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Infinite Geraghty type extensions and their applications on integral equations

R. Bardhan[1], C. Ozel[2], L. Guran[3], H. Aydi[4,5,6*] and Choonkil Park[7*]

*Correspondence:
hassen.aydi@isima.rnu.tn;
baak@hanyang.ac.kr
[4]Institut Supérieur d'Informatique et
des Techniques de Communication,
Université de Sousse, H. Sousse
4000, Tunisia
[7]Research Institute for Natural
Sciences, Hanyang University, Seoul
04763, Korea
Full list of author information is
available at the end of the article

Abstract

In this article, we discuss about a series of infinite dimensional extensions of some theorems given in (Shumrani et al. in SER Math. Inform. 33(2):197–202, 2018), (Fisher in Math. Mag. 48(4):223–225, 1975), and (Fogh, Behnamian and Pashaie in Int. J. Maps in Mathematics 2(41):1–13, 2019). We also prove a similar Geraghty type construction for Fisher (Math. Mag. 48(4):223–225, 1975) in an infinite dimension using similar techniques as in (Shumrani et al. in SER Math. Inform. 33(2):197–202, 2018) and (Fogh, Behnamian and Pashaie in Int. J. Maps in Mathematics 2(41):1–13, 2019). As an application, we ensure the existence of solutions for infinite dimensional Fredholm integral equation and Uryshon type integral equation.

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1 Introduction and preliminaries

The fixed point theory is an essential tool not only in the field of nonlinear analysis, but also in almost all branches of mathematics. After the Banach contraction principle [4], there were multiple generalizations that have been made. For instance, see [5–23]. Among these generalizations, we have Geraghty's theorem [24]. Moreover, in [1] a k-dimensional extension of Geraghty's result is also provided. Also, Kannan and Fisher gave two independent types of contraction operators (see [2, 25]) which are completely independent of Banach's contraction. Also, in the recent paper [3], Fogh *et al.* introduced the concept of Kannan–Geraghty contractions.

In this paper, we extend the k-dimensional extension of Geraghty's theorem stated in [1] to an infinite dimension. We also introduce and develop a new notion of H_k contractions for Geraghty–Kannan type operators from [3] to provide both k dimensional extension and infinite dimensional extension. We also provide the same family of k-dimensional extensions for Fisher type contractions and also develop similar infinite dimensional extension in this case. As application, we define the notion of two independent types of infinite integral equations, namely infinite dimensional Fredholm equation and Uryshon type integral equation, and ensure the existence of solutions.

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Next, we discuss some of the preliminaries which will be needed later for proving our main theorems. The following result is due to Geraghty [24].

Theorem 1.1 ([24]) *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \le \beta\big(d(x, y)\big)d(x, y),$$

where $\beta : [0, \infty) \to [0, 1)$ is a function satisfying the condition

$$\lim_{n \to \infty} \beta(t_n) = 1 \quad \text{implies} \quad \lim_{n \to \infty} t_n = 0.$$

Then T admits a unique fixed point $u \in X$ and $\{T^n x\}$ converges to u for each $x \in X$.

We denote by $\mathbb{G}$ the set of all functions β given in Theorem 1.1. We denote by $\mathbb{N}$ (resp. $\mathbb{N}_0$) the set of positive (nonnegative) integers.

Over the years, the study of a Geraghty type contraction has been the interest of many works in literature. This contraction was extended and generalized in many directions as Kannan–Geraghty type contractions, Fisher–Geraghty type contractions, etc. For more details, we refer the readers to [26–38].

Further, let us recall some results and definitions useful in our main results.

Theorem 1.2 ([25]) *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. If T verifies*

$$d(Tx, Ty) \le c\big(d(Tx, x) + (Ty, y)\big)$$

for all $x, y \in X$, where $c \in (0, 1/2)$, then T has a unique fixed point $u \in X$.

Theorem 1.3 [2] *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. If T verifies*

$$d(Tx, Ty) \le c\big(d(Tx, y) + d(Ty, x)\big)$$

for all $x, y \in X$, where $c \in (0, 1/2)$, then T possesses a unique fixed point $u \in X$.

Definition 1.1 (Kannan–Geraghty map, see [3]) Let T be a self-map on a metric space (X, d). Such T is said to be a Kannan–Geraghty contraction if and only if it satisfies the following:

$$d(Tx, Ty) \le \frac{\beta(d(x, y))}{2} \big(d(Tx, x) + d(Ty, y)\big)$$

for all $x, y \in X$, where $\beta \in \mathbb{G}$.

Theorem 1.4 (see [3]) *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. If T is a Kannan–Geraghty contraction on (X, d), then T possesses a unique fixed point $u \in X$, and for any $x_0 \in X$, $\{T^n x_0\}$ converges to u.*

Theorem 1.5 ([1]) *Let (X, d) be a complete metric space and $k \in \mathbb{N}$. Let $T : X^k \to X$ be such that*

$$d\big(T(x_1, \ldots, x_k), T(x_2, \ldots, x_{k+1})\big)$$
$$\leq M\big((x_1, \ldots, x_k), (x_2, \ldots, x_{k+1}))\big)M\big((x_1, \ldots, x_k), (x_2, \ldots, x_{k+1})\big),$$

for all $x_1, x_2, \ldots, x_{k+1} \in X$, where $\beta \in \mathbb{G}$ and $M : X^k \times X^k \to [0, \infty)$ is given as

$$M\big((x_1, \ldots, x_k), (x_2, \ldots, x_{k+1})\big)$$
$$= \max\big\{d(x_k, x_{k+1}), \big(x_k, T(x_1, \ldots, x_k)\big), d\big(x_{k+1}, T(x_2, \ldots, x_{k+1})\big)\big\}.$$

Then there exists a point $u \in X$ such that $T(u, u, \ldots, u) = u$.

The aim of the next section is to generalize and extend Theorem 1.1, Theorem 1.4, and Theorem 1.5 as well as the k-dimensional extension of the result given in [24] to an infinite dimension. We denote the infinite tuples of points $(x_1, x_2, \ldots)$ by $(x_k)_{i=1}^{\infty}$ and the infinite tuples $(x_1, x_2, \ldots, x_{k-1}, x_k, x_k, \ldots)$ with the kth point repeated by $(x_{j,\hat{k}})_{j=1}^{\infty}$.

2 Main results

From now on, let us consider that (X, d) is a *complete metric space* and $k \in \mathbb{N}$.

Definition 2.1 (Extended M_k function) For any $(u_i)_{i=1}^{\infty}, (v_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X$ and $T : \prod_{i=1}^{\infty} X \to X$, we define the extended function $M_k : \prod_{l=1}^{\infty} X \times \prod_{l=1}^{\infty} X \to X$ as follows:

$$M_k\big((u_i)_{i=1}^{\infty}, (v_i)_{i=1}^{\infty}\big)$$
$$= \begin{cases} \max\{\sup_{l \geq k} d(u_l, v_l), \sup_{l \geq k} d(u_l, T(u_{i,\hat{l}})_{i=1}^{\infty}), \sup_{l \geq k} d(v_l, T(v_{i,\hat{l}})_{i=1}^{\infty})\}, \\ \quad \text{if all supremum exists} \\ \max\{d(u_k, v_k), d(u_k, T(u_{i,\hat{k}})_{i=1}^{\infty}), d(v_k, T(v_{i,\hat{k}})_{i=1}^{\infty})\}, \\ \quad \text{if one fails to exist} \end{cases}$$

Example 2.1 Consider $X = [0, 1]$. For $k = 1$, take the operator $T : \prod_{i=1}^{\infty} X \to X$ defined by $T((u_i)_{i=1}^{\infty}) = u_k = u_1$ and $M_k((u_i)_{i=1}^{\infty}, (v_i)_{i=1}^{\infty}) = |u_k - v_k| = |u_1 - v_1|$.

Definition 2.2 (H_k contraction) An operator $T : \prod_{l=1}^{\infty} X \to X$ is called an H_k *contraction* if and only if it satisfies the following inequality:

$$d\big(T\big((u_{i,\hat{k}})_{i=1}^{\infty}\big), T\big((v_{i,\hat{k}})_{i=1}^{\infty}\big)\big) \leq \beta\big(M_k\big((u_{i,\hat{k}})_{i=1}^{\infty}, (v_{i,\hat{k}})_{i=1}^{\infty}\big)\big)M_k\big((u_{i,\hat{k}})_{i=1}^{\infty}, (v_{i,\hat{k}})_{i=1}^{\infty}\big) \tag{1}$$

for all $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k \in X$, where $\beta \in \mathbb{G}$.

Example 2.2 Consider $X = [0, 1]$. The operator $T : \prod_{i=1}^{\infty} X \to X$ defined by $T((x_i)_{i=1}^{\infty}) := cx_k$ is an H_k contraction, where $k \in \mathbb{N}$ is any fixed number and $c \in (0, 1)$.

Definition 2.3 (Kannan–Geraghty H_k contraction) An operator $T : \prod_{l=1}^{k} X \to X$ is called a Kannan–Geraghty contraction of dimension k if and only if it satisfies the following

inequality:

$$d\big(T\big((u_i)_{i=1}^k\big), T\big((v_i)_{i=1}^k\big)\big) \le \frac{\beta(d(u_k, v_k))}{2}\big(d\big(T\big((u_i)_{i=1}^k\big), u_k\big) + d\big(T\big((v_i)_{i=1}^k\big), v_k\big)\big)$$

for all $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k \in X$, where $\beta \in \mathbb{G}$.

Definition 2.4 (Extended Kannan–Geraghty H_k contraction) An operator $T : \prod_{l=1}^\infty X \to X$ is called an extended Kannan–Geraghty H_k contraction if and only if it satisfies the following:

$$d\big(T\big((u_{i,\hat{k}})_{i=1}^\infty\big), T\big((v_{i,\hat{k}})_{i=1}^\infty\big)\big) \le \frac{\beta(d(u_k, v_k))}{2}\big(d\big(T\big((u_{i,\hat{k}})_{i=1}^\infty\big), u_k\big) + d\big(T\big((v_{i,\hat{k}})_{i=1}^\infty\big), v_k\big)\big)$$

for all $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k \in X$, where $\beta \in \mathbb{G}$.

Definition 2.5 (Fisher–Geraghty H_k contraction) An operator $T : \prod_{l=1}^k X \to X$ is called a Fisher–Geraghty H_k contraction of dimension k if and only if it satisfies the following:

$$d\big(T\big((u_i)_{i=1}^k\big), T\big((v_i)_{i=1}^k\big)\big) \le \frac{\beta(d(u_k, v_k))}{2}\big(d\big(T\big((u_i)_{i=1}^k\big), v_k\big) + d\big(T\big((v_i)_{i=1}^k\big), u_k\big)\big)$$

for all $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k \in X$, where $\beta \in \mathbb{G}$.

Definition 2.6 (Extended Fisher–Geraghty H_k contraction) An operator $T : \prod_{l=1}^\infty X \to X$ is called an extended Kannan–Geraghty H_k contraction if and only if it satisfies the following:

$$d\big(T\big((u_{i,\hat{k}})_{i=1}^\infty\big), T\big((v_{i,\hat{k}})_{i=1}^\infty\big)\big) \le \frac{\beta(d(u_k, v_k))}{2}\big(d\big(T\big((u_{i,\hat{k}})_{i=1}^\infty\big), v_k\big) + d\big(T\big((v_{i,\hat{k}})_{i=1}^\infty\big), u_k\big)\big)$$

for all $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k \in X$, where $\beta \in \mathbb{G}$.

Example 2.3 As an example for Definition 2.3 and Definition 2.5, we can consider, for $X = [0, 1]$, the operator $T : \prod_{i=1}^k X \to X$ defined by $T((x_i)_{i=1}^k) = cx_k$ for some $c \in (0, \frac{1}{2})$.

Example 2.4 As an example for Definition 2.4 and Definition 2.6, for $X = [0, 1]$, take the operator $T : \prod_{i=1}^\infty X \to X$ defined by $T((x_i)_{i=1}^\infty) = cx_k$ for some $c \in (0, \frac{1}{2})$.

The theory of Picard operators was developed by Rus and is used a lot in proving the existence and uniqueness of a solution of different types of integral or differential equations. For more details, see [39, 40].

Definition 2.7 (Picard operator) Let (X, d) be a metric space. An operator $T : X \to X$ is a Picard operator if there exists $x^* \in X$ such that $Fix T = \{x^*\}$ and the sequence $(T^n(x_0))_{n \in \mathbb{N}}$ converges to x^* for all $x_0 \in X$.

Inspired by the Picard operator definition, we extend this notion for the k-dimensional and infinite dimensional cases as follows.

Definition 2.8 (k-Picard sequence with respect to the operator T) Let $T : \prod_{i=1}^{k} X \to X$ be any operator, and let us choose $x_1, x_2, \ldots, x_k \in X$. The k-Picard sequence with respect to the operator T based on the base point set $\{x_1, x_2, \ldots, x_k\}$ is defined as $x_{n+k} := T((x_{n+i-1})_{i=1}^{k})$ for all $n \geq k$.

Example 2.5 If we fix $k = 1$, then the base point set is singleton and the 1-Picard sequence with respect to T based on $\{x_0\}$ is basically the Picard sequence of T based on the base point $\{x_0\}$ defined by $x_n := T(x_{n-1})$ for all $n \geq 1$ and some $x_0 \in X$.

Now, we define the following notions.

Definition 2.9 (Infinite k-Picard sequence with respect to the operator T) Let $T : \prod_{i=1}^{\infty} X \to X$ be any operator, and let us choose $x_1, x_2, \ldots, x_k, \ldots \in X$. The k-Picard sequence with respect to the operator T based on the base point set $\{x_1, x_2, \ldots, x_k, \ldots\}$ is defined as $x_{n+k} := T((x_{\widehat{n+i-1, n+k-1}})_{i=1}^{\infty})$ for all $n \geq k$.

Example 2.6 If we fix $k = 1$, then the base point set is singleton and the infinite 1-Picard sequence with respect to T based on the base point $\{x_0\}$ is basically the sequence defined by $x_n := T((x_{n-1})_{n=1}^{\infty})$ $\forall n \geq 1$ for some $x_0 \in X$.

Let us give our first main fixed point result, which is a generalization of the Banach contraction principle with respect to the infinite-dimensional notion introduced in our paper.

Theorem 2.1 *Let* (X, d) *be a complete metric space.* $T : \prod_{l=1}^{\infty} X \to X$ *is an* H_k *contraction for some* $k \in \mathbb{N}$. *Then there exists* $u \in X$ *such that* $T((u)_{i=1}^{\infty}) = u$ *and the infinite* k-*Picard sequence for* T *converges to* u.

Proof Let $x_1, x_2 \ldots, x_k \in X$. Then an infinite k-Picard sequence is defined as follows: $\forall n \in \mathbb{N}$ we define

$$x_{n+k} := T\big((x_{\widehat{n+i-1, n+k-1}})_{i=1}^{\infty}\big). \tag{2}$$

We claim that $\{d(x_{n+k}, x_{n+k+1})\}_{n \in \mathbb{N}}$ is convergent. From Definition 2.2, one writes

$$d(x_{n+k+1}, x_{n+k+2}) = d\big(T\big((x_{\widehat{n+i, n+k}})_{i=1}^{\infty}\big), T\big((x_{\widehat{n+i+1, n+1+k}})_{i=1}^{\infty}\big)\big)$$
$$\leq \beta\big(M_k\big((x_{\widehat{n+i, n+k}})_{i=1}^{\infty}, (x_{\widehat{n+1+i, n+1+k}})_{i=1}^{\infty}\big)\big) M_k\big((x_{\widehat{n+i, n+k}})_{i=1}^{\infty}, (x_{\widehat{n+1+i, n+1+k}})_{i=1}^{\infty}\big).$$

Let

$$M_k\big((x_{\widehat{n+i, n+k}})_{i=1}^{\infty}, (x_{\widehat{n+i+1, n+1+k}})_{i=1}^{\infty}\big)$$
$$= \max\Big\{ \sup_{l \geq k} d(u_l, v_l), \sup_{l \geq k} d\big(u_l, T(u_{i,\widehat{l}})_{i=1}^{\infty}\big), \sup_{l \geq k} d\big(v_l, T(v_{i,\widehat{l}})_{i=1}^{\infty}\big) \Big\},$$

where

$$u_l := x_{n+l}, \quad \forall 1 \leq l \leq k, \tag{3}$$

$$u_l := x_{n+k}, \quad \forall l > k, \tag{4}$$

$$v_l := x_{n+l+1}, \quad \forall 1 \le l \le k, \tag{5}$$

$$u_l := x_{n+k+1}, \quad \forall l > k. \tag{6}$$

But

$$\sup_{l \ge k} d\big((u_l, v_l)\big) = d\big((x_{n+k}, x_{n+k+1})\big),$$

$$\sup_{l \ge k} d\big((u_l, T((u_{i,\widehat{l}})_{i=1}^\infty)\big) = d\big((x_{n+k}, T((x_{n+i,\widehat{n+k}})_{i=1}^\infty))\big),$$

$$\sup_{l \ge k} d\big((v_l, T((v_{i,\widehat{l}})_{i=1}^\infty)\big) = d\big((x_{n+k+1}, T((x_{n+i+1,\widehat{n+k+1}})_{i=1}^\infty))\big).$$

We have

$$M_k\big((x_{n+i,\widehat{n+k}})_{i=1}^\infty, (x_{n+i+1,\widehat{n+k+1}})_{i=1}^\infty\big)$$
$$= \max\big\{d\big((x_{n+k}, x_{n+k+1})\big), d\big((x_{n+k}, T((x_{n+i,\widehat{n+k}})_{i=1}^\infty))\big), d\big((x_{n+k+1}, T((x_{n+1+i,\widehat{n+k+1}})_{i=1}^\infty))\big)\big\}.$$

That is,

$$M_k\big((x_{n+i,\widehat{n+k}})_{i=1}^\infty, (x_{n+1+i,\widehat{n+k+1}})_{i=1}^\infty\big)$$
$$= \max\big\{d((x_{n+k}, x_{n+k+1})), d((x_{n+k}, x_{n+k+1})), d((x_{n+k+1}, x_{n+k+2}))\big\}$$
$$= \max\big\{d((d_{n+k}, d_{n+k+1})), d((x_{n+k+1}, x_{n+k+2}))\big\}.$$

Without loss of generality, assume that $d((x_{n+k+1}, x_{n+k+2})) > 0$ for each k.

If, for some k, we have $M_k((x_{n+i,\widehat{n+k}})_{i=1}^\infty, (x_{n+i+1,\widehat{n+k+1}})_{i=1}^\infty) = d((x_{n+k+1}, x_{n+k+2}))$, then, from Definition 2.2, we get

$$0 < d((x_{n+k+1}, x_{n+k+2}) = d\big((T((x_{n+i,\widehat{n+k}})_{i=1}^\infty), T((x_{n+i+1,\widehat{n+k+1}})_{i=1}^\infty)\big)$$
$$\le \beta\big(M_k((x_{n+i,\widehat{n+k}})_{i=1}^\infty, (x_{n+i+1,\widehat{n+k+1}})_{i=1}^\infty)\big) M_k\big((x_{n+i,\widehat{n+k}})_{i=1}^\infty, (x_{n+i+1,\widehat{n+k+1}})_{i=1}^\infty\big)$$
$$= \beta(d((x_{n+k+1}, x_{n+k+2}))) d((x_{n+k+1}, x_{n+k+2}))$$
$$< d((x_{n+k+1}, x_{n+k+2})).$$

Then we get a contradiction. Hence, for each k, $M_k((x_{n+i,\widehat{n+k}})_{i=1}^\infty, (x_{n+i+1,\widehat{n+k+1}})_{i=1}^\infty) = d((x_{n+k}, x_{n+k+1}))$. One writes

$$d((x_{n+k+1}, x_{n+k+2})) \le \beta(d((x_{n+k}, x_{n+k+1}))) d((x_{n+k}, x_{n+k+1})) < d((x_{n+k}, x_{n+k+1})).$$

We deduce that $\{x_{n+k}\}_{n \in \mathbb{N}}$ is a strictly decreasing sequence. So there is $(r \ge 0) \in \mathbb{R}$ so that

$$\lim_{n \to \infty} d(x_{n+k}, x_{n+k+1}) = r.$$

We claim $\lim_{n \to \infty} d(x_{n+k}, x_{n+k+1}) = 0$. Suppose on the contrary that $r > 0$. We have

$$d((x_{n+k+1}, x_{n+k+2})) \le \beta(d((x_{n+k}, x_{n+k+1}))) d((x_{n+k}, x_{n+k+1})).$$

That is,

$$\frac{d((x_{n+k+1},x_{n+k+2})}{d((x_{n+k},x_{n+k+1})} \leq \beta(d((x_{n+k},x_{n+k+1}))).$$

This implies that $\lim_{n\to\infty} \beta(d((x_{n+k},x_{n+k+1}))) \geq 1$.

Since $\beta \in \mathbb{G}$, necessarily $\lim_{n\to\infty} \beta(d((x_{n+k},x_{n+k+1}))) = 1$, and so

$$\lim_{n\to\infty} d((x_{n+k},x_{n+k+1}) = 0. \tag{7}$$

We now claim that $\{x_{n+k}\}_{n\in\mathbb{N}}$ is Cauchy, and we prove it by contradiction. Suppose on the contrary that there is $\varepsilon > 0$ such that we can find some subsequences $\{x_{m(q)+k}\}_{p\in\mathbb{N}}$, $\{x_{n(q)+k}\}_{p\in\mathbb{N}}$ with $m(q) > n(q) > q$ such that, for every q, we have

$$d((x_{m(q)+k},x_{n(q)+k}) \geq \varepsilon. \tag{8}$$

Moreover, corresponding to each $n(q)$, we can choose least of such $m(q)$ satisfying (8). Then

$$d((x_{m(q)+k-1},x_{n(q)+k}) < \varepsilon. \tag{9}$$

From (7),(9) and using the triangle inequality, we get

$$d((x_{m(q)+k-1},x_{n(q)+k-1}) \leq d((x_{m(q)+k-1},x_{n(q)+k}) + d((x_{n(q)+k-1},x_{n(q)+k})$$
$$< \varepsilon + d((x_{n(q)+k-1},x_{n(q)+k}) \tag{10}$$

and

$$\varepsilon \leq d((x_{n(q)+k},x_{m(q)+k})$$
$$\leq d((x_{n(q)+k},x_{n(q)+k-1}) + d((x_{n(q)+k-1},x_{m(q)+k-1}) + d((x_{m(q)+k-1},x_{m(q)+k}). \tag{11}$$

Letting $q \to \infty$ in (10) and using (11), we get

$$\lim_{q\to\infty} d((x_{m(q)+k-1},x_{n(q)+k-1}) = \varepsilon. \tag{12}$$

On the other hand, if

$$M_k\left((x_{n(q)+i-1,\widehat{n(q)+k-1}})_{i=1}^\infty, (x_{m(q)+i-1,\widehat{m(q)+k-1}})_{i=1}^\infty\right)$$
$$= \max\left\{\sup_{l\geq k-1} d((u_l,v_l), \sup_{l\geq k-1} d((u_l, T((u_{i,\widehat{l}})_{i=1}^\infty)), \sup_{l\geq k-1} d((v_l, T((v_{i,\widehat{l}})_{i=1}^\infty))\right\},$$

where

$$u_l := x_{n(q)+l-1}, \quad \forall 1 \leq l \leq k, \tag{13}$$

$$u_l := x_{n(q)+k-1}, \quad \forall l > k, \tag{14}$$

$$v_l := x_{m(q)+l-1}, \quad \forall 1 \leq l \leq k, \tag{15}$$

$$u_l := x_{m(q)+k-1}, \quad \forall l > k \tag{16}$$

then

$$M_k\big((x_{n(q)+i-1,\widehat{n(q)+k-1}})_{i=1}^{\infty}, (x_{m(q)+i-1,\widehat{m(q)+k-1}})_{i=1}^{\infty}\big)$$

$$= \max\big\{\xi\big((x_{n(q)+k-1}, x_{m(q)+k-1}), d\big((x_{n(q)+k-1}, T\big((x_{n(q)+i-1,\widehat{n(q)+k-1}})_{i=1}^{\infty}\big)\big),$$

$$d\big((x_{m(q)+k-1}, T\big((x_{m(q)+i-1,\widehat{m(q)+k-1}})_{i=1}^{\infty}\big)\big)\big)\big\}$$

$$= \max\big\{d((x_{n(q)+k-1}, x_{m(q)+k-1}), d(x_{n(q)+k-1}, x_{n(q)+k}), d(x_{m(q)+k-1}, x_{m(q)+k})\big\}.$$

Using (7) and (12) we get

$$\lim_{q \to \infty} M_k\big((x_{n(q)+i-1,\widehat{n(q)+k-1}})_{i=1}^{\infty}, (x_{m(q)+i-1,\widehat{m(q)+k-1}})_{i=1}^{\infty}\big) = \varepsilon. \tag{17}$$

By (1) and (17) we get

$$\varepsilon \leq d((x_{n(q)+k}, x_{m(q)+k}) = d\big((T(x_{n(q)+i-1,\widehat{n(q)+k-1}})_{i=1}^{\infty}, T(x_{m(q)+i-1,\widehat{m(q)+k-1}})_{i=1}^{\infty}\big)\big)$$

$$\leq \beta\big(M_k\big((x_{n(q)+i-1,\widehat{n(q)+k-1}})_{i=1}^{\infty}, (x_{m(q)+i-1,\widehat{m(q)+k-1}})_{i=1}^{\infty}\big)\big)$$

$$\times M_k\big((x_{n(q)+i-1,\widehat{n(q)+k-1}})_{i=0}^{\infty}, (x_{m(q)+i-1,\widehat{m(q)+k-1}})_{i=1}^{\infty}\big)$$

$$< M_k\big((x_{n(q)+i-1,\widehat{n(q)+k-1}})_{i=1}^{\infty}, (x_{m(q)+i-1,\widehat{m(q)+k-1}})_{i=1}^{\infty}\big). \tag{18}$$

If we suppose

$$\sup_{l \geq k} d((u_l, v_l) = d((x_{n(q)+k-1}, x_{m(q)+k-1}),$$

$$\sup_{l \geq k} d((u_l, T(u_{i,\widehat{l}})_{i=1}^{\infty}) = d\big((x_{n(q)+k-1}, T\big((x_{n(q)+i-1,\widehat{n(q)+k-1}})_{i=1}^{\infty}\big)\big),$$

$$\sup_{l \geq k} d((v_l, T(v_{i,\widehat{l}})_{i=1}^{\infty}) = d\big((x_{m(q)+k-1}, T\big((x_{m(q)+i-1,\widehat{m(q)+k-1}})_{i=1}^{\infty}\big)\big),$$

then

$$M_k\big((x_{n(q)+i-1,\widehat{n(q)+k-1}})_{i=1}^{\infty}, (x_{m(q)+i-1,\widehat{m(q)+k-1}})_{i=1}^{\infty}\big)$$

$$= \max\big\{d((x_{n(q)+k}, x_{m(q)+k}), d\big((x_{n(q)+k-1}, T\big((x_{n(q)+i-1,\widehat{n(q)+k-1}})_{i=1}^{\infty}\big)\big),$$

$$d\big((x_{m(q)+k-1}, T\big((x_{m(q)+i-1,\widehat{m(q)+k-1}})_{i=1}^{\infty}\big)\big)\big)\big\}$$

$$= \max\big\{d((x_{n(q)+k}, x_{m(q)+k}), d(x_{n(q)+k-1}, x_{n(q)+k}), d(x_{m(q)+k-1}, x_{m(q)+k})\big\}.$$

Letting $q \to \infty$ and using (1), (7), (17), and (18), we get

$$\lim_{q \to \infty} \beta\big(M_k\big((x_{n(q)+i-1,\widehat{n(q)+k-1}})_{i=1}^{\infty}, (x_{m(q)+i-1,\widehat{m(q)+k-1}})_{i=1}^{\infty}\big)\big)$$

and

$$\lim_{q \to \infty} M_k\big((x_{n(q)+i-1,\widehat{n(q)+k-1}})_{i=1}^{\infty}, (x_{m(q)+i-1,\widehat{m(q)+k-1}})_{i=1}^{\infty}\big) \geq \varepsilon.$$

From (17)

$$\beta\big(M_k\big((x_{n(q)+i-1,\widehat{n(q)+k-1}})_{i=1}^{\infty}, (x_{m(q)+i-1,\widehat{m(q)+k-1}})_{i=1}^{\infty}\big)\big) \geq 1.$$

Since $\beta \in \mathbb{G}$, we get

$$\lim_{q \to \infty} \beta\big(M_k\big((x_{n(q)+i-1,\widehat{n(q)+k-1}})_{i=1}^{\infty}, (x_{m(q)+i-1,\widehat{m(q)+k-1}})_{i=1}^{\infty}\big)\big) = 1.$$

Then

$$\lim_{q \to \infty} M_k\big((x_{n(q)+i-1,\widehat{n(q)+k-1}})_{i=1}^{\infty}, (x_{m(q)+i-1,\widehat{m(q)+k-1}})_{i=1}^{\infty}\big) = 0.$$

We get a contradiction to (17). Hence, $\{x_{n+k}\}_{n \in \mathbb{N}}$ is a Cauchy sequence.

Since (X, d) is complete, there exists $u \in X$ such that

$$\lim_{n \to \infty} x_{n+k} = u. \tag{19}$$

We claim that $d((T((u)_{i=1}^{\infty}), u) = 0$. If we suppose on the contrary that $d((u, T((u)_{i=1}^{\infty})) > 0$, then we have

$$M\big((x_{n+i-1,\widehat{n+k-1}})_{i=1}^{\infty}, ((u)_{i=1}^{\infty})\big)$$

$$= \max\big\{d(x_{n+k-1}, u), d\big((x_{n+k-1}, T(x_{n+i-1,\widehat{n+k-1}})_{i=1}^{\infty}), d\big((u, T((u)_{i=1}^{\infty}))\big\}$$

$$= \max\big\{d((x_{n+k-1}, u), d((x_{n+k}, x_{n+k-1}), d\big((u, T((u)_{i=1}^{\infty}))\big\}.$$

Letting $n \to \infty$ and using (7), (19), we get

$$\lim_{n \to \infty} M\big((x_{n+i-1,\widehat{n+k-1}})_{i=1}^{\infty}, (u)_{i=1}^{\infty}\big) = d\big((u, T((u)_{i=1}^{\infty})\big) \neq 0.$$

Then we have

$$d\big((u, T((u)_{i=1}^{\infty})\big)$$

$$\leq d((x_{n+k}, u) + d\big((x_{n+k}, T((u)_{i=1}^{\infty}))$$

$$= d((x_{n+k}, u) + d\big((T((x_{n+i-1,\widehat{n+k-1}})_{i=1}^{\infty}), T((u)_{i=1}^{\infty}))$$

$$\leq d((x_{n+k}, u) + \beta\big(M((x_{n+i-1,\widehat{n+k-1}})_{i=1}^{\infty}, (u)_{i=1}^{\infty})\big)M\big((x_{n+i-1,\widehat{n+k-1}})_{i=1}^{\infty}, (u)_{i=1}^{\infty}\big)$$

$$\leq d((x_{n+k}, u) + \beta(d\big((u, T((u)_{i=1}^{\infty})))d\big((u, T((u)_{i=1}^{\infty})).$$

Letting $n \to \infty$ and using (19), we get $\lim_{n \to \infty} \beta(d((u, T((u)_{i=1}^{\infty}))) \geq 1$.

Since $\beta \in \mathbb{G}$, one writes $\lim_{n \to \infty} \beta(d((u, T((u)_{i=1}^{\infty}))) = 1$. Then we have $\lim_{n \to \infty} d((u, T((u)_{i=1}^{\infty})) = 0$, which implies $d((u, T((u)_{i=1}^{\infty})) = 0$. So we get a contradiction. Hence, $d((u, T((u)_{i=1}^{\infty})) = 0$, that is, $T((u)_{i=1}^{\infty}) = u$. $\qquad\qquad\square$

Remark 2.1 Theorem 2.1 is a proper generalization of Theorem 1.5 since in the case of the simplest operator on $\prod_{i=1}^{\infty} X \to X$ the contraction condition of Theorem 1.5 is not applicable; on the other hand, the H_k contraction (see Definition 2.2) is easily applicable for the infinite case. Also, if we restrict the operator to any finite k dimension through an easy calculation, it is obvious that it is an equivalent statement of Theorem 1.5.

Theorem 2.2 . *Let (X, d) be a complete metric space and $T : \prod_{l=1}^{\infty} X \to X$ be an extended Kannan–Geraghty H_k contraction for some $k \in \mathbb{N}$. Then there is $u \in X$ such that $T((u)_{i=1}^{\infty}) = u$, and for any $x_1, \ldots, x_k \in X$, the infinite k-Picard sequence converges to u.*

Proof Let $x_1, \ldots, x_k \in X$. For all $n \in \mathbb{N}$, we define the infinite k-Picard sequence as follows:

$$x_{n+k} := T\big((x_{n+i-1,\widehat{n+k-1}})_{i=1}^{\infty}\big). \tag{20}$$

We claim that $\lim_{n\to\infty} d(x_{n+k}, x_{n+k+1}) = 0$. Then we have

$$
\begin{aligned}
d&((x_{n+k+1}, x_{n+k+2}) \\
&= d\big((T\big((x_{n+i,\widehat{n+k}})_{i=1}^{\infty}\big), T\big((x_{n+i+1,\widehat{n+1+k}})_{i=1}^{\infty}\big)\big) \\
&\leq \frac{\beta(d(x_{n+k}, x_{n+k+1}))}{2}\big(d\big(T\big((x_{n+i,\widehat{n+k}})_{i=1}^{\infty}\big), x_{n+k}\big) + d\big(T\big((x_{n+i+1,\widehat{n+k+1}})_{i=1}^{\infty}\big), x_{n+k+1}\big)\big) \\
&< \frac{1}{2}\big(d(x_{n+k}, x_{n+k+1}) + d(x_{n+k+1}, x_{n+k+2})\big).
\end{aligned}
$$

Thus, $d(x_{n+k+1}, x_{n+k+2}) < d(x_{n+k}, x_{n+k+1})$. This sequence is decreasing, so there is $r \geq 0$ so that

$$\lim_{n\to 0} d(x_{n+k}, x_{n+k+1}) = r. \tag{21}$$

We claim that $r = 0$. If we consider on the contrary and suppose $r > 0$, we have

$$
\begin{aligned}
\lim_{n\to\infty} d(x_{n+k+1}, x_{n+k+2}) \leq {}& \frac{\lim_{n\to\infty} \beta(d(x_{n+k}, x_{n+k+1}))}{2}\Big(\lim_{n\to\infty} d(x_{n+k}, x_{n+k+1}) \\
& + \lim_{n\to\infty} d(x_{n+k+1}, x_{n+k+2})\Big).
\end{aligned}
$$

Therefore,

$$\frac{2\lim_{n\to\infty} d(x_{n+k+1}, x_{n+k+2})}{\lim_{n\to\infty} d(x_{n+k}, x_{n+k+1}) + \lim_{n\to\infty} d(x_{n+k+1}, x_{n+k+2})} \leq \lim_{n\to\infty} \beta\big(d(x_{n+k}, x_{n+k+1})\big).$$

From (21) we get $\frac{2r}{2r} \leq \beta(d(x_{n+k}, x_{n+k+1}))$, and so $\lim_{n\to\infty} \beta(d(x_{n+k}, x_{n+k+1})) \geq 1$. Since $\beta \in \mathbb{G}$, we have $\lim_{n\to\infty} \beta(d(x_{n+k}, x_{n+k+1})) \leq 1$. Using the well-known "sandwich theorem", we obtain

$$\lim_{n\to\infty} \beta\big(d(x_{n+k}, x_{n+k+1})\big) = 1 \quad \Longrightarrow \quad \lim_{n\to\infty} d(x_{n+k}, x_{n+k+1}) = 0. \tag{22}$$

Further, we have

$$d(x_{n+k}, x_{m+k}) = d\big(T\big((x_{n+i-1,\widehat{n+k-1}})_{i=1}^{\infty}\big), T\big((x_{m+i-1,\widehat{m+k-1}})_{i=1}^{\infty}\big)\big)$$

$$\leq \frac{\beta(d(x_{n+k-1}, x_{m+k-1}))}{2}\big(d\big(T\big((x_{n+i-1,\widehat{n+k-1}})_{i=1}^{\infty}\big), x_{n+k-1}\big)\big)$$
$$+ d\big(T\big((x_{m+i-1,\widehat{m+k-1}})_{i=1}^{\infty}\big), x_{m+k-1}\big)\big)$$
$$< \frac{1}{2}\big(d(x_{n+k-1}, x_{n+k}) + d(x_{m+k-1}, x_{m+k})\big).$$

For large enough $n, m \in \mathbb{N}$, one has

$$d(x_{n+k}, x_{m+k}) < \varepsilon$$

for fixed $\varepsilon > 0$. Then $\{x_{n+k}\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_{n+k} = u. \tag{23}$$

We claim that $T((u)_{i=1}^{\infty}) = u$. If we suppose on the contrary that $d(u, T((u)_{i=1}^{\infty})) > 0$. Then, by (21) and (23), for arbitrary $\varepsilon > 0$ and sufficiently large n, we get

$$
\begin{aligned}
d\big((u, T\big((u)_{i=1}^{\infty}\big)\big) &\leq d((x_{n+k}, u) + d\big(x_{n+k}, T\big((u)_{i=1}^{\infty}\big)\big) \\
&= d((x_{n+k}, u) + d\big(\big(T\big((x_{n+i-1,\widehat{n+k-1}})_{i=1}^{\infty}\big), T\big((u)_{i=1}^{\infty}\big)\big) \\
&\leq d((x_{n+k}, u) + \frac{\beta(d(u, x_{n+k-1}))}{2}\big(d\big(T\big((x_{n+i-1,\widehat{n+k-1}})_{i=1}^{\infty}\big), x_{n+k-1}\big) \\
&\quad + d\big(T\big((u)_{i=1}^{\infty}\big), u\big)\big) \\
&\leq \frac{1}{2} d\big((u, T\big((u)_{i=1}^{\infty}\big)\big) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \frac{1}{2} d(u, T\big((u)_{i=1}^{\infty}\big) + \varepsilon \\
&\Longrightarrow \frac{1}{2} d\big((u, T\big((u)_{i=1}^{\infty}\big)\big) \leq \varepsilon. \quad \text{This is a contradiction.}
\end{aligned}
$$

Hence, $d(T((u)_{i=1}^{\infty}), u) = 0$, that is, $T((u)_{i=1}^{\infty}) = u$. □

Next, we will provide a new result for a multivalued proper extension of Theorem 1.4, which is also a generalization of Kannan (Theorem 1.3) as a result of Theorem 2.2.

Corollary 2.1 *Let (X, d) be a complete metric space and T be a Kannan–Geraghty H_k contraction. Then T has a fixed point and every k-Picard sequence for T converges to u.*

Proof Let us choose $x_0, \dots, x_k \in X$. We define a k-Picard sequence by

$$x_{n+k} := T(x_n, \dots, x_{n+k-1}). \tag{24}$$

If we follow the same steps as in the proof of Theorem 2.2, we get the required result. □

Remark 2.2 For $k = 1$, we get Theorem 1.4 in [3] which proves that Corollary 2.1 is a proper generalization of Theorem 1.4.

Theorem 2.3 *Let (X,d) be a complete metric space and $T : \prod_{l=1}^{\infty} X \to X$ be an extended Fisher–Geraghty H_k contraction for some $k \in \mathbb{N}$. Then there exists $u \in X$ such that $T((u)_{i=1}^{\infty}) = u$, and for any $x_1, x_2, \ldots x_k$, the infinite k-Picard sequence converges to u.*

Proof Let $x_1, \ldots, x_k \in X$. For all $n \in \mathbb{N}$, we define an infinite k- Picard sequence by

$$x_{n+k} := T\big((x_{n+i-1,\widehat{n+k-1}})_{i=1}^{\infty}\big). \tag{25}$$

We claim $\lim_{n\to\infty} d(x_{n+k}, d_{n+k+1}) = 0$.

Then we have

$$
\begin{aligned}
d((x_{n+k+1}, x_{n+k+2}) &= d\big(\big(T\big((x_{n+i,\widehat{n+k}})_{i=1}^{\infty}\big), T\big((x_{n+i+1,\widehat{n+1+k}})_{i=1}^{\infty}\big)\big) \\
&\leq \frac{\beta(d(x_{n+k}, x_{n+k+1}))}{2}\big(d\big(T\big((x_{n+i,\widehat{n+k}})_{i=1}^{\infty}\big), x_{n+k+1}\big) \\
&\quad + d\big(T\big((x_{n+i,\widehat{n+k+1}})_{i=1}^{\infty}\big), x_{n+k}\big)\big) \\
&< \frac{1}{2}\big(d(x_{n+k+1}, x_{n+k+1}) + d(x_{n+k}, x_{n+k+2})\big) \\
&< \frac{1}{2}\big(d(x_{n+k}, x_{n+k+1}) + d(x_{n+k+1}, x_{n+k+2})\big).
\end{aligned}
$$

We have $d(x_{n+k+1}, x_{n+k+2}) < d(x_{n+k}, x_{n+k+1})$. This sequence is decreasing, so there is $r \geq 0$ such that

$$\lim_{n\to 0} d(x_{n+k}, x_{n+k+1}) = r. \tag{26}$$

We claim that $r = 0$. If we suppose on the contrary that $r > 0$. One has

$$
\begin{aligned}
&\lim_{n\to\infty} d(x_{n+k+1}, x_{n+k+2}) \\
&\quad \leq \frac{\lim_{n\to\infty} \beta(d(x_{n+k}, x_{n+k+1}))}{2}\Big(\lim_{n\to\infty} d(x_{n+k}, x_{n+k+1}) + \lim_{n\to\infty} d(x_{n+k+1}, x_{n+k+2})\Big).
\end{aligned}
$$

Thus,

$$\frac{2\lim_{n\to\infty} d(x_{n+k+1}, x_{n+k+2})}{\lim_{n\to\infty} d(x_{n+k}, x_{n+k+1}) + \lim_{n\to\infty} d(x_{n+k+1}, x_{n+k+2})} \leq \lim_{n\to\infty} \beta\big(d(x_{n+k}, x_{n+k+1})\big).$$

Using (26), we obtain $\frac{2r}{2r} \leq \lim_{n\to\infty} \beta(d(x_{n+k}, x_{n+k+1}))$. Hence,

$$\lim_{n\to\infty} \beta\big(d(x_{n+k}, x_{n+k+1})\big) \geq 1.$$

Since $\beta \in \mathbb{G}$, we have $\lim_{n\to\infty} \beta(d(x_{n+k}, x_{n+k+1})) \leq 1$.

At the limit, we have

$$\lim_{n\to\infty} \beta\big(d(x_{n+k}, x_{n+k+1})\big) = 1. \tag{27}$$

Then $\lim_{n\to\infty} d(x_{n+k}, x_{n+k+1}) = 0$. We claim that $\{x_{n+k}\}_{n\in\mathbb{N}}$ is Cauchy, and we want to prove this by contradiction. Then, using the contrary, there exists $\varepsilon > 0$ such that we can find

subsequences $\{x_{m(q)+k}\}_{p\in\mathbb{N}}$, $\{x_{n(q)+k}\}_{p\in\mathbb{N}}$ with $m(q) > n(q) > q$ such that, for every q, we have

$$d((x_{m(q)+k}, x_{n(q)+k}) \geq \varepsilon. \tag{28}$$

Moreover, corresponding to each $n(q)$ we can choose $m(q)$ satisfying (28) so that

$$d((x_{m(q)+k-1}, x_{n(q)+k}) < \varepsilon. \tag{29}$$

Using (28), (29) and the triangle inequality, we get

$$d((x_{m(q)+k-1}, x_{n(q)+k-1}) \leq d((x_{m(q)+k-1}, x_{n(q)+k}) + d((x_{n(q)+k-1}, x_{n(q)+k})$$
$$< \varepsilon + \xi((x_{n(q)+k-1}, x_{n(q)+k}) \tag{30}$$

and

$$\varepsilon \leq d((x_{n(q)+k}, x_{m(q)+k})$$
$$\leq d((x_{n(q)+k}, x_{n(q)+k-1}) + d((x_{n(q)+k-1}, x_{m(q)+k-1}) + d((x_{m(q)+k-1}, x_{m(q)+k}).$$

Letting $q \to \infty$ in (30) and (27), we get

$$\lim_{q\to\infty} d((x_{m(q)+k-1}, x_{n(q)+k-1}) = \varepsilon, \tag{31}$$

$$d(x_{n(q)+k}, x_{m(q)+k})$$
$$= d\big(T\big((x_{n(q)+i-1,\widehat{n(q)+k-1}})_{i=1}^{\infty}\big), T\big((x_{m(q)+i-1,\widehat{m(q)+k-1}})_{i=1}^{\infty}\big)\big)$$
$$\leq \frac{\beta(d(x_{n(q)+k-1}, x_{m(q)+k-1}))}{2}\big(d\big(T\big((x_{n(q)+i-1,\widehat{n(q)+k-1}})_{i=1}^{\infty}\big), x_{m(q)+k-1}\big)$$
$$\quad + d\big(T\big((x_{m(q)+i-1,\widehat{m(q)+k-1}})_{i=1}^{\infty}\big), x_{n(q)+k-1}\big)\big)$$
$$< \frac{1}{2}\big(d(x_{n(q)+k}, x_{m(q)+k-1}) + d(x_{m(q)+k}, x_{n(q)+k-1})\big)$$
$$\leq \frac{1}{2}\big(\varepsilon + d(x_{m(q)+k}, x_{m(q)+k-1}) + d(x_{m(q)+k-1}, x_{n(q)+k}) + d(x_{n(q)+k}, x_{n(q)+k-1})\big)$$
$$\leq \frac{1}{2}\big(2\varepsilon + d(x_{m(q)+k}, x_{m(q)+k-1}) + d(x_{n(q)+k}, x_{n(q)+k-1})\big).$$

Then from (31) and (27) we get a contradiction. Then

$$\{x_{n+k}\}_{n\in\mathbb{N}} \quad \text{is a Cauchy sequence.} \tag{32}$$

Since (X, d) is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_{n+k} = u. \tag{33}$$

We claim that $T((u)_{i=1}^{\infty}) = u$. If we suppose on the contrary that $d(u, T((u)_{i=1}^{\infty})) > 0$, then, by (26) and (33), for arbitrary $\varepsilon > 0$ and sufficiently large n, we get

$$d\big((u, T\big((u)_{i=1}^{\infty}\big)\big)$$

$$\leq d((x_{n+k}, u) + d\big(x_{n+k}, T\big((u)_{i=1}^{\infty}\big)\big)$$

$$= d((x_{n+k}, u) + d\big(\big(T\big((x_{n+i-1,\widehat{n+k-1}})_{i=1}^{\infty}\big), T\big((u)_{i=1}^{\infty}\big)\big)$$

$$\leq d\bigg((x_{n+k}, u) + \frac{\beta(\xi(u, x_{n+k-1}))}{2}\big(d\big(T\big((x_{n+i-1,\widehat{n+k-1}})_{i=1}\big)^{\infty}\big), u\big) + d\big(T\big((u)_{i=1}^{\infty}\big), x_{n+k-1}\big)\bigg)$$

$$\leq d\bigg((x_{n+k}, u) + \frac{\beta(d(u, x_{n+k-1}))}{2}\big(d\big(T\big((x_{n+i-1,\widehat{n+k-1}})_{i=1}\big)^{\infty}\big), u\big)$$

$$\quad + d\big(T\big((u)_{i=1}^{\infty}\big), u\big) + d(u, x_{n+k-1})\bigg)$$

$$\leq \frac{1}{2}\, d\big((u, T\big((u)_{i=1}^{\infty}\big)\big) + \frac{\varepsilon}{3} + \frac{\varepsilon}{6}$$

$$\leq \frac{1}{2}\, d\big(u, T\big((u)_{i=1}^{\infty}\big)\big) + \frac{1}{2}\varepsilon.$$

That is, $d((u, T((u)_{i=1}^{\infty})) \leq \varepsilon$ for any arbitrary $\varepsilon > 0$. Hence, $d(T((u)_{i=1}^{\infty}), u) = 0$, which implies $T((u)_{i=1}^{\infty}) = u$. □

Corollary 2.2 *Let (X, d) be a complete metric space and T be a Fisher–Geraghty H_k contraction. Then T has a fixed point.*

Proof Choose any $x_0, \ldots, x_k \in X$. Define

$$x_{n+k} := T(x_n, \ldots, x_{n+k-1}). \tag{34}$$

Using the same steps as in the proof of Theorem 2.3, we get the conclusion. □

Remark 2.3 For $k = 1$, we get a new type of extension of Theorem 1.3 proved in [2], and for any k, Corollary 2.2 also gives the multidimensional extension of the same theorem stated in [2].

Remark 2.4 The Banach fixed point theorem [4], Theorem 1.1, Theorem 1.2, Theorem 1.3, Theorem 1.4, and Theorem 1.5 are all applicable only in complete metric spaces as well as the theorems described in [1], but Theorem 2.1, Theorem 2.2, Theorem 2.3, which we have proved, are talking about the space$_{i=1}^{\infty}X$ (where X is a complete metric space) which is indeed metrizable, but might not be complete.

3 Applications to integral equations

In [41], Singh et al. introduced the notion of multi-dimensional Fredholm integral equations by taking into account $n \in \mathbb{N}$ as the dimension of the equation

$$u(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$$
$$+ \underbrace{\int_0^1 \int_0^1 \ldots \int_0^1}_{n\text{-times}} K(x_1, \ldots, x_n, y_1, \ldots, y_n) u(y_1, \ldots, y_n)\, dy_1 \ldots dy_n,$$

where $(x_1, \ldots, x_n) \in D = \underbrace{([0,1] \times [0,1] \times \cdots \times [0,1])}_{n\text{-times}}$ and $f(x_1, \ldots, x_n)$, $K(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are known continuous functions defined on D and D^2, respectively, where $u(x_1, \ldots, x_n)$ is an unknown function.

The Fredholm integral equations play an important role in modeling of physics phenomena described by two or three dimensions. In the same way, they have applications in astrophysics models thinking of the four dimensions of a neutron star or a black hole.

Thinking of these aspects, if we extend to infinity the dimension "n" of the previous multi-dimensional integral equation, we introduce a new notion, *the infinite dimensional Fredholm integral equation*, as follows:

$$
u\big((t_{i,\widehat{k}})_{i=1}^{\infty}\big) = f\big((t_{i,\widehat{k}})_{i=1}^{\infty}\big)
$$
$$
+ \int_0^1 \dots \int_0^1 \dots K\big((t_{i,\widehat{k}})_{i=1}^{\infty}, (s_{i,\widehat{k}})_{i=1}^{\infty}\big) u\big((s_{i,\widehat{k}})_{i=1}^{\infty}\big)\, ds_1 \cdots ds_n \cdots, \tag{35}
$$

where $f((t_{i,\widehat{k}})_{i=1}^{\infty}) : [0,1] \to \mathbb{R}$ and $K : [0,1] \times \mathbb{R} \to \mathbb{R}$ are two known continuous functions and $u((t_{i,\widehat{k}})_{i=1}^{\infty})$ is an unknown function.

Further, let us give our first application of the main results of this paper by proving the existence of a solution of infinite dimensional Fredholm integral equation (35).

Theorem 3.1 *Let $X = C([0,1], \mathbb{R})$ be the set of real continuous functions on $[0,1]$, and let $d : X \times X \to \mathbb{R}_+$ be given by*

$$
d\big((u_{i,\widehat{k}})_{i=1}^{\infty}, T(u_{i,\widehat{k}})_{i=1}^{\infty}\big) = \sup_{t \in [0,1]} |\, (u_{i,\widehat{k}})_{i=1}^{\infty} - T(u_{i,\widehat{k}})_{i=1}^{\infty}\,|. \tag{36}
$$

Define $T : \prod_{i=1}^{\infty} X \to X$ by

$$
Tu\big((t_{i,\widehat{k}})_{i=1}^{\infty}\big) = f\big((t_{i,\widehat{k}})_{i=1}^{\infty}\big)
$$
$$
+ \int_0^1 \dots \int_0^1 \dots K\big((t_{i,\widehat{k}})_{i=1}^{\infty}, (s_{i,\widehat{k}})_{i=1}^{\infty}\big) u\big((s_{i,\widehat{k}})_{i=1}^{\infty}\big)\, ds_1 \cdots ds_n \cdots. \tag{37}
$$

Assume that the following hold:
  (i) $|u((t_{i,\widehat{k}})_{i=1}^{\infty}) - Tu((t_{i,\widehat{k}})_{i=1}^{\infty})| \le \frac{1}{2}(|T((u_{i,\widehat{k}})_{i=1}^{\infty}) - u_k| + |T((u_{i+1,\widehat{k+1}})_{i=1}^{\infty}) - u_{k+1}|);$
  (ii) *There exists a constant $\delta \in (0,1)$ so that $K((t_{i,\widehat{k}})_{i=1}^{\infty}, (s_{i,\widehat{k}})_{i=1}^{\infty}) < \delta$;*
  (iii) *Let $\beta : [0,\infty) \to [0,1)$ be a lower bounded function and $\gamma > 1$ be large enough such that $\frac{1}{\gamma} < \beta(z)$ for every $z \in X$.*
Then the infinite dimensional Fredholm integral equation (35) has a solution.

Proof We can prove the existence of a solution of infinite dimensional Fredholm integral equation if we show that the operator T defined by (37) has a fixed point.

We can easy remark that the space $X = (C[0,1], \mathbb{R})$ endowed with the metric d defined by relation (36) forms a complete metric space. Then we shall show that all the hypotheses of Theorem 2.2 are verified.

We have the following estimation:

$$
\big|Tu\big((t_{i,\widehat{k}})_{i=1}^{\infty}\big) - T^2 u\big((t_{i,\widehat{k}})_{i=1}^{\infty}\big)\big|
$$
$$
= \bigg| \int_0^1 \dots \int_0^1 \dots K\big((t_{i,\widehat{k}})_{i=1}^{\infty}, (s_{i,\widehat{k}})_{i=1}^{\infty}\big) u\big((s_{i,\widehat{k}})_{i=1}^{\infty}\big)\, ds_1 \dots ds_n \dots
$$
$$
- \int_0^1 \dots \int_0^1 \dots K\big((t_{i,\widehat{k}})_{i=1}^{\infty}, (s_{i,\widehat{k}})_{i=1}^{\infty}\big) Tu\big((s_{i,\widehat{k}})_{i=1}^{\infty}\big)\, ds_1 \dots ds_n \dots \bigg|
$$

$$\leq \int_0^1 \dots \int_0^1 \dots K\big((t_{i,\widehat{k}})_{i=1}^\infty, (s_{i,\widehat{k}})_{i=1}^\infty\big) \big| u\big((s_{i,\widehat{k}})_{i=1}^\infty\big) - Tu\big((s_{i,\widehat{k}})_{i=1}^\infty\big) \big| ds_1 \dots ds_n \dots$$

$$\leq \frac{\delta}{2\gamma}\big(\big| T\big((u_{i,\widehat{k}})_{i=1}^\infty\big) - u_k \big| + \big| T\big((u_{i+1,\widehat{k+1}})_{i=1}^\infty\big) - u_{k+1} \big|\big) \int_0^1 \dots \int_0^1 \dots ds_1 \dots ds_n \dots$$

$$= \frac{\delta}{2\gamma}\big(\big| T\big((u_{i,\widehat{k}})_{i=1}^\infty\big) - u_k \big| + \big| T\big((u_{i+1,\widehat{k+1}})_{i=1}^\infty\big) - u_{k+1} \big|\big).$$

Taking supremum on both sides, we get

$$\sup_{t \in [0,1]} \big| Tu\big((t_{i,\widehat{k}})_{i=1}^\infty\big) - T^2 u\big((t_{i,\widehat{k}})_{i=1}^\infty\big) \big|$$

$$\leq \frac{\delta}{2\gamma} \sup_{t \in [0,1]} \big(\big| T\big((u_{i,\widehat{k}})_{i=1}^\infty\big) - u_k \big| + \big| T\big((u_{i+1,\widehat{k+1}})_{i=1}^\infty\big) - u_{k+1} \big|\big).$$

Further, since $\frac{1}{\gamma} < \beta(z)$, for $z = d(u_k, u_{k+1})$ and $\delta \in (0,1)$, we obtain

$$d\big(\big(Tu\big((t_{i,\widehat{k}})_{i=1}^\infty\big), T^2 u\big((t_{i,\widehat{k}})_{i=1}^\infty\big)\big)$$

$$\leq \frac{\delta\beta(d(u_k, u_{k+1}))}{2}\big(d\big(\big(T\big((u_{i,\widehat{k}})_{i=1}^\infty\big), u_k\big) + d\big(\big(T\big((u_{i+1,\widehat{k+1}})_{i=1}^\infty\big), u_{k+1}\big)\big)$$

$$\leq \frac{\beta(d(u_k, u_{k+1}))}{2}\big(d\big(\big(T\big((u_{i,\widehat{k}})_{i=1}^\infty\big), u_k\big) + d\big(\big(T\big((u_{i+1,\widehat{k+1}})_{i=1}^\infty\big), u_{k+1}\big)\big).$$

In conclusion, all the hypotheses of Theorem 2.2 are accomplished. Then the operator T has a fixed point, which means the infinite dimensional Fredholm integral equation (35) has a solution. □

The following application involves another type of integral equations: Urysohn type integral equations. We extend the known cases of this type of integral equations to *infinite dimensional Urysohn integral equations*:

$$u\big((t_{i,\widehat{k}})_{i=1}^\infty\big) = f\big((t_{i,\widehat{k}})_{i=1}^\infty\big)$$

$$+ \int_a^b \dots \int_a^b \dots P\big((t_{i,\widehat{k}})_{i=1}^\infty, (s_{i,\widehat{k}})_{i=1}^\infty, u\big((s_{i,\widehat{k}})_{i=1}^\infty\big)\big) ds_1 \cdots ds_n \cdots, \tag{38}$$

where $f\big((t_{i,\widehat{k}})_{i=1}^\infty\big) : [a,b] \to \mathbb{R}$ and $P : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ are two known continuous functions and $u\big((t_{i,\widehat{k}})_{i=1}^\infty\big)$ is an unknown function.

For this new type of Urysohn integral equation, let us give the following result.

Theorem 3.2 *Let $X = C([a,b], \mathbb{R})$ be the set of real continuous functions on $[a,b]$ endowed with the Bielecki norm $d : X \times X \to \mathbb{R}_+$ given by*

$$d\big(((u_{i,\widehat{k}})_{i=1}^\infty, T(u_{i,\widehat{k}})_{i=1}^\infty\big) = \big\| (u_{i,\widehat{k}})_{i=1}^\infty \big\|_B = \sup_{t \in [a,b]} | u(t_{i,\widehat{k}})_{i=1}^\infty | \cdot e^{-\tau t}, \quad with \ \tau > 1. \tag{39}$$

Define $T : \prod_{i=1}^\infty X \to X$ by

$$Tu\big((t_{i,\widehat{k}})_{i=1}^\infty\big) = f\big((t_{i,\widehat{k}})_{i=1}^\infty\big)$$

$$+ \int_a^b \dots \int_a^b \dots P\big((t_{i,\widehat{k}})_{i=1}^\infty, (s_{i,\widehat{k}})_{i=1}^\infty, u\big((s_{i,\widehat{k}})_{i=1}^\infty\big)\big) ds_1 \cdots ds_n \cdots . \tag{40}$$

Assume that there is $\varpi > 0$ so that

$$\left| P\big((t_{i,\widehat{k}})_{i=1}^\infty, (s_{i,\widehat{k}})_{i=1}^\infty, u\big((s_{i,\widehat{k}})_{i=1}^\infty\big)\big) - P\big((t_{i,\widehat{k}})_{i=1}^\infty, (s_{i,\widehat{k}})_{i=1}^\infty, Tu\big((s_{i,\widehat{k}})_{i=1}^\infty\big)\big) \right|$$

$$\leq \frac{1}{\varpi} \left| (u_{i,\widehat{k}})_{i=1}^\infty - T(u_{i,\widehat{k}})_{i=1}^\infty \right|.$$

Let $\beta(z) : [0,\infty) \to [0,1)$ be a lower bounded function such that, for $\varpi > 1$ large enough, $\frac{1}{\varpi} < \beta(z)$ for $z \in X$.

Then the infinite dimensional Urysohn integral equation (38) *has a solution.*

Proof It is easy to check that the space $X = (C[0,1], \mathbb{R})$ endowed with the metric d defined by relation (39) is a complete metric space.

To prove the existence of a solution of infinite dimensional Urysohn integral equation, we shall show that the operator T defined by (40) has a fixed point.

We have the following estimation:

$$\left| Tu\big((t_{i,\widehat{k}})_{i=1}^\infty\big) - T^2 u\big((t_{i,\widehat{k}})_{i=1}^\infty\big) \right|$$

$$= \left| \int_a^b \dots \int_a^b \dots P\big((t_{i,\widehat{k}})_{i=1}^\infty, (s_{i,\widehat{k}})_{i=1}^\infty, u\big((s_{i,\widehat{k}})_{i=1}^\infty\big)\big)\, ds_1 \dots ds_n \dots \right.$$

$$\left. - \int_a^b \dots \int_a^b \dots P\big((t_{i,\widehat{k}})_{i=1}^\infty, (s_{i,\widehat{k}})_{i=1}^\infty, Tu\big((s_{i,\widehat{k}})_{i=1}^\infty\big)\big)\, ds_1 \dots ds_n \dots \right|$$

$$\leq \int_a^b \dots \int_a^b \dots \left| P\big((t_{i,\widehat{k}})_{i=1}^\infty, (s_{i,\widehat{k}})_{i=1}^\infty, u\big((s_{i,\widehat{k}})_{i=1}^\infty\big)\big) \right.$$

$$\left. - P\big((t_{i,\widehat{k}})_{i=1}^\infty, (s_{i,\widehat{k}})_{i=1}^\infty, Tu\big((s_{i,\widehat{k}})_{i=1}^\infty\big)\big) \right| ds_1 \dots ds_n \dots$$

$$\leq \frac{1}{\varpi} \int_a^b \dots \int_a^b \dots \left| (u_{i,\widehat{k}})_{i=1}^\infty - T(u_{i,\widehat{k}})_{i=1}^\infty \right| ds_1 \dots ds_n \dots$$

$$\leq \frac{e^{\tau t}}{\varpi} \left| (u_{i,\widehat{k}})_{i=1}^\infty - T(u_{i,\widehat{k}})_{i=1}^\infty \right| e^{-\tau t} \int_a^b \dots \int_a^b \dots ds_1 \dots ds_n \dots$$

$$\leq \frac{e^{\tau t}}{\varpi} \left| (u_{i,\widehat{k}})_{i=1}^\infty - T(u_{i,\widehat{k}})_{i=1}^\infty \right| e^{-\tau t}.$$

Applying maximum on both sides, we get

$$\max_{t \in [a,b]} \left| Tu\big((t_{i,\widehat{k}})_{i=1}^\infty\big) - T^2 u\big((t_{i,\widehat{k}})_{i=1}^\infty\big) \right| e^{-\tau t} \leq \frac{1}{\varpi} \max_{t \in [a,b]} \left| (u_{i,\widehat{k}})_{i=1}^\infty - T(u_{i,\widehat{k}})_{i=1}^\infty \right| e^{-\tau t}.$$

Then we have

$$d\big(\big(Tu\big((t_{i,\widehat{k}})_{i=1}^\infty\big), T^2 u\big((t_{i,\widehat{k}})_{i=1}^\infty\big)\big)\big) \leq \frac{1}{\varpi} d\big(((u_{i,\widehat{k}})_{i=1}^\infty, T(u_{i,\widehat{k}})_{i=1}^\infty\big).$$

Since $\frac{1}{\varpi} < \beta(z)$ and for $z = M_k((u_{i,\widehat{k}})_{i=1}^\infty, (u_{i+1,\widehat{k+1}})_{i=1}^\infty)$, we obtain

$$d\big(\big(Tu\big((t_{i,\widehat{k}})_{i=1}^\infty\big), T^2 u\big((t_{i,\widehat{k}})_{i=1}^\infty\big)\big)$$

$$\leq \beta\big(M_k\big((u_{i,\widehat{k}})_{i=1}^\infty, (u_{i+1,\widehat{k+1}})_{i=1}^\infty\big)\big) d\big(((u_{i,\widehat{k}})_{i=1}^\infty, T(u_{i,\widehat{k}})_{i=1}^\infty\big)$$

$$\leq \beta\big(M_k\big((u_{i,\widehat{k}})_{i=1}^\infty, (u_{i+1,\widehat{k+1}})_{i=1}^\infty\big)\big) M_k\big((u_{i,\widehat{k}})_{i=1}^\infty, (u_{i+1,\widehat{k+1}})_{i=1}^\infty\big).$$

Then all the conditions of Theorem 2.1 are satisfied, so the operator T has a fixed point. Then the infinite dimensional Urysohn type integral equation (38) has a solution.   □

4 Conclusion

Here, in this article we have shown some metric combinatorial arguments that can be profitably applied to extend Geraghty's theorem, Kannan–Geraghty theorem, and Fisher's theorem as well as a particular extension for the infinite cases. This idea could also motivate to find a similar infinite extension of the other contractions and their related operators, to find fixed points or coincidence points. We also defined two new types of infinite integral equations which would encourage the study of the infinite integral equations and their solutions. From the discussion we have some interesting problems as follows:

1. Can a similar concept be extended to the notion of coincidence points?
2. We have discussed the fixed point theorem for the contraction from countable products of the complete metric space X (with respect to uniform metric topology) to the space X. Can it further be extended to an uncountable product of a complete metric space which is not necessarily a metric space?

Author details
[1]Tezpur University, Napaam, Tezpur Assam 784028, India. [2]Department of Mathematics, Faculty of Sciences, King Abdulaziz University, Jeddah, Saudi Arabia. [3]Department of Pharmaceutical Sciences, "Vasile Goldiş" Western University of Arad, Arad, Romania. [4]Institut Supérieur d'Informatique et des Techniques de Communication, Université de Sousse, H. Sousse 4000, Tunisia. [5]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa. [6]China Medical University Hospital, China Medical University, Taichung, 40402, Taiwan. [7]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea.

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