Research Article

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Numerical exploration of thin film flow of MHD pseudo-plastic fluid in fractional space: Utilization of fractional calculus approach

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Abstract: In this article, thin film flow of non-Newtonian pseudo-plastic fluid is investigated on a vertical wall through homotopy-based scheme along with fractional calculus. Three cases were examined after considering (i) partial fractional differential equation (PFDE) by altering first-order derivative to fractional derivative in the interval (0, 1), (ii) PFDE by altering second-order derivative to fractional derivative in the interval (1, 2), and (iii) fully FDE by altering first-order derivative to fractional derivative in (0, 1) and second-order derivative to fractional derivative in (1, 2). Different physical quantities such as the velocity profile and volume flux were computed and analyzed. Validity of obtained results was checked by finding residuals. Moreover, consequence of different parameters on the velocity were also explored in fractional space.

Keywords: magneto hydro dynamic, homotopy perturbation method, fractional differential equation, pseudo-plastic fluid

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1 Introduction

Thin film stream can be seen in various normally happening wonders like development of downpour beads down a window sheet, eye tears, and magma streams. Modern and designing applications are found in oil refining measures, chip production, atomic reactors, development and common works, water system, laser cutting, painting, among others [1–4]. Since stream conditions can be essentially influenced by various naturally visible dangers, critical measure of hypothetical and exploratory work has been performed to comprehend the crucial stream qualities. Aman *et al.* [5] investigating the natural convection flow of fluids using the Atangana–Baleanu fractional model developed a modified fractional model for magneto hydrodynamic (MHD) fluid flow in which they used the Atangana–Baleanu fractional derivative (ABFD). The partial differential equations are translated into ordinary differential equations (ODEs). The Laplace transformation method is used for its inversion to complete the exact solutions of the momentum and heat equations. It is found that the normal fluid flows faster compared to the fractional fluid. Moreover, Al-Mdallal *et al.* [6] defined a new approach of Caputo-Fabrizio fractional derivative (CFFD) for use in the mathematical formation of the problem of heat and mass transfer with MHD flow over a vertical oscillating plate embedded in a porous medium. The problem is solved analytically and solutions of mass concentration, temperature distribution, and velocity field are found in the presence and absence of porous and magnetic field influences. In addition, Aman *et al.* [7] investigated the flow of Maxwell's fractional nano-gluten with a second-order slip. The numerical technique of the Laplace Stehfest translation algorithm was applied. Various parameters are investigated and effected. Aman and Al-Mdallal [8] which is investigated in deriving a non-integer order, Maxwell's fractional ferrofluid flow is examined with a second-order slip. The effect of volume fraction on flow behavior is thoroughly analyzed and discussed. In addition, Aman *et al.* [9] investigated the effect of second-order slip on MHD flow of Maxwell's fractional fluid on a moving plate together with a comparison of two numerical algorithms. The influences of a fractional parameter, the magnetic force, the porosity parameter, and the slip parameters are analyzed and discussed. Also, Aman *et al.* [10] used a new Caputo time fractional model to improve the heat transfer of water-based graphene nanofluid with an emphasis on its application to solar energy. It is found that the heat transfer rate increases as the volume fraction of the nanoparticles and fractional time parameters of Caputo increase. This body of work is briefly outlined in the remainder of this article.

Beginning work on the issue was completed based on Newtonian liquids [3], where the terms of speed increase were precluded and the subsequent interaction brought about a harmony among thick and gravitational powers. The methodology was legitimate for enormous time ranges, nonlinear investigation needed in non-Newtonian liquids like cements, gels, liquid plastics, greases containing polymer added substances, blood, and staples like nectar, pot was deficient [11]. Thus, after tackling the thin film flow problem for Phan-Thien-Tanner fluid and third-grade fluid, over a tilted plane [12,13], Siddiqui *et al.* [14] also tackled the problem by using fourth-grade fluids on vertical cylinders. Alam *et al.* [15] studied the thin-film flow of MHD pseudo-plastic fluid on a vertical wall. Bazighifan and Ramos [16] studied the asymptotic analysis by engaging generalized Riccati transformations. They derived the criterion for oscillatory behavior of the differential system solutions. Imran and Abdel-Salam [17] analyzed the comportment of slip on peristaltic transport. In ref. [18], comportment of chemical reactions is reported. Studies mentioned in ref. [19] contain the inspirations of Hall effects. In ref. [20], authors studied the transport mechanism under peristaltic phenomenon in a flexible channel. In ref. [21], utilization of Darcy's law is securitized. The studies reported in ref. [22] contain the theoretical analysis for Sutterby fluid model in ref. [23]. The author proposed a powerful tool to obtain the solutions of linear differential systems. In ref. [24], authors discussed the solution of nonlinear model of propagating waves. In ref. [25], modification in perturbation approach is discussed in detail. In ref. [26], boundary value problem is solved. Yildirim [27] used the perturbation method for fourth-order equations. Applications of HPM on squeezing flow was studied by Qayyum *et al.* [28]. For flow types, Yih applied an initial analysis of laminar flows [29], while Landau [30] and Stuart [31] extended the analysis to turbulent flows. Nakaya [32] and Lin [33] performed a stability analysis taking into account the surface tension. Ahmad *et al.* [34] performed the comparative study for fractional Jeffery's model by considering natural convection and heat source. In another survey, Ahmad *et al.* [35] modeled the fractional hybrid nanofluid model based on viscoelastic nature of Maxwell fluid. Imran *et al.* [36] studied the characteristics of thermal and mass transportation in second-grade fluid. Ahmad *et al.* [37] studied the MHD viscous incompressible flow model with thermal transport in fractional sense. Sohail *et al.* [38] studied the nonlinear couple stress model with variable properties using optimal procedure. The dynamics of non-Newtonian Oldroyd-B model over a paraboloid surface with thermal transport were examined by Ali *et al.* [39]. Ali *et al.* [40] focused on fractional Casson fluid model with magnetic effect in an axisymmetric cylinder. They solved the modeled equations using Hankel transform coupled with Laplace procedure. Some important contributions on fractional models are covered in refs. [41–53] and studies reported therein.

In this contemplation, we stretch out to partial space crafted by ref. [9] by addressing the issue as various kinds of fractional differential equations (FDEs), and got an answer utilizing a half and half methodology of blending fragmentary analytics with homotopy perturbation method (HPM) which was at first proposed by He [23,24]. The HPM consolidates both old style homotopy with bother procedures, and has been effectively applied to address numerous straight [23] and non-direct issues [24–28]. The FDEs are speculations of customary differential conditions to non-whole number self-assertive request. The use of FDEs permit us to demonstrate and notice more mind boggling spatial and transient marvels concerning the liquid stream by considering non-neighborhood relations.

As far as author could possibly know, the examination of the issue in partial space has not yet been performed. In our future attempts, we will consider the association of on the asymptotic and oscillatory conduct of the arrangements of a class of defer differential conditions on consider circumstance and we will screen their belongings.

2 Basic definitions of fractional calculus

FDEs have been the focal point of many examinations in material science, science, designing, signal preparing, control hypothesis, and money in view of its capacity to catch complex nonlinear wonders that are not normally acknowledged by ordinary differential conditions. Before we continue on to different areas portraying a slender film stream model in partial space, we present a couple of essential definitions and properties of fragmentary math that will be utilized in later segments.

2.1 Definition

A real function $h(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in R$, if there exists a real number $p > \mu$ such that $h(t) = t^p h_1(t)$, where $h_1(t) \in C(0, \infty)$, and it is said to be in the space C_μ^n if and only if $h^n \in C_\mu$, $n \in N$.

2.2 Definition

The fractional derivative (D^α) of $h(t)$, in Caputo sense is defined as

$$D^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} h^n(\tau) d\tau , \tag{1}$$

for $n-1 < \alpha < n$, $n \in N$, $t > 0$, $h \in C_{-1}^n$.

The following are two basic properties of the Caputo fractional derivative [10].

(2.2.1) Let $h \in C_{-1}^n$, $n \in N$. Then, $D^\alpha h$, $0 \le \alpha \le n$ is well defined and $D^\alpha h \in C_{-1}$.

(2.2.2) Let $n-1 \le \alpha \le n, n \in N$ and $h \in C_\mu^n, \mu \ge -1$. Then,

$$(J^\alpha D^\alpha) h(t) = h(t) - \sum_{k=0}^{n-1} h^k(0^+) \frac{t^k}{k!}.$$

3 Mathematical formulation

Consider an instance of steady, laminar, and uniform electrically conductive thin film streaming down a boundless vertical divider. The film thickness is accepted as δ and gravity acts along the descending way. The essential conditions administering the movement of isothermal, homogeneous, and uncompressed liquids are [14]

$$\operatorname{div} \mathbf{V} = 0, \tag{2}$$

$$\rho \frac{D\boldsymbol{V}}{Dt} = \rho \boldsymbol{f} - \operatorname{grad} P + \operatorname{div} \boldsymbol{S}, , \tag{3}$$

where $\boldsymbol{V}$ is the speed vector, ρ is the consistent thickness, $\boldsymbol{f}$ is the body power per unit mass, P indicates the unique pressing factor, and $\frac{D}{Dt}$ signifies the material subsidiary.

$$\boldsymbol{S} + \lambda_1 \boldsymbol{S}^\nabla + \frac{1}{2}(\lambda_1 - \mu_1)(\boldsymbol{A}_1 \boldsymbol{S} + \boldsymbol{S}\boldsymbol{A}_1) = \eta_0 \boldsymbol{A}_1, \tag{4}$$

where η_0 is the zero shear consistency, λ_1 is the unwinding time, μ_1 is the material steady, and

$$\boldsymbol{A}_1 = (\operatorname{grad} \ \boldsymbol{V}) + (\operatorname{grad} \ \boldsymbol{V})^T, \tag{5}$$

and

$$\boldsymbol{S}^\nabla = \frac{D\boldsymbol{S}}{Dt} - \{(\operatorname{grad} \boldsymbol{V})^T \boldsymbol{S} + \boldsymbol{S}(\operatorname{grad} \ \boldsymbol{V})\}. \tag{6}$$

The boundary conditions are

$$\text{at} \quad x = 0, \quad w = 0. \tag{7}$$

$$\text{at} \quad x = \delta, \quad S_{xz} = 0. \tag{8}$$

The speed field with attractive impact for the expressed issue characterizes as

$$\boldsymbol{V} = (0, \ 0, \ -\sigma B_0{}^2 w(x)), \tag{9}$$

and extra stress tensor as

$$\boldsymbol{S} = \boldsymbol{S}(x). \tag{10}$$

By inserting equation (9) in equations (2) and (10), the continuity equation (2) is identically satisfied and equation (3) takes the following form

$$0 = \frac{dS_{xx}}{dx}, \tag{11}$$

$$0 = \frac{dS_{zx}}{dx} - \rho g - \sigma B_0{}^2 w(x), \tag{12}$$

where

$$S_{xx} = \frac{-\eta_0(\lambda_1 - \mu_1)\left(\frac{dw}{dx}\right)^2}{1 + (\lambda_1{}^2 - \mu_1{}^2)\left(\frac{dw}{dx}\right)^2},$$

$$S_{zz} = \frac{\eta_0(\lambda_1 + \mu_1)\left(\frac{dw}{dx}\right)^2}{1 + (\lambda_1{}^2 - \mu_1{}^2)\left(\frac{dw}{dx}\right)^2}, \tag{13}$$

$$S_{zx} = \frac{\eta_0 \frac{dw}{dx}}{1 + (\lambda_1{}^2 - \mu_1{}^2)\left(\frac{dw}{dx}\right)^2}.$$

Substituting S_{zx} in equation (12) and utilizing $w^* = \frac{w}{U_0}$ and $x^* = \frac{x}{\delta}$, we get the following three sets of problems

$$\frac{d^2 w(x)}{dx^2} - \beta\left(\frac{dw(x)}{dx}\right)^2 \frac{d^2 w(x)}{dx^2} - \beta^2 S_t \left(\frac{dw(x)}{dx}\right)^4 - 2\beta S_t \left(\frac{dw(x)}{dx}\right)^2 - M^2 \beta^2 w(x) \left(\frac{dw(x)}{dx}\right)^4 - 2\beta M^2 w(x) \left(\frac{dw(x)}{dx}\right)^2 - M^2 w(x) = S_t, \tag{14}$$

with

$$\frac{dw}{dx} = 0 \quad \text{at } x = 1 \quad \text{and} \quad w = 0 \quad \text{at } x = 0, \tag{15}$$

where non-Newtonian parameter is $\beta = \frac{(\lambda_1{}^2 - \mu_1{}^2)U_0{}^2}{\delta^2}$ and stokes number is $S_t = \frac{\rho g \delta^2}{\mu_{\text{eff}} U_0}$, while MHD parameter is $M^2 = \frac{\sigma B_0{}^2 \delta}{\mu_{\text{eff}}}$.

Equation (14) subject to the conditions (15) is highly nonlinear BVP. After using basic definitions of fractional calculus following three cases of FDEs along with boundary conditions are obtained from equations (14) and (15).

3.1 Case I

$$\begin{aligned} &\frac{d^2w(x)}{dx^2} - \beta(D^\alpha w(x))^2 \frac{d^2w(x)}{dx^2} - \beta^2 S_t (D^\alpha w(x))^4 \\ &- 2\beta S_t 2(D^\alpha w(x))^2 - M^2\beta^2 w(x)(D^\alpha w(x))^4 \\ &- 2\beta M^2 w(x)(D^\alpha w(x))^2 - M^2 w(x) = S_t, \end{aligned} \tag{16}$$

$$w(0) = 0, \; w'(1) = 0, \quad 0 < \alpha < 1. \tag{17}$$

3.2 Case II

$$\begin{aligned} &D^\gamma w(x) - \beta\left(\frac{dw(x)}{dx}\right)^2 D^\gamma w(x) - \beta^2 S_t \left(\frac{dw(x)}{dx}\right)^4 \\ &- 2\beta S_t 2\left(\frac{dw(x)}{dx}\right)^2 - M^2\beta^2 w(x)\left(\frac{dw(x)}{dx}\right)^4 \\ &- 2\beta M^2 w(x)\left(\frac{dw(x)}{dx}\right)^2 - M^2 w(x) = S_t, \end{aligned} \tag{18}$$

$$w(0) = 0, \quad w'(1) = 0, \quad 1 < \gamma < 2. \tag{19}$$

3.3 Case III

$$\begin{aligned} &D^\gamma w(x) - \beta(D^\alpha w(x))^2 D^\gamma w(x) - \beta^2 S_t (D^\alpha w(x))^4 \\ &- 2\beta S_t 2(D^\alpha w(x))^2 - M^2\beta^2 w(x)(D^\alpha w(x))^4 \\ &- 2\beta M^2 w(x)(D^\alpha w(x))^2 - M^2 w(x) = S_t, \end{aligned} \tag{20}$$

$$w(0) = 0, \quad w'(1) = 0, \quad 0 < \alpha < 1, \quad 1 < \gamma < 2. \tag{21}$$

4 Application of HPM to Case I

Let us define homotopy for equation (16) as follows:

$$\begin{aligned} (1-p)\frac{d^2w}{dx^2} + p\Bigg[&\frac{d^2w}{dx^2} - \beta(D^\alpha)^2 \frac{d^2w}{dx^2} \\ &- \beta^2 S_t (D^\alpha)^4 - 2\beta S_t (D^\alpha)^2 - M^2\beta^2 w(x)(D^\alpha)^4 \\ &- 2\beta M^2 w(x)(D^\alpha)^2 - M^2 w(x) - S_t\Bigg] = 0. \end{aligned} \tag{22}$$

4.1 Zeroth-order problem

Various order problems using (20) and (21) are

$$w'_0(1) = 0, \quad w''_0(x) = 0, \quad w_0(0) = 0. \tag{23}$$

4.2 First-order problem

$$\begin{aligned} &-S_t - M^2 w_0(x) - 2S_t\beta(D^\alpha w_0(x))^2 - 2M^2\beta(D^\alpha w_0(x))^3 \\ &- S_t\beta^2(D^\alpha w_0(x))^4 - M^2\beta^2(D^\alpha w_0(x))^5 - \beta(D^\alpha w_0(x))^2 w''_0(x) \\ &+ w''_0(x) = 0, w_1(0) = 0, \quad w'_0(1) = 0. \end{aligned} \tag{24}$$

4.3 Second-order problem

$$\begin{aligned} &-M^2 w_1(x) - 4S_t\beta(D^\alpha w_0(x))^2 w_1(x) \\ &- 6M^2\beta(D^\alpha w_0(x))^2 w_1(x) - 4S_t\beta^2(D^\alpha w_0(x))^4 w_1(x) \\ &- 5(D^\alpha w_0(x))^4 M^2\beta^2 w_1(x) - 2\beta(D^\alpha w_0(x))^2 w_1(x) w''_0(x) \\ &- (D^\alpha w_0(x))^2 w''_1(x) + w''_2(x) = 0, \\ &w_2(0) = 0, \quad w'_2(1) = 0. \end{aligned} \tag{25}$$

4.4 Third-order problem

$$\begin{aligned} &-2S_t\beta(D^\alpha w_1(x))^2 - 6\beta(D^\alpha w_1(x))^2 M^2 w_0(x) \\ &- 6S_t\beta^2(D^\alpha w_0(x))^4 w_1(x) - 10M^2\beta^2(D^\alpha w_0(x))^4(w_1(x))^2 \\ &- M^2 w_2(x) - 4S_t\beta(D^\alpha w_0(x))^2 S\beta w_2(x) \\ &- 6M^2\beta(D^\alpha w_0(x))^2 w_2(x) - 4S_t\beta^2(D^\alpha w_0(x))^4 w_2(x) \\ &- 5M^2\beta^2(D^\alpha w_0(x))^4 w_2(x) - (D^\alpha w_0(x))^2\beta w''_0(x) \\ &- 2(D^\alpha w_2(x))^2\beta w_0(x) w''_0(x) - 2\beta(D^\alpha w_0(x))^2 w_1(x) w''_1(x) \\ &- \beta(D^\alpha w_0(x))^2 w''_2(x) + w''_3(x) = 0, \\ &w_3(0) = 0, \quad w'_3(1) = 0. \end{aligned} \tag{26}$$

4.5 Fourth-order problem

$$\begin{aligned}&-2M^2\beta(D^\alpha w_1(x))^2-4S_t\beta^2(D^\alpha w_1(x))^4w_0(x)\\&-10M^2\beta^2(D^\alpha w_1(x))^4w_0(x)-4S_t\beta(D^\alpha w_1(x))^2w_2(x)\\&-12M^2\beta(D^\alpha w_0(x))^2w_1(x)w_2(x)\\&-12S_t\beta^2(D^\alpha w_0(x))^4w_1(x)w_2(x)\\&-20M^2\beta^2(D^\alpha w_0(x))^4w_1(x)w_2(x)-M^2w_3(x)\\&-4S_t\beta(D^\alpha w_0(x))^2w_3(x)-6M^2\beta(D^\alpha w_0(x))^2M^2\beta w_3(x)\\&-4S_t\beta^2(D^\alpha w_0(x))^4w_3(x)-5M^2\beta^2(D^\alpha w_0(x))^4w_3(x)\\&-2\beta(D^\alpha w_2(x))^2w_1(x)w_0''(x)-2\beta(D^\alpha w_3(x))^2w_0(x)w_0''(x)\\&-(D^\alpha w_1(x))^2w_1''(x)-2\beta(D^\alpha w_0(x))^2w_2(x)w_1''(x)\\&-2\beta(D^\alpha w_0(x))^2w_1(x)w_2''(x)\\&-\beta(D^\alpha w_0(x))^2w_3''(x)+w_4''(x)=0,\\&w_4(0)=0,\ w_4'(1)=0.\end{aligned}\tag{27}$$

Fixing $\beta = 0.1$, $\alpha = 0.99$, $S_t = 0.1$ and $M = 0.1$, fourth order solution is

$$\begin{aligned}W(x)&=½(-0.2x+0.1x^2)+\frac{1}{24}(0.008x\\&-0.004x^3+0.001x^4)\\&+\left(\frac{1}{x^{1.98}}\right)0.000110654(-0.605635x^{2.98}\\&+0.887264x^4+0.00502064x^{4.98}\\&-0.59341x^5+0.147614x^6\\&-0.000753097x^{6.98}+0.000125516x^{7.98})\\&-\left(\frac{1}{x^{1.98}}\right)4.34454\times10^{-8}(737.514x^{2.98}\\&-1114.85x^4+2.5709x^{4.98}+758.295x^5\\&-203.137x^6-0.00639373x^{6.98}\\&+8.30092x^7-1.37889x^8\\&+0.1000456695x^{8.98}-0.0000570869\,x^{9.98}).\end{aligned}\tag{28}$$

The residual of the problem is

$$\begin{aligned}\text{Residula1}=&\frac{d^2W(x)}{dx^2}-\beta(D^\alpha W(x))^2\frac{d^2W(x)}{dx^2}\\&-\beta^2S_t(D^\alpha W(x))^4-2\beta S_t2(D^\alpha W(x))^2\\&-M^2\beta^2W(x)(D^\alpha W(x))^4-2\beta M^2W(x)(D^\alpha W(x))^2\\&-M^2W(x)-S_t.\end{aligned}\tag{29}$$

5 Flow rate and average velocity

Flow rate (FR) per unit width is

$$\text{FR}=\int_0^1 w(x)dx,\tag{30}$$

$$\begin{aligned}\text{FR}=\;&S_t(945S^2(-9+2\alpha)(6(-3+\alpha)^2\\&\times(88-207\alpha+176\alpha^2-64\alpha^3+8\alpha^4)\\&+M^2(1{,}188-2{,}574\alpha+2{,}401\alpha^2-1{,}206\alpha^3\\&+347\alpha^4-56\alpha^5+4\alpha^6))\beta[5-\alpha]^2\\&-2\Gamma[4-\alpha]^2(945S^2(12-7\alpha+\alpha^2)\\&\times(3(6{,}048-17{,}184\alpha+21{,}034\alpha^2-14{,}107\alpha^3\\&+5{,}537\alpha^4-1{,}266\alpha^5+156\alpha^6-8\alpha^7)\\&+2M^2(-29{,}484+87{,}369\alpha-109{,}265\alpha^2\\&+73{,}862\alpha^3-28{,}941\alpha^4+6{,}558\alpha^5-796\alpha^6\\&+40\alpha^7))\beta-2(-945+378M^2-153M^4+62M^6)\\&\times(945-1{,}488\alpha+824\alpha^2-192\alpha^3\\&+16\alpha^4)\Gamma[5-\alpha]^2)))/(11,340(-9+2\alpha)\\&\times(-105+142\alpha-60\alpha^2+8\alpha^3)\Gamma[4-\alpha]^2\Gamma[5-\alpha]^2\end{aligned}\tag{31}$$

The average velocity $\bar{V}$ is

$$\bar{V}=Q.\tag{32}$$

6 Application of HPM to Case II

Let us define homotopy for equation (18) as follows:

$$\begin{aligned}&(1-p)(D^\gamma w(x))+p[D^\gamma w(x)-\beta\left(\frac{dw(x)}{dx}\right)^2D^\gamma w(x)\\&-\beta^2S_t\left(\frac{dw(x)}{dx}\right)^4-2\beta S_t2\left(\frac{dw(x)}{dx}\right)^2-M^2\beta^2w(x)\left(\frac{dw(x)}{dx}\right)^4\\&-2\beta M^2w(x)\left(\frac{dw(x)}{dx}\right)^2-M^2w(x)-S_t=0.\end{aligned}\tag{33}$$

6.1 Zeroth-order problem

Various order problems using (18) and (19) are

$$D^\gamma w_0(x)=0,\ w_0(0)=0,\ w_0'(1)=0.\tag{34}$$

6.2 First-order problem

$$\begin{aligned}&-S_t+D^\gamma(w_1(x))-M^2w_0(x)-2S_t\beta(w_0'(x))^2\\&-\beta(D^\gamma(w_0(x)))(w_0'(x))^2-2M^2\beta(w_0(x))(w_0'(x))^2\\&-S_t\beta^2(w_0'(x))^4-M^2\beta^2(w_0(x))(w_0'(x))^4=0,\\&w_1(0)=0,\ w_1'(1)=0.\end{aligned}\tag{35}$$

6.3 Second-order problem

$$\begin{aligned} &D^{\gamma}(w_2(x)) - M^2w_1(x) - \beta D^{\gamma}(w_1(x))(w_0'(x))^2 \\ &- 2M^2\beta w_1(x)(w_0'(x))^2 - M^2\beta^2 w_1(x)(w_0'(x))^4 \\ &- 4S_t\beta(w_0'(x))(w_1'(x)) - 2\beta D^{\gamma}(w_0(x))(w_0'(x))(w_1'(x)) \\ &- 4M^2\beta w_0(x)(w_0'(x))(w_1'(x)) - 4S_t\beta^2(w_0'(x))^3(w_1'(x)) \\ &- 4M^2\beta^2 w_0(x)(w_0'(x))^3(w_1'(x)) = 0, \\ &w_2(0) = 0, w_2'(1) = 0. \end{aligned} \tag{36}$$

6.4 Third-order problem

$$\begin{aligned} &D^{\gamma}(w_3(x)) - M^2w_2(x) - \beta D^{\gamma}(w_2(x))(w_0'(x))^2 \\ &- 2M^2\beta(w_2(x))(w_0'(x))^2 - M^2\beta^2(w_2(x))(w_0'(x))^4 \\ &- 2\beta D^{\gamma}(w_1(x))(w_0'(x))(w_1'(x)) - 4M^2\beta(w_1(x))(w_0'(x))(w_1'(x)) \\ &- M^2\beta^2(w_2(x))(w_0'(x))^4 - 2\beta D^{\gamma}(w_1(x))(w_0'(x))(w_1'(x)) \\ &- 4M^2\beta(w_1(x))(w_0'(x))(w_1'(x)) - 2M^2\beta(w_0'(x))(w_0'(x))^2 \\ &- 6S_t\beta^2(w_0'(x))^2(w_1'(x))^2 - 6M^2\beta^2(w_0'(x))(w_0'(x))^2(w_1'(x))^2 \\ &- 4S_t\beta(w_0'(x))(w_2'(x)) - 2\beta D^{\gamma}(w_0(x))(w_0'(x))(w_2'(x)) \\ &- 4M^2\beta(w_0(x))(w_0'(x))(w_2'(x)) - 4S_t\beta^2(w_0'(x))^3(w_2'(x)) \\ &- 4M^2\beta^2(w_0'(x))(w_0'(x))^3(w_2'(x)) = 0, \\ &w_3(0) = 0, w_3'(1) = 0. \end{aligned} \tag{37}$$

6.5 Fourth-order problem

$$\begin{aligned} &-M^2w_3(x) + D^{\gamma}(w_4(x)) - \beta D^{\gamma}(w_3(x))(w_0'(x))^2 \\ &- 2M^2\beta(w_3(x))(w_0'(x))^2 - M^2\beta^2(w_3(x))(w_0'(x))^4 \\ &- 2\beta D^{\gamma}(w_2(x))(w_0'(x))(w_1'(x)) - 4M^2\beta(w_0'(x))(w_1'(x)) \\ &- 4M^2\beta^2(w_2(x))(w_0'(x))^3(w_1'(x)) \\ &- \beta D^{\gamma}(w_1(x))(w_1'(x))^2 - 2M^2\beta(w_1(x))(w_1'(x))^2 \\ &- 6M^2\beta^2(w_1(x))(w_0'(x))^2(w_1'(x))^2 - 4S_t\beta^2(w_0'(x))(w_1'(x))^3 \\ &- 4M^2\beta^2(w_0(x))(w_0'(x))(w_1'(x))^3 - 2\beta D^{\gamma}(w_1(x))(w_0'(x))(w_2'(x)) \\ &- 4M^2\beta(w_1(x))(w_0'(x))(w_2'(x)) - 4M^2\beta^2(w_1(x))(w_0'(x))^3(w_2'(x)) \\ &- 4S_t\beta(w_1'(x))(w_2'(x)) - 2\beta D^{\gamma}(w_0(x))(w_1'(x))(w_2'(x)) \\ &- 4M^2\beta(w_0(x))(w_1'(x))(w_2'(x)) - 12S_t\beta^2(w_0'(x))^2(w_1'(x))(w_2'(x)) \\ &- 12M^2\beta^2(w_0(x))(w_0'(x))^2(w_1'(x))(w_2'(x)) - 4S_t\beta(w_0'(x))(w_3'(x)) \\ &- 2\beta D^{\gamma}(w_0(x))(w_0'(x))(w_3'(x)) - 4M^2\beta(w_0(x))(w_0'(x))(w_3'(x)) \\ &- 4S_t\beta^2(w_0'(x))^3(w_3'(x)) - 4M^2\beta^2(w_0(x))(w_0'(x))^3(w_3'(x)) = 0, \\ &w_4(0) = 0, w_4'(1) = 0. \end{aligned} \tag{38}$$

Fixing $\beta = 0.2$, $\gamma = 1.99$, $S_t = 0.01$ and $M = 0$. 1 fourth order solution is

$$\begin{aligned} W(x) &= -0.0100002x + 0.00504696x^{1.99} \\ &+ 0.0050005x^2 - 1.36726 \times 10^{-7}x^{2.98} \\ &- 0.0000178199x^{2.99} - 4.00144 \times 10^{-7}x^3 \\ &+ 4.36363 \times 10^{-6}x^{3.98} + 4.57125 \times 10^{-6}x^{3.99} \\ &+ 1.00136 \times 10^{-7}x^4 - 9.79591 \times 10^{-9}x^{4.98} \\ &- 3.16703 \times 10^{-9}x^{4.99} - 6 \times 10^{-11}x^5 + 1.70948 \\ &\times 10^{-9}x^{5.98} + 5.28971 \times 10^{-10}x^{5.99} \\ &+ 1 \times 10^{-11}x^6 - 1.36242 \times 10^{-12}x^{6.98} \\ &- 1.08198 \times 10^{-13}x^{6.99} + 1.71212 \times 10^{-13}x^{7.98} \\ &+ 1.35449 \times 10^{-14}x^{7.99} - 3.95961 \times 10^{-17}x^{8.98} \\ &- 1.06448 \times 10^{-18}x^{8.99} + 3.9692 \times 10^{-18}x^{9.98} \\ &+ 1.06555 \times 10^{-19}x^{9.99} - 4.52311 \times 10^{-22}x^{10.98} \\ &+ 3.7764 \times 10^{-23}x^{11.98} - 1.95805 \times 10^{-27}x^{12.98} \\ &+ 1.40061 \times 10^{-28}x^{13.98}. \end{aligned} \tag{39}$$

The residual of the problem is

$$\begin{aligned} \text{Residual2} = {} & D^{\gamma}W(x) - \beta\left(\frac{\mathrm{d}W(x)}{\mathrm{d}x}\right)^2 D^{\gamma}W(x) \\ & - \beta^2 S_t\left(\frac{W(x)}{dx}\right)^4 - 2\beta S_t 2\left(\frac{\mathrm{d}W(x)}{\mathrm{d}x}\right)^2 \\ & - M^2\beta^2 w(x)\left(\frac{\mathrm{d}W(x)}{\mathrm{d}x}\right)^4 - 2\beta M^2 W(x)\left(\frac{\mathrm{d}W(x)}{\mathrm{d}x}\right)^2 \\ & - M^2W_1(x) - S_t. \end{aligned} \tag{40}$$

7 Application of HPM to Case III

Let us define homotopy for equation (21) as follows:

$$\begin{aligned} &(1-p)(D^{\gamma}w(x)) + p[D^{\gamma}w(x) - \beta(D^{\alpha}w(x))^2 D^{\gamma}w(x) \\ &- \beta^2 S_t(D^{\alpha}w(x))^4 - 2\beta S_t 2(D^{\alpha}w(x))^2 \\ &- M^2\beta^2 w(x)(D^{\alpha}w(x))^4 - 2\beta M^2 w(x)(D^{\alpha}w(x))^2 \\ &- M^2 w(x) - S_t] = 0. \end{aligned} \tag{41}$$

7.1 Zeroth-order problem

Various order problems using (20) and (21) are

$$D^{\gamma}w_0(x) = 0, w_0(0) = 0, w_0'(1) = 0. \tag{42}$$

7.2 First-order problem

$$-S_t - 2S_t\beta(D^\alpha w_0(x))^2 - S_t\beta^2(D^\alpha w_0(x))^4 - \beta(D^\alpha w_0(x))^2(D^\gamma(w_0(x))) + D^\gamma(w_1(x)) - M^2(w_0(x)) - 2M^2\beta(D^\alpha w_0(x))^2(w_0(x)) - M^2\beta^2(D^\alpha w_0(x))^4(w_0(x)) = 0,\ w_1(0) = 0, w_1'(1) = 0. \quad (43)$$

7.3 Second-order problem

$$-4S_t - \beta(D^\alpha w_0(x))(D^\alpha w_1(x)) - 4S_t - \beta^2(D^\alpha w_0(x))^3(D^\alpha w_1(x)) - 2\beta(D^\alpha w_0(x))(D^\alpha w_1(x))(D^\gamma(w_0(x))) - \beta(D^\alpha w_0(x))^2(D^\gamma(w_1(x))) + (D^\gamma(w_2(x))) - 4M^2\beta(D^\alpha w_0(x))(D^\alpha w_1(x))(w_0(x)) - 4M^2\beta^2(D^\alpha w_0(x))^3(D^\alpha w_1(x))(w_0(x)) - M^2(w_1(x)) - 2M^2\beta(D^\alpha w_0(x))^2(w_1(x)) - M^2\beta^2(D^\alpha w_0(x))^4(w_1(x)) = 0,\ w_2(0) = 0, w_2'(1) = 0. \quad (44)$$

7.4 Third-order problem

$$-2S_t\beta(D^\alpha w_1(x))^2 - 6S_t\beta^2 D^\alpha w_0(x))^2(D^\alpha w_1(x))^2 - 4S_t\beta(D^\alpha w_0(x))(D^\alpha w_2(x)) - 4S_t\beta^2(D^\alpha w_0(x))^3(D^\alpha w_2(x)) - \beta(D^\alpha w_1(x))^2(D^\gamma(w_0(x))) - 2\beta(D^\alpha w_0(x))(D^\alpha w_2(x))(D^\gamma(w_0(x))) - 2\beta(D^\alpha w_0(x))(D^\alpha w_1(x))(D^\gamma(w_1(x))) - \beta(D^\alpha w_1(x))^2(D^\gamma(w_2(x))) + D^\gamma(w_3(x)) - 2M^2\beta(D^\alpha w_1(x))^2(w_0(x)) - 6M^2\beta^2(D^\alpha w_0(x))^2(D^\alpha w_1(x))^2(w_0(x)) - 4M^2\beta(D^\alpha w_0(x))(D^\alpha w_2(x))(w_0(x)) - 4M^2\beta^2(D^\alpha w_0(x))^3(D^\alpha w_2(x))(w_0(x)) - 4M^2\beta(D^\alpha w_0(x))(D^\alpha w_1(x))(w_1(x)) - 4M^2\beta^2(D^\alpha w_0(x))^3(D^\alpha w_1(x))(w_1(x)) - M^2(w_2(x)) - 2M^2\beta(D^\alpha w_0(x))^2(w_2(x)) - M^2\beta^2(D^\alpha w_0(x))^4(w_2(x)) = 0,\ w_3(0) = 0, w_3'(1) = 0. \quad (45)$$

7.5 Fourth-order problem

$$- 4S_t\beta^2(D^\alpha w_0(x))(D^\alpha w_1(x))^3 - 4S_t\beta(D^\alpha w_1(x))(D^\alpha w_2(x)) - 12S_t\beta^2(D^\alpha w_0(x))^2(D^\alpha w_1(x))(D^\alpha w_2(x)) - 4S_t\beta(D^\alpha w_0(x))(D^\alpha w_3(x)) - 4S_t\beta^2(D^\alpha w_0(x))^3(D^\alpha w_3(x)) - 2\beta(D^\alpha w_1(x))(D^\alpha w_2(x))(D^\gamma(w_0(x))) - 2\beta(D^\alpha w_0(x))(D^\alpha w_3(x))(D^\gamma(w_0(x))) - \beta(D^\alpha w_1(x))^2(D^\gamma(w_1(x))) - 2\beta(D^\alpha w_0(x))(D^\alpha w_2(x))(D^\gamma(w_1(x))) - 2\beta(D^\alpha w_0(x))(D^\alpha w_1(x))(D^\gamma(w_2(x))) - \beta(D^\alpha w_0(x))^2(D^\gamma(w_3(x))) + D^\gamma(w_4(x)) - 4M^2\beta^2(D^\alpha w_0(x))(D^\alpha w_1(x))^3(w_0(x))) - 4M^2\beta(D^\alpha w_1(x))(D^\alpha w_2(x))(w_0(x)) - 12M^2\beta^2(D^\alpha w_0(x))^2(D^\alpha w_1(x))(D^\alpha w_2(x))(w_0(x)) - 4M^2\beta(D^\alpha w_0(x))(D^\alpha w_3(x))(w_0(x)) - 4M^2\beta^2(D^\alpha w_0(x))^3(D^\alpha w_3(x))(w_0(x)) - 2M^2\beta(D^\alpha w_1(x))^2(w_1(x)) - 6M^2\beta^2(D^\alpha w_0(x))^2(D^\alpha w_1(x))^2(w_1(x)) - 4M^2\beta(D^\alpha w_0(x))(D^\alpha w_2(x))(w_1(x)) - 4M^2\beta^2(D^\alpha w_0(x))^3(D^\alpha w_2(x))(w_1(x)) - 4M^2\beta(D^\alpha w_0(x))(D^\alpha w_1(x))(w_2(x)) - 4M^2\beta^2(D^\alpha w_0(x))^3(D^\alpha w_1(x))(w_2(x)) - M^2(w_3(x)) - 2M^2\beta(D^\alpha w_0(x))^2(w_3(x)) - M^2\beta^2(D^\alpha w_0(x))^4(w_3(x)) = 0, \quad (46)$$

$w_4(0) = 0,\ w_4'(1) = 0.$

Fixing $\alpha = 0.99,\ \gamma = 1.99,\ \beta = 0.1,\ S_t = 0.01$ and $M = 0.1$ fourth order solution is

$$W(x) = 0.00504625\, x^{1.99} + 4.2939 \times 10^{-6}\, x^{3.98} + 0.01\,(1.46905 \times 10^{-6} + 1.69194 \times 10^{-6}/x^{1.98})\, x^{5.97}. \quad (47)$$

The residual of the problem is

$$\text{Residual3} = D^\gamma W(x) - \beta(D^\alpha W(x))^2 D^\gamma w(x) - \beta^2 S_t(D^\alpha W(x))^4 - 2\beta S_t 2(D^\alpha W(x))^2 - M^2\beta^2 W(x)(D^\alpha W(x))^4 - 2\beta M^2 W(x)(D^\alpha W(x))^2 - M^2 W(x) - S_t. \quad (48)$$

8 Results and discussion

In this article, homotopy-based fractional analysis of thin film flow of pseudo-plastic fluid on a vertical wall has been performed. The problem is solved as a partially fractional differential equation (PFDEs) and fully fractional differential equation (FFDE). Three cases are considered: (i) PFDE by altering first-order derivative to fractional derivative in $(0,1)$, (ii) PFDE by altering second-order derivative to fractional derivative in $(1,2)$, and (iii) FFDE by altering first- and second-order derivatives to fractional derivatives in $(0,1)$ and $(1,2)$, respectively. These cases are solved for different values of considered parameters and results are presented in tables and graphs. Tables 1–4

Table 1: Solutions and residuals for different α, while $\beta = 0.1$, $S_t = 0.01$, $M = 0.1$ in Case I

x	$\alpha = 0.4$		$\alpha = 0.8$		$\alpha = 0.99$	
	Sol.	Error	Sol.	Error	Sol.	Error
0.1	0.000946702	7.27×10^{-11}	0.000946704	3.38×10^{-10}	0.000946705	6.55×10^{-10}
0.2	0.0017935	1.60×10^{-10}	0.0017935	4.42×10^{-10}	0.0017935	6.38×10^{-10}
0.3	0.00254047	2.47×10^{-10}	0.00254047	4.85×10^{-10}	0.00254047	5.77×10^{-10}
0.4	0.0031877	3.22×10^{-10}	0.0031877	4.85×10^{-10}	0.0031877	4.90×10^{-10}
0.5	0.00373524	3.78×10^{-10}	0.00373524	4.51×10^{-10}	0.00373524	3.92×10^{-10}
0.6	0.00418316	4.12×10^{-10}	0.00418316	3.94×10^{-10}	0.0048315	2.95×10^{-10}
0.7	0.00453149	4.23×10^{-10}	0.00453149	3.26×10^{-10}	0.00453148	2.08×10^{-10}
0.8	0.00478027	4.12×10^{-10}	0.00478027	2.55×10^{-10}	0.00478027	1.39×10^{-10}
0.9	0.00492953	3.81×10^{-10}	0.00492953	1.89×10^{-10}	0.00492952	9.52×10^{-11}
1	0.00497928	3.36×10^{-10}	0.00497928	1.37×10^{-10}	0.00497928	7.81×10^{-11}

Table 2: Solutions and residuals for different β, while $\alpha = 0.99$, $S_t = 0.01$, $M = 0.3$ in Case I

x	$\beta = 0.3$		$\beta = 0.7$		$\beta = 1$	
	Sol.	Error	Sol.	Error	Sol.	Error
0.1	0.000918127	1.10×10^{-8}	0.000918158	3.28×10^{-8}	0.000918181	4.91062×10^{-8}
0.2	0.00173716	5.05×10^{-9}	0.00173721	2.58×10^{-8}	0.00173725	4.13872×10^{-8}
0.3	0.0024579	-1.68×10^{-9}	0.00245798	1.68×10^{-8}	0.00245803	3.06414×10^{-8}
0.4	0.00308111	-8.56×10^{-9}	0.00308119	6.91×10^{-9}	0.00308125	1.8503×10^{-8}
0.5	0.00360738	-1.51×10^{-8}	0.00360746	-2.90×10^{-9}	0.00360753	6.2414×10^{-9}
0.6	0.00403725	-2.09×10^{-8}	0.00403733	-1.19×10^{-8}	0.0040374	-5.13147×10^{-9}
0.7	0.00437114	-2.58×10^{-8}	0.00437123	-1.95×10^{-8}	0.0043713	-1.48221×10^{-8}
0.8	0.0046094	-2.95×10^{-8}	0.00460949	-2.53×10^{-8}	0.00460956	-2.22307×10^{-8}
0.9	0.00475226	-3.18×10^{-8}	0.00475235	-2.90×10^{-8}	0.00475242	-2.69346×10^{-8}
1	0.00479986	-3.26×10^{-8}	0.00479995	-3.03×10^{-8}	0.00480002	-2.86763×10^{-8}

Table 3: Solutions and residuals for different S_t, while $\alpha = 0.98$, $\beta = 0.2$, $M = 0.2$ in Case I

x	$S_t = 0.001$		$S_t = 0.01$		$S_t = 0.2$	
	Sol.	Error	Sol.	Error	Sol.	Error
0.1	0.0000936941	-8.93×10^{-12}	0.000936957	4.66×10^{-9}	0.0188698	-3.46×10^{-5}
0.2	0.000177425	-2.24×10^{-11}	0.00177428	4.45×10^{-9}	0.0357116	-1.02×10^{-5}
0.3	0.000251227	-3.56×10^{-11}	0.00251231	3.89×10^{-9}	0.0505386	5.11×10^{-6}
0.4	0.000315129	-4.80×10^{-11}	0.00315133	3.15×10^{-9}	0.063363	1.29×10^{-5}
0.5	0.000369157	-5.92×10^{-11}	0.00369161	2.33×10^{-9}	0.0741956	1.52×10^{-5}
0.6	0.000413332	-6.89×10^{-11}	0.00413337	1.52×10^{-9}	0.0830458	1.42×10^{-5}
0.7	0.000447672	-7.67×10^{-11}	0.00447677	8.11×10^{-10}	0.0899212	1.14×10^{-5}
0.8	0.000472191	-8.25×10^{-11}	0.00472196	2.45×10^{-10}	0.0948277	8.26×10^{-6}
0.9	0.000486898	-8.61×10^{-11}	0.00486903	-1.29×10^{-10}	0.0977698	5.81×10^{-6}
1	0.0004918	-8.73×10^{-11}	0.00491805	-2.84×10^{-10}	0.0987502	4.73×10^{-6}

Table 6: Solutions and residuals for different α, while $\gamma = 1.98$, $M = 0.1$, $\beta = 0.2$, $S_t = 0.001$ in Case III

x	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.99$	
	Sol.	**Error**	**Sol.**	**Error**	**Sol.**	**Error**
0.1	0.00000533279	-1.98011×10^{-3}	0.00000533279	-1.98011×10^{-3}	0.00000533279	-1.98011×10^{-3}
0.2	0.0000210381	-1.98042×10^{-3}	0.0000210381	-1.98042×10^{-3}	0.0000210381	-1.98042×10^{-3}
0.3	0.0000469554	-1.98093×10^{-3}	0.0000469554	-1.98093×10^{-3}	0.0000469554	-1.98093×10^{-3}
0.4	0.0000830025	-1.98164×10^{-3}	0.0000830025	-1.98164×10^{-3}	0.0000830025	-1.98164×10^{-3}
0.5	0.000129124	-1.98256×10^{-3}	0.000129124	-1.98256×10^{-3}	0.000129124	-1.98256×10^{-3}
0.6	0.00018528	-1.98367×10^{-3}	0.00018528	-1.98367×10^{-3}	0.00018528	-1.98367×10^{-3}
0.7	0.000251438	-1.98498×10^{-3}	0.000251438	-1.98498×10^{-3}	0.000251438	-1.98498×10^{-3}
0.8	0.000327576	-1.98649×10^{-3}	0.000327576	-1.98649×10^{-3}	0.000327576	-1.98649×10^{-3}
0.9	0.000413673	-1.98819×10^{-3}	0.000413673	-1.98819×10^{-3}	0.000413673	-1.98819×10^{-3}
1	0.000509716	-1.99009×10^{-3}	0.000509716	-1.99009×10^{-3}	0.000509716	-1.99009×10^{-3}

Table 5: Solutions and residuals for different γ, while $M = 0.1$, $\beta = 0.1$, $S_t = 0.001$ in Case II

x	$\gamma = 1.5$		$\gamma = 1.8$		$\gamma = 1.99$	
	Sol.	**Error**	**Sol.**	**Error**	**Sol.**	**Error**
0.1	0.0000712193	-1.01713×10^{-3}	0.0000855495	-1.04881×10^{-3}	0.0000898378	-1.08657×10^{-3}
0.2	0.000112754	-1.04078×10^{-3}	0.000147101	-1.09814×10^{-3}	0.0001595	-1.15489×10^{-3}
0.3	0.000131483	-1.06249×10^{-3}	0.000186758	-1.138×10^{-3}	0.000209073	-1.20404×10^{-3}
0.4	0.000129857	-1.07862×10^{-3}	0.000205487	-1.16532×10^{-3}	0.000238604	-1.23383×10^{-3}
0.5	0.000109286	-1.08678×10^{-3}	0.000203909	-1.17837×10^{-3}	0.000248125	-1.24412×10^{-3}
0.6	0.0000707202	-1.08516×10^{-3}	0.000182478	-1.17598×10^{-3}	0.00023766	-1.23485×10^{-3}
0.7	0.0000148502	-1.07228×10^{-3}	0.000141545	-1.15726×10^{-3}	0.000207225	-1.20596×10^{-3}
0.8	0.000577914	-1.04689×10^{-3}	0.0000813935	-1.12151×10^{-3}	0.000156834	-1.15742×10^{-3}
0.9	0.00014678	-1.00788×10^{-3}	0.00000225869	-1.06816×10^{-3}	0.0000864945	-1.08918×10^{-3}
1	0.000251767	-9.54261×10^{-4}	0.0000956603	-9.96704×10^{-4}	0.00000378926	-1.00122×10^{-3}

demonstrate the solution and errors for various values of α, β, S_t, and M, respectively, in Case I. Table 5 presents solution and error for various values of γ, while fixing other parameters in Case II. Tables 6 and 7 show solution and error for various α and γ keeping other parameters fixed in Case III. All the tables clearly show that solutions are acceptable. Furthermore, the parameters' effect on the speed profiles are examined graphically. Figure 1 shows the effect

Table 4: Solutions and residuals for different M, while $\alpha = 0.95$, $\beta = 0.2$, $S_t = 0.01$ in Case I

x	$M = 0.1$		$M = 0.4$		$M = 0.6$	
	Sol.	**Error**	**Sol.**	**Error**	**Sol.**	**Error**
0.1	0.000946713	1.13×10^{-9}	0.00090012	-2.17×10^{-8}	0.000845297	-8.84×10^{-7}
0.2	0.00179352	1.18×10^{-9}	0.00170166	-5.61×10^{-8}	0.00159361	-1.76×10^{-6}
0.3	0.00254049	1.12×10^{-9}	0.00240591	-8.99×10^{-8}	0.00224765	-2.60×10^{-6}
0.4	0.00318772	9.91×10^{-10}	0.003014	-1.21×10^{-7}	0.00280978	-3.38×10^{-6}
0.5	0.00373526	8.23×10^{-10}	0.0035269	-1.50×10^{-7}	0.00328203	-4.07×10^{-6}
0.6	0.00418318	6.43×10^{-10}	0.00394542	-1.75×10^{-7}	0.00366611	-4.66×10^{-6}
0.7	0.00453151	4.72×10^{-10}	0.00427025	-1.96×10^{-7}	0.0039634	-5.13×10^{-6}
0.8	0.00478029	3.29×10^{-10}	0.0045019	-2.11×10^{-7}	0.00417499	-5.48×10^{-6}
0.9	0.00492955	2.25×10^{-10}	0.00464074	-2.20×10^{-7}	0.00430164	-5.69×10^{-6}
1	0.0049793	1.72×10^{-10}	0.00468699	-2.23×10^{-7}	0.00434381	-5.77×10^{-6}

Table 7: Solutions and residuals for different γ, while $\alpha = 0.99$, $M = 0.1$, $\beta = 0.1$, $S_t = 0.001$ in Case III

x	$\gamma = 1.3$		$\gamma = 1.5$		$\gamma = 1.99$	
	Sol.	**Error**	**Sol.**	**Error**	**Sol.**	**Error**
0.1	4.2964×10^{-5}	-1.30056×10^{-3}	1.29173×10^{-5}	-1.70022×10^{-3}	0.00000516384	−0.0019901
0.2	1.05814×10^{-4}	-1.30138×10^{-3}	4.19714×10^{-5}	-1.70071×10^{-3}	0.0000205132	-1.99041×10^{-3}
0.3	1.79299×10^{-4}	-1.30233×10^{-3}	8.36279×10^{-5}	-1.70142×10^{-3}	0.0000459699	-1.99091×10^{-3}
0.4	260693×10^{-4}	-1.30339×10^{-3}	1.36396×10^{-4}	-1.70232×10^{-3}	0.0000814945	-1.99162×10^{-3}
0.5	3.48539×10^{-4}	-1.30453×10^{-3}	1.99348×10^{-4}	-1.70339×10^{-3}	0.000127061	-1.99253×10^{-3}
0.6	4.41911×10^{-4}	-1.30575×10^{-3}	2.71828×10^{-4}	-1.70462×10^{-3}	0.000182652	-1.99363×10^{-3}
0.7	5.40159×10^{-4}	-1.30702×10^{-3}	3.53336×10^{-4}	-1.70601×10^{-3}	0.000248254	-1.99494×10^{-3}
0.8	6.42797×10^{-4}	-1.30836×10^{-3}	4.43472×10^{-4}	-1.70754×10^{-3}	0.000323858	-1.99644×10^{-3}
0.9	7.49446×10^{-4}	-1.30974×10^{-3}	5.41908×10^{-4}	-1.70921×10^{-3}	0.00040946	-1.99815×10^{-3}
1	8.59805×10^{-4}	-1.31118×10^{-3}	6.48368×10^{-4}	-1.71102×10^{-3}	0.00505055	-2.00005×10^{-3}

of M on the velocity profiles. It is seen that velocity decreases as M increases. Higher estimation of magnetic field produced causes the velocity field to retard. Figures 2 and 3 show that S_t and β have direct relation with velocity. Figure 4 shows the effect of α on the dimensionless velocity. It has been noticed that the velocity profile slightly decreases with increase in α.

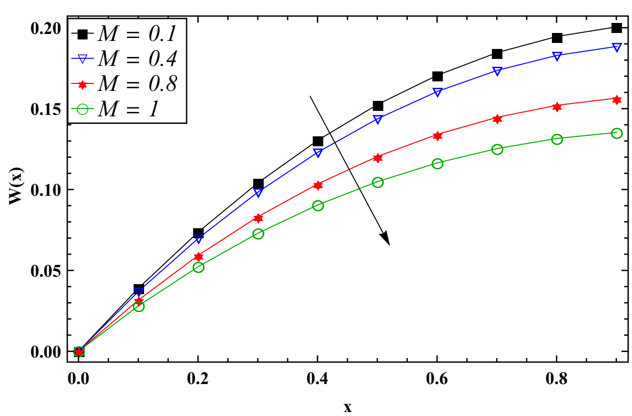

Figure 1: Effect of M on velocity field in Case I.

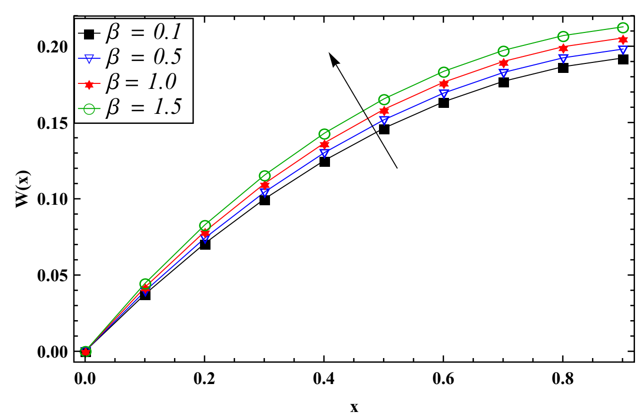

Figure 3: Effect of β on velocity in Case I.

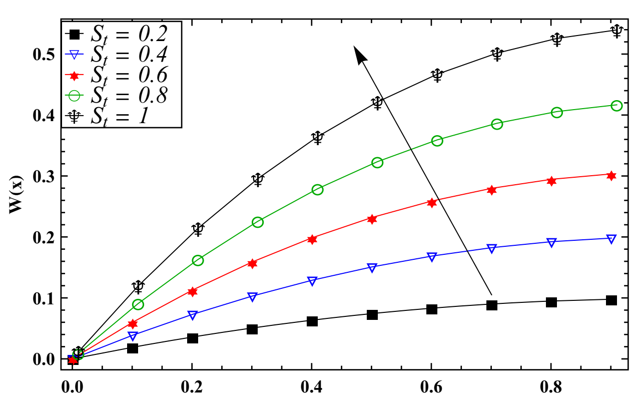

Figure 2: Effect of S_t on fluid velocity in Case I.

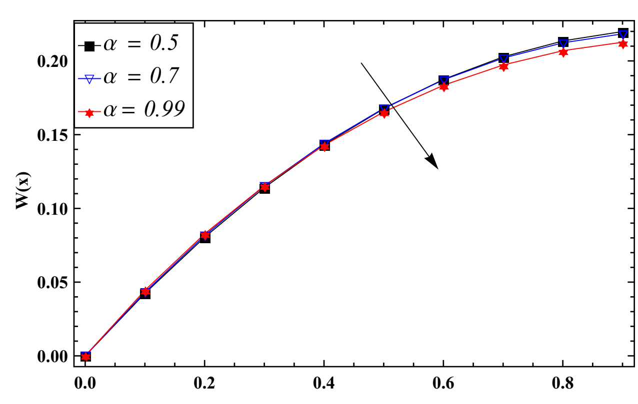

Figure 4: Effect of α on velocity in Case I.

Nomenclature

w	velocity vector
D^{α}	α order derivative
δ	thickness of the film
$\boldsymbol{f}$	body force
$\boldsymbol{S}$	stress tensor
β	non-Newtonian parameter
S_t	Stokes number
M^2	MHD parameter
α, γ	fractional parameter
ρ	fluid density
P	pressure
Q	flow rate
$\bar{V}$	average velocity

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