mathematics

MDPI

Article

Indefinite Linear-Quadratic Stochastic Control Problem for Jump-Diffusion Models with Random Coefficients: A Completion of Squares Approach

Jun Moon [1] and Jin-Ho Chung [2,*]

[1] Department of Electrical Engineering, Hanyang University, Seoul 04763, Korea; junmoon@hanyang.ac.kr
[2] School of IT Convergence, University of Ulsan, Ulsan 44610, Korea
* Correspondence: jinho@ulsan.ac.kr

Abstract: In this paper, we study the indefinite linear-quadratic (LQ) stochastic optimal control problem for stochastic differential equations (SDEs) with jump diffusions and random coefficients driven by both the Brownian motion and the (compensated) Poisson process. In our problem setup, the coefficients in the SDE and the objective functional are allowed to be random, and the jump-diffusion part of the SDE depends on the state and control variables. Moreover, the cost parameters in the objective functional need not be (positive) definite matrices. Although the solution to this problem can also be obtained through the stochastic maximum principle or the dynamic programming principle, our approach is simple and direct. In particular, by using the Itô-Wentzell's formula, together with the integro-type stochastic Riccati differential equation (ISRDE) and the backward SDE (BSDE) with jump diffusions, we obtain the equivalent objective functional that is quadratic in control u under the positive definiteness condition, where the approach is known as the completion of squares method. Then the explicit optimal solution, which is linear in state characterized by the ISRDE and the BSDE jump diffusions, and the associated optimal cost are derived by eliminating the quadratic term of u in the equivalent objective functional. We also verify the optimality of the proposed solution via the verification theorem, which requires solving the stochastic HJB equation, a class of stochastic partial differential equations with jump diffusions.

Keywords: stochastic systems with jump diffusions and random coefficients; completion of squares method; stochastic HJB equation with jump diffusions; verification theorem

check for updates

Citation: Moon, J.; Chung, J.-H. Indefinite Linear-Quadratic Stochastic Control Problem for Jump-Diffusion Models with Random Coefficients: A Completion of Squares Approach. *Mathematics* **2021**, *9*, 2918. https://doi.org/10.3390/math9222918

Academic Editor: Mariano Torrisi

Received: 15 October 2021
Accepted: 12 November 2021
Published: 16 November 2021

1. Introduction

We first provide the precise problem statement. The detailed literature review and the comparison of our paper with the existing results are then provided.

1.1. Problem Statement

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. For $x, y \in \mathbb{R}^n$, $x^\top$ denotes the transpose of x, $\langle x, y\rangle$ is the inner product, and $|x| := \langle x, x\rangle^{1/2}$. Let $\mathbb{S}^n$ be the set of $n \times n$ symmetric matrices. Let $\mathrm{Tr}(\cdot)$ be the trace operator.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with the natural filtration $\mathbb{F} := \{\mathcal{F}_s,\ 0 \leq s \leq t\}$ generated by the following two mutually independent stochastic processes and augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$: (i) an r-dimensional standard Brownian motion $W := \begin{bmatrix} W_1 & \cdots & W_r \end{bmatrix}^\top$ defined on $[0, T]$ and (ii) an E_j-marked right continuous Poisson random measure (process) N_j defined on $E_j \times [0, T]$ with $j = 1, \ldots, l$, where $E_j := \bar{E}_j \setminus \{0\}$ with $\bar{E}_j \subset \mathbb{R}$ is a Borel subset of $\mathbb{R}$ equipped with its Borel σ-field $\mathcal{B}(E_j)$. Let $\widetilde{N}_j(\mathrm{d}e_j, \mathrm{d}s) := N_j(\mathrm{d}e_j, \mathrm{d}s) - \lambda_j(\mathrm{d}e_j)\mathrm{d}s$, $j = 1, \ldots, l$, be the associated $\mathcal{F}_s$-martingale compensated Poisson process of N_j, where λ_j is a σ-finite Lévy measure on $(E_j, \mathcal{B}(E_j))$ satisfying $\int_{E_j}(1 \wedge |e_j|^2)\lambda_j(\mathrm{d}e_j) < \infty$ for $j = 1, \ldots, l$ [1] (Chapter 2).

Let $\mathcal{C}^2_{\mathbb{F}}(\mathbb{R}^n)$ be the space of $\mathcal{F}_t$-adapted $\mathbb{R}^n$-valued stochastic processes, which is càdlàg and satisfies $\mathbb{E}[\sup_{t\in[0,T]}|x(t)|^2]^{\frac{1}{2}} < \infty$ for $x \in \mathcal{C}^2_{\mathbb{F}}(\mathbb{R}^n)$. Let $\mathcal{L}^2_{\mathbb{F},p}(\mathbb{R}^n)$ be the space of $\mathcal{F}_t$-predictable (The stochastic processis $\mathcal{F}_t$-*predictable* if it is $\mathcal{F}_t$-measurable for each $t \in [0,T]$ and left continuous [1] (page 216).) $\mathbb{R}^n$-valued stochastic processes satisfying $\mathbb{E}[\int_0^T |x(t)|^2 \mathrm{d}t]^{\frac{1}{2}} < \infty$ for $x \in \mathcal{L}^2_{\mathbb{F},p}(\mathbb{R}^n)$. For $j = 1,\ldots,l$, let $G^2(E_j,\mathcal{B}(E_j),\lambda_j;\mathbb{R}^n)$ be the space of square integrable functions such that for $k \in G^2(E_j,\mathcal{B}(E_j),\lambda_j;\mathbb{R}^n)$, $k : E_j \to \mathbb{R}^n$ satisfies $\|k\|_{G^2} := (\int_E |k(e_j)|^2 \lambda_j(\mathrm{d}e_j))^{\frac{1}{2}} < \infty$. For $j = 1,\ldots,l$, let $\mathcal{G}^2_{\mathbb{F},p}(E_j,\mathcal{B}(E_j),\lambda_j;\mathbb{R}^n)$ be the space of stochastic processes such that for $k \in \mathcal{G}^2_{\mathbb{F},p}(E_j,\mathcal{B}(E_j),\lambda_j;\mathbb{R}^n)$, $k : \Omega \times [0,T] \times E_j \to \mathbb{R}^n$ is an $\mathcal{P} \times \mathcal{B}(E_j)$-measurable $\mathbb{R}^n$-valued $\mathcal{F}_t$-predictable process satisfying $\mathbb{E}[\int_0^T \|k(t)\|_{G^2} \mathrm{d}t]^{\frac{1}{2}} < \infty$, where $\mathcal{P}$ denotes the σ-algebra of $\mathcal{F}_t$-predictable subsets of $\Omega \times [0,T]$.

In this paper, we study the linear-quadratic (LQ) stochastic optimal control problem for jump-diffusion models with random coefficients. Specifically, let $\mathcal{U} := \mathcal{L}^2_{\mathbb{F},p}(\mathbb{R}^m)$ be the set of admissible controls. Then our problem is to minimize the following objective functional over $u \in \mathcal{U}$:

$$J(u) = \frac{1}{2}\mathbb{E}\left[\int_0^T \left\langle \begin{bmatrix} x(s) \\ u(s) \end{bmatrix}, \begin{bmatrix} L(s) & S(s) \\ S(s)^\top & R(s) \end{bmatrix} \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} \right\rangle \mathrm{d}s + \langle x(T), Mx(T)\rangle \right], \tag{1}$$

subject to the stochastic differential equation (SDE) with jump diffusions given by:

$$\begin{cases} \mathrm{d}x(s) = \Big[A(s)x(s-) + B(s)u(s) + f(s)\Big]\mathrm{d}s \\ \qquad + \displaystyle\sum_{i=1}^{r} \Big[C_i(s)x(s-) + D_i(s)u(s) + \sigma_i(s)\Big]\mathrm{d}W_i(s) \\ \qquad + \displaystyle\sum_{j=1}^{l} \int_{E_j} \Big[F_j(e_j,s)x(s-) + G_j(e_j,s)u(s) + \phi_j(e_j,s)\Big]\widetilde{N}_j(\mathrm{d}e_j,\mathrm{d}s) \\ x(0) = a, \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$ is the controlled state process with the initial condition $x(0) = a$ and $u \in \mathbb{R}^m$ is the control process. Note that the SDE in (2) is a stochastic system driven by both the Brownian motion and the (compensated) Poisson process. We note that there are various applications of SDEs (with deterministic and/or random coefficients). Specifically, SDEs arise in diverse fields of applications, such as mathematical finance, economics, biology, mechanical systems, communication networks, analysis of (stochastic and deterministic) PDEs, and interacting large-scale mean-field systems. For various applications of SDEs (with deterministic and/or random coefficients), see [2–15] and the references therein.

Equivalently, the problem studied in this paper is given by:

$$\textbf{(P)}: \inf_{u\in\mathcal{U}} J(u), \text{ subject to (2)}.$$

In (1) and (2), we have the following standing assumptions:

(H.1) For $i = 1,\ldots,r$ and $j = 1\ldots,l$, $A, C_i : \Omega \times [0,T] \to \mathbb{R}^n$, $B, D_i : \Omega \times [0,T] \to \mathbb{R}^m$, $F_j : \Omega \times [0,T] \to G^2(E_j,\mathcal{B}(E_j),\lambda_j;\mathbb{R}^n)$, $G_j : \Omega \times [0,T] \to G^2(E_j,\mathcal{B}(E_j),\lambda_j;\mathbb{R}^n)$, $f, \sigma_i : \Omega \times [0,T] \to \mathbb{R}^n$, and $\phi_j : \Omega \times [0,T] \times E_j \to \mathbb{R}^n$;

(H.2) For $i = 1,\ldots,r$ and $j = 1\ldots,l$, A, B, C_i, D_i, F_j and G_j are $\mathcal{F}_s$-predictable stochastic processes, which are uniformly bounded in a.e. $(\omega,s) \in \Omega \times [0,T]$. Moreover, $f, \sigma_i \in \mathcal{L}^2_{\mathbb{F},p}(\mathbb{R}^n)$ and $\phi_j \in \mathcal{G}^2_{\mathbb{F},p}(E_j,\mathcal{B}(E_j),\lambda_j;\mathbb{R}^n)$ for $i = 1,\ldots,r$ and $j = 1\ldots,l$;

(H.3) $L : \Omega \times [0,T] \to \mathbb{S}^n$, $R : \Omega \times [0,T] \to \mathbb{S}^m$, $S : \Omega \times [0,T] \to \mathbb{R}^{n\times m}$ and $M : \Omega \to \mathbb{S}^n$;

(H.4) L, R and S are $\mathcal{F}_s$-predictable stochastic processes, which are uniformly bounded in a.e. $(\omega,s) \in \Omega \times [0,T]$. Moreover, M is uniformly bounded in a.e. $\omega \in \Omega$.

Note that based on [1], (Theorem 6.2.3) (see also [16], (Theorem 1.19) and [17]), (Lemma 2.1) under (H.1)–(H.4), there exists a unique solution of (2) in $\mathcal{C}_{\mathbb{F}}^2(\mathbb{R}^n)$.

Remark 1. *(a) In the SDE of (2), the jump-diffusion part depends on the control and state variables due to the presence of C_i, D_i, F_j and G_j. Moreover, the additive signal terms, f, σ_i and ϕ_j, are included in the SDE.*

(b) In view of (1)–(2) and (H.1)–(H.4), ***(P)*** *can be referred to as the indefinite linear-quadratic (LQ) stochastic optimal control for jump-diffusion models with random coefficients. Here, the "indefinite" means that the cost parameters (L, R and M) in (1) are not needed to be (positive) definite matrices. Furthermore, the "random coefficients" implies that the coefficients in (1) and (2) are explicitly dependent on Ω from (H.1)–(H.4).*

(c) If the coefficients in (1) and (2) are independent of Ω, ***(P)*** *is reduced to the case of the deterministic coefficients, which is a simplified problem of* ***(P)****.*

To solve **(P)**, in Section 2.1, we first develop the direct method, also known as the completion of squares method, which constructs the equivalent objective functional of Equation (1), which is quadratic in u under the positive definiteness condition (see Theorem 1). Specifically, by using the Itô-Wentzell's formula for general Lévy type stochastic integrals (see [18]), together with the integro-type stochastic Riccati differential equation (ISRDE) with jump diffusions (see Equation (3)) and the backward stochastic differential equation (BSDE) with jump diffusions (see Equation (4)), we are able to obtain the equivalent objective functional of J in (1) (see the equivalent expression of J in (14)). Then under the positive definiteness condition in (5), we can see that the equivalent objective functional of J in Equation (14) is quadratic in u. Hence, the explicit optimal solution for **(P)** (see Equation (6)) can be characterized by eliminating the quadratic term of u in the equivalent objective functional. Moreover, the explicit optimal cost can be derived easily by checking the optimality of the (quadratic) equivalent objective functional under the optimal solution of **(P)**. Note that in our direct approach, the additional positive definiteness condition is induced due to the indefiniteness of the cost parameters and the dependence of the control on the jump-diffusion term. We also note that the corresponding optimal solution for **(P)** in Equation (6) is a linear state-feedback controller, where the associated parameters of the optimal solution are determined by the solutions of the ISRDE and the BSDE with jump diffusions.

Next, in Section 2.2, we verify the optimality of the proposed solution in Theorem 1 via the verification theorem (see Appendix A), which requires solving a certain class of stochastic Hamilton-Jacobi-Bellman (SHJB) equations with jump diffusions (see Equation (16) and Proposition 3). This means that unlike the direct approach (equivalently, the completion of squares method) developed in Section 2.1, Proposition 3 needs to solve the complex SHJB equation with jump diffusions (see Remark 7). Note that the SHJB equation of this paper is a class of integro-type stochastic partial differential equations (SPDEs) with jump diffusions. Based on the verification theorem (see Appendix A), if the SHJB equation with jump diffusions in Equation (16) admits a unique (smooth) solution, then the corresponding solution constitutes the optimal cost of **(P)**, i.e., it is the value function of **(P)**. In addition, the associated minimizer of the Hamiltonian is the optimal solution of **(P)**. We show that the minimizer of the Hamiltonian is equivalent to the optimal solution of **(P)** in Theorem 1, i.e., it is equivalent to Equation (6). Moreover, we find the explicit solution of the SHJB equation that is quadratic in x, which leads to the optimal cost of **(P)** (see Proposition 3).

Finally, we apply Theorem 1 to three different examples (see Examples 1–3). Note that by Theorem 1, we are able to characterize the explicit optimal solutions for these examples in terms of the ISRDE and the BSDE with jump diffusions.

Now, the main results of this paper can be summarized as follows:

(a) Under the positive definiteness condition, we construct the quadratic equivalent objective functional, by which the explicit optimal solution for **(P)** and the associated optimal cost are characterized via the completion of squares method (see Theorem 1).

The optimal solution is linear in state, and its parameters are determined by the solutions of the ISRDE and the BSDE with jump diffusions;

(b) We use the verification theorem to validate the optimality of the proposed solution. We first solve the SHJB equation with jump diffusions and find the explicit solution that minimizes the Hamiltonian. In Proposition 3, we show that the corresponding solution is equivalent to that obtained by the completion of squares method in Theorem 1.

We should mention that the approach in (a) is more convenient than that of (b), where (a) does not need to solve the complex SHJB equation. A detailed comparison between (a) and (b) can be found in Remark 7.

The organization of the paper is as follows. A detailed literature review is given in Section 1.2. We provide the main results of **(P)** in Section 2. Two different examples of **(P)**, including the modified mean-variance portfolio optimization problem, are also discussed in Section 2. We conclude the paper in Section 3. Appendix A provides the verification theorem of **(P)**.

1.2. Literature Review and Comparison

Linear-quadratic (LQ) stochastic optimal control with random coefficients and its applications to mathematical finance, engineering and science have been studied extensively in the literature; see [2,19–33] and the references therein. We should note that the aforementioned references considered the case when the SDE in Equation (2) is driven by Brownian motion only (equivalently, $F_j = G_j = \phi_j = 0$ for $j = 1, \ldots, l$) with/without additive signal terms (equivalently, $f = \sigma_i = \phi_j = 0$ for $i = 1, \ldots, r$ and $j = 1, \ldots, l$). We can easily observe that both problem formulations are special cases of **(P)** studied in our paper. We also note that nonlinear stochastic control problems with random coefficients were studied in [34,35], where their SDEs are also driven by the Brownian motion only. Moreover, the LQ mean-field type game problems were studied in [36,37], where their coefficients are deterministic.

Recently, LQ stochastic optimal control for jump-diffusion models with random coefficients was studied in [38,39]. However, the corresponding SDE does not have the additive signal terms (equivalently, $f = \sigma_i = \phi_j = 0$ for $i = 1, \ldots, r$ and $j = 1, \ldots, l$). Moreover, [38,39] did not consider the characterization of the explicit state-feedback optimal solution using the completion of squares approach and the validation of the proposed solution via the verification theorem. We mention that the problem formulation and the approaches established in our paper are different from those of [38,39]. Furthermore, the completion of squares method and the validation of the optimal solution for **(P)** via the verification theorem have not been studied in the existing literature, which we address in our paper.

We note that our paper can be viewed as a generalization of [32,38]. The main comparisons of **(P)** with [32,38] are stated as follows:

(i) Unlike [38], we consider the explicit characterization of the optimal solution for **(P)** via the completion of squares approach. Moreover, different from [38], we consider the case, where the SDE in Equation (2) also includes the additive signal terms of f, σ_i and ϕ_j for $i = 1, \ldots, r$ and $j = 1, \ldots, l$. In addition, our objective functional in Equation (1) includes the state-control coupling term with S, whereas [38] did not consider any coupling nature in the corresponding objective functional;

(ii) Unlike [32], we consider the jump-diffusion model, where the jump-diffusion part also depends on the state and control variables. Note that [32] considered the situation when the SDE is driven by the Brownian motion only (equivalently, $F_j = G_j = \phi_j = 0$ for $j = 1, \ldots, l$ in (2)).

We note that the generalization stated in (i) is not trivial, since by including the additive signal terms in Equation (2), the associated optimal solution is not of the pure state-feedback form, and the additional bias term is induced (see Equation (6) in Theorem 1). Note that this additional bias term is characterized by the solution of the BSDE with random coefficients.

When there are no additive signal terms in Equation (2) (equivalently, $f = \sigma_i = \phi_j = 0$ for $i = 1, \ldots, r$ and $j = 1, \ldots, l$), the solution of the BSDE with random coefficients is zero and the additional bias term does not appear in the optimal solution (see Remark 4). Moreover, [38] focused on the general solvability of the ISRDE with jump diffusions, but did not consider the explicit characterization of the optimal solution via the completion of squares method. We should note that the completion of squares method is an effective approach to characterize the optimal solution of LQ problems, since the approach relies on neither the dynamic programming principle nor the stochastic maximum principle. In fact, various completion of squares approaches have been developed for several LQ problems; see [32,33,37,40–42] and the references therein. Furthermore, the completion of squares method for nonlinear stochastic control problems was developed in [43–45].

The extension stated in (ii) is also not straightforward due to the inclusion of the jump-diffusion part in Equation (2) and its dependence on the state and control variables. In particular, unlike [32], we need to handle the integro-type SRDE (ISRDE) and the BSDE with jump diffusions, meaning that constructing the equivalent objective functional in the completion of squares method is more involved than the case without jumps in [32]. In addition, finding the positive definiteness condition for the existence of the optimal solution in Theorem 1 is more involved than [32] due to the dependence of the state and control variables on the jump-diffusion part. Finally, due to the jump-diffusion process, we need to solve the SHJB equation with jump diffusions in the verification theorem (see Equation (16) and Proposition 3). In fact, solving the SHJB equation with jump diffusions in our paper (see Equation (16)) is more challenging than the case without jumps in [32] due the presence of the nonlocal integral part in terms of the Lévy measure as seen from Equation (16).

It should be noted that a similar problem of **(P)** was studied in [46]. However, the optimal solution in [46] is of the *open-loop* type in terms of the forward-backward SDE, which was obtained by the stochastic maximum principle. Notice that our optimal solution in Theorem 1 is of the *state-feedback* type in terms of the ISRDE with jump diffusions. Hence, unlike [46], the optimal solution in Theorem 1 is more applicable for various practical situations. Moreover, the approach developed in our paper is different from that used in [46].

The efficiency of the proposed method compared with [32,38,39,46] is summarized in Table 1.

Table 1. Efficiency of the Proposed Method.

Feature	[32]	[38]	[39]	[46]	This Work
Brownian motion & Jump diffusions		✓	✓	✓	✓
Additive signal terms $(f, \sigma_1, \ldots, \sigma_r, \phi_1, \ldots, \phi_l) \neq 0$	✓			✓	✓
State-control coupling ($S \neq 0$)	✓			✓	✓
Integro-type SRDE		✓	✓		✓
Completion of Squares (Direct) approach	✓				✓
Verification approach	✓	✓	✓		✓
State-feedback type optimal control	✓	✓	✓		✓
Simulation results					✓

We finally mention that there are several examples of LQ stochastic control problems. In particular, the LQ pollution level control problem was considered in [47]. Ref. [28] studied the LQ mean-variance portfolio optimization problem. The stochastic control problem for electric water heating loads was studied in [10]. In [9], the stochastic power adjustment problem of wireless communication networks was studied. Several different applications of LQ control problems can be found in [2] and the references therein.

2. Main Results

In this section, we first characterize the optimal solution for **(P)** via the completion of squares method. Then we verify the optimality of the proposed solution by the verification theorem solving the stochastic Hamilton-Jacobi-Bellman equation with jump diffusions.

2.1. Characterization of the Optimal Solution for *(P)*

We first introduce the following integro-type stochastic Riccati differential equation (ISRDE) with jump diffusions:

$$\begin{cases} \mathrm{d}P(s) = -\Bigg[A(s)^\top P(s-) + P(s-)A(s) + L(s) \\ \qquad + \sum_{i=1}^{r}\Big[C_i(s)^\top P(s-)C_i(s) + C_i(s)^\top Q_i(s) + Q_i(s)C_i(s)\Big] \\ \qquad + \sum_{j=1}^{l}\int_{E_j}\Big[F_j(e_j,s)^\top P(s-)F_j(e_j,s) + F_j(e_j,s)^\top Z_j(e_j,s)F_j(e_j,s) \\ \qquad\quad + F_j(e_j,s)^\top Z_j(e_j,s) + Z_j(e_j,s)F_j(e_j,s)\Big]\lambda_j(\mathrm{d}e_j) \\ \qquad - \Big(\mathcal{K}_1(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{K}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\Big) \\ \qquad\quad \times \Big(\mathcal{R}_1(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{R}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\Big)^{-1} \\ \qquad\quad \times \Big(\mathcal{K}_1(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{K}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\Big)^{\top}\Bigg]\mathrm{d}s \\ \qquad + \sum_{i=1}^{r}Q_i(s)\mathrm{d}W_i(s) + \sum_{j=1}^{l}\int_{E_j}Z_j(e_j,s)\widetilde{N}(\mathrm{d}e_j,\mathrm{d}s) \\ P(T) = M, \end{cases} \tag{3}$$

where:

$$\begin{aligned} \mathcal{R}_1(s) &:= R(s) + \sum_{i=1}^{r}D_i(s)^\top P(s-)D_i(s) \\ \mathcal{R}_{2,j}(e_j,s) &:= G_j(e_j,s)^\top(P(s-) + Z_j(e_j,s))G_j(e_j,s)\lambda_j(\mathrm{d}e_j) \\ \mathcal{K}_1(s) &:= P(s-)B(s) + \sum_{i=1}^{r}[Q_i(s)D_i(s) + C_i(s)^\top P(s-)D_i(s)] + S(s) \\ \mathcal{K}_{2,j}(e_j,s) &:= Z_j(e_j,s)G_j(e_j,s) + F_j(e_j,s)^\top(P(s-) + Z_j(e_j,s))G_j(e_j,s). \end{aligned}$$

Moreover, the linear BSDE with jump diffusions and random coefficients is given by:

$$\begin{cases} \mathrm{d}p(s) = -\Big[\widehat{A}(s)^\top p(s-) + P(s-)f(s) + \sum_{i=1}^{r} \widehat{C}_i(s)q_i(s) \\ \qquad + \sum_{j=1}^{l} \int_{E_j} \widehat{F}_j(e_j,s)z_j(e_j,s)\lambda_j(\mathrm{d}e_j) + \sum_{i=1}^{r} \mathcal{K}_{3,i}(s)\sigma_i(s) \\ \qquad + \sum_{j=1}^{l} \int_{E_j} \mathcal{K}_{4,j}(e_j,s)\phi_j(e_j,s)\lambda_j(\mathrm{d}e_j)\Big]\mathrm{d}s \\ \qquad + \sum_{i=1}^{r} q_i(s)\mathrm{d}W_i(s) + \sum_{j=1}^{l} \int_{E_j} z_j(e_j,s)\widetilde{N}(\mathrm{d}e_j,\mathrm{d}s) \\ p(T) = 0, \end{cases} \tag{4}$$

where:

$$\begin{aligned} \widehat{A}(s) &:= A(s) - B(s)(\mathcal{R}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{R}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j))^{-1} \\ &\quad \times (\mathcal{K}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{K}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j))^\top \\ \widehat{C}_i(s) &:= C_i(s) - D_i(s)(\mathcal{R}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{R}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j))^{-1} \\ &\quad \times (\mathcal{K}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{K}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j))^\top \\ \widehat{F}_j(e_j,s) &:= F_j(e_j,s) - G_j(e_j,s)(\mathcal{R}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{R}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j))^{-1} \\ &\quad \times (\mathcal{K}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{K}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j))^\top \\ \mathcal{K}_{3,i}(s) &:= \sum_{i=1}^{l} [C_i(s)^\top P(s-) + Q_i(s)] - (\mathcal{K}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{K}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)) \\ &\quad \times (\mathcal{R}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{R}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j))^{-1} D_i(s)^\top P(s-) \\ \mathcal{K}_{4,j}(e_j,s) &:= F_j(e_j,s)^\top P(s-) + F_j(e_j,s)^\top Z_j(e_j,s) + Z_j(e_j,s) \\ &\quad - (\mathcal{K}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{K}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j))(\mathcal{R}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{R}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j))^{-1} \\ &\quad \times (G_j(e_j,s)^\top P(s-) + G_j(e_j,s)^\top Z_j(e_j,s)). \end{aligned}$$

Remark 2. *(a) The solution of the ISRDE with jump diffusions in Equation (3) corresponds to the tuple $(P, Q_1, \ldots, Q_r, Z_1, \ldots, Z_l)$ satisfying Equation (3). By (H.1)–(H.4), $(P, Q_1, \ldots, Q_r, Z_1, \ldots, Z_l)$, the solution of Equation (3), is symmetric and a class of $\mathbb{S}^n$-valued (matrix-valued) BSDE with jump diffusions and random coefficients.*

(b) We also note that Equation (4) is the linear BSDE with jump diffusions and random coefficients, whose coefficients are explicitly dependent on the solution of the ISRDE with jump diffusions.

Based on [38] (Theorems 4.1 and 5.2), (see also [39] (Theorem 5.3)), we first state the sufficient condition for the solvability of the ISRDE with jump diffusions in Equation (3).

Proposition 1. *Assume that (H.1)–(H.4) hold. Suppose that R is uniformly positive definite for a.e. $(\omega, s) \in \Omega \times [0, T]$, L is positive semidefinite for a.e. $(\omega, s) \in \Omega \times [0, T]$, $S = 0$ and M is positive semidefinite for a.e. $\omega \in \Omega$. Then, $\mathcal{R}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{R}_{2,j}(e_j, s)\lambda_j(\mathrm{d}e_j)$ is uniformly positive definite for a.e. $(\omega, s) \in \Omega \times [0, T]$, and the ISRDE with jump diffusions in (3) admits a unique solution of $(P, Q_1, \ldots, Q_r, Z_1, \ldots, Z_l) \in \mathcal{C}^2_{\mathbb{F}}(\mathbb{S}^n) \times \mathcal{L}^2_{\mathbb{F},p}(\mathbb{S}^n \times \cdots \times \mathbb{S}^n) \times \mathcal{G}^2_{\mathbb{F},p}(E_1, \mathcal{B}(E_1), \lambda_1; \mathbb{S}^n) \times \cdots \times \mathcal{G}^2_{\mathbb{F},p}(E_l, \mathcal{B}(E_l), \lambda_l; \mathbb{S}^n)$.*

Then in view of [48] (Lemma 2.4) (see also [49] (Lemma 3.1)), the solvability of the BSDE with jump diffusions in Equation (4) is given as follows:

Proposition 2. *Assume that (H.1)–(H.4) hold. Suppose that there is a unique solution of the ISRDE with jump diffusions in Equation (3) satisfying $(P, Q_1, \ldots, Q_r, Z_1, \ldots, Z_l) \in \mathcal{C}^2_{\mathbb{F}}(\mathbb{S}^n) \times \mathcal{L}^2_{\mathbb{F},p}(\mathbb{S}^n \times \cdots \times \mathbb{S}^n) \times \mathcal{G}^2_{\mathbb{F},p}(E_1, \mathcal{B}(E_1), \lambda_1; \mathbb{S}^n) \times \cdots \times \mathcal{G}^2_{\mathbb{F},p}(E_l, \mathcal{B}(E_l), \lambda_l; \mathbb{S}^n)$. Then the BSDE with jump diffusions in Equation (4) admits a unique solution satisfying $(p, q_1, \ldots, q_r, z_1, \ldots, z_l) \in \mathcal{C}^2_{\mathbb{F}}(\mathbb{R}^n) \times \mathcal{L}^2_{\mathbb{F},p}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n) \times \mathcal{G}^2_{\mathbb{F},p}(E_1, \mathcal{B}(E_1), \lambda_1; \mathbb{R}^n) \times \cdots \times \mathcal{G}^2_{\mathbb{F},p}(E_l, \mathcal{B}(E_l), \lambda_l; \mathbb{R}^n)$.*

Remark 3. *(a) Suppose that the coefficients in Equations (3) and (4) are deterministic, i.e., they are independent of Ω. Then in the ISRDE of Equation (3), $Q_i = Z_j = 0$ for all $i = 1, \ldots, r$ and $j = 1, \ldots, l$. In this case, the ISRDE is reduced to the integro-type deterministic Riccati differential equation. Moreover, in the BSDE of Equation (4), we have $q_i = z_j = 0$ for all $i = 1, \ldots, r$ and $j = 1, \ldots, l$, and since $p(T) = 0$, we can easily see that $(p, q_1, \ldots, q_r, z_1, \ldots, z_l) = (0, 0, \ldots, 0, 0, \ldots, 0)$ is the unique solution of the BSDE with jump diffusions in Equation (4).*

(b) If the SDE in Equation (2) does not have the additive signal terms (equivalently, $f = \sigma_i = \phi_j = 0$ for $i = 1, \ldots, r$ and $j = 1, \ldots, l$), then since $p(T) = 0$ in Equation (4), we can show that the solution of the BSDE with jump diffusions in Equation (4) satisfies $(p, q_1, \ldots, q_r, z_1, \ldots, z_l) = (0, 0, \ldots, 0, 0, \ldots, 0)$.

We now state the main result of this paper.

Theorem 1. *Assume that (H.1)–(H.4) hold. Suppose that the tuple given by $(P, Q_1, \ldots, Q_r, Z_1, \ldots, Z_l) \in \mathcal{C}^2_{\mathbb{F}}(\mathbb{S}^n) \times \mathcal{L}^2_{\mathbb{F},p}(\mathbb{S}^n \times \cdots \times \mathbb{S}^n) \times \mathcal{G}^2_{\mathbb{F},p}(E_1, \mathcal{B}(E_1), \lambda_1; \mathbb{S}^n) \times \cdots \times \mathcal{G}^2_{\mathbb{F},p}(E_l, \mathcal{B}(E_l), \lambda_l; \mathbb{S}^n)$ is the solution of the ISRDE with jump diffusions in Equation (3), and the tuple $(p, q_1, \ldots, q_r, z_1, \ldots, z_l) \in \mathcal{C}^2_{\mathbb{F}}(\mathbb{R}^n) \times \mathcal{L}^2_{\mathbb{F},p}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n) \times \mathcal{G}^2_{\mathbb{F},p}(E_1, \mathcal{B}(E_1), \lambda_1; \mathbb{R}^n) \times \cdots \times \mathcal{G}^2_{\mathbb{F},p}(E_l, \mathcal{B}(E_l), \lambda_l; \mathbb{R}^n)$ is the solution to the BSDE with jump diffusions in Equation (4). Assume that:*

$$\mathcal{R}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{R}_{2,j}(e_j, s)\lambda_j(\mathrm{d}e_j) > 0, \text{ a.e, } (\omega, s) \in \Omega \times [0, T], \tag{5}$$

*i.e., $\mathcal{R}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{R}_{2,j}(e_j, s)\lambda_j(\mathrm{d}e_j)$ is uniformly positive definite in a.e. $(\omega, s) \in \Omega \times [0, T]$. Then the optimal solution for **(P)** can be written as:*

$$
\begin{aligned}
u^*(s) = &-\Big(\mathcal{R}_1(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{R}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\Big)^{-1} \\
&\times \Big(\mathcal{K}_1(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{K}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\Big)^{\top} x(s-) \\
&-\Big(\mathcal{R}_1(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{R}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\Big)^{-1} \\
&\times \Big(B(s)^{\top}p(s-) + \sum_{i=1}^{r}D_i(s)^{\top}q_i(s) + \sum_{j=1}^{l}\int_{E_j}G_j(e_j,s)^{\top}z_j(e_j,s)\lambda_j(\mathrm{d}e_j) \\
&+ \sum_{i=1}^{r}D_i(s)^{\top}P(s-)\sigma_i(s) + \sum_{j=1}^{l}\int_{E_j}G_j(e_j,s)^{\top}(P(s-)+Z_j(e_j,s))\phi_j(e_j,s)\Big).
\end{aligned} \tag{6}
$$

Moreover, the corresponding optimal cost under Equation (6) is given by:

$$
\begin{aligned}
J(u^*) = \frac{1}{2}\mathbb{E}\Bigg[&\langle a, P(0)a\rangle + 2\langle a, p(0)\rangle \\
&+ \int_0^T \sum_{i=1}^{r}\Big[\langle \sigma_i(s), P(s-)\sigma_i(s)\rangle + 2\langle f(s), p(s-) + 2\langle \sigma_i(s), q_i(s)\rangle\rangle\Big]\mathrm{d}s \\
&+ \int_0^T \sum_{j=1}^{l}\int_{E_j}\langle \phi_j(e_j,s), (P(s-)+Z_j(e_j,s))\phi_j(e_j,s)\rangle\lambda_j(\mathrm{d}e_j)\mathrm{d}s \\
&+ 2\int_0^T \sum_{j=1}^{l}\int_{E_j}\langle \phi_j(e_j,s), z_j(e_j,s)\rangle\lambda_j(\mathrm{d}e_j)\mathrm{d}s \\
&- \int_0^T \langle \widehat{\mathcal{K}}(s), \Big(\mathcal{R}_1(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{R}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\Big)^{-1}\widehat{\mathcal{K}}(s)\rangle\mathrm{d}s\Bigg],
\end{aligned} \tag{7}
$$

where:

$$
\begin{aligned}
\widehat{\mathcal{K}}(s) := &B(s)^{\top}p(s-) + \sum_{i=1}^{r}D_i(s)^{\top}q_i(s) + \sum_{j=1}^{l}\int_{E_j}G_j(e_j,s)^{\top}z_j(e_j,s)\lambda_j(\mathrm{d}e_j) \\
&+ \sum_{i=1}^{r}D_i(s)^{\top}P(s-)\sigma_i(s) + \sum_{j=1}^{l}\int_{E_j}G_j(e_j,s)^{\top}(P(s-)+Z_j(e_j,s))\phi_j(e_j,s).
\end{aligned}
$$

Proof. We prove the theorem by establishing the completion of squares method using the solutions of the ISRDE and the BSDE with jump diffusions.

By applying the Itô-Wentzell's formula for general Lévy-type stochastic integrals (see [18]), we have:

$$
\begin{aligned}
\mathrm{d}x(s)x(s)^\top = {} & \Big[A(s)x(s-) + B(s)u(s) + f(s)\Big]x(s-)^\top \mathrm{d}s \\
& + \sum_{i=1}^{r}\Big[C_i(s)x(s-) + D_i(s)u(s) + \sigma_i(s)\Big]x(s-)^\top \mathrm{d}W_i(s) \\
& + \sum_{j=1}^{l}\int_{E_j}\Big[F_j(e_j,s)x(s-) + G_j(e_j,s)u(s) + \phi_j(e_j,s)\Big]x(s-)^\top \widetilde{N}_j(\mathrm{d}e_j,\mathrm{d}s) \\
& + x(s-)\Big[A(s)x(s-) + B(s)u(s) + f(s)\Big]^\top \mathrm{d}s \\
& + \sum_{i=1}^{r}x(s-)\Big[C_i(s)x(s-) + D_i(s)u(s) + \sigma_i(s)\Big]^\top \mathrm{d}W_i(s) \\
& + x(s-)\sum_{j=1}^{l}\int_{E_j}\Big[F_j(e_j,s)x(s-) + G_j(e_j,s)u(s) + \phi_j(e_j,s)\Big]^\top \widetilde{N}_j(\mathrm{d}e_j,\mathrm{d}s) \\
& + \sum_{i=1}^{r}\Big[C_i(s)x(s-) + D_i(s)u(s) + \sigma_i(s)\Big]\Big[C_i(s)x(s-) + D_i(s)u(s) + \sigma_i(s)\Big]^\top \mathrm{d}s \\
& + \sum_{j=1}^{l}\int_{E_j}\Big[F_j(e_j,s)x(s-) + G_j(e_j,s)u(s) + \phi_j(e_j,s)\Big] \\
& \quad \times \Big[F_j(e_j,s)x(s-) + G_j(e_j,s)u(s) + \phi_j(e_j,s)\Big]^\top \lambda_j(\mathrm{d}e_j)\mathrm{d}s \\
& + \sum_{j=1}^{l}\int_{E_j}\Big[F_j(e_j,s)x(s-) + G_j(e_j,s)u(s) + \phi_j(e_j,s)\Big] \\
& \quad \times \Big[F_j(e_j,s)x(s-) + G_j(e_j,s)u(s) + \phi_j(e_j,s)\Big]^\top \widetilde{N}_j(\mathrm{d}e_j,\mathrm{d}s).
\end{aligned} \tag{8}
$$

Using the solution of the ISRDE with jump diffusions in Equation (3) and the expression in Equation (8), we apply the Itô-Wentzell's formula to get (note that $\mathrm{Tr}(\cdot)$ denotes the trace operator):

$$
\begin{aligned}
&\frac{1}{2}\mathrm{d}\langle x(s), P(s)x(s)\rangle = \frac{1}{2}\mathrm{d}\,\mathrm{Tr}\Big(P(s)x(s)x(s)^\top\Big) \\
&= \langle B(s)u(s)+f(s), P(s-)x(s-)\rangle \mathrm{d}s - \frac{1}{2}\sum_{i=1}^{r}\langle x(s-), C_i(s)P(s-)C_i(s)x(s-)\rangle \mathrm{d}s \\
&\quad + \frac{1}{2}\sum_{i=1}^{r}\Big[C_i(s)x(s-)+D_i(s)u(s)+\sigma_i(s)\Big]^\top P(s-)\Big[C_i(s)x(s-)+D_i(s)u(s)+\sigma_i(s)\Big]\mathrm{d}s \\
&\quad + \frac{1}{2}\sum_{j=1}^{l}\int_{E_j}\Big[F_j(e_j,s)x(s-)+G_j(e_j,s)u(s)+\phi_j(e_j,s)\Big]^\top (P(s-)+Z_j(e_j,s)) \\
&\qquad \times \Big[F_j(e_j,s)x(s-)+G_j(e_j,s)u(s)+\phi_j(e_j,s)\Big]\lambda_j(\mathrm{d}e_j)\mathrm{d}s \\
&\quad - \frac{1}{2}\sum_{j=1}^{l}\langle F_j(e_j,s)x(s-), (P(s-)+Z_j(e_j,s))F_j(e_j,s)x(s-)\rangle \lambda_j(\mathrm{d}e_j)\mathrm{d}s \\
&\quad + \sum_{i=1}^{r}\langle D_i u(s)+\sigma_i(s), Q_i(s)x(s-)\rangle \mathrm{d}s \\
&\quad + \sum_{j=1}^{l}\int_{E_j}\langle G_j(e_j,s)u(s)+\phi_j(e_j,s), Z_j(e_j,s)x(s-)\rangle \lambda_j(\mathrm{d}e_j)\mathrm{d}s \\
&\quad + \big(\mathcal{K}_1(s)+\sum_{j=1}^{l}\int_{E_j}\mathcal{K}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\big)\big(\mathcal{R}_1(s)+\sum_{j=1}^{l}\int_{E_j}\mathcal{R}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\big)^{-1} \\
&\qquad \times \big(\mathcal{K}_1(s)+\sum_{j=1}^{l}\int_{E_j}\mathcal{K}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\big)^\top \mathrm{d}s \\
&\quad - \frac{1}{2}\langle x(s-), L(s)x(s-)\rangle \mathrm{d}s \\
&\quad + \sum_{i=1}^{r}\Big[\frac{1}{2}\langle x(s-), Q_i(s)x(s-)\rangle + \langle P(s-)x(s-), C_i(s)x(s-)+D_i(s)u(s)+\sigma_i(s)\rangle\Big]\mathrm{d}W_i(s) \\
&\quad + \sum_{j=1}^{l}\int_{E_j}\langle (P(s-)+Z_j(e_j,s))x(s-), F_j(e_j,s)x(s-)+G_j(e_j,s)u(s)+\phi_j(e_j,s)\rangle \widetilde{N}_j(\mathrm{d}e_j,\mathrm{d}s) \\
&\quad + \sum_{j=1}^{l}\int_{E_j}\langle x(s-), Z_j(e_j,s)x(s-)\rangle \widetilde{N}_j(\mathrm{d}e_j,\mathrm{d}s) \\
&\quad + \sum_{j=1}^{l}\int_{E_j}\Big[F_j(e_j,s)x(s-)+G_j(e_j,s)u(s)+\phi_j(e_j,s)\Big]Z_j(e_j,s) \\
&\qquad \times \Big[F_j(e_j,s)x(s-)+G_j(e_j,s)u(s)+\phi_j(e_j,s)\Big]^\top \widetilde{N}_j(\mathrm{d}e_j,\mathrm{d}s).
\end{aligned} \tag{9}
$$

Moreover, using a similar approach, we can show that:

$$
\begin{aligned}
\mathrm{d}\langle x(s), p(s)\rangle = & \Big[A(s)x(s-) + B(s)u(s) + f(s)\Big]^\top p(s-)\mathrm{d}s \\
& - x(s-)^\top \Big[\widehat{A}(s)^\top p(s-) \\
& \quad + \sum_{i=1}^{r} \widehat{C}_i(s)q_i(s) + \sum_{j=1}^{l} \int_{E_j} \widehat{F}_j(e_j, s)z_j(e_j, s)\lambda_j(\mathrm{d}e_j) \\
& \quad + P(s-)f(s) + \sum_{i=1}^{r} \mathcal{K}_{3,i}(s)\sigma_i(s) \\
& \quad + \sum_{j=1}^{l} \int_{E_j} \mathcal{K}_{4,j}(e_j, s)\phi_j(e_j, s)\lambda_j(\mathrm{d}e_j)\Big]\mathrm{d}s \\
& + \sum_{i=1}^{r} \langle q_i(s), C_i(s)x(s-) + D_i(s)u(s) + \sigma_i(s)\rangle \mathrm{d}s \\
& + \sum_{j=1}^{l} \int_{E_j} \langle z_j(e_j, s), F_j(e_j, s)x(s-) + G_j(e_j, s)u(s) + \phi_j(e_j, s)\rangle \lambda_j(\mathrm{d}e_j)\mathrm{d}s \\
& + \sum_{i=1}^{r} \Big[C_i(s)x(s-) + D_i(s)u(s) + \sigma_i(s)\Big]^\top p(s-)\mathrm{d}W_i(s) \\
& + \sum_{j=1}^{l} \int_{E_j} \Big[F_j(e_j, s)x(s-) + G_j(e_j, s)u(s) + \phi_j(e_j, s)\Big]^\top p(s-)\widetilde{N}_j(\mathrm{d}e_j, \mathrm{d}s) \\
& + \sum_{j=1}^{l} \int_{E_j} \langle z_j(e_j, s), F_j(e_j, s)x(s-) + G_j(e_j, s)u(s) + \phi_j(e_j, s)\rangle \widetilde{N}_j(\mathrm{d}e_j, \mathrm{d}s) \\
& + \sum_{i=1}^{r} x(s-)^\top q_i(s)\mathrm{d}W_i(s) + \sum_{j=1}^{l} \int_{E_j} x(s-)^\top z_j(e_j, s)\widetilde{N}_j(\mathrm{d}e_j, \mathrm{d}s).
\end{aligned} \tag{10}
$$

Let,

$$
\widehat{J}(u) := \frac{1}{2} \int_0^T \left\langle \begin{bmatrix} x(s) \\ u(s) \end{bmatrix}, \begin{bmatrix} L(s) & S(s) \\ S(s)^\top & R(s) \end{bmatrix} \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} \right\rangle \mathrm{d}s + \frac{1}{2}\langle x(T), Mx(T)\rangle. \tag{11}
$$

Note that $J(u) = \mathbb{E}[\widehat{J}(u)]$. Moreover, $P(T) = M$, $p(T) = 0$ and $x(0) = a$. Then integrating Equations (9) and (10) from 0 to T and summing them yields:

$$
\begin{aligned}
\widehat{J}(u) = & \frac{1}{2}\langle a, P(0)a\rangle + \langle a, p(0)\rangle + \langle B(s)u(s) + f(s), P(s-)x(s-)\rangle \mathrm{d}s \\
& - \frac{1}{2}\sum_{i=1}^{r}\langle x(s-), C_i(s)P(s-)C_i(s)x(s-)\rangle \mathrm{d}s \\
& + \frac{1}{2}\sum_{i=1}^{r}\Big[C_i(s)x(s-) + D_i(s)u(s) + \sigma_i(s)\Big]^\top \\
& \quad \times P(s-)\Big[C_i(s)x(s-) + D_i(s)u(s) + \sigma_i(s)\Big]\mathrm{d}s \\
& + \frac{1}{2}\sum_{j=1}^{l}\int_{E_j}\Big[F_j(e_j,s)x(s-) + G_j(e_j,s)u(s) + \phi_j(e_j,s)\Big]^\top (P(s-) + Z_j(e_j,s)) \\
& \quad \times \Big[F_j(e_j,s)x(s-) + G_j(e_j,s)u(s) + \phi_j(e_j,s)\Big]\lambda_j(\mathrm{d}e_j)\mathrm{d}s \\
& - \frac{1}{2}\sum_{j=1}^{l}\langle F_j(e_j,s)x(s-), (P(s-) + Z_j(e_j,s))F_j(e_j,s)x(s-)\rangle\lambda_j(\mathrm{d}e_j)\mathrm{d}s \\
& + \sum_{i=1}^{r}\langle D_i u(s) + \sigma_i(s), Q_i(s)x(s-)\rangle \mathrm{d}s \\
& + \sum_{j=1}^{l}\int_{E_j}\langle G_j(e_j,s)u(s) + \phi_j(e_j,s), Z_j(e_j,s)x(s-)\rangle\lambda_j(\mathrm{d}e_j)\mathrm{d}s \\
& + \Big(\mathcal{K}_1(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{K}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\Big)\Big(\mathcal{R}_1(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{R}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\Big)^{-1} \\
& \times \Big(\mathcal{K}_1(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{K}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\Big)^\top \mathrm{d}s - \frac{1}{2}\langle x(s-), L(s)x(s-)\rangle \mathrm{d}s \\
& + \Big[A(s)x(s-) + B(s)u(s) + f(s)\Big]^\top p(s-)\mathrm{d}s \\
& - x(s-)^\top\Big[\widehat{A}(s)^\top p(s-) + \sum_{i=1}^{r}\widehat{C}_i(s)q_i(s) + \sum_{j=1}^{l}\int_{E_j}\widehat{F}_j(e_j,s)z_j(e_j,s)\lambda_j(\mathrm{d}e_j) \\
& \quad + P(s-)f(s) + \sum_{i=1}^{r}\mathcal{K}_{3,i}(s)\sigma_i(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{K}_{4,j}(e_j,s)\phi_j(e_j,s)\lambda_j(\mathrm{d}e_j)\Big]\mathrm{d}s \\
& + \sum_{i=1}^{r}\langle q_i(s), C_i(s)x(s-) + D_i(s)u(s) + \sigma_i(s)\rangle \mathrm{d}s \\
& + \sum_{j=1}^{l}\int_{E_j}\langle z_j(e_j,s), F_j(e_j,s)x(s-) + G_j(e_j,s)u(s) + \phi_j(e_j,s)\rangle\lambda_j(\mathrm{d}e_j)\mathrm{d}s \\
& + \sum_{i=1}^{r}\mathcal{W}_i(s)\mathrm{d}W_i(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{N}_j(e_j,s)\tilde{N}_j(\mathrm{d}e_j,\mathrm{d}s),
\end{aligned}
\tag{12}
$$

where $\mathcal{W}_i$, $i = 1,\ldots,r$, and $\mathcal{N}_j$, $j = 1,\ldots,l$, are components in the stochastic integrals, which can be obtained from Equations (9) and (10).

We are now completing the expression of $\widehat{J}$ in Equation (12) in terms of u. Let $\|x\|_S^2 := x^\top Sx$ for $S \in \mathbb{S}^n$. Then it follows from the definition of $\widehat{J}$ in Equation (11) that:

$$
\begin{aligned}
\widehat{J}(u) = &\frac{1}{2}\langle a, P(0)a\rangle + \langle a, p(0)\rangle \\
&+ \frac{1}{2}\int_0^T \Bigg\| u(s) + \Big(\mathcal{R}_1(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{R}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\Big)^{-1} \\
&\qquad \times \Big(\mathcal{K}_1(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{K}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\Big)^{\top} x(s-) \\
&\qquad + \Big(\mathcal{R}_1(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{R}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\Big)^{-1}\widehat{\mathcal{K}}(s)\Bigg\|_{\mathcal{R}_1(s)+\sum_{j=1}^{l}\int_{E_j}\mathcal{R}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)} \mathrm{d}s \\
&+ \int_0^T \Big[\sum_{i=1}^{r}\frac{1}{2}\langle\sigma_i(s), P(s-)\sigma_i(s)\rangle + \langle f(s), p(s-)\rangle + \sum_{i=1}^{r}\langle\sigma_i(s), q_i(s)\rangle \\
&\quad + \sum_{j=1}^{l}\int_{E_j}\langle\phi_j(e_j,s), (P(s-) + Z_j(e_j,s))\phi_j(e_j,s)\rangle\lambda_j(\mathrm{d}e_j) \\
&\quad + \sum_{j=1}^{l}\int_{E_j}\langle\phi_j(e_j,s), z_j(e_j,s)\rangle\lambda_j(\mathrm{d}e_j) \\
&\quad - \frac{1}{2}\langle\widehat{\mathcal{K}}(s), \Big(\mathcal{R}_1(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{R}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\Big)^{-1}\widehat{\mathcal{K}}(s)\rangle\Big]\mathrm{d}s \\
&+ \sum_{i=1}^{r}\mathcal{W}_i(s)\mathrm{d}W_i(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{N}_j(e_j,s)\widetilde{N}_j(\mathrm{d}e_j,\mathrm{d}s).
\end{aligned}
\tag{13}
$$

Note that the stochastic integrals in Equation (13) are $\mathcal{F}_s$-martingales. This implies that their expectations are zero, i.e., $\mathbb{E}\Big[\sum_{i=1}^{r}\mathcal{W}_i(s)\mathrm{d}W_i(s)\Big] = 0$ and $\mathbb{E}\Big[\sum_{j=1}^{l}\int_{E_j}\mathcal{N}_j(e_j,s)\widetilde{N}_j(\mathrm{d}e_j,\mathrm{d}s)\Big] = 0$. Hence, it follows that:

$$
\begin{aligned}
J(u) = \mathbb{E}[\widehat{J}(u)] = \mathbb{E}\Bigg[&\frac{1}{2}\langle a, P(0)a\rangle + \langle a, p(0)\rangle \\
&+ \frac{1}{2}\int_0^T \Bigg\| u(s) + \Big(\mathcal{R}_1(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{R}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\Big)^{-1} \\
&\qquad \times \Big(\mathcal{K}_1(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{K}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\Big)^{\top} x(s-) \\
&\qquad + \Big(\mathcal{R}_1(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{R}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\Big)^{-1}\widehat{\mathcal{K}}(s)\Bigg\|_{\mathcal{R}_1(s)+\sum_{j=1}^{l}\int_{E_j}\mathcal{R}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)} \mathrm{d}s \\
&+ \int_0^T \Big[\sum_{i=1}^{r}\frac{1}{2}\langle\sigma_i(s), P(s-)\sigma_i(s)\rangle + \langle f(s), p(s-)\rangle \\
&\quad + \sum_{i=1}^{r}\langle\sigma_i(s), q_i(s)\rangle + \sum_{j=1}^{l}\int_{E_j}\langle\phi_j(e_j,s), (P(s-) + Z_j(e_j,s))\phi_j(e_j,s)\rangle\lambda_j(\mathrm{d}e_j) \\
&\quad + \sum_{j=1}^{l}\int_{E_j}\langle\phi_j(e_j,s), z_j(e_j,s)\rangle\lambda_j(\mathrm{d}e_j) \\
&\quad - \frac{1}{2}\langle\widehat{\mathcal{K}}(s), \Big(\mathcal{R}_1(s) + \sum_{j=1}^{l}\int_{E_j}\mathcal{R}_{2,j}(e_j,s)\lambda_j(\mathrm{d}e_j)\Big)^{-1}\widehat{\mathcal{K}}(s)\rangle\Big]\mathrm{d}s\Bigg].
\end{aligned}
\tag{14}
$$

Note that $\mathcal{R}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{R}_{2,j}(e_j, s)\lambda_j(\mathrm{d}e_j)$ is uniformly positive definite in a.e. $(\omega, s) \in \Omega \times [0, T]$. This implies that J in Equation (14) is quadratic in u, which is the equivalent expression of the original objective functional in Equation (1). Hence, with $u = u^*$ given in Equation (6), in view of the fact that J in Equation (14) is quadratic in u, it follows that for any $u \in \mathcal{U}$,

$$\begin{aligned} J(u) \geq J(u^*) = \mathbb{E}\Bigg[&\frac{1}{2}\langle a, P(0)a\rangle + \langle a, p(0)\rangle \\ &+ \int_0^T \Big[\sum_{i=1}^{r} \frac{1}{2}\langle \sigma_i(s), P(s-)\sigma_i(s)\rangle + \langle f(s), p(s-)\rangle + \sum_{i=1}^{r}\langle \sigma_i(s), q_i(s)\rangle \\ &+ \sum_{j=1}^{l} \int_{E_j} \langle \phi_j(e_j, s), z_j(e_j, s)\rangle \lambda_j(\mathrm{d}e_j) \\ &+ \sum_{j=1}^{l} \int_{E_j} \langle \phi_j(e_j, s), (P(s-) + Z_j(e_j, s))\phi_j(e_j, s)\rangle \lambda_j(\mathrm{d}e_j) \\ &- \frac{1}{2}\langle \widehat{\mathcal{K}}(s), \Big(\mathcal{R}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{R}_{2,j}(e_j, s)\lambda_j(\mathrm{d}e_j)\Big)^{-1} \widehat{\mathcal{K}}(s)\rangle \Big] \mathrm{d}s \Bigg]. \end{aligned} \tag{15}$$

Note that Equation (15) shows that $u^* \in \mathcal{U}$ given in Equation (6) is the optimal solution of **(P)** and Equation (7) is the corresponding optimal cost. This completes the proof of the theorem. □

Remark 4. *(a)* *As mentioned in (a) of Remark 3, when all the coefficients are deterministic, i.e., they are independent of Ω, the second and third components of the ISRDE in Equation (3) and the BSDE in Equation (4) become zero. In this case, the optimal solution in Equation (6) is as follows:*

$$\begin{aligned} u^*(s) = &-\Big(R(s) + \sum_{i=1}^{r} D_i(s)^\top P(s-)D_i(s) \\ &\quad + \sum_{j=1}^{l} \int_{E_j} G_j(e_j, s)^\top P(s-)G_j(e_j, s)\lambda_j(\mathrm{d}e_j)\Big)^{-1} \\ &\quad \times \Big(P(s-)B(s) + \sum_{i=1}^{r} C_i(s)^\top P(s-)D_i(s) + S(s) \\ &\quad + \sum_{j=1}^{l} \int_{E_j} F_j(e_j, s)^\top P(s-)G_j(e_j, s)\lambda_j(\mathrm{d}e_j)\Big)^\top x(s-) \\ &- \Big(R(s) + \sum_{i=1}^{r} D_i(s)^\top P(s-)D_i(s) \\ &\quad + \sum_{j=1}^{l} \int_{E_j} G_j(e_j, s)^\top P(s-)G_j(e_j, s)\lambda_j(\mathrm{d}e_j)\Big)^{-1} \\ &\quad \times \Big(B(s)^\top p(s-) + \sum_{i=1}^{r} D_i(s)^\top P(s-)\sigma_i(s) + \sum_{j=1}^{l} \int_{E_j} G_j(e_j, s)^\top P(s-)\phi_j(e_j, s)\Big). \end{aligned}$$

(b) *When the SDE in Equation (2) does not have the additive signal terms (equivalently, $f = \sigma_i = \phi_j = 0$ for $i = 1, \ldots, r$ and $j = 1, \ldots, l$ as seen from (b) of Remark 3), the optimal solution in Equation (6) becomes as follows:*

$$u^*(s) = -\Big(\mathcal{R}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{R}_{2,j}(e_j, s)\lambda_j(\mathrm{d}e_j)\Big)^{-1} \times \Big(\mathcal{K}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{K}_{2,j}(e_j, s)\lambda_j(\mathrm{d}e_j)\Big)^{\top} x(s-).$$

In fact, with the absence of the additive signal terms, the optimal solution is of the pure state-feedback form.

(c) *When the SDE in Equation (2) does not have the jump-diffusion part (equivalently, $F_j = G_j = \phi_j = 0$ for $j = 1, \ldots, l$), the results of this paper (including Theorem 1) are reduced to those for the case without jumps studied in [32].*

Remark 5. *In Equation (6), we define:*

$$\begin{aligned}
\widehat{\mathcal{R}}(s, P, Q, Z) &:= \mathcal{R}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{R}_{2,j}(e_j, s)\lambda_j(\mathrm{d}e_j) \\
\widehat{\mathcal{K}}(s, P, Q, Z) &:= \mathcal{K}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{K}_{2,j}(e_j, s)\lambda_j(\mathrm{d}e_j) \\
\widetilde{\mathcal{K}}(s, P, Q, Z, p, q, z; f, \sigma, \phi) &:= B(s)^{\top} p(s-) + \sum_{i=1}^{r} D_i(s)^{\top} q_i(s) \\
&\quad + \sum_{j=1}^{l} \int_{E_j} G_j(e_j, s)^{\top} z_j(e_j, s)\lambda_j(\mathrm{d}e_j) \\
&\quad + \sum_{i=1}^{r} D_i(s)^{\top} P(s-)\sigma_i(s) \\
&\quad + \sum_{j=1}^{l} \int_{E_j} G_j(e_j, s)^{\top} (P(s-) + Z_j(e_j, s))\phi_j(e_j, s).
\end{aligned}$$

Then Equation (6) can be written as:

$$\begin{aligned}
u^*(s) &= -\widehat{\mathcal{R}}(s, P, Q, Z)^{-1}\widehat{\mathcal{K}}(s, P, Q, Z)x(s-) \\
&\quad - \widehat{\mathcal{R}}(s, P, Q, Z)^{-1}\widetilde{\mathcal{K}}(s, P, Q, Z, p, q, z; f, \sigma, \phi).
\end{aligned}$$

Note that this expression means that the parameters of Equation (6) are dependent on the solutions of the ISRDE in Equation (3) and the BSDE with jump diffusions in Equation (4).

2.2. Verification

In this subsection, we verify that $u^* \in \mathcal{U}$ given in Equation (6) is the optimal solution of **(P)** using the verification theorem obtained from the dynamic programming principle.

By the verification theorem in Theorem A1 of Appendix A (see also [16] (Theorem 3.1) and [2] (Theorem 5.1, Chapter 5)) , we need to solve the following stochastic Hamilton-Jacobi-Bellman (SHJB) equation with jump diffusions:

$$\begin{cases} \mathrm{d}V(s,x) = -H(s,x,(V,DV,D^2V,\Lambda,D\Lambda)(s,x),\Gamma(s,x,\cdot))\mathrm{d}s \\ \qquad + \sum_{i=1}^{r} \Lambda_i(s,x)\mathrm{d}W_i(s) + \sum_{j=1}^{l} \int_{E_j} \Gamma_j(s,x,e_j)\widetilde{N}_j(\mathrm{d}e_j,\mathrm{d}s),\ (s,x) \in [t,T) \times \mathbb{R}^n \\ V(T,x) = \frac{1}{2}\langle x, Mx\rangle,\ x \in \mathbb{R}^n, \end{cases} \tag{16}$$

where $H : \Omega \times [t,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \times \mathbb{R}^{1\times r} \times \mathbb{R}^{r\times n} \times G^2(E,\mathcal{B}(E),\lambda;\mathbb{R}^l) \to \mathbb{R}$ is the Hamiltonian with random coefficients defined by:

$$\begin{aligned} &H(s,x,(V,DV,D^2V,\Lambda,D\Lambda)(s,x),\Gamma(s,x,\cdot)) \\ &:= \inf_u \Bigg\{ \langle DV(s,x), Ax+Bu\rangle + \mathrm{Tr}\left(\sum_{i=1}^{r}[C_ix+D_iu+\sigma_i]D\Lambda_i(s,x)\right) \\ &\quad + \frac{1}{2}\mathrm{Tr}\left(\sum_{i=1}^{r}[C_ix+D_iu+\sigma_i][C_ix+D_iu+\sigma_i]^\top D^2V(s,x)\right) \\ &\quad + \sum_{j=1}^{l}\int_{E_j}[V(t,x+\sum_{j=1}^{l}[F_j(e_j)x+G_j(e_j)+\phi_j(e_j)]) \\ &\qquad - V(s,x) - \langle DV(s,x), \sum_{j=1}^{l}[F_j(e_j)x+G_j(e_j)+\phi_j(e_j)]\rangle]\lambda_j(\mathrm{d}e_j) \\ &\quad + \sum_{j=1}^{l}\int_{E_j}[\Gamma_j(s,x+\sum_{j=1}^{l}[F_j(e_j)x+G_j(e_j)+\phi_j(e_j)],e_j) - \Gamma_j(s,x,e_j)]\lambda_j(\mathrm{d}e_j) \\ &\quad + \frac{1}{2}\left\langle \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} L & S \\ S^\top & R \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}\right\rangle \Bigg\}. \end{aligned} \tag{17}$$

Remark 6. *(a)* *Note that the SHJB with jump diffusions in Equation (16) is a class of integro-type second-order stochastic partial differential equations, where $(V,\Lambda_1,\ldots,\Lambda_r,\Gamma_1,\ldots,\Gamma_l)$ define the solution of Equation (16). Moreover, as in (a) of Remark 3, if the coefficients in Equation (16) (or (1) and (2)) are deterministic, then $\Lambda_i = 0$ and $\Gamma_j = 0$ for $i = 1,\ldots,r$ and $j = 1,\ldots,l$.*

(b) *In view of the verification theorem in Theorem A1 of Appendix A, if we solve the SHJB with jump diffusions in Equation (16), then the minimizer of the Hamiltonian in Equation (17) is the optimal control for* ***(P)*** *and V is the corresponding optimal cost. In fact, V constitutes the value function for* ***(P)*** *in the dynamic programming principle.*

We now conjecture that the solution of Equation (16) is quadratic in x, i.e., V, Λ_i and Γ_j are quadratic in x, where:

$$\begin{cases} V(s,x) = \frac{1}{2}\langle x, \mathcal{T}(s)x\rangle + \langle x(s), \widehat{p}(s)\rangle + \eta(s) \\ \Lambda_i(s,x) = \frac{1}{2}\langle x, \mathcal{Q}_i(s)x\rangle + \langle x(s), \widehat{q}_i(s)\rangle + \widehat{\eta}_i(s),\ i = 1,\ldots,r \\ \Gamma_j(s,x,e_j) = \frac{1}{2}\langle x, \mathcal{Z}_j(e_j,s)x\rangle + \langle x(s), \widehat{z}_j(e_j,s)\rangle + \widetilde{\eta}_j(e_j,s),\ j = 1,\ldots,l. \end{cases} \tag{18}$$

Now, based on Equation (18) and using Equation (17), we can show that if:

$$R(s)+\sum_{i=1}^{r}D_i(s)^\top\mathcal{T}(s-)D_i(s) \tag{19}$$
$$+\sum_{j=1}^{l}\int_{E_j}G_j(e_j,s)^\top(\mathcal{T}(s-)+\mathcal{Z}_j(e_j,s))G_j(e_j,s)\lambda_j(\mathrm{d}e_j)>0,\ \text{a.e,}\ (\omega,s)\in\Omega\times[0,T],$$

that is, it is uniformly positive definite in a.e. $(\omega,s)\in\Omega\times[0,T]$, then the unique minimizer of the Hamiltonian in Equation (17) exists, which can be written as follows (see the notation in Remark 5):

$$\begin{aligned}\widehat{u}^*(s)=&-\widehat{\mathcal{R}}(s,\mathcal{T},\mathcal{Q},\mathcal{Z})^{-1}\widehat{\mathcal{K}}(s,\mathcal{T},\mathcal{Q},\mathcal{Z})x(s-)\\&-\widehat{\mathcal{R}}(s,\mathcal{T},\mathcal{Q},\mathcal{Z})^{-1}\widetilde{\mathcal{K}}(s,\mathcal{T},\mathcal{Q},\mathcal{Z},\widehat{p},\widehat{q},\widehat{z};f,\sigma,\phi).\end{aligned} \tag{20}$$

Moreover, by substituting Equation (20) into (17), we can show that if:

$$\begin{aligned}(\mathcal{T},\mathcal{Q}_1,\ldots,\mathcal{Q}_r,\mathcal{Z}_1,\ldots,\mathcal{Z}_l)\in\ &\mathcal{C}^2_{\mathbb{F}}(\mathbb{S}^n)\times\mathcal{L}^2_{\mathbb{F},p}(\mathbb{S}^n\times\cdots\times\mathbb{S}^n)\\&\times\mathcal{G}^2_{\mathbb{F},p}(E_1,\mathcal{B}(E_1),\lambda_1;\mathbb{S}^n)\times\cdots\times\mathcal{G}^2_{\mathbb{F},p}(E_l,\mathcal{B}(E_l),\lambda_l;\mathbb{S}^n)\end{aligned} \tag{21}$$

satisfies the ISRDE with jump diffusions in Equation (3), and the BSDE with jump diffusions in Equation (4) holds with the following expressions:

$$\begin{aligned}(\widehat{p},\widehat{q}_1,\ldots,\widehat{q}_r,\widehat{z}_1,\ldots,\widehat{z}_l)\in\ &\mathcal{C}^2_{\mathbb{F}}(\mathbb{R}^n)\times\mathcal{L}^2_{\mathbb{F},p}(\mathbb{R}^n\times\cdots\times\mathbb{R}^n)\\&\times\mathcal{G}^2_{\mathbb{F},p}(E_1,\mathcal{B}(E_1),\lambda_1;\mathbb{R}^n)\times\cdots\times\mathcal{G}^2_{\mathbb{F},p}(E_l,\mathcal{B}(E_l),\lambda_l;\mathbb{R}^n),\end{aligned} \tag{22}$$

then Equation (18) solves the SHJB equation with jump diffusions Equation (16). That is, Equation (18) is the solution of the SHJB equation with jump diffusions in Equation (16), provided that Equation (21) satisfies (3), and the BSDE (4) holds with the expressions in Equation (22), where the unique minimizer of the Hamiltonian Equation (17) is given in Equation (20) under the condition that Equation (19) is uniformly positive definite in a.e. $(\omega,s)\in\Omega\times[0,T]$.

By the verification theorem in Theorem A1, this implies that Equation (20) is the optimal control for **(P)** and V in Equation (18) is the corresponding optimal cost (see (b) of Remark 6). Since Equation (21) satisfies Equations (3) and (22) and holds (4), we can easily check that Equation (20) is equivalent to (6) in Theorem 1, where Equation (5) is equivalent to (19). Moreover, it can be observed that $V(0,a)$ in Equation (18) is equivalent to (7). This verifies that under the condition that Equation (19) is uniformly positive definite in a.e. $(\omega,s)\in\Omega\times[0,T]$ (equivalent to (5)), (20) (equivalently (6)) is the optimal solution of **(P)** and V in (18) is the corresponding optimal cost, which is equivalent to Equation (7).

In summary, we state the following result:

Proposition 3. *Assume that (H.1)–(H.4) hold. Suppose that Equation (21) satisfies the ISRDE with jump diffusions in Equations (3) and (22) hold the BSDE with jump diffusions in Equation (4). Assume that Equation (19) is uniformly positive definite in a.e. $(\omega,s)\in\Omega\times[0,T]$. Then Equation (18) with the minimizer of the Hamiltonian (17) given in (20) solves the SHJB equation with jump diffusions in Equation (16). Hence, Equation (20), which is equivalent to Equation (6), is the optimal solution for* **(P)***, and $V(0,a)$ in Equation (18) is the corresponding optimal cost, which is equivalent to Equation (7).*

Remark 7. *The superiority of the direct approach established in Theorem 1, compared with the verification theorem approach in Proposition 3, is that in Theorem 1 we do not need to solve the complex stochastic HJB equation with jump diffusions in Equation (16). Note that to solve the SHJB equation in Equation (16), we need prior knowledge of the structure of the solution to the SHJB equation as in Equation (18). In the direct approach, this initial guess is not required as seen* *from the proof of Theorem 1. Furthermore, Theorem 1 explicitly shows that Equation (6) is the optimal solution provided that Equation (5) holds, by constructing the equivalent cost functional in Equation (14) that is quadratic in u under the condition in Equation (5).*

2.3. Examples

We now discuss three examples of **(P)**. We assume that $r = l = 1$ and drop the subscript i and j in (2) to simplify the notation.

Example 1. *The modified LQ problem studied in [47] (Example 3.1) and [32] (Example 1) to jump-diffusion models is given by:*

$$J(u) = \frac{1}{2}\mathbb{E}\Big[\int_0^T [L(s)x(s)^2 + R(s)u(s)^2]\mathrm{d}s + W(T)x(T)^2\Big],$$

where the corresponding SDE with jump diffusions is as follows:

$$\begin{cases} \mathrm{d}x(s) = f(s)\mathrm{d}s + [x(s-) + u(s)]\mathrm{d}W(s) + \int_E [x(s-) + u(s) + \phi(e,s)]\widetilde{N}(\mathrm{d}e, \mathrm{d}s) \\ x(0) = a. \end{cases}$$

Based on Theorem 1, the corresponding optimal solution can be written as:

$$\begin{aligned} u^*(s) = &-\Big(R(s) + P(s-) + \int_E (P(s-) + Z(e,s))\lambda(\mathrm{d}e)\Big)^{-1} \\ &\quad \times (Q(s) + P(s-) + \int_E [2Z(e,s) + P(s-)]\lambda(\mathrm{d}e))x(s-) \\ &- \Big(R(s) + P(s-) + \int_E (P(s-) + Z(e,s))\lambda(\mathrm{d}e)\Big)^{-1} \\ &\quad \times (q(s) + \int_E z_j(e,s)\lambda_j(\mathrm{d}e) + \int_E (P(s-) + Z(e,s))\phi(e,s)\lambda(\mathrm{d}e)), \end{aligned}$$

provided that $R(s) + P(s-) + \int_E (P(s-) + Z(e,s))\lambda(\mathrm{d}e)$ is uniformly positive definite in a.e. $(\omega, s) \in \Omega \times [0,T]$ from (5). Here, the ISRDE with jump diffusions is given by:

$$\begin{cases} \mathrm{d}P(s) = -\Bigg[L(s) + 2Q(s) + \int_E (3Z(e,s) + P(s-))\lambda(\mathrm{d}e) \\ \qquad\qquad - \dfrac{(Q(s) + P(s-) + \int_E [2Z(e,s) + P(s-)]\lambda(\mathrm{d}e))^2}{R(s) + P(s-) + \int_E (P(s-) + Z(e,s))\lambda(\mathrm{d}e)}\Bigg]\mathrm{d}s \\ \qquad\qquad + Q(s)\mathrm{d}W(s) + \int_E Z(e,s)\widetilde{N}(\mathrm{d}e, \mathrm{d}s) \\ P(T) = W(T), \end{cases}$$

where it admits a unique solution when R is uniformly positive definite in a.e. $(\omega, s) \in \Omega \times [0,T]$ in view of Proposition 1. Moreover, the BSDE with jump diffusion and the corresponding optimal cost can be derived from Equations (4) and (7), respectively. Note that by Proposition 1, the positive definiteness condition holds when R and L are uniformly positive definite in a.e. $(\omega, s) \in \Omega \times [0,T]$.

Example 2. *Next, we extend the (indefinite) mean-variance portfolio optimization problem of [28] (Example 4.7) and [32] (Example 2) to the case for jump-diffusion models. In particular, we have:*

$$J(u) = \frac{1}{2}\mathbb{E}\Big[\int_0^1 x(s)^2\mathrm{d}s + x(1)^2\Big]$$

and:

$$\begin{cases} \mathrm{d}x(s) = [x(s-) + B(s)u(s)]\mathrm{d}s + u(s)\mathrm{d}W(s) + \int_E x(s-)\widetilde{N}(\mathrm{d}e, \mathrm{d}s) \\ x(0) = a. \end{cases} \tag{23}$$

Note that the state process captures the wealth process, *where the jump-diffusion part describes the random jumps in the stock market. By Theorem 1 and (b) of Remark 4, the optimal portfolio strategy is given by:*

$$u^*(s) = \frac{B(s)P(s-) + Q(s)}{P(s-)} x(s-), \tag{24}$$

provided that P is uniformly positive definite in a.e. $(\omega, s) \in \Omega \times [0, T]$ from (5), where (P, Q, Z) satisfy the ISRDE given by:

$$\begin{cases} \mathrm{d}P(s) = -\Bigg[1 + 2P(s-) + \int_E (P(s-) + 3Z(e,s))\lambda(\mathrm{d}e) \\ \qquad\qquad - \dfrac{(P(s-)B(s) + Q(s))^2}{P(s-)}\Bigg]\mathrm{d}s + Q(s)\mathrm{d}W(s) + \int_E Z(e,s)\widetilde{N}(\mathrm{d}e, \mathrm{d}s) \\ P(1) = 1. \end{cases} \tag{25}$$

Assume that $a = 1$ and $B(s) = W(s)$ in (23), i.e., B is the random coefficient. Furthermore, assume that $E = \{1\}$ with $\lambda = 1$, i.e., the Poisson process has a unit magnitude of jumps with the rate of 1 ($\lambda = 1$). The simulation results of Equation (25) and the controlled state process in Equation (23) under the optimal control in Equation (24) are depicted in Figure 1. Note that P in Figure 1(left) is positive. The plots were obtained by using the finite difference method for BSDEs based on Monte Carlo simulations as well as the (stochastic) Euler's method [5,50–52]. Note that the drift term of Equation (25) has the quadratic growth in Q. Hence, in some situations, the explicit Euler scheme used in this simulation to obtain Figure 1 may show some numerical issues (e.g., blowing up in finite time and/or negative solutions for positive processes). Fortunately, in our case, since the simulation time is short, the terminal condition of P is a constant, and the quadratic growth in Q is normalized by P, we did not observe such numerical issues in Figure 1.

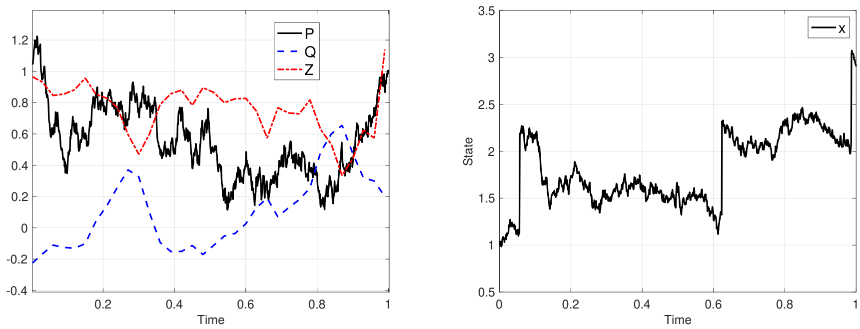

Figure 1. The simulation results of Example 2: the sample path of the ISRDE with jump diffusions in Equation (25) (**left**); the sample path of the controlled state process in Equation (23) under the optimal control in Equation (24) (**right**).

Example 3. *We study the two-dimensional problem of **(P)**. Consider the minimization of:*

$$J(u) = \mathbb{E}\Big[\int_0^2 [x_1(s)^2 + x_2(s)^2 + u(s)^2]\mathrm{d}s + x_1(2)^2 + M_2 x_2(2)^2\Big]$$

subject to:

$$\begin{cases} \mathrm{d}\begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \left[\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(s-) \\ x_2(s-) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(s)\right]\mathrm{d}s + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\mathrm{d}W(s) + \int_E \begin{bmatrix} 0 \\ 1 \end{bmatrix}\widetilde{N}(\mathrm{d}e, \mathrm{d}s) \\ x_0 = a. \end{cases} \tag{26}$$

Here, M_2 is an $\mathbb{R}$-valued random variable that is positive semidefinite and uniformly bounded in a.e. $\omega \in \Omega$. Note that this problem holds the conditions in Proposition 1 and Equation (5). We can easily see that the corresponding ISRDE is the $\mathbb{S}^2$-valued stochastic process with $(P, Q, Z) = \left(\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}, \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix}, \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{bmatrix}\right)$, and the BSDE is the $\mathbb{R}^2$-valued stochastic process with $(p, q, z) = \left(\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right)$.

The corresponding ISRDE is given by:

$$\begin{cases} \mathrm{d}P_{11}(s) = -\left[1 - P_{12}(s)^2\right]\mathrm{d}s + Q_{11}(s)\mathrm{d}W(s) + \int_E Z_{11}(e,s)\widetilde{N}(\mathrm{d}e, \mathrm{d}s) \\ \mathrm{d}P_{12}(s) = -\left[P_{11}(s) - P_{12}(s)P_{22}(s)\right]\mathrm{d}s + Q_{12}(s)\mathrm{d}W(s) + \int_E Z_{12}(e,s)\widetilde{N}(\mathrm{d}e, \mathrm{d}s) \\ \mathrm{d}P_{22}(s) = -\left[2P_{12}(s) + 1 - P_{22}(s)^2\right]\mathrm{d}s + Q_{22}(s)\mathrm{d}W(s) + \int_E Z_{22}(e,s)\widetilde{N}(\mathrm{d}e, \mathrm{d}s) \\ P_{11}(2) = 0,\ P_{12}(2) = 0,\ P_{22}(2) = M_2, \end{cases} \tag{27}$$

and the BSDE with jump diffusions as follows:

$$\begin{cases} \mathrm{d}p_1(s) = -\left[-P_{12}(s)p_2(s) + Q_{12}(s) + \int_E Z_{12}(e,s)\lambda(\mathrm{d}e)\right]\mathrm{d}s \\ \qquad\qquad + q_1(s)\mathrm{d}W(s) + \int_E z_1(e,s)\widetilde{N}(\mathrm{d}e, \mathrm{d}s) \\ \mathrm{d}p_2(s) = -\left[p_1(s) - P_{22}(s)p_2(s) + Q_{22}(s) + \int_E Z_{22}(e,s)\lambda(\mathrm{d}e)\right]\mathrm{d}s \\ \qquad\qquad + q_2(s)\mathrm{d}W(s) + \int_E z_2(e,s)\widetilde{N}(\mathrm{d}e, \mathrm{d}s) \\ p_1(2) = 0,\ p_2(2) = 0. \end{cases} \tag{28}$$

By Propositions 1 and 2, Equations (27) and (28) admit unique solutions. Then, due to the terminal conditions in Equations (27) and (28), it can be easily seen that $(P_{11}, Q_{11}, Z_{11}) = (0, 0, 0)$, $(P_{12}, Q_{12}, Z_{12}) = (0, 0, 0)$, and $(p_1, q_1, z_1) = (0, 0, 0)$. Hence, by Theorem 1, the optimal solution to the above problem can be obtained by:

$$u^*(s) = -P_{22}(s-)x_2(s-) - p_2(s-). \tag{29}$$

Assume that $a = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$, $M_2 = W(2)^2$ and $E = \{1\}$ with $\lambda = 2$. The simulation results of (P_{22}, Q_{22}, Z_{22}) in Equation (27), (p_2, q_2, z_2) in Equation (28), and the controlled sample path in Equation (26) under the optimal control in Equation (29) are shown in Figure 2. Similar to Example 2, the drift term of Equation (27) has the quadratic growth in P_{22}. Fortunately, in our case, since the simulation time is short and all the parameters are constants, we did not observe numerical issues in Figure 2.

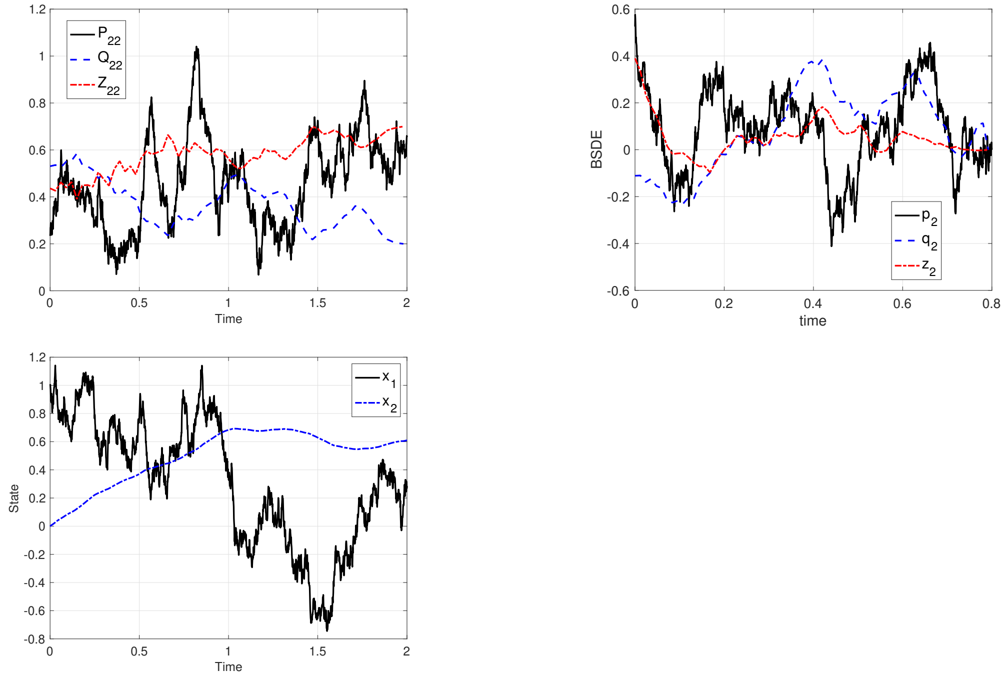

Figure 2. The simulation results of Example 3: the sample path of the ISRDE with jump diffusions (P_{22}, Q_{22}, Z_{22}) in Equation (27) (**left**); the sample path of the BSDE with jump diffusions (p_2, q_2, z_2) in Equation (28) (**middle**); the sample path of the controlled state process in Equation (26) under the optimal control in Equation (29) (**right**).

3. Concluding Remarks

In this paper, we have established the direct approach, the completion of squares method, to find the explicit optimal solution and the optimal cost of the LQ stochastic optimal control problem for jump-diffusion models with random coefficients. The approach constructs an equivalent objective functional that is quadratic in u under the positive definiteness condition dependent on the solution of the ISRDE. Using the equivalent objective functional, the explicit optimal solution and the optimal cost have been obtained, where the corresponding optimal control is linear in the state characterized by the ISRDE and the BSDE with jump diffusions. The optimality of the proposed solution has been validated via the verification theorem, which requires solving the complex SHJB equation with jump diffusions.

One immediate future research problem is to generalize the SDE in Equation (2) to the Markov regime-switching jump-diffusion model. Another problem would be the case where there are multiple players, which can be treated by the stochastic zero-sum game framework. Moreover, it would also be interesting to consider the general noise model in the SDE, such as the fractional Brownian motion. Finally, it is necessary to study more general conditions than those in Proposition 1 for the existence and uniqueness of the solution to the ISRDE with jump diffusions in Equation (3). As mentioned in [38] (Section 1.3), this problem is challenging, compared to the case without jumps studied in the earlier references (e.g., [22] (Section 3), [26] (Sections 4 and 5), [29]) (Section 3), since $(\mathcal{R}_1(s) + \sum_{j=1}^{l} \int_{E_j} \mathcal{R}_{2,j}(e_j, s)\lambda_j(\mathrm{d}e_j))^{-1}$ in Equation (3) includes not only the first component of the solution of Equation (3) (that is P) but also the last components (jump-diffusion part) of the solution (that is Z), where Z is only the square integrable. This problem will be addressed in the near future.

Author Contributions: J.M.: problem formulation, analysis, general writing and editing, revising the paper, simulations; J.-H.C.: analysis, general writing and editing, revising the paper. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported in part by the National Research Foundation of Korea (NRF) Grant funded by the Ministry of Science and ICT, South Korea (NRF-2021R1A2C2094350, NRF-2017R1A5A1015311) and in part by the Institute of Information and communications Technology Planning and Evaluation (IITP) grant funded by the Korea government (MSIT) (No. 2020-0-01373, Artificial Intelligence Graduate School Program (Hanyang University)).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the two anonymous reviewers for their valuable comments on the earlier version of the manuscript.

Conflicts of Interest: The authors declare no potential conflict of interests.

Appendix A. Verification Theorem

In this appendix, we state and prove the verification theorem of **(P)** used in Section 2.2. For convenience, we rewrite Equation (17) as follows:

$$
\begin{aligned}
&H(s,x,(V,DV,D^2V,\Lambda,D\Lambda)(s,x),\Gamma(s,x,\cdot)) \\
&:= \inf_u \widehat{H}(s,x,(V,DV,D^2V,\Lambda,D\Lambda)(s,x),\Gamma(s,x,\cdot);u) \\
&:= \inf_u \Big\{ \widehat{H}_1(s,x,(V,DV,D^2V,\Lambda,D\Lambda)(s,x),\Gamma(s,x,\cdot);u) + \frac{1}{2}\Big\langle \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} L & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \Big\rangle \Big\}.
\end{aligned}
$$

Then the verification theorem of **(P)** can be stated as follows:

Theorem A1. *Suppose that the $(r+l+1)$-tuple $(V,\Lambda_1,\ldots,\Lambda_r,\Gamma_1,\ldots,\Gamma_l)$ solve the SHJB equation with jump diffusions in Equation (16). Then it holds that:*

$$V(0,x) \leq J(u),\ \forall x \in \mathbb{R}^n,\ u \in \mathcal{U}.$$

Moreover, assume that $\widehat{u}(\cdot) \in \mathbb{R}^m$ with $\widehat{u} := (\widehat{u}(s))_{s\in[0,T]} \in \mathcal{U}$ is the minimizing solution of the Hamiltonian in Equation (17) for $s \in [0,T]$, $\mathbb{P}$-a.s., i.e.,

$$
\begin{aligned}
&H(s,x(s),(V,DV,D^2V,\Lambda,D\Lambda)(s,x(s)),\Gamma(s,x(s),\cdot)) \\
&= \widehat{H}(s,x(s),(V,DV,D^2V,\Lambda,D\Lambda)(s,x(s)),\Gamma(s,(s),\cdot);\widehat{u}(s)),\ \forall s \in [0,T],\ \mathbb{P}\text{-a.s.}
\end{aligned}
$$

Then $\widehat{u} := (\widehat{u}(s))_{s\in[0,T]} \in \mathcal{U}$ is the optimal control of **(P)** *and the optimal cost of* **(P)** *is $V(0,x) = J(\widehat{u})$ for $x \in \mathbb{R}^n$.*

Proof. For $\widehat{u} \in \mathcal{U}$, by applying the Itô-Wentzell's formula, we have:

$$
\begin{aligned}
&\mathbb{E}[V(T, x(T))] \\
&= V(0,x) + \mathbb{E}\bigg[\int_0^T [\widehat{H}_1(s, x(s), (V, DV, D^2V, \Lambda, D\Lambda)(s, x(s)), \Gamma(s, x(s), \cdot); \widehat{u}(s)) \\
&\qquad - H(s, x(s), (V, DV, D^2V, \Lambda, D\Lambda)(s, x(s)), \Gamma(s, (s), \cdot))]\mathrm{d}s\bigg] \\
&= V(0,x) + \mathbb{E}\bigg[\int_0^T [\widehat{H}_1(s, x(s), (V, DV, D^2V, \Lambda, D\Lambda)(s, x(s)), \Gamma(s, x(s), \cdot); \widehat{u}(s)) \\
&\qquad - \widehat{H}(s, x(s), (V, DV, D^2V, \Lambda, D\Lambda)(s, x(s)), \Gamma(s, (s), \cdot); \widehat{u}(s))]\mathrm{d}s\bigg] \\
&= V(0,x) - \frac{1}{2}\mathbb{E}\bigg[\int_0^T \Big\langle \begin{bmatrix} x(s) \\ \widehat{u}(s) \end{bmatrix}, \begin{bmatrix} L(s) & S(s) \\ S(s)^\top & R(s) \end{bmatrix} \begin{bmatrix} x(s) \\ \widehat{u}(s) \end{bmatrix} \Big\rangle \mathrm{d}s\bigg].
\end{aligned}
$$

This, together with the terminal condition of the SHJB equation in (16), implies:

$$
\begin{aligned}
V(0,x) &= \frac{1}{2}\mathbb{E}\bigg[\int_0^T \Big\langle \begin{bmatrix} x(s) \\ \widehat{u}(s) \end{bmatrix}, \begin{bmatrix} L(s) & S(s) \\ S(s)^\top & R(s) \end{bmatrix} \begin{bmatrix} x(s) \\ \widehat{u}(s) \end{bmatrix} \Big\rangle \mathrm{d}s + \langle x(T), Mx(T) \rangle \bigg] \qquad \text{(A1)} \\
&= J(\widehat{u}).
\end{aligned}
$$

Furthermore, note that for any $u \in \mathcal{U}$,

$$
\begin{aligned}
&H(s, x(s), (V, DV, D^2V, \Lambda, D\Lambda)(s, x(s)), \Gamma(s, x(s), \cdot)) \qquad \text{(A2)} \\
&\leq \widehat{H}(s, x(s), (V, DV, D^2V, \Lambda, D\Lambda)(s, x(s)), \Gamma(s, (s), \cdot); u(s)).
\end{aligned}
$$

Then by applying the Itô-Wentzell's formula and using (A2),

$$
\begin{aligned}
&\mathbb{E}[V(T, x(T))] \\
&= V(0,x) + \mathbb{E}\bigg[\int_0^T [\widehat{H}_1(s, x(s), (V, DV, D^2V, \Lambda, D\Lambda)(s, x(s)), \Gamma(s, x(s), \cdot); u(s)) \\
&\qquad - H(s, x(s), (V, DV, D^2V, \Lambda, D\Lambda)(s, x(s)), \Gamma(s, (s), \cdot))]\mathrm{d}s\bigg] \\
&\geq V(0,x) + \mathbb{E}\bigg[\int_0^T [\widehat{H}_1(s, x(s), (V, DV, D^2V, \Lambda, D\Lambda)(s, x(s)), \Gamma(s, x(s), \cdot); u(s)) \\
&\qquad - \widehat{H}(s, x(s), (V, DV, D^2V, \Lambda, D\Lambda)(s, x(s)), \Gamma(s, (s), \cdot); u(s))]\mathrm{d}s\bigg] \\
&= V(0,x) - \frac{1}{2}\mathbb{E}\bigg[\int_0^T \Big\langle \begin{bmatrix} x(s) \\ u(s) \end{bmatrix}, \begin{bmatrix} L(s) & S(s) \\ S(s)^\top & R(s) \end{bmatrix} \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} \Big\rangle \mathrm{d}s\bigg].
\end{aligned}
$$

Hence, with the terminal condition of the SHJB equation in (16), we have for $u \in \mathcal{U}$,

$$
\begin{aligned}
V(0,x) &\leq \frac{1}{2}\mathbb{E}\bigg[\int_0^T \Big\langle \begin{bmatrix} x(s) \\ u(s) \end{bmatrix}, \begin{bmatrix} L(s) & S(s) \\ S(s)^\top & R(s) \end{bmatrix} \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} \Big\rangle \mathrm{d}s + \langle x(T), Mx(T) \rangle \bigg] \qquad \text{(A3)} \\
&= J(u).
\end{aligned}
$$

Note that in (A3), the inequality becomes equality as in (A1) when $u = \widehat{u} \in \mathcal{U}$. Then (A1) and (A3) show the desired result. This completes the proof. $\square$

References

1. Applebaum, D. *Lévy Processes and Stochastic Calculus*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2009.
2. Yong, J.; Zhou, X.Y. *Stochastic Controls: Hamiltonian Systems and HJB Equations*; Springer: New York, NY, USA, 1999.
3. Touzi, N. *Optimal Stochastic Control, Stochastic Target Problems, and Backward SDE*; Springer: Berlin/Heidelberg, Germany, 2013.
4. Fleming, W.; Soner, H.M. *Controlled Markov Processes and Viscosity Solutions*, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 2006.
5. Hanson, F.B. *Applied Stochastic Processes and Control for Jump-Diffusions: Modeling, Analysis and Computation*; SIAM: Philadelphia, PA, USA, 2007.
6. Privault, N. Notes on Stochastic Finance. Available online: https://personal.ntu.edu.sg/nprivault/MA5182/stochastic_finance.pdf (accessed on 11 November 2021).
7. Huang, M.; Malhame, R.; Caines, P.E. Computationally Tractable Stochastic Power Control Laws in Wireless Communications. *IEEE Trans. Autom. Control* **2005**, *50*, 263–268. [CrossRef]
8. Başar, T.; Olsder, G.J. *Dynamic Noncooperative Game Theory*, 2nd ed.; SIAM: Philadelphia, PA, USA, 1999.
9. Huang, M.; Caines, P.E.; Malhame, R.P. Uplink Power Adjustment in Wireless Communication Systems: A Stochastic Control Analysis. *IEEE Trans. Autom. Control* **2004**, *49*, 1693–1708. [CrossRef]
10. Kizilkale, A.C.; Malhame, R.P. Collective Target Tracking Mean Field Control for Markovian Jump-Driven Models of Electric Water Heating Loads. In Proceedings of the 19th World Congress, The International Federation of Automatic Control, Cape Town, South Africa, 24–29 August 2014; pp. 1867–1872.
11. Kizilkale, A.C.; Salhab, R.; Malhame, R.P. An integral control formulation of mean field game based large scale coordination of loads in smart grids. *Automatica* **2019**, *100*, 312–322. [CrossRef]
12. Mehri, S.; Scheutzow, M.; Stannat, W.; Zangeneh, B.Z. Propagation of chaos for stochastic spatially structured neuronal networks with delay driven by jump diffusions. *Ann. Appl. Probab.* **2020**, *30*, 175–207. [CrossRef]
13. Chin, E.; Chieng, D.; Teh, V.; Natkaniec, M.; Loziak, K.; Gozdecki, J. Wireless link prediction and triggering using modified Ornstein-Uhlenbeck jump diffusion process. *Wirel. Netw.* **2014**, *20*, 379–396.
14. Lasry, J.M.; Lions, P.L. Mean Field Games. *Jpn. J. Math.* **2007**, *2*, 229–260.
15. Kou, S.G. A Jump-Diffusion Model for Option Pricing. *Manag. Sci.* **2002**, *48*, 1086–1101. [CrossRef]
16. Oksendal, B.; Sulem, A. *Applied Stochastic Control of Jump Diffusions*, 2nd ed.; Springer: Berlin Heidelberg, Germany, 2006.
17. Moon, J. Backward Reachability Approach to State-Constrained Stochastic Optimal Control Problems for Jump Diffusion Systems. Available online: https://arxiv.org/pdf/2006.05577.pdf (accessed on 11 November 2021) .
18. Oksendal, B.; Zhang, T. The Itô-Wentzell Formula and Forward Stochastic Differential Equations Driven by Poisson Random Measures. *Osaka J. Math* **2007**, *44*, 207–230.
19. Bismut, J.M. Linear-Quadratic Optimal Stochastic Control with Random Coefficients. *SIAM J. Control Optim.* **1976**, *14*, 419–444.
20. Peng, S. Stochastic Hamilton-Jacobi-Bellman Equations. *SIAM J. Control Optim.* **1992**, *30*, 284–304.
21. Chen, S.; Yong, J. Stochastic Linear Quadratic Optimal Control Problems. *Appl. Math. Optim.* **2001**, *43*, 21–45. [CrossRef]
22. Chen, S.; Yong, J. Stochastic Linear Quadratic Optimal Control Problems with Random Coefficients. *Chin. Ann. Math.* **2000**, *21*, 323–338. [CrossRef]
23. Tang, S. General Linear Quadratic Optimal Stochastic Control Problems with Random Coefficients: Linear Stochastic Hamiltonian Systems and Backward Stochastic Riccati Equations. *SIAM J. Control Optim.* **2003**, *42*, 53–75. [CrossRef]
24. Kohlmann, M.; Tang, S. Minimization of Risk and Linear Quadratic Optimal Control Theory. *SIAM J. Control Optim.* **2003**, *42*, 1118–1142. [CrossRef]
25. Kohlmann, M.; Tang, S. Multidimensional Backward Stochastic Riccati Equations and Applications. *SIAM J. Control Optim.* **2003**, *41*, 1696–1721. [CrossRef]
26. Hu, Y.; Zhou, X.Y. Indefinite Stochastic Riccati Equations. *SIAM J. Control Optim.* **2003**, *42*, 123–137. [CrossRef]
27. Qian, Z.; Zhou, X. Existence of Solutions to a Class of Indefinite Stochastic Riccati Equations. *SIAM J. Control Optim.* **2013**, *51*, 221–229. [CrossRef]
28. Huang, J.; Yu, Z. Solvability of indefinite stochastic Riccati equations and linear quadratic optimal control problems. *Syst. Control Lett.* **2014**, *68*, 68–75. [CrossRef]
29. Du, K. Solvability Conditions for Indefinite Linear Quadratic Optimal Stochastic Control Problems and Associated Stochastic Riccati Equations. *SIAM J. Control Optim.* **2015**, *53*, 3673–3689. [CrossRef]
30. Tang, S. Dynamic Programming for General Linear Quadratic Optimal Stochastic Control with Random Coefficients. *SIAM J. Control Optim.* **2015**, *53*, 1082–1106. [CrossRef]
31. Qiu, J. Viscosity Solutions of Stochastic Hamilton-Jacobi-Bellman Equations. *SIAM J. Control Optim.* **2018**, *56*, 3708–3730. [CrossRef]
32. Moon, J.; Duncan, T.E. A Simple Proof of Indefinite Linear-Quadratic Stochastic Optimal Control with Random Coefficients. *IEEE Trans. Autom. Control* **2020**, *65*, 5422–5428. [CrossRef]

33. Moon, J. A Feedback Nash Equilibrium for Affine-Quadratic Zero-Sum Stochastic Differential Games with Random Coefficients. *IEEE Control Syst. Lett.* **2020**, *4*, 868–873. [CrossRef]
34. Yang, Z.; Tang, S. Dynkin Game of Stochastic Differential Equations with Random Coefficients and Associated Backward Stochastic Partial Differential Variational Inequality. *SIAM J. Control Optim.* **2013**, *51*, 64–95. [CrossRef]
35. Wang, H.; Wu, Z. Time-Inconsistent Optimal Control Problem with Random Coefficients and Stochastic Equilibrium HJB Equation. *Math. Control Relat. Fields* **2015**, *5*, 651–678. [CrossRef]
36. Duncan, T.E.; Tembine, H. Linear–Quadratic Mean-Field-Type Games: A Direct Method. *Games* **2018**, *9*, 1–18. [CrossRef]
37. Barreiro-Gomez, J.; Duncan, T.E.; Tembine, H. Linear-Quadratic Mean-Field-Type Games: Jump-Diffusion Process With Regime Switching. *IEEE Trans. Autom. Control* **2019**, *64*, 4329–4336. [CrossRef]
38. Zhang, F.; Dong, Y.; Meng, Q. Backward Stochastic Riccati Equation with Jumps Associated with Stochastic Linear Quadratic Optimal Control with Jumps and Random Coefficients. *SIAM J. Control Optim.* **2020**, *58*, 393–424. [CrossRef]
39. Meng, Q. General Linear Quadratic Optimal Stochastic Control Problem Driven by a Brownian Motion and Poisson Random Martingale Measure with Random Coefficients. *Stoch. Anal. Appl.* **2014**, *32*, 88–109. [CrossRef]
40. Mitsui, K.; Tabata, Y. A Stochastic Linear-Quadratic Problem with Lévy Processes and its Application to Finance. *Stoch. Process. Their Appl.* **2008**, *118*, 120–152. [CrossRef]
41. Duncan, T.E. Linear Exponential Quadratic Stochastic Differential Games. *IEEE Trans. Autom. Control* **2016**, *61*, 2550–2552. [CrossRef]
42. Duncan, T.E. Linear-Exponential-Quadratic Gaussian Control. *IEEE Trans. Autom. Control* **2013**, *58*, 2910–2911. [CrossRef]
43. Zhang, W.; Chen, B.S.; Tang, H.; Sheng, L.; Gao, M. Some Remarks on General Nonlinear Stochastic H_∞ Control with State, Control, and Disturbance-Dependent Noise. *IEEE Trans. Autom. Control* **2014**, *59*, 237–242. [CrossRef]
44. Zhang, W.; Chen, B.S. Stochastic Affine Quadratic Regulator with Applications to Tracking control of Quantum Systems. *Automatica* **2008**, *44*, 2869–2875. [CrossRef]
45. Zhang, W.; Xie, L.; Chen, B.S. *Stochastic H_2/H_∞ Control: A Nash Game Approach*; CRC Press: Boca Raton, FL, USA, 2014.
46. Li, N.; Yu, Z. Recursive Stochastic Linear Quadratic Optimal Control and Nonzero-Sum Differential Game Problem with Random Jumps. *Adv. Differ. Equ.* **2015**, *2015*, 144. [CrossRef]
47. Chen, S.; Li, X.; Zhou, X. Stochastic Linear Quadratic Regulators with Indefinite Control Weight Costs. *SIAM J. Control Optim.* **1998**, *36*, 1685–1702. [CrossRef]
48. Tang, S.; Li, X. Necessary Conditions for Optimal Control of Stochastic Systems with Random Jumps. *SIAM J. Control Optim.* **1994**, *32*, 1447–1475. [CrossRef]
49. Shen, Y.; Siu, T.K. The Maximum Principle for a Jump-Diffusion Mean-Field Model and Its Application to the Mean-Variance Problem. *Nonlinear Anal.* **2013**, *86*, 58–73. [CrossRef]
50. Peng, S.; Xu, M. Numerical Algorithms for Backward Stochatic Differential Equations with 1-D Brownian Motion: Convergence and Simulations. *ESIAM Math. Model. Numer. Anal.* **2011**, *45*, 335–360. [CrossRef]
51. Guo, G. Finite Difference Methods for the BSDEs in Finance. *Int. J. Financ. Stud.* **2018**, *6*, 26. [CrossRef]
52. Steinicke, A. Backward Stochastic Differential Equations and Applications. Vienna Seminar in Mathematical Finance and Probability. Available online: https://www.wu.ac.at/fileadmin/wu/d/i/statmath/Research_Seminar/SS_2017/steinicke_slides.pdf (accessed on 11 November 2021).