}q(x,t) &= 2\omega\sqrt{\frac{3\delta(6\sqrt{(m-1)m\omega^4}+(2m-1)\omega^2-1)}{\gamma((32m^2-32m-1)\omega^4+(4m-2)\omega^2-1)}}\\ &\times \left(\mp\sqrt{m-1}\mathrm{cn}^{-1}\left(\frac{\omega}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right.\right.\\ &\left. - \frac{2\delta\omega(6\sqrt{(m-1)m\omega^4}+(2m-1)\omega^2-1)}{\beta((32m^2-32m-1)\omega^4+(4m-2)\omega^2-1)}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\\ &\pm \sqrt{m}\mathrm{cn}\left(\frac{\omega}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \frac{2\delta\omega(6\sqrt{(m-1)m\omega^4}+(2m-1)\omega^2-1)}{\beta((32m^2-32m-1)\omega^4+(4m-2)\omega^2-1)}\right.\\ &\left.\left.\times\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right).\end{aligned} \tag{41}$$

Set 4:

$$\{\alpha_{-1} = -2\omega\sqrt{\frac{3\delta(m-1)\left(-6\sqrt{(m-1)m\omega^4}+(2m-1)\omega^2-1\right)}{\gamma\left(\left(32m^2-32m-1\right)\omega^4+(4m-2)\omega^2-1\right)}}, \alpha_0 = 0,$$

$$\alpha_1 = -2\omega\sqrt{\frac{3\delta m\left(-6\sqrt{(m-1)m\omega^4}+(2m-1)\omega^2-1\right)}{\gamma\left(\left(32m^2-32m-1\right)\omega^4+(4m-2)\omega^2-1\right)}},$$

$$\lambda = \frac{2\delta\omega\left(6\sqrt{(m-1)m\omega^4}+(1-2m)\omega^2+1\right)}{\left(32m^2-32m-1\right)\omega^4+(4m-2)\omega^2-1}\}. \tag{42}$$

By using Eqs. (42) and (35) into Eq. (5), we get

$$q(x,t) = -2\omega\sqrt{\frac{3\delta(-6\sqrt{(m-1)m\omega^4}+(2m-1)\omega^2-1)}{\gamma((32m^2-32m-1)\omega^4+(4m-2)\omega^2-1)}}(\sqrt{m-1}\mathrm{cn}^{-1} \times (\frac{\omega}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{2\delta\omega(6\sqrt{(m-1)m\omega^4}+(1-2m)\omega^2+1)}{\beta((32m^2-32m-1)\omega^4+(4m-2)\omega^2-1)}(t+\frac{1}{\Gamma(\beta)})^{\beta}) + \sqrt{m}\mathrm{cn}(\frac{\omega}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{2\delta\omega(6\sqrt{(m-1)m\omega^4}+(1-2m)\omega^2+1)}{\beta((32m^2-32m-1)\omega^4+(4m-2)\omega^2-1)} \times (t+\frac{1}{\Gamma(\beta)})^{\beta})). \tag{43}$$

Set 5:

$$\{\alpha_{-1} = 2\omega\sqrt{\frac{3\delta(m-1)\left(-6\sqrt{(m-1)m\omega^4}+(2m-1)\omega^2-1\right)}{\gamma\left(\left(32m^2-32m-1\right)\omega^4+(4m-2)\omega^2-1\right)}}, \alpha_0 = 0,$$

$$\alpha_1 = 2\omega\sqrt{\frac{3\delta m\left(-6\sqrt{(m-1)m\omega^4}+(2m-1)\omega^2-1\right)}{\gamma\left(\left(32m^2-32m-1\right)\omega^4+(4m-2)\omega^2-1\right)}},$$

$$\lambda = \frac{2\delta\omega\left(6\sqrt{(m-1)m\omega^4}+(1-2m)\omega^2+1\right)}{\left(32m^2-32m-1\right)\omega^4+(4m-2)\omega^2-1}\}. \tag{44}$$

By using Eqs. (44) and (35) into Eq. (5) yield

$$q(x,t) = 2\omega\sqrt{\frac{3\delta(-6\sqrt{(m-1)m\omega^4}+(2m-1)\omega^2-1)}{\gamma((32m^2-32m-1)\omega^4+(4m-2)\omega^2-1)}} \times (\sqrt{m-1}\mathrm{cn}^{-1}(\frac{\omega}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{2\delta\omega(6\sqrt{(m-1)m\omega^4}+(1-2m)\omega^2+1)}{\beta((32m^2-32m-1)\omega^4+(4m-2)\omega^2-1)}(t+\frac{1}{\Gamma(\beta)})^{\beta}) + \sqrt{m}\mathrm{cn}(\frac{\omega}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{2\delta\omega(6\sqrt{(m-1)m\omega^4}+(1-2m)\omega^2+1)}{\beta((32m^2-32m-1)\omega^4+(4m-2)\omega^2-1)} \times (t+\frac{1}{\Gamma(\beta)})^{\beta})). \tag{45}$$

Soliton solutions

When $n \to 1$ then Eqs. (37), (39), (41), (43) and (45) produce the bright soliton solutions as:

$$q(x,t) = \mp\frac{2\sqrt{3}\sqrt{\delta}\omega}{\sqrt{\gamma\left(-\omega^2+1\right)}}sech(\frac{\omega}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{2\delta\omega}{\beta(-\omega^2+1)}(t+\frac{1}{\Gamma(\beta)})^{\beta}). \tag{46}$$

$$q(x,t) = \pm 2\omega\sqrt{\frac{3\delta(\omega^2-1)}{\gamma(-\omega^4+2\omega^2-1)}}sech(\frac{\omega}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{2\delta\omega(\omega^2-1)}{\beta(-\omega^4+2\omega^2-1)}(t+\frac{1}{\Gamma(\beta)})^{\beta}). \tag{47}$$

Since Eq. (46) produces the bright soliton solution and the following $3D$ graph displays the solution dynamics by taking $\gamma = 0.6$ and $\delta = 0.6$.

Case 3: If $J = J(\xi) = dn(\xi, m)$, then Eq. (17) becomes:

$$Y(\eta) = \alpha_{-1}dn^{-1}(\eta,m) + \alpha_0 + \alpha_1 dn(\eta,m). \tag{48}$$

By putting Eq. (48) into the Eq. (16), we obtain the successive solution sets:

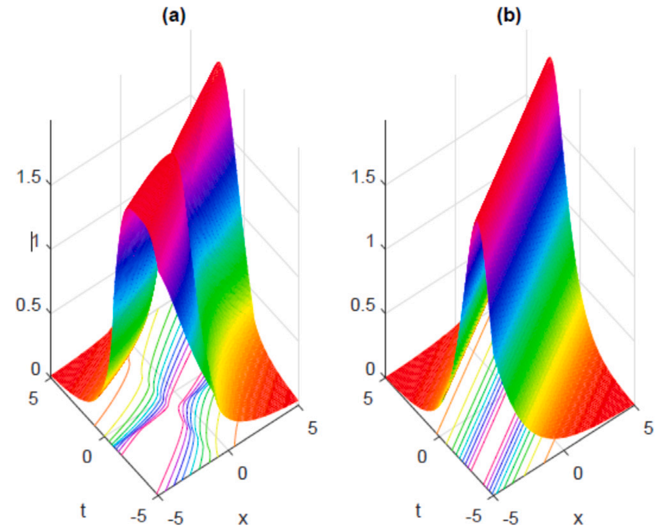

Fig. 4. Bright soliton wave simulation of Eq. (46) is presented in (a) and (b) for fractional parameter $\beta = 0.75, 1.0$ and $\omega = 0.5$.

Set 1;

$$\left\{\alpha_{-1} = \mp\frac{2i\sqrt{3}\sqrt{\delta}\sqrt{m-1}\omega}{\sqrt{\gamma\left((m-2)\omega^2+1\right)}}, \alpha_0 = 0, \alpha_1 = 0, \lambda = -\frac{2\delta\omega}{(m-2)\omega^2+1}\right\}. \tag{49}$$

By using Eqs. (49) and (48) into Eq. (5), we secure

$$q(x,t) = \mp\frac{2i\sqrt{3}\sqrt{\delta}\sqrt{m-1}\omega}{\sqrt{\gamma\left((m-2)\omega^2+1\right)}}\mathrm{dn}^{-1}(\frac{\omega}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{2\delta\omega}{\beta((m-2)\omega^2+1)}(t+\frac{1}{\Gamma(\beta)})^{\beta}). \tag{50}$$

Set 2:

$$\left\{\alpha_{-1} = 0, \alpha_0 = 0, \alpha_1 = \mp\frac{2\sqrt{3}\sqrt{\delta}\omega}{\sqrt{\gamma\left((m-2)\omega^2+1\right)}}, \lambda = -\frac{2\delta\omega}{(m-2)\omega^2+1}\right\}. \tag{51}$$

By using Eqs. (51) and (48) into Eq. (5), we have

$$q(x,t) = \mp\frac{2\sqrt{3}\sqrt{\delta}\omega}{\sqrt{\gamma\left((m-2)\omega^2+1\right)}}\mathrm{dn}(\frac{\omega}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{2\delta\omega}{\beta((m-2)\omega^2+1)}(t+\frac{1}{\Gamma(\beta)})^{\beta}). \tag{52}$$

Set 3:

$$\{\alpha_{-1} = \mp i\omega 2\sqrt{\frac{3\delta(m-1)\left(-6\sqrt{-(m-1)\omega^4}+(m-2)\omega^2+1\right)}{\gamma\left(\left(m^2+32m-32\right)\omega^4+2(m-2)\omega^2+1\right)}}, \alpha_0 = 0,$$

$$\alpha_1 = \omega \pm 2\sqrt{\frac{3\delta\left(-6\sqrt{-(m-1)\omega^4}+(m-2)\omega^2+1\right)}{\gamma\left(\left(m^2+32m-32\right)\omega^4+2(m-2)\omega^2+1\right)}},$$

$$\lambda = -\frac{2\delta\omega\left(-6\sqrt{-(m-1)\omega^4}+(m-2)\omega^2+1\right)}{\left(m^2+32m-32\right)\omega^4+2(m-2)\omega^2+1}\}. \tag{53}$$

By using Eqs. (53) and (48) into Eq. (5), we obtain

$$q(x,t) = 2\omega\sqrt{\frac{3\delta(-6\sqrt{-(m-1)\omega^4}+(m-2)\omega^2+1)}{\gamma((m^2+32m-32)\omega^4+2(m-2)\omega^2+1)}}$$

$$\times\ (\mp i\sqrt{m-1}\mathrm{dn}^{-1}(\frac{\omega}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{2\delta\omega(-6\sqrt{-(m-1)\omega^4})+(m-2)\omega^2+1}{\beta((m^2+32m-32)\omega^4+2(m-2)\omega^2+1)}(t+\frac{1}{\Gamma(\beta)})^{\beta}) \pm \mathrm{dn}(\frac{\omega}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{2\delta\omega(-6\sqrt{-(m-1)\omega^4})+(m-2)\omega^2+1}{\beta((m^2+32m-32)\omega^4+2(m-2)\omega^2+1)} \times (t+\frac{1}{\Gamma(\beta)})^{\beta})). \tag{54}$$

Set 4:

$$\{\alpha_{-1} = -2i\omega\sqrt{\frac{3\delta(m-1)\left(6\sqrt{-(m-1)\omega^4}+\left((m-2)\omega^2+1\right)\right)}{\gamma\left(\left(m^2+32m-32\right)\omega^4+2(m-2)\omega^2+1\right)}}, \alpha_0 = 0,$$
$$\alpha_1 = -2\omega\sqrt{\frac{3\delta\left(6\sqrt{-(m-1)\omega^4}+\left((m-2)\omega^2+1\right)\right)}{\gamma\left(\left(m^2+32m-32\right)\omega^4+2(m-2)\omega^2+1\right)}},$$
$$\lambda = -\frac{2\delta\omega\left(6\sqrt{-(m-1)\omega^4}+(m-2)\omega^2+1\right)}{\left(m^2+32m-32\right)\omega^4+2(m-2)\omega^2+1}\}. \tag{55}$$

By using Eqs. (55) and (48) into Eq. (5), we get

$$q(x,t) = -2\omega\sqrt{\frac{3\delta(6\sqrt{-(m-1)\omega^4}+(m-2)\omega^2+1)}{\gamma((m^2+32m-32)\omega^4+2(m-2)\omega^2+1)}} \times (i\sqrt{m-1}\mathrm{dn}^{-1}(\frac{\omega}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{2\delta\omega(6\sqrt{-(m-1)\omega^4})+(m-2)\omega^2+1}{\beta((m^2+32m-32)\omega^4+2(m-2)\omega^2+1)}(t+\frac{1}{\Gamma(\beta)})^{\beta}) + \mathrm{dn}(\frac{\omega}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{2\delta\omega(6\sqrt{-(m-1)\omega^4})+(m-2)\omega^2+1}{\beta((m^2+32m-32)\omega^4+2(m-2)\omega^2+1)} \times (t+\frac{1}{\Gamma(\beta)})^{\beta})). \tag{56}$$

Set 5:

$$\{\alpha_{-1} = 2i\omega\sqrt{\frac{3(m-1)\delta\left(6\sqrt{-(m-1)\omega^4}+(m-2)\omega^2+1\right)}{\gamma\left(\left(m^2+32m-32\right)\omega^4+2(m-2)\omega^2+1\right)}}, \alpha_0 = 0,$$
$$\alpha_1 = 2\omega\sqrt{\frac{3\delta\left(6\sqrt{-(m-1)\omega^4}+(m-2)\omega^2+1\right)}{\gamma\left(\left(m^2+32m-32\right)\omega^4+2(m-2)\omega^2+1\right)}},$$
$$\lambda = -\frac{2\delta\omega\left(6\sqrt{-(m-1)\omega^4}+(m-2)\omega^2+1\right)}{\left(m^2+32m-32\right)\omega^4+2(m-2)\omega^2+1}\}. \tag{57}$$

By inserting Eqs. (57) and (48) into Eq. (5), we achieve

$$q(x,t) = 2\omega\sqrt{\frac{3\delta(6\sqrt{-(m-1)\omega^4}+(m-2)\omega^2+1)}{\gamma((m^2+32m-32)\omega^4+2(m-2)\omega^2+1)}} \times (i\sqrt{m-1}\mathrm{dn}^{-1}(\frac{\omega}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{2\delta\omega(6\sqrt{-(m-1)\omega^4})+(m-2)\omega^2+1}{\beta((m^2+32m-32)\omega^4+2(m-2)\omega^2+1)}(t+\frac{1}{\Gamma(\beta)})^{\beta}) + \mathrm{dn}(\frac{\omega}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{2\delta\omega(6\sqrt{-(m-1)\omega^4})+(m-2)\omega^2+1}{\beta((m^2+32m-32)\omega^4+2(m-2)\omega^2+1)} \times (t+\frac{1}{\Gamma(\beta)})^{\beta})). \tag{58}$$

Solutions with Kudryashov method

Since from Eq. (6) we secure $m = 1$, then Eq. (11) reduces to:

$$Y(\eta) = \alpha_0 + \alpha_1\phi(\eta) \tag{59}$$

Here α_0 and α_1 are unknowns. By inserting the Eq. (59) along Eq. (13) into the Eq. (6) and summing up the coefficients of each power of $\phi(\eta)$, yields an algebraic system of expressions containing α_0, α_1 and other parameters. With the help of symbolic software, we have the successive solution sets:

Set 1:

$$\left\{\alpha_0 = 0, \alpha_1 = -\frac{2\sqrt{3}\sqrt{d}\sqrt{\delta\omega}}{\sqrt{\gamma\left(1-\omega^2\right)}}, \lambda = \frac{2\delta\omega}{\omega^2-1}\right\}. \tag{60}$$

By using Eqs. (60) and (59) into Eq. (5) yield

$$q(\eta) = -\frac{2\sqrt{3}\sqrt{d}\sqrt{\delta\omega}}{\sqrt{\gamma-\gamma\omega^2}((a-b)\sinh(\eta)+(a+b)\cosh(\eta))}. \tag{61}$$

where η is given by Eq. (5).

Set 2:

$$\left\{\alpha_0 = 0, \alpha_1 = \frac{2\sqrt{3}\sqrt{d}\sqrt{\delta\omega}}{\sqrt{\gamma\left(1-\omega^2\right)}}, \lambda = \frac{2\delta\omega}{\omega^2-1}\right\}. \tag{62}$$

By using Eqs. (62) and (59) into Eq. (5), we get

$$q(\eta) = \frac{2\sqrt{3}\sqrt{d}\sqrt{\delta\omega}}{\sqrt{\gamma-\gamma\omega^2}((a-b)\sinh(\eta)+(a+b)\cosh(\eta))}. \tag{63}$$

where η is given by Eq. (5). Since Eq. (61) produces a general form of bright solitons and the following Figs. 5–6 describe the solution dynamics corresponding to $\gamma = 0.6 = \delta$, $d = 1$ and $\omega = 0.5$ (see Fig. 4).

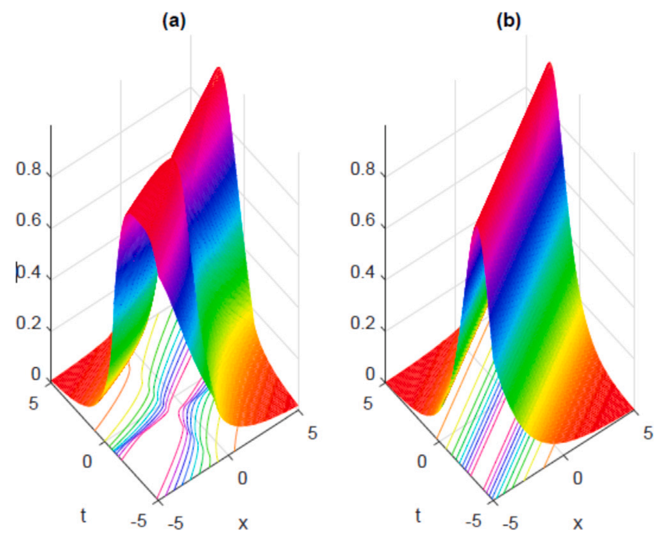

Fig. 5. Bright soliton (61), $3D$ simulation is presented in (a) and (b) for fractional parameter $\beta = 0.75, 1.0$ and $\beta_0 = 1 = \beta_1$.

Solutions with Exp_a function method

As we know $m = 1$ for Eq. (16), then Eq. (14) reduces to:

$$Y(\eta) = \frac{\alpha_0+\alpha_1 D^{\eta}}{\beta_0+\beta_1 D^{\eta}} \tag{64}$$

where α_0, α_1, β_0 and β_1 are parameters. By putting the Eq. (64) into the Eq. (16) and collecting the coefficients of each power of D^{η}, yields the system of algebraic expressions containing α_0, α_1, β_0, β_1 and other parameters. By solving this system, we gain the successive solution sets:

Set 1:

$$\left\{\alpha_0 = -\frac{\sqrt{6}\beta_0\sqrt{\delta\omega}\ln(D)}{\sqrt{-\gamma\left(\omega^2\ln^2(D)+2\right)}}, \alpha_1 = \frac{\sqrt{6}\beta_1\sqrt{\delta\omega}\ln(D)}{\sqrt{-\gamma\left(\omega^2\ln^2(D)+2\right)}}, \lambda = -\frac{4\delta\omega}{\omega^2\ln^2(D)+2}\right\}. \tag{65}$$

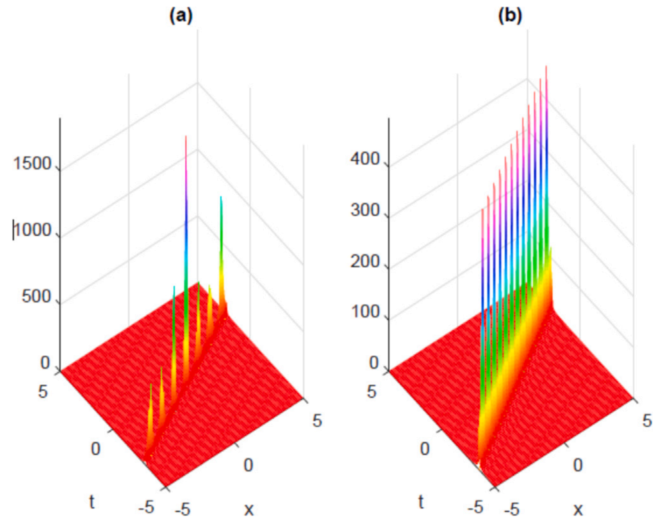

Fig. 6. Multiple solitons (61), $3D$ simulation is presented in (a) and (b) for fractional parameter $\beta = 0.75, 1.0$ and $\beta_0 = 1,\ \beta_1 = -1$.

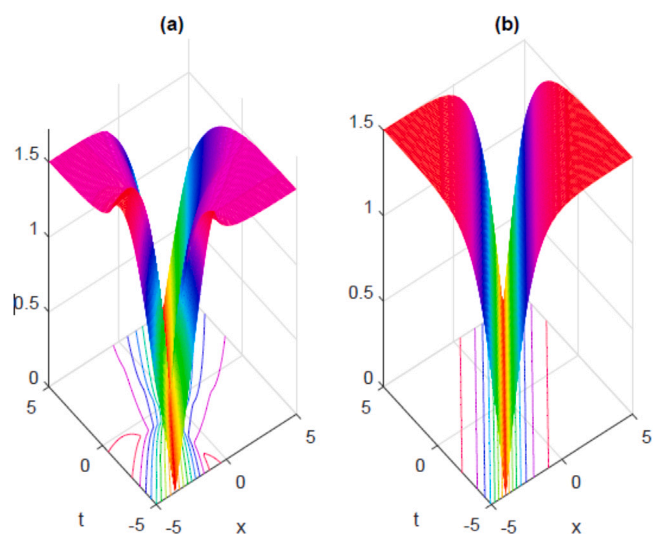

Fig. 7. $3D$ wave simulation of Eq. (66) is presented in (a) and (b) for fractional parameter $\beta = 0.75, 1.0$ and $\omega = 1$.

By putting Eqs. (65) and (64) into Eq. (5), we get

$$q() = -\frac{\sqrt{6}\sqrt{\delta}\omega\ln(D)}{\sqrt{-\gamma(\omega^2\ln^2(D)+2)}}\left(\frac{\beta_0 - \beta_1 D^{\left(\frac{\omega}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^{\beta}+\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}}{\beta_0 + \beta_1 D^{\left(\frac{\omega}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^{\beta}+\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}}\right). \tag{66}$$

Set 2:

$$\left\{\alpha_0 = \frac{\sqrt{6}\beta_0\sqrt{\delta\omega}\ln(D)}{\sqrt{-\gamma\left(\omega^2\ln^2(D)+2\right)}}, \alpha_1 = -\frac{\sqrt{6}\beta_1\sqrt{\delta\omega}\ln(D)}{\sqrt{-\gamma\left(\omega^2\ln^2(D)+2\right)}}, \lambda = -\frac{4\delta\omega}{\omega^2\ln^2(D)+2}\right\}. \tag{67}$$

By using the Eqs. (67) and (64) into Eq. (5), we get

$$q(x,t) = \frac{\sqrt{6}\sqrt{\delta}\omega\ln(D)}{\sqrt{-\gamma(\omega^2\ln^2(D)+2)}}\left(\frac{\beta_0 - \beta_1 D^{\left(\frac{\omega}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^{\beta}+\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}}{\beta_0 + \beta_1 D^{\left(\frac{\omega}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^{\beta}+\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}}\right). \tag{68}$$

The Eq. (66) produce the dark soliton and the following $3D$ graph displays its dynamics corresponding to $\gamma = 0.6 = \delta$ and $D = 1$ (see Fig. 7).

Conclusion

We have succeeded to explore a water wave equation namely dispersive modified Camassa–Holm equation for a variety of solitary wave solutions with space–time fractional derivative. We have acquired novel solutions in the form of Jacobi's elliptic, hyperbolic, periodic and rational exponential functions via three prolific schemes. The obtained results have been verified and also depicted with the help of numerical simulations. This work has much importance because the methods applied and the Beta-derivative both have been exercised for the first time to the aforesaid equation. The obtained results may have an impact on further investigations for the nonlinear fractional physical model equations.

CRediT authorship contribution statement

Asim Zafar: Formal analysis, Methodology. **M. Raheel:** Software, Writing – original draft. **Kamyar Hosseini:** Conceptualization, Supervision. **Mohammad Mirzazadeh:** Original draft, Validation. **Soheil Salahshour:** Writing – review and editing. **Choonkil Park:** Supervision, Validation. **Dong Yun Shin:** Methodology, Review.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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