Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(3): 3379–3394.
DOI: 10.3934/math.2022188
Received: 16 September 2021
Accepted: 11 November 2021
Published: 30 November 2021

Research article

Additive and Fréchet functional equations on restricted domains with some characterizations of inner product spaces

Choonkil Park[1]**, Abbas Najati**[2,*]**, Batool Noori**[2] **and Mohammad B. Moghimi**[2,*]

[1] Department of Mathematics, Hanyang University, Korea

[2] Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil 56199-11367, Iran

* **Correspondence:** Email: a.nejati@yahoo.com, mbfmoghimi@yahoo.com.

Abstract: In this paper, we investigate the Hyers-Ulam stability of additive and Fréchet functional equations on restricted domains. We improve the bounds and thus the results obtained by S. M. Jung and J. M. Rassias. As a consequence, we obtain asymptotic behaviors of functional equations of different types. One of the objectives of this paper is to bring out the involvement of functional equations in various characterizations of inner product spaces.

Keywords: Hyers-Ulam stability; quadratic functional equation; Fréchet functional equation; asymptotic behavior
Mathematics Subject Classification: 39B82, 39B52, 39B62, 46C15

1. Introduction

Ulam [33] gave a fascinating and famous lecture in 1940 that encouraged the study of stability problems for various functional equations. He discussed a number of important unsolved problems in mathematics. Among them, a question of the stability of group homomorphisms seemed too abstract for anyone to come to any conclusion. In fact, he asked the following question about the stability of homomorphisms:

Let $(G_1, *)$ be a group and let $(G_2, \diamond)$ be a metric group with a metric d. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(x * y), f(x) \diamond f(y)) < \delta$ for all $x, y \in G_1$; then there is a homomorphism $h : G_1 \to G_2$ with $d(f(x), h(x)) < \varepsilon$ for all $x \in G_1$?

If the answer is affirmative, the functional equation of homomorphisms is called stable. In 1941, Hyers [11] was able to give a partial solution to the Ulam' question, which was the first important step forward and a step towards further solutions in this field. He was the first mathematician to present the result about the stability of functional equations. He masterly answered the question of Ulam for the case in which it is assumed that G_1 and G_2 are Banach spaces. Aoki [3] and Th. M. Rassias [29] extended the Hyers' theorem by considering an unbounded Cauchy difference. They tried to weaken the condition for the bound of the norm of the Cauchy difference as follows:

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^p), \quad x, y \in X.$$

where $0 \leqslant p < 1$. In 1994, Găvruta [9] provided a generalization of Rassias' theorem by replacing the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. In recent decades, several stability problems for various functional equations and also for mappings with more general domains and ranges have been investigated by a number of mathematicians. We refer the interested reader the following books and surveys [5, 6, 12, 16, 17] and the references therein for more detailed information.

It will also be interesting to study the stability problems of additive and quadratic functional equations on restricted domains. More precisely, the goal is whether there is a true additive (resp. quadratic) function in the neighborhood of a function f which only satisfies $\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon$ (resp.$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leqslant \varepsilon$) in a restricted domain. Skof [32] was the first person to address the stability on a bounded domain. She proved the following theorem and applied the result to the study of an asymptotic behavior of additive functions.

Theorem 1.1. *Let E be a Banach space, and let $d > 0$ be a given constant. Suppose a function $f : \mathbb{R} \to E$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon, \quad |x| + |y| > d.$$

for some $\varepsilon \geqslant 0$. Then there exists a unique additive function $A : \mathbb{R} \to E$ such that

$$\|f(x) - A(x)\| \leqslant 9\varepsilon, \quad x \in \mathbb{R}.$$

Using this theorem, Skof [32] investigated an interesting asymptotic behavior of additive functions, as we see in the following theorem.

Theorem 1.2. *Let X and $\mathcal{Y}$ be a normed space and a Banach space, respectively. Suppose z is a fixed point of $\mathcal{Y}$. For a function $f : X \to \mathcal{Y}$ the following two conditions are equivalent:*

(*i*) $f(x+y) - f(x) - f(y) \to z$ *as* $\|x\| + \|y\| \to \infty$;
(*ii*) $f(x+y) - f(x) - f(y) = z$ *for all* $x, y \in X$.

Z. Kominek [18] introduced a stability result for the Jensen's equation on a bounded domain. Another stability result of the Jensen's equation on an unbounded and restricted domain was obtained by S. M. Jung [14]. He was able to prove an asymptotic property of the additive functions which may be regarded as a modification of Skof's result mentioned above.

Among the normed linear spaces, inner product spaces play an important role. In an inner product space E the parallelogram law is an algebraic identity, i. e.,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x, y \in E.$$

This translates into a functional equation well known as the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad x, y \in X.$$

where X is a linear space. Most mathematicians may be interested in the study of the quadratic functional equation since the quadratic functions are applied to almost every field of mathematics. Skof [32] was the first person who proved the Hyers–Ulam stability of the quadratic functional equation for the functions $f : X \to \mathcal{Y}$, where X is a normed space and $\mathcal{Y}$ is a Banach space. In 1998, Jung [13] investigated the Hyers–Ulam stability of the quadratic and Fréchet functional equations on the unbounded restricted domains. He also investigated the asymptotic behavior of quadratic and Fréchet functional equations. J. M. Rassias [30] improved the bounds and thus the stability results obtained by S. M. Jung. Besides, he established the Ulam stability for more general functional equations on a restricted domain. For more detailed information on the stability of the Cauchy and quadratic functional equations, we can refer to [10, 19, 21–28, 30, 31].

In this paper, we investigate the Hyers–Ulam stability of additive and Fréchet functional equations on some restricted domains. Moreover, we improve the bounds and thus the results obtained by S. M. Jung and J. M. Rassias. As a consequence, we obtain asymptotic behaviors of functional equations of different types. One of the objectives of this paper is to bring out the involvement of functional equations in various characterizations of inner product spaces.

Throughout this paper, X, Y are normed linear spaces and $\mathcal{Y}$ is a Banach space.

2. Stability of additive functional equation on some restricted domains

Theorem 2.1. *Let $f : X \to Y$ be an even function. If*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon, \quad \|x+y\| \geqslant d. \tag{2.1}$$

Then f is bounded, i.e., $\|f(x)\| \leqslant 3\varepsilon$ for all $x \in X$. Especially, $\lim_{n\to\infty} \frac{f(nx)}{n} = 0$ for all $x \in X$.

Proof. Let $\widetilde{Y}$ be the completion of Y. Letting $y = x$ in (2.1), we get

$$\|f(2x) - 2f(x)\| \leqslant \varepsilon, \quad \|x\| \geqslant d. \tag{2.2}$$

Therefore

$$\left\| \frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^m x)}{2^m} \right\| \leqslant \sum_{k=m}^{n} \frac{\varepsilon}{2^{k+1}}, \quad \|x\| \geqslant d, \quad n \geqslant m \geqslant 0. \tag{2.3}$$

This implies that the sequence $\{\frac{f(2^n x)}{2^n}\}_n$ is Cauchy for all $x \in X$. Define $T : X \to \widetilde{Y}$ by

$$T(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}, \quad x \in X.$$

Since f is even, T is even. In view of the definition of T, (2.1) implies that $T(x+y) = T(x) + T(y)$ for all $x, y \in X$ with $x + y \neq 0$. Since T is even, we obtain

$$T(x-y) = T(x) + T(y) = T(x+y), \quad x \pm y \neq 0. \tag{2.4}$$

By the definition of T, we have $T(2x) = 2T(x)$ for all $x \in X$. Setting $x = 3y$, the last Eq (2.4) yields $T(2y) = T(4y)$ for all $y \in X$ (notice that $T(0) = 0$). Then $T(y) = 0$ for all $y \in X$. Letting $m = 0$ and allowing n tending to infinity in (2.3), we get

$$\|f(x)\| \leqslant \varepsilon, \quad \|x\| \geqslant d. \tag{2.5}$$

Let $x \in X$ be an arbitrary element, and choose $y \in X$ such that $\|y\| \geqslant d + \|x\|$. It is clear that $\|x+y\| \geqslant d$. So (2.5) implies that

$$\|f(y)\| \leqslant \varepsilon \quad \text{and} \quad \|f(x+y)\| \leqslant \varepsilon.$$

These together with (2.1) give $\|f(x)\| \leqslant 3\varepsilon$. This completes the proof. □

Theorem 2.2. *Let X be a linear normed space and $\mathcal{Y}$ a Banach space. Suppose that $f : X \to \mathcal{Y}$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon, \quad \|x+y\| \geqslant d. \tag{2.6}$$

for some $d > 0$. Then there exists a unique additive function $A : X \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \leqslant 3\varepsilon, \quad x \in X. \tag{2.7}$$

Proof. We know the function $f : X \to Y$ can be written as $f(x) = f_e(x) + f_o(x)$ for all $x \in X$, where $f_e(x) = \frac{f(x)+f(-x)}{2}$ is called the even part of f and $f_o(x) = \frac{f(x)-f(-x)}{2}$ is called the odd part of f. It is clear that f_e is even and f_o is odd.

It is easy to see that f_e and f_o satisfy

$$\|f_e(x+y) - f_e(x) - f_e(y)\| \leqslant \varepsilon, \quad \|x+y\| \geqslant d, \tag{2.8}$$
$$\|f_o(x+y) - f_o(x) - f_o(y)\| \leqslant \varepsilon, \quad \|x+y\| \geqslant d, \tag{2.9}$$

By (2.8) and Theorem 2.1, we infer that $\lim_{n\to\infty} \frac{f_e(nx)}{n} = 0$ for all $x \in X$.

Letting $y = x$ in (2.9), we get

$$\|f_o(2x) - 2f_o(x)\| \leqslant \varepsilon, \quad \|x\| \geqslant d.$$

Therefore

$$\left\| \frac{f_o(2^{n+1}x)}{2^{n+1}} - \frac{f_o(2^m x)}{2^m} \right\| \leqslant \sum_{k=m}^{n} \frac{\varepsilon}{2^{k+1}}, \quad \|x\| \geqslant d, \ \ n \geqslant m \geqslant 0. \tag{2.10}$$

This implies that the sequence $\{\frac{f_o(2^n x)}{2^n}\}_n$ is Cauchy for all $x \in X$. Define $A : X \to \mathcal{Y}$ by

$$A(x) := \lim_{n\to\infty} \frac{f_o(2^n x)}{2^n}, \quad x \in X.$$

In view of the definition of A, (2.6) implies that $A(x+y) = A(x) + A(y)$ for all $x, y \in X$ with $x + y \neq 0$. Since A is odd (we notice that f_o is odd), we conclude that A is additive.

It is clear that

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = \lim_{n\to\infty} \frac{f_o(2^n x) + f_e(2^n x)}{2^n} = \lim_{n\to\infty} \frac{f_o(2^n x)}{2^n} = A(x), \quad x \in X.$$

It follows from (2.6) that

$$\left\| \frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^m x)}{2^m} \right\| \leqslant \sum_{k=m}^{n} \frac{\varepsilon}{2^{k+1}}, \quad \|x\| \geqslant d, \ \ n \geqslant m \geqslant 0. \tag{2.11}$$

Letting $m = 0$ and allowing n tending to infinity in (2.11), we get

$$\|f(x) - A(x)\| \leqslant \varepsilon, \quad \|x\| \geqslant d. \tag{2.12}$$

To extend (2.12) to the whole X, let $x \in X$ and choose $y \in X$ such that $\|y\| \geqslant d + \|x\|$. Then $\|x + y\| \geqslant d$, and (2.12) yields

$$\|f(y) - A(y)\| \leqslant \varepsilon \quad \text{and} \quad \|f(x + y) - A(x + y)\| \leqslant \varepsilon.$$

Using these inequalities together with (2.6), we obtain

$$\|A(x + y) - A(y) - f(x)\| \leqslant 3\varepsilon.$$

Since A is additive, we get (2.7). The uniqueness of A follows easily from (2.7). □

Corollary 2.3. *Suppose that $f : X \to \mathcal{Y}$ satisfies*

$$\|f(x + y) - f(x) - f(y)\| \leqslant \varepsilon, \quad \|x\| + \|y\| \geqslant d.$$

for some $d > 0$. Then there exists a unique additive function $A : X \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \leqslant 3\varepsilon, \quad x \in X.$$

Proof. Because $\{(x, y) \in X \times X : \|x + y\| \geqslant d\} \subseteq \{(x, y) \in X \times X : \|x\| + \|y\| \geqslant d\}$, the result follows by Theorem 2.2. □

Remark 2.4. We improved the bounds and thus the results of Losonczi [20] and S. M. Jung [15] by obtaining sharper estimates.

Theorem 1.2 is a consequence of Theorem 2.2.

Corollary 2.5. *Let X and Y be linear normed spaces. Suppose z is a fixed point of Y. For a function $f : X \to Y$ the following conditions are equivalent:*

(*i*) $\lim_{\|x+y\|\to\infty}[f(x + y) - f(x) - f(y)] = z$;
(*ii*) $\lim_{\|x\|+\|y\|\to\infty}[f(x + y) - f(x) - f(y)] = z$;
(*iii*) $f(x + y) - f(x) - f(y) = z, \quad x, y \in X$.

Proof. It is clear that if f satisfies (*ii*), then f satisfies (*i*). To prove (*i*) ⇒ (*iii*), let f satisfy (*i*). Define $g(x) := f(x) + z$ for all $x \in X$. Then

$$\lim_{\|x+y\|\to\infty}[g(x + y) - g(x) - g(y)] = 0.$$

Let $\varepsilon > 0$ be an arbitrary real number. By (*i*) there exists $d_\varepsilon > 0$ such that

$$\|g(x + y) - g(x) - g(y)\| \leqslant \varepsilon, \quad \|x + y\| \geqslant d_\varepsilon.$$

Let $\mathcal{Y}$ be the completion of Y. In view of Theorem 2.2 there exists a unique additive function $A_\varepsilon : X \to \mathcal{Y}$ such that

$$\|g(x) - A_\varepsilon(x)\| \leqslant 3\varepsilon, \quad x \in X.$$

Then

$$\begin{aligned}\|g(x+y)-g(x)-g(y)\| &\leqslant \|g(x+y)-A_\varepsilon(x+y)\|+\|g(x)-A_\varepsilon(x)\|+\|g(y)-A_\varepsilon(y)\|\\ &\leqslant 9\varepsilon, \quad x,y\in X.\end{aligned}$$

Since ε is arbitrary, we get g is additive. Then

$$f(x+y)-f(x)-f(y)=g(x+y)-g(x)-g(y)+z=z, \quad x,y\in X.$$

This implies (iii). The implication $(iii)\Rightarrow(ii)$ is obvious. Hence the proof is complete. □

Corollary 2.6. *Let X and Y be linear normed spaces and let $\varphi : X\times X\to[0,+\infty)$. Suppose z is a fixed point of Y. A function $f : X\to Y$ satisfies*

$$f(x+y)-f(x)-f(y)=z, \quad x,y\in X$$

if one of the following conditions holds:

(i) $\lim\limits_{\|x+y\|\to\infty}\varphi(x,y)=+\infty, \quad \limsup\limits_{\|x+y\|\to\infty}\varphi(x,y)\|f(x+y)-f(x)-f(y)-z\|<\infty$*;*

(ii) $\lim\limits_{\|x\|+\|y\|\to\infty}\varphi(x,y)=+\infty, \quad \limsup\limits_{\|x\|+\|y\|\to\infty}\varphi(x,y)\|f(x+y)-f(x)-f(y)-z\|<\infty.$

Proof. It is obvious that (ii) implies (i). According to (i), there exist constants $d>0$ and $M>0$ such that

$$\varphi(x,y)\|f(x+y)-f(x)-f(y)-z\|<M, \quad \|x+y\|\geqslant d.$$

Since $\lim_{\|x+y\|\to\infty}\varphi(x,y)=+\infty$, we infer that

$$\lim_{\|x+y\|\to\infty}\|f(x+y)-f(x)-f(y)-z\|=0.$$

Hence by Corollary 2.5 we conclude that $f(x+y)-f(x)-f(y)=z$ for all $x,y\in X$. □

Theorem 2.7. *Let X be a linear normed space and $\mathcal{Y}$ a Banach space. Suppose that $f : X\to\mathcal{Y}$ satisfies*

$$\|f(x+y)-f(x)-f(y)\|\leqslant\varepsilon, \quad \min\{\|x\|,\|y\|\}\geqslant d. \tag{2.13}$$

for some $d>0$. Then there exists a unique additive function $A : X\to\mathcal{Y}$ such that

$$\|f(0)\|\leqslant 7\varepsilon \quad \textit{and} \quad \|f(x)-A(x)\|\leqslant 3\varepsilon, \quad x\in X\setminus\{0\}.$$

Proof. Letting $y=x$ in (2.13), we get

$$\|f(2x)-2f(x)\|\leqslant\varepsilon, \quad \|x\|\geqslant d.$$

Therefore

$$\left\|\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^m x)}{2^m}\right\|\leqslant\sum_{k=m}^{n}\frac{\varepsilon}{2^{k+1}}, \quad \|x\|\geqslant d, \ \ n\geqslant m\geqslant 0. \tag{2.14}$$

This implies that the sequence $\{\frac{f(2^n x)}{2^n}\}_n$ is Cauchy for all $x \in X$. Define $A : X \to \mathcal{Y}$ by

$$A(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}, \quad x \in X.$$

In view of the definition of A, (2.13) implies that $A(x + y) = A(x) + A(y)$ for all $x, y \in X \setminus \{0\}$. Since $A(0) = 0$, we conclude that A is additive. Letting $m = 0$ and allowing n tending to infinity in (2.14), we get

$$\|f(x) - A(x)\| \leqslant \varepsilon, \quad \|x\| \geqslant d. \tag{2.15}$$

To extend (2.15) to the whole X, let $x \in X \setminus \{0\}$ and choose a positive integer such that $\|nx\| \geqslant d$. Then (2.15) yields

$$\|f(-nx) - A(-nx)\| \leqslant \varepsilon \quad \text{and} \quad \|f((n + 1)x) - A((n + 1)x)\| \leqslant \varepsilon.$$

On the other hand, (2.13) implies

$$\|f(x) - f((n + 1)x) - f(-nx)\| \leqslant \varepsilon.$$

Using these inequalities, we obtain

$$\|f(x) - A(-nx) - A((n + 1)x)\| \leqslant 3\varepsilon.$$

Since A is additive, we get $\|f(x) - A(x)\| \leqslant 3\varepsilon$ for all $x \in X \setminus \{0\}$.

Now, let $x \in X \setminus \{0\}$. Then (2.13) yields $\|f(0) - f(x) - f(-x)\| \leqslant \varepsilon$. Hence

$$\begin{aligned}\|f(0)\| &\leqslant \|f(0) - f(x) - f(-x)\| + \|f(x) - A(x)\| + \|f(-x) - A(-x)\|\\ &\leqslant \varepsilon + 3\varepsilon + 3\varepsilon = 7\varepsilon.\end{aligned}$$

The uniqueness of A follows easily from (2.7). □

Corollary 2.8. *Let X and Y be linear normed spaces. Suppose z is a fixed point of Y. For a function $f : X \to Y$ the following conditions are equivalent:*

(*i*) $\lim_{\min\{\|x\|,\|y\|\}\to\infty}[f(x + y) - f(x) - f(y)] = z$;
(*ii*) $\lim_{\max\{\|x\|,\|y\|\}\to\infty}[f(x + y) - f(x) - f(y)] = z$;
(*iii*) $f(x + y) - f(x) - f(y) = z, \quad x, y \in X$.

Corollary 2.9. *Let X and Y be linear normed spaces and let $\varphi : X\times X \to [0, +\infty)$. Suppose z is a fixed point of Y. A function $f : X \to Y$ satisfies*

$$f(x + y) - f(x) - f(y) = z, \quad x, y \in X.$$

if one of the following conditions holds:

(*i*) $\displaystyle\lim_{\min\{\|x\|,\|y\|\}\to\infty} \varphi(x, y) = +\infty, \quad \limsup_{\min\{\|x\|,\|y\|\}\to\infty} \varphi(x, y)\|f(x + y) - f(x) - f(y) - z\| < \infty$;

(*ii*) $\displaystyle\lim_{\max\{\|x\|,\|y\|\}\to\infty} \varphi(x, y) = +\infty, \quad \limsup_{\max\{\|x\|,\|y\|\}\to\infty} \varphi(x, y)\|f(x + y) - f(x) - f(y) - z\| < \infty.$

Proof. It is obvious that (ii) implies (i). According to (i), there exist constants $d > 0$ and $M > 0$ such that

$$\varphi(x, y)\|f(x + y) - f(x) - f(y) - z\| < M, \quad \min\{\|x\|, \|y\|\} \geqslant d.$$

Since $\lim_{\min\{\|x\|,\|y\|\}\to\infty} \varphi(x, y) = +\infty$, we infer that

$$\lim_{\min\{\|x\|,\|y\|\}\to\infty} \|f(x + y) - f(x) - f(y) - z\| = 0.$$

Hence by Corollary 2.8, we have $f(x + y) - f(x) - f(y) = z$ for all $x, y \in X$. □

Corollary 2.10. *Let X and Y be linear normed spaces, $\varepsilon \geqslant 0$ and let $p < 0$. Suppose z is a fixed point of Y and $f : X \to Y$ satisfies*

$$\|f(x + y) - f(x) - f(y) - z\| \leqslant \varepsilon(\|x\|^p + \|y\|^p), \quad x, y \in X \setminus \{0\}.$$

Then $f(x + y) - f(x) - f(y) = z$ for all $x, y \in X \setminus \{0\}$.

3. Stability of Fréchet functional equation on some restricted domains

A function $f : X \to Y$ between linear spaces X and Y satisfies the Fréchet equation if

$$f(x + y + z) + f(x) + f(y) + f(z) = f(x + y) + f(y + z) + f(x + z), \quad x, y, z \in X. \tag{3.1}$$

Theorem 3.1. *Suppose that $\varepsilon \geqslant 0$ and $f : X \to \mathcal{Y}$ is an odd function satisfies*

$$\|f(x + y + z) + f(x) + f(y) + f(z) - f(x + y) - f(y + z) - f(x + z)\| \leqslant \varepsilon. \tag{3.2}$$

for all $x, y, z \in X$ with $\|x + y + z\| \geqslant d$, where $d > 0$ is a constant. Then there exist a unique additive function $A : X \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \leqslant 4\varepsilon, \quad x \in X. \tag{3.3}$$

Proof. Letting $z = -y$ in (3.2), yields

$$\|f(x + y) + f(x - y) - 2f(x)\| \leqslant \varepsilon, \quad \|x\| \geqslant d. \tag{3.4}$$

Settings $y = x$ in (3.4), yields

$$\|f(2x) - 2f(x)\| \leqslant \varepsilon, \quad \|x\| \geqslant d.$$

Therefore

$$\left\|\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^m x)}{2^m}\right\| \leqslant \sum_{k=m}^{n} \frac{\varepsilon}{2^{k+1}}, \quad \|x\| \geqslant d, \;\; n \geqslant m \geqslant 0. \tag{3.5}$$

Then it is easy to infer that the sequence $\{\frac{f(2^n x)}{2^n}\}_n$ is Cauchy for all $x \in X$. Define $A : X \to \mathcal{Y}$ by

$$A(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}, \quad x \in X.$$

Obviously, A is odd. In view of the definition of A, (3.4) implies that

$$A(x + y) + A(x - y) = 2A(x), \quad x \neq 0.$$

Since A is odd, we conclude that A is additive. Letting $m = 0$ and allowing n tending to infinity in (3.5), we get

$$\|f(x) - A(x)\| \leqslant \varepsilon, \quad \|x\| \geqslant d. \tag{3.6}$$

It is natural to expect to extend this to the whole X. Let $z \in X \setminus \{0\}$ and choose a positive integer n such that $\|nz\| \geqslant d$. Setting $x = (n+1)z, y = nz$ in (3.4) yields

$$\|f((2n+1)z) + f(z) - 2f((n+1)z)\| \leqslant \varepsilon.$$

On the other hand, (3.6) implies

$$\|A((2n+1)z) - f((2n+1)z)\| \leqslant \varepsilon \quad \text{and} \quad \|2f((n+1)z) - 2A((n+1)z)\| \leqslant 2\varepsilon.$$

Using these inequalities, we obtain

$$\|f(z) + A((2n+1)z) - 2A((n+1)z)\| \leqslant 4\varepsilon.$$

Since A is additive, we get $\|f(z) - A(z)\| \leqslant 4\varepsilon$ for all $z \in X \setminus \{0\}$. Since $A(0) = f(0) = 0$, the last inequality yields (3.3).

The uniqueness of A follows easily from (3.3) □

Theorem 3.2. *Suppose that $\varepsilon \geqslant 0, d > 0$ and $f : X \to \mathcal{Y}$ is an even function satisfies* (3.2) *for all $x, y, z \in X$ with $\|x + y + z\| \geqslant d$. Then there exist a unique quadratic function $Q : X \to \mathcal{Y}$ such that*

$$\left\|Q(x) - f(x) + \frac{f(0)}{2}\right\| \leqslant \frac{7\varepsilon}{6}, \quad \|Q(x) - f(x)\| \leqslant \frac{5\varepsilon}{3}, \quad x \in X. \tag{3.7}$$

Proof. Letting $z = -y$ in (3.2), yields

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y) + f(0)\| \leqslant \varepsilon, \quad \|x\| \geqslant d. \tag{3.8}$$

Settings $y = x$ in (3.8), yields

$$\|f(2x) - 4f(x) + 2f(0)\| \leqslant \varepsilon, \quad \|x\| \geqslant d.$$

Therefore

$$\left\|\frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^m x)}{4^m} + 2\sum_{k=m}^{n} \frac{f(0)}{4^{k+1}}\right\| \leqslant \sum_{k=m}^{n} \frac{\varepsilon}{4^{k+1}}, \quad \|x\| \geqslant d, \ n \geqslant m \geqslant 0. \tag{3.9}$$

Then it is easy to infer that the sequence $\{\frac{f(2^n x)}{2^n}\}_n$ is Cauchy for all $x \in X$. Define $Q : X \to \mathcal{Y}$ by

$$Q(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n}, \quad x \in X.$$

Obviously, Q is even. In view of the definition of Q, (3.8) implies that

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y), \quad x \neq 0.$$

Since Q is even and $Q(0) = 0$, we conclude that Q is quadratic. Letting $m = 0$ and allowing n tending to infinity in (3.9), we get

$$\left\|Q(x) - f(x) + \frac{2}{3}f(0)\right\| \leqslant \frac{\varepsilon}{3}, \quad \|x\| \geqslant d. \tag{3.10}$$

Now we extend (3.10) to all of X. Let $y \in X$ and choose $x \in X$ such that $\|x\| \geqslant d + \|y\|$. Clearly, $\|x \pm y\| \geqslant d$. By (3.10), we obtain

$$\left\|Q(x+y) - f(x+y) + \frac{2}{3}f(0)\right\| \leqslant \frac{\varepsilon}{3};$$
$$\left\|Q(x-y) - f(x-y) + \frac{2}{3}f(0)\right\| \leqslant \frac{\varepsilon}{3};$$
$$\left\|2f(x) - 2Q(x) - \frac{4}{3}f(0)\right\| \leqslant \frac{2\varepsilon}{3}.$$

Adding these inequalities and (3.8), gives

$$\|Q(x+y) + Q(x-y) - 2Q(x) - 2f(y) + f(0)\| \leqslant \frac{7\varepsilon}{3}.$$

Since Q is quadratic, the last inequality yields

$$\|2Q(y) - 2f(y) + f(0)\| \leqslant \frac{7\varepsilon}{3}, \quad y \in X.$$

Besides from(3.8) with $y = 0$, we get that $\|f(0)\| \leqslant \varepsilon$. This gives the desired result (3.7).

The uniqueness of Q follows easily from (3.7) □

Theorem 3.3. *Suppose that $\varepsilon \geqslant 0$ and $f : X \to Y$ is a function satisfies* (3.2) *for some $d > 0$. Then there exist a unique additive $A : X \to \mathcal{Y}$ and a unique quadratic function $Q : X \to \mathcal{Y}$ such that*

$$\|A(x) + Q(x) - f(x)\| \leqslant \frac{17\varepsilon}{3}, \quad x \in X.$$

Proof. We know every function $f : X \to Y$ can be written as $f(x) = f_e(x) + f_o(x)$ for all $x \in X$, where $f_e(x) = \frac{f(x)+f(-x)}{2}$ is called the even part of f and $f_o(x) = \frac{f(x)-f(-x)}{2}$ is called the odd part of f. It is clear that f_e is even and f_o is odd.

It is easy to see that f_e and f_o satisfy (3.2). By Theorems 3.1 and 3.2, there exist a unique additive $A : X \to \mathcal{Y}$ and a unique quadratic function $Q : X \to \mathcal{Y}$ such that

$$\|Q(x) - f_e(x))\| \leqslant \frac{5\varepsilon}{3} \quad \text{and} \quad \|A(x) - f_o(x)\| \leqslant 4\varepsilon, \quad x \in X.$$

Then

$$\|A(x) + Q(x) - f(x)\| \leqslant \frac{17\varepsilon}{3}, \quad x \in X.$$

□

Remark 3.4. We note that

$$\{(x, y, z) \in X \times X \times X : \|x + y + z\| \geqslant d\} \subseteq \{(x, y, z) \in X \times X \times X : \|x\| + \|y\| + \|z\| \geqslant d\},$$

the above results remain valid if the condition $\|x+y+z\| \geqslant d$ in (3.2) is replaced by $\|x\|+\|y\|+\|z\| \geqslant d$.

It should be noted that S. M. Jung [13] obtained the bound 21ε in inequalities (3.3) and (3.7). Later, J. M. Rassias [30] improved this bound by replacing the bound 21ε with the bound 15ε. Obviously, our bounds in inequalities (3.3) and (3.7) are also sharper than the corresponding inequalities of S. M. Jung [13] and J. M. Rassias [30].

Remark 3.5. If the condition $\|x+y+z\| \geqslant d$ in (3.2) is replaced by $\min\{\|x\|, \|y\|, \|z\|\} \geqslant d$, the results of Theorems 3.1, 3.2 and 3.3 are still valid. In fact, the proofs are quite similar and the arguments can be easily completed.

Now, we can prove the following corollary concerning an asymptotic property of the functional Eq (3.1). For convenience, let

$$Df(x,y,z) := f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(x+z), \quad x,y,z \in X.$$

for a given $f : X \to Y$.

Corollary 3.6. *Let $b \in Y$ be a fixed element. Suppose that $f : X \to Y$ satisfies one of the following asymptotic behaviors*

(*i*) $\lim_{\|x\|+\|y\|+\|z\|\to\infty} Df(x,y,z) = b;$
(*ii*) $\lim_{\|x+y+z\|\to\infty} Df(x,y,z) = b;$
(*iii*) $\lim_{\min\{\|x\|,\|y\|,\|z\|\}\to\infty} Df(x,y,z) = b;$
(*iv*) $\lim_{\max\{\|x\|,\|y\|,\|z\|\}\to\infty} Df(x,y,z) = b.$

Then f has the form $f = b + A + Q$, where $A : X \to Y$ is additive and $Q : X \to Y$ is quadratic.

Corollary 3.7. *Let $\varphi : X \times X \times X \to [0, +\infty)$ and b be a fixed point of Y. A function $f : X \to Y$ satisfies*

$$Df(x,y,z) = b, \quad x,y,z \in X.$$

if one of the following conditions holds:

(*i*) $$\lim_{\min\{\|x\|,\|y\|,\|z\|\}\to\infty} \varphi(x,y,z) = +\infty,$$
$$\limsup_{\min\{\|x\|,\|y\|,\|z\|\}\to\infty} \varphi(x,y,z)\|Df(x,y,z) - b\| < \infty;$$

(*ii*) $$\lim_{\max\{\|x\|,\|y\|,\|z\|\}\to\infty} \varphi(x,y,z) = +\infty,$$
$$\limsup_{\max\{\|x\|,\|y\|,\|z\|\}\to\infty} \varphi(x,y,z)\|Df(x,y,z) - b\| < \infty;$$

(*iii*) $$\lim_{\|x\|+\|y\|+\|z\|\to\infty} \varphi(x,y,z) = +\infty,$$
$$\limsup_{\|x\|+\|y\|+\|z\|\to\infty} \varphi(x,y,z)\|Df(x,y,z) - b\| < \infty;$$

(*iv*) $$\lim_{\|x+y+z\|\to\infty} \varphi(x,y,z) = +\infty,$$
$$\limsup_{\|x+y+z\|\to\infty} \varphi(x,y,z)\|Df(x,y,z) - b\| < \infty.$$

Corollary 3.8. *Let $p < 0$ and b is a fixed point of Y. Suppose a function $f : X \to Y$ satisfies*

$$\|Df(x,y,z) - b\| \leqslant \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p), \quad x,y,z \in X \setminus \{0\}.$$

Then $Df(x,y,z) = b$ for all $x,y,z \in X \setminus \{0\}$.

4. Some characterizations of inner product spaces

Some known characterizations of inner product spaces and their generalizations can be found in [1, 2, 4] and references therein. In this section we give various characterizations of inner product spaces. Eq (3.1) was applied by Fréchet [8] in a characterization of the inner product spaces.

Theorem 4.1 (Fréchet)**.** *A normed linear space $(X, \|.\|)$ is an inner product space if and only if*

$$\|x + y + z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 = \|x + y\|^2 + \|y + z\|^2 + \|x + z\|^2, \quad x, y, z \in X.$$

Theorem 4.2. *Let $X \neq \{0\}$ be a real normed linear space such that*

$$\|x + y + z\|^p + \|x\|^q + \|y\|^s + \|z\|^t = \|x + y\|^\alpha + \|y + z\|^\beta + \|x + z\|^\gamma, \quad x, y, z \in X. \tag{4.1}$$

for some real numbers $p, q, s, t, \alpha, \beta, \gamma \in (0, +\infty)$. Then X is an inner product space.

Proof. Letting $y = z = 0$ and choosing $\|x\| = 2$ in (4.1), we get $2^p + 2^q = 2^\alpha + 2^\gamma$. Letting $z = 0$ and $y = x$ with $\|x\| = 1$ in (4.1), we obtain $2^p = 2^\alpha$. Then $p = \alpha$ and $q = \gamma$. Letting $x = 0$ and $z = y$ with $\|y\| = 1$ in (4.1), we infer that $p = \beta$. Letting $y = 0$ and $z = x$ with $\|x\| = 1$ in (4.1), we infer that $p = \gamma$. Hence $p = q = \alpha = \beta = \gamma$. Setting $x = z = 0$ and choosing $\|y\| = 2$ in (4.1), we obtain $s = \beta$. Finally, Setting $x = y = 0$ and choosing $\|z\| = 2$ in (4.1), we obtain $t = \gamma$. Therefore $p = q = s = t = \alpha = \beta = \gamma$. Putting $y = x$ and $z = -x$ with $\|x\| = 1$ in (4.1), we get $\alpha = 2$. Then $p = q = s = t = \alpha = \beta = \gamma = 2$. Hence Theorem 4.1 provides the desired result. □

Let us recall the following result from [2].

Theorem 4.3. *Let X be a normed linear space and $\phi : \mathbb{R} \to \mathbb{R}$ be continuous with $\phi(0) = 0, \phi(1) = 1$ and satisfy*

$$\phi(\|x + y\|) + \phi(\|x - y\|) = 2\phi(\|x\|) + 2\phi(\|y\|), \quad x, y \in X.$$

Then X is an inner product space.

Now we consider a generalization of Theorem 4.3.

Theorem 4.4. *Let X be a normed linear space and $\varphi_i : \mathbb{R} \to \mathbb{R}$ $(1 \leqslant i \leqslant 4)$. Suppose that φ_i is continuous with $\varphi_i(0) = 0, \varphi_i(1) = 1$ for some i, and*

$$\varphi_1(\|x + y\|) + \varphi_2(\|x - y\|) = \varphi_3(\|x\|) + \varphi_4(\|y\|), \quad x, y \in X.$$

Then X is an inner product space.

Proof. We may assume without loss of generality that φ_1 is continuous with $\varphi_1(0) = 0$ and $\varphi_1(1) = 1$. Define $f_i : X \to \mathbb{R}$ by

$$f_i(x) := \varphi_i(\|x\|), \quad x \in X, \ \ i = 1, 2, 3, 4.$$

Each f_i is even and we have

$$f_1(x + y) + f_2(x - y) = f_3(x) + f_4(y), \quad x, y \in X. \tag{4.2}$$

By [17, Theorem 4.28], each $f_i - f_i(0)$ is quadratic and

$$f_2 = f_1 + f_2(0), \quad f_3 = 2f_1 + f_3(0), \quad f_4 = 2f_1 + f_4(0).$$

Then
$$f_1(x+y)+f_1(x-y)+f_2(0)=2f_1(x)+2f_1(y)+f_3(0)+f_4(0),\quad x,y\in X.$$
Since $f_1(0)=0$, it follows from (4.2) that $f_2(0)=f_3(0)+f_4(0)$. Therefore
$$f_1(x+y)+f_1(x-y)=2f_1(x)+2f_1(y),\quad x,y\in X.$$
This means
$$\varphi_1(\|x+y\|)+\varphi_1(\|x-y\|)=2\varphi_1(\|x\|)+2\varphi_1(\|y\|),\quad x,y\in X.$$
Hence X is an inner product space by Theorem 4.3. □

It is proven that if for all $x,y\in X$ ($y\neq 0$), the corresponding function $\varphi:\mathbb{R}\to\mathbb{R}$ given by $\varphi(t)=\|x+ty\|^2$ is a polynomial in t of degree 2, then X is an inner product space (see [2]). Now we prove the following result.

Proposition 4.5. *Suppose that the continuous function $f:\mathbb{R}\to\mathbb{R}$ is such that, for each $x,y\in X$ ($y\neq 0$), the function $\varphi:\mathbb{R}\to\mathbb{R}$ given by $\varphi(t)=f(\|x+ty\|^2)+f(\|x-ty\|^2)$ is a polynomial in t of degree 2. Then X is an inner product space.*

Proof. Let
$$\varphi(t)=f(\|x+ty\|^2)+f(\|x-ty\|^2)=at^2+bt+c,$$
where a,b,c are functions of x and y. It is clear that $c=\varphi(0)=2f(\|x\|^2)$ and
$$f(\|x+y\|^2)+f(\|x-y\|^2)=\frac{1}{2}[\varphi(1)+\varphi(-1)]=a+c=2f(\|x\|^2)+a,\quad x,y\in X.$$
Letting $x=0$, the equation above yields $a=2f(\|y\|^2)-2f(0)$. Define $\psi:X\to\mathbb{R}$ by $\psi(x):=f(\|x\|^2)-f(0)$ for all $x\in X$. Then
$$\psi(x+y)+\psi(x-y)=2\psi(x)+2\psi(y),\quad x,y\in X. \tag{4.3}$$
Since f is continuous, ψ is continuous. Therefore (4.3) yields $\psi(tx)=t^2\psi(x)$ for all $x\in X$ and $t\in\mathbb{R}$. Setting $x\in X$ with $\|x\|=1$, we get
$$\begin{aligned}f(t)-f(0)&=f\left(\|\sqrt{t}x\|^2\right)-f(0)\\&=\psi\left(\sqrt{t}x\right)\\&=t\psi(x)=t[f(\|x\|^2)-f(0)]=t[f(1)-f(0)],\quad t\geqslant 0.\end{aligned}$$
So we conclude that $\psi(x)=\|x\|^2[f(1)-f(0)]$ for all $x\in X$. Since $\varphi(t)$ is a polynomial in t of degree 2, we infer $f(1)-f(0)\neq 0$, and (4.3) implies
$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2,\quad x,y\in X.$$
Thus X is an inner product space. □

Corollary 4.6. *Let $\|x+ty\|^2+\|x-ty\|^2$ be a polynomial in t of degree 2 for $x,y\in X$ ($y\neq 0$). Then X is an inner product space.*

Now, we recall a result from [2].

Theorem 4.7. *Let X be a real normed linear space such that*

$$\|x+y+z\|^2 + \|x+y-z\|^2 - \|x-y+z\|^2 - \|x-y-z\|^2, \quad x, y, z \in X.$$

is independent of z. Then X is an inner product space.

Theorem 4.8. *Let $p, q, r, s > 0$ be real numbers and $X \neq \{0\}$ be a real normed linear space such that*

$$\|x+y+z\|^p + \|x+y-z\|^q - \|x-y+z\|^r - \|x-y-z\|^s, \quad x, y, z \in X. \tag{4.4}$$

is independent of z. Then X is an inner product space.

Proof. Setting $z = x + y$ and $z = 0$ in (4.4), we obtain

$$2^p\|x+y\|^p - 2^r\|x\|^r - 2^s\|y\|^s = \|x+y\|^p + \|x+y\|^q - \|x-y\|^r - \|x-y\|^s, \quad x, y \in X. \tag{4.5}$$

Letting $y = 0$ and choosing x with $\|x\| = 1$ in (4.5), we infer $p = r$. Similarly by letting $x = 0$ and choosing y with $\|y\| = 1$ in (4.5), we infer $p = s$. Hence $p = r = s$. Letting $y = x$ with $\|x\| = 1$ and $\|x\| = 2$ in (4.5), respectively, we obtain

$$4^p - 3 \times 2^p = 2^q \quad \text{and} \quad 8^p - 3 \times 4^p = 4^q.$$

Therefore

$$4^q = 8^p - 3 \times 4^p = 2^p[4^p - 3 \times 2^p] = 2^p 2^q \quad \Longrightarrow p = q.$$

Hence $4^p - 3 \times 2^p = 2^p$ implies $p = 2$. Then $p = q = r = s = 2$, and we conclude

$$\|x+y+z\|^2 + \|x+y-z\|^2 - \|x-y+z\|^2 - \|x-y-z\|^2, \quad x, y, z \in X$$

is independent of z. Now, by Theorem 4.7, we obtain the result sought. □

5. Conclusions

In this work, we established a new strategy to study the Hyers-Ulam stability of additive and Fréchet functional equations on restricted domains. We also improved the bounds and thus the results obtained by S. M. Jung and J. M. Rassias. As a consequence, we obtained asymptotic behaviors of functional equations of different types. One of the objectives of this paper was to bring out the involvement of functional equations in various characterizations of inner product spaces.

Acknowledgments

The authors would like to thank the referees for some valuable comments and useful suggestions.

Conflict of interest

The authors declare that they have no competing interests.

References

1. C. Alsina, J. Sikorska, M. Santos Tomás, *Norm derivatives and characterizations of inner product spaces*, World Scientific, Hackensack, 2010.
2. D. Amir, *Characterizations of inner product spaces*, Birkhäuser, Basel, 1986.
3. T. Aoki, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan*, **2** (1950), 64–66. doi: 10.2969/jmsj/00210064.
4. J. H. Bae, B. Noori, M. B. Moghimi, A. Najati, Inner product spaces and quadratic functional equations, *Adv. Differ. Equ.*, (2021). doi: 10.1186/s13662-021-03307-x.
5. S. Czerwik, *Functional equations and inequalities in several variables*, World Scientific Publishing Co., Inc., River Edge, NJ, 2002.
6. G. L. Forti, Hyers-Ulam stability of functional equations in several variables, *Aequationes Math.*, **50** (1995), 143–190. doi: 10.1007/BF01831117.
7. M. Fréchet, Une définition fonctionelle des polynômes, *Nouv. Ann.*, **49** (1909), 145–162.
8. M. Fréchet, Sur la définition axiomatique d'une classe d'espaces vectoriels distanciés applicables vectoriellement sur l'espace de Hilbert, *Ann. Math.*, **36** (1935), 705–718. doi: 10.2307/1968652.
9. P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.*, **184** (1994), 431–436. doi: 10.1006/jmaa.1994.1211.
10. H. Gharib, M. B. Moghimi, A. Najati, J. H. Bae, Asymptotic stability of the Pexider-Cauchy functional equation in non-Archimedean spaces, *Mathematics*, **9** (2021), 2197. doi: 10.3390/math9182197.
11. D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci. USA*, **27** (1941), 222–224. doi: 10.1073/pnas.27.4.222.
12. D. H. Hyers, G. Isac, Th. M. Rassias, *Stability of functional equations in several variables*, Birkhäuser, Boston, 1998.
13. S. M. Jung, On the Hyers-Ulam stability of the functional equations that have the quadratic property, *J. Math. Anal. Appl.*, **222** (1998), 126–137. doi: 10.1006/jmaa.1998.5916.
14. S. M. Jung, Hyers-Ulam-Rassias stability of Jensen's equation and its application, *Proc. Am. Math. Soc.*, **126** (1998), 3137–3143. doi: 10.1090/S0002-9939-98-04680-2.
15. S. M. Jung, Local stability of the additive functional equation, *Glas. Mat. Ser. III*, **38** (2003), 45–55.
16. S. M. Jung, *Hyers-Ulam-Rassias stability of functional equations in nonlinear analysis*, Springer, New York, Dordrecht, Heidelberg, London, 2011.
17. Pl. Kannappan, *Functional equations and inequalities with applications*, Springer, New York, 2009.

18. Z. Kominek, On a local stability of the Jensen functional equation, *Demonstratio Math.*, **22** (1989), 499–507. doi: 10.1515/dema-1989-0220.

19. Q. Liu, S. Zhuang, Y. Li, Additive double ρ-functional inequalities in β-homogeneous F-spaces, *J. Math. Inequal.*, **15** (2021), 605–613. doi: 10.7153/jmi-2021-15-44.

20. L. Losonczi, On the stability of Hosszús' functional equation, *Results Math.*, **29** (1996), 305–310. doi: 10.1007/BF03322226.

21. M. B. Moghimi, A. Najati, C. Park, A functional inequality in restricted domains of Banach modules, *Adv. Differ. Equ.*, (2009). doi: 10.1155/2009/973709.

22. D. Molaei, A. Najati, Hyperstability of the general linear equation on restricted domains, *Acta Math. Hungar.*, **149** (2016), 238–253. doi: 10.1007/s10474-016-0609-y.

23. A. Najati, S. M. Jung, Approximately quadratic mappings on restricted domains, *J. Inequal. Appl.*, (2010). doi: 10.1155/2010/503458.

24. A. Najati, Th. M. Rassias, Stability of the Pexiderized Cauchy and Jensen's equations on restricted domains, *Commun. Math. Anal.*, **8** (2010), 125–135.

25. A. Najati, G. Zamani Eskandani, A fixed point method to the generalized stability of a mixed additive and quadratic functional equation in Banach modules, *J. Differ. Equ. Appl.*, **16** (2010), 773–788. doi: 10.1080/10236190802448609.

26. C. Park, B. Noori, M. B. Moghimi, A. Najati, J. M. Rassias, Local stability of mappings on multi-normed spaces, *Adv. Differ. Equ.*, (2020). doi: 10.1186/s13662-020-02858-9.

27. C. Park, K. Tamilvanan, B. Noori, M. B. Moghimi, A. Najati, Fuzzy normed spaces and stability of a generalized quadratic functional equation, *AIMS Math.*, **5** (2020), 7161–7174. doi: 10.3934/math.2020458.

28. J. Senasukh, S. Saejung, On the hyperstability of the Drygas functional equation on a restricted domain, *Bull. Aust. Math. Soc.*, **102** (2020), 126–137. doi: 10.1017/S0004972719001096.

29. Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Am. Math. Soc.*, **72** (1978), 297–300. doi: 10.1090/S0002-9939-1978-0507327-1.

30. J. M. Rassias, On the Ulam stability of mixed type mappings on restricted domains, *J. Math. Anal. Appl.*, **276** (2002), 747–762. doi: 10.1016/S0022-247X(02)00439-0.

31. J. M. Rassias, M. J. Rassias, On the Ulam stability of Jensen and Jensen type mappings on restricted domains, *J. Math. Anal. Appl.*, **281** (2003), 516–524. doi: 10.1016/S0022-247X(03)00136-7.

32. F. Skof, Proprietá locali e approssimazione di operatori, *Rend. Sem. Mat. Fis. Milano*, **53** (1983), 113–129. doi: 10.1007/BF02924890.

33. S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1960.

AIMS Press