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Stability of an additive-quartic functional equation in modular spaces

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Abstract

In this paper, we prove the Ulam-Hyers stability of the following additive-quartic functional equation

$$f\left(\frac{u+v}{2}-w\right)+f\left(\frac{v+w}{2}-u\right)+f\left(\frac{w+u}{2}-v\right)=\frac{25}{32}\left(f(u-v)+f(v-w)+f(w-u)\right)-\frac{7}{32}\left(f(v-u)+f(w-v)+f(u-w)\right)$$

in modular spaces by using the direct method.

Keywords: Ulam-Hyers stability, additive functional equation, quartic functional equation, modular space.

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1. Introduction

Stability of the functional equation has been growing tremendously over the last seventy years. Ulam [43], who was responsible for this, raised the question of stability at a conference in 1940. The following year, a mathematician named Hyers [11] gave an answer based on Ulam's question. Aoki [2] published a paper in1950 generalizing Hyers' response. Then in 1978, Rassias [36] introduced a new stability result as sum of two norms. Rassias [37] converted to the multiplication of two norms in 1982. In 1994, Găvruta [9] provided a further generalization of Rassias theorem in which he replaced the bound by a general control function. In 2008, Ravi [40] established mixed type stability by adding sum of two norms and product of two norms. Subsequent authors have given flexible results using a lot of functional equations in modular spaces [4, 10, 22, 32, 34, 35, 44, 45].

The additive functional equation is

$$f(x+y)=f(x)+f(y). \tag{1.1}$$

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Since $f(x) = kx$ is the solution of the functional equation (1.1), every solution of the additive functional equation is called an additive mapping. The functional equation

$$f(2x+y)+f(2x-y) = 4f(x+y)+4f(x-y)+24f(x)-6f(y) \tag{1.2}$$

is called a quartic functional equation. Since the function $f(x) = x^4$ is a solution of (1.2), every solution of the quartic functional equation is called a quartic mapping.

In this paper, we present the Ulam-Hyers stability of the additive-quartic mixed type functional equation of the form

$$\begin{aligned} & f\left(\frac{u+v}{2}-w\right)+f\left(\frac{v+w}{2}-u\right)+f\left(\frac{w+u}{2}-v\right) \\ & = \frac{25}{32}\left(f(u-v)+f(v-w)+f(w-u)\right)-\frac{7}{32}\left(f(v-u)+f(w-v)+f(u-w)\right) \end{aligned} \tag{1.3}$$

in modular spaces by using the direct method.

2. Basic concepts on modular space

The research on modulars and modular spaces was begun by Nakano [31] as generalizations of normed spaces. Since the 1950s, many prominent mathematicians like Luxemburg, Mazur, Musielak, and Orlicz [25, 26, 29, 30] developed it extensively. Modulars and modular spaces have broad branches of applications, e.g., interpolation theory and Orlicz spaces. We start by considering some basic relevant notions.

Definition 2.1. Let X be a vector space over a field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A generalized function $\rho : X \to [0, \infty]$ is called a modular if for any $\alpha, \beta \in \mathbb{K}$ and $x, y \in X$,

(1) $\rho(x) = 0$ if and only if $x = 0$;
(2) $\rho(\alpha x) = \rho(x)$ for every α with $|\alpha| = 1$;
(3) $\rho(\alpha x + \beta y) \leqslant \rho(x) + \rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geqslant 0$.

If the condition (3) is replaced with

(4) $\rho(\alpha x + \beta y) \leqslant \alpha^s \rho(x) + \beta^s \rho(y)$ if $\alpha s + \beta s = 1$ and $\alpha, \beta \geqslant 0$ with an $s \in (0, 1]$,

then ρ is called an s-convex modular. We call 1-convex modulars as convex modulars.

A modular ρ on X generates a linear subspace X_ρ of X naturally defined by

$$X_\rho = \left\{ x \in X | \lim_{\lambda \to 0} \rho(\lambda x) = 0 \right\}.$$

X_ρ is called a modular space.

Definition 2.2. Let X_ρ be a modular space and $\{x_n\}$ be a sequence in X_ρ.

(1) $\{x_n\}$ is ρ-convergent to a point $x \in X_\rho$ if $\rho(x_n \to x_m) \to 0$ as $n \to \infty$. The point x is called a ρ-limit of the sequence $\{x_n\}$.
(2) $\{x_n\}$ is called a ρ-Cauchy sequence if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
(3) X_ρ is called ρ-complete if every ρ-Cauchy sequence in X_ρ is ρ-convergent.

Remark 2.3. If ρ is a convex modular and $0 \leqslant \lambda \leqslant 1$, then we have $\rho(\lambda x) \leqslant \lambda \rho(x)$ for all $x \to X_\rho$. If ρ is a convex modular, and $\lambda_i \geqslant 0, i = 1, 2, \ldots, n$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n \leqslant 1$, then $\rho(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) \leqslant \lambda_1 \rho(x_1) + \lambda_2 \rho(x_2) + \cdots + \lambda_n \rho(x_n)$. If $\{x_n\}$ is ρ-convergent to x, then $\{\alpha x_n\}$ is ρ-convergent to αx, where $0 \leqslant \alpha \leqslant 1$. But the ρ-convergence of a sequence $\{x_n\}$ to x does not imply that $\{c x_n\}$ is ρ-convergent to cx for scalars c with $|c| > 1$.

There are two notions that play important roles when we study modulars. A modular ρ is said to have the Fatou property if $\rho(x) \leqslant \liminf_{n\to\infty} \rho(x_n)$ for every sequence $\{x_n\}$ that is ρ-convergent to x. ρ is said to satisfy the Δ_2-condition if there exists a constant $k \geqslant 0$ such that $\rho(2x) \leqslant k\rho(x)$ for all $x \to X_\rho$.

Example 2.4. For a measure space $(\Omega, \sum, \mu)$, let $L^0(\mu)$ be the collection of all measurable functions on Ω. Let

$$L^{\Phi}(\mu) = \left\{ f \to L^0(\mu) | \int_{\Omega} \varphi\left(|\lambda f(x)|\right) d\mu(x) \to 0 \text{ as } \lambda \to 0 \right\},$$

where $\phi : [0, \infty) \to R$ is assumed to be a continuous, positive, convex and nondecreasing function increasing to infinity with $\phi(0) = 0$. We can take, e.g., $\phi(t) = e^{t^2} - 1$. $L^{\Phi}(\mu)$ is called an Orlicz space. Define for $f \to L^{\Phi}(\mu)$,

$$\rho_{\Phi}(f) = \int_{\Omega} \varphi\left(|f|\right) d\mu.$$

Then ρ_{Φ} is a complete modular.

3. Solution of the functional equation (1.3)

In this section, we solve the mixed type functional equation (1.3). Throughout this section let E and H be real vector spaces.

Theorem 3.1. *An odd mapping $f : E \to H$ satisfies the functional equation (1.3) for all $u, v, w \in E$ if and only if $f : E \to H$ satisfies the functional equation (1.1) for all $x, y \in E$.*

Proof. Since f is an odd mapping, one can deduce from (1.3) that we have

$$f\left(\frac{u+v}{2} - w\right) + f\left(\frac{v+w}{2} - u\right) + f\left(\frac{w+u}{2} - v\right) = f(u-v) + f(v-w) + f(w-u) \tag{3.1}$$

for all $u, v, w \in E$. Setting $v = u, w = -u$ in (3.1) and finally replacing u by $-x$, we obtain

$$2f(x) = f(2x) \tag{3.2}$$

for all $x \in E$. Replacing x by $\frac{x}{2}$ in (3.2), we get

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x)$$

for all $x \in E$. Letting $u = x, v = y, w = 0$ in (3.1) and using the oddness, we obtain

$$f(x+y) + f(y-2x) + f(x-2y) = 2f(x-y) + 2f(y) - 2f(x) \tag{3.3}$$

for all $x, y \in E$. Interchanging x and y in (3.3) and using the oddness, we get

$$f(x+y) + f(x-2y) + f(-2x+y) = -2f(x-y) + 2f(x) - 2f(y) \tag{3.4}$$

for all $x, y \in E$. Subtracting (3.4) from (3.3), we get

$$f(x-y) = f(x) - f(y)$$

for all $x, y \in E$. Replacing y by $-y$ and using the oddness, we have (1.1).

Conversly, replacing (x, y) by (u, u) in (1.1), we get $f(2u) = 2f(u)$ for all $u \in E$.

Replacing (x, y) by $\left(\frac{u+w}{2}, v\right)$ in (1.1) and using the oddness, we get

$$f\left(\frac{u+w}{2} - v\right) = \frac{1}{2}f(u) + \frac{1}{2}f(w) - f(v) \tag{3.5}$$

for all $u, v, w \in E$. Replacing (x, y) by $\left(\frac{v+w}{2}, u\right)$ in (1.1), we get

$$f\left(\frac{v+w}{2}-u\right) = \frac{1}{2}f(v) + \frac{1}{2}f(w) - f(u) \tag{3.6}$$

for all $u, v, w \in E$. Replacing (x, y) by $\left(\frac{u+v}{2}, w\right)$ in (1.1), we get

$$f\left(\frac{u+v}{2}-w\right) = \frac{1}{2}f(u) + \frac{1}{2}f(v) - f(w) \tag{3.7}$$

for all $u, v, w \in E$. Adding (3.5), (3.6), (3.7), and using the oddness, we obtain (3.1) for all $u, v, w \in E$. □

Lemma 3.2. *Let $f : E \to H$ satisfy the functional equation (1.2) for all $x, y \in E$. Then $f : E \to H$ satisfies the functional equation*

$$f(2x+y) + f(2x-y) + f(x+2y) + f(x-2y) = 8f(x+y) + 8f(x-y) + 18f(x) + 18f(y) \tag{3.8}$$

for all $x, y \in E$.

Proof. Let $f : E \to H$ be a mapping satisfying the functional equation (1.2). Interchanging x and y in (1.2) and using eveness of f, we arrive at

$$f(x+2y) + f(x-2y) = 4(x+y) + 4f(x-y) + 24f(y) - 6f(x). \tag{3.9}$$

Adding (1.2) and (3.9), we get (3.8). □

Theorem 3.3. *Let an even mapping $f : E \to H$ satisfies the functional equation (1.3) for all $x, y \in E$. Then $f : E \to H$ satisfies the functional equation (3.8) for all $x, y, z \in E$.*

Proof. Since f is an even mapping, one can deduce from (1.3) that we have

$$f\left(\frac{u+v}{2}-w\right) + f\left(\frac{v+w}{2}-u\right) + f\left(\frac{w+u}{2}-v\right) = \frac{9}{16}\left(f(u-v) + f(v-w) + f(w-u)\right) \tag{3.10}$$

for all $u, v, w \in E$. Setting $u = v = w = 0$ in (3.10), we get $f(0) = 0$. Replacing (u, v, w) by $(x, x, -x)$ in (3.10), we get

$$f(2x) = 16f(x) \tag{3.11}$$

for all $x \in E$. Setting x by $\frac{x}{2}$ in (3.11), we have

$$f\left(\frac{x}{2}\right) = \frac{1}{16}f(x) \tag{3.12}$$

for all $x \in E$. Replacing (u, v, w) by $(x, y, 0)$ in (3.10) using (3.12) and evenness, we obtain

$$f(x+y) + f(2x-y) + f(x-2y) = 9\left(f(x-y) + f(y) + f(x)\right) \tag{3.13}$$

for all $x, y \in E$. Replacing y by $-y$ in (3.13) and using evenness, we obtain

$$f(x-y) + f(2x+y) + f(x+2y) = 9\left(f(x+y) + f(y) + f(x)\right) \tag{3.14}$$

for all $x, y \in E$. Adding (3.13) and (3.14), we arrive (3.8) for all $x, y \in E$. By Lemma 3.2, f is quartic. □

4. Additive-quartic mixed type stability results: direct method

Throughout this paper, let V and X be linerar spaces, ρ be a convex modular, and X_ρ be a ρ-complete modular space. Define a mappings $Df, Df_o, Df_{q_2} : V^3 \to X_\rho$ respectively by

$$\begin{aligned} D\ f(u,v,w) =& f\left(\frac{u+v}{2}-w\right)+f\left(\frac{v+w}{2}-u\right)+f\left(\frac{w+u}{2}-v\right) \\ &-\frac{25}{32}\left(f(u-v)+f(v-w)+f(w-u)\right)+\frac{7}{32}\left(f(v-u)+f(w-v)+f(u-w)\right) \end{aligned}$$

for all $u,v,w \in V$,

$$D\ f_o(u,v,w) = f\left(\frac{u+v}{2}-w\right)+f\left(\frac{v+w}{2}-u\right)+f\left(\frac{w+u}{2}-v\right)-\left(f(u-v)+f(v-w)+f(w-u)\right)$$

for all $u,v,w \in V$, and

$$D\ f_{q_2}(u,v,w) = f\left(\frac{u+v}{2}-w\right)+f\left(\frac{v+w}{2}-u\right)+f\left(\frac{w+u}{2}-v\right)-\frac{9}{16}\left(f(u-v)+f(v-w)+f(w-u)\right)$$

for all $u,v,w \in V$.

4.1. *Additive stability of (1.3) in modular space without Δ_2-conditions*

In this subsection, we present the Ulam-Hyers stability of the additive-quartic functional equation (1.3) using Hyers' direct method in modular space without Δ_2-conditions.

Theorem 4.1. *Let $j \in \{-1,1\}$. Let $\Gamma : V^3 \to [0,\infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\Gamma\left(2^{kj}u, 2^{kj}v, 2^{kj}w\right)}{2^{kj}} \quad \text{converges to } \mathbb{R} \tag{4.1}$$

for all $u,v,w \in V$ and $f_o : V \to X_\rho$ be an odd mapping that satisfies the inequality

$$\rho\left(D\ f_o(u,v,w)\right) \leqslant \Gamma\left(u,v,w\right) \tag{4.2}$$

for all $u,v,w \in V$. Then there exists a unique additive function $A : V \to X_\rho$ such that

$$\rho\left(f_o(u)-A(u)\right) \leqslant \sum_{i=\frac{1-j}{2}}^{\infty} \frac{\Gamma(2^{ij}u, 2^{ij}u, -2^{ij}u)}{2^{ij}} \tag{4.3}$$

for all $u \in V$. The mapping $A(u)$ is defined by

$$A(u) = \rho - \lim_{k\to\infty} \frac{f_o(2^{kj}u)}{2^{kj}} \tag{4.4}$$

for all $u \in V$.

Proof. Assume $j = 1$. Replacing (u,v,w) by $(u,u,-u)$ in (4.2) and dividing by 2, we get

$$\rho\left(\frac{f_o(2u)}{2}-f_o(u)\right) \leqslant \frac{1}{2}\Gamma(u,u,-u) \tag{4.5}$$

for all $u \in V$, since f is an odd mapping. Replacing u by 2u in (4.5) and dividing by 2, we get

$$\rho\left(\frac{f_o(2^2u)}{2^2}-\frac{f_o(2u)}{2^2}\right) \leqslant \frac{1}{2^2}\Gamma(2u,2u,-2u) \tag{4.6}$$

for all $u \in V$. Combining (4.5) and (4.6), we obtain

$$\rho\left(\frac{f_o(2^2u)}{2^4} - f_o(u)\right) \leqslant \left[\frac{1}{2}\Gamma(u,u,-u) + \frac{1}{2^2}\Gamma(2u,2u,-2u)\right]$$

for all $u \in V$. Using induction on a positive integer k, we obtain that

$$\rho\left(\frac{f_o(2^ku)}{2^k} - f_o(u)\right) \leqslant \sum_{i=0}^{k-1}\frac{1}{2^{i+1}}\Gamma\left(2^iu,2^iu,-2^iu\right) \leqslant \sum_{i=0}^{\infty}\frac{1}{2^{i+1}}\Gamma\left(2^iu,2^iu,-2^iu\right) \tag{4.7}$$

for all $u \in V$. Let m and n be nonnegative integers with $n > m$. By (4.7), we have

$$\begin{aligned}
\rho\left(\frac{f_o(2^nu)}{2^n} - \frac{f_o(2^mu)}{2^m}\right) &= \rho\left(\frac{1}{2^m}\left(\frac{f_o(2^{n-m}.2^mu)}{2^{n-m}} - f_o(2^mu)\right)\right)\\
&\leqslant \frac{1}{2^m}\sum_{i=0}^{n-m}\frac{1}{2^{i+1}}\Gamma\left(2^i.2^mu,2^i.2^mu,-2^i.2^mu\right)\\
&= \sum_{i=0}^{n-m}\frac{1}{2^{i+m+1}}\Gamma\left(2^{i+m}u,2^{i+m}u,-2^{i+m}u\right)\\
&= \sum_{i=m+1}^{n}\frac{1}{2^i}\Gamma\left(2^{i-1}u,2^{i-1}u,-2^{i-1}u\right)
\end{aligned} \tag{4.8}$$

for all $u \in V$. Then (4.1) and (4.8) yield that $\left\{\frac{f_o(2^ku)}{2^k}\right\}$ is a ρ-Cauchy sequence in X_ρ. The ρ-completeness of X_ρ guarantees its ρ-convergence. Hence, there exists a mapping $A : V \to X_\rho$ defined by

$$A(u) = \rho - \lim_{n\to\infty}\frac{f_o(2^nu)}{2^n}, \quad \forall\ u \in V. \tag{4.9}$$

Then we see that

$$\begin{aligned}
\rho\left(\frac{A(2u) - 2A(u)}{2^3}\right) &= \rho\left(\frac{1}{2^3}\left(A(2u) - \frac{f_o\left(2^{n+1}u\right)}{2^n}\right) + \frac{1}{2}\left(\frac{1}{2}.\frac{f_o\left(2^{n+1}\right)}{2^{n+1}} - \frac{1}{2}A(u)\right)\right)\\
&\leqslant \frac{1}{2^3}\rho\left(A(2u) - \frac{2^{n+1}}{2^n}\right) + \frac{1}{4}\rho\left(\frac{f_o\left(2^{n+1}\right)}{2^{n+1}} - A(u)\right)
\end{aligned} \tag{4.10}$$

for all $u \in V$. Then by (4.9), the right hand side of (4.10) tends to 0 as $n \to \infty$. Therefore, it follows that

$$A(2u) = 2A(u), \quad \forall\ u \in V.$$

Next, we calculate $\rho\left(A(u) - f_o(u)\right)$. Note that for every $n \in N$, by (4.10) we write

$$\begin{aligned}
\rho\left(A(u) - f_o(u)\right) &= \rho\left(\sum_{k=1}^{n}\frac{f_o(2^ku) - 2f_o(2^{k-1}u)}{2^k} + \left(A(u) - \frac{f_o(2^nu)}{2^n}\right)\right)\\
&= \rho\left(\sum_{k=1}^{n}\frac{f_o(2^ku) - 2f_o(2^{k-1}u)}{2^k} + \frac{1}{2}\left(A(2u) - \frac{f_o(2^{n-1}.2u)}{2^{n-1}}\right)\right).
\end{aligned} \tag{4.11}$$

Since $\sum\limits_{k=1}^{n}\frac{1}{2^k} + \frac{1}{2} < 1$, it follows from (4.3) and (4.11) that

$$\begin{aligned}
\rho\left(A(u) - f_o(u)\right) &\leqslant \sum_{k=1}^{n}\frac{1}{2^k}\rho\left(f_o\left(2^ku\right) - 2f\left(2^{k-1}\right)\right) + \frac{1}{2}\rho\left(A(2u) - \frac{f_o\left(2^{n-1}.2u\right)}{2^{n-1}}\right)\\
&\leqslant \sum_{k=1}^{n}\frac{1}{2^k}\Gamma\left(2^{k-1}u,2^{k-1}u,-2^{k-1}u\right) + \frac{1}{2}\rho\left(A(2u) - \frac{f_o\left(2^{n-1}.2u\right)}{2^{n-1}}\right)
\end{aligned} \tag{4.12}$$

for all $u \in V$. Letting $n \to \infty$ in (4.12), we obtain

$$\rho\left(A(u) - f_o(u)\right) \leqslant \Gamma(u, u, -u), \quad \forall\ u \in V.$$

Therefore, we get (4.3).

Now, we prove that A is additive. We note that

$$\rho\left(\frac{1}{2^n} D\ f_o(2^n u, 2^n v, 2^n w)\right) \leqslant \frac{1}{2^n}\Gamma(2^n u, 2^n u, -2^n u) \to 0 \text{ as } n \to \infty$$

for all $u, v, w \in V$. Thus, we observe by convexity of ρ that

$$\begin{aligned}
&\rho\left(\frac{1}{7}A\left(\frac{u+v}{2} - w\right) + \frac{1}{7}A\left(\frac{v+w}{2} - u\right) + \frac{1}{7}A\left(\frac{w+u}{2} - v\right)\right. \\
&\quad \left. - \left(\frac{1}{7}A(u-v) + \frac{1}{7}A(v-w) + \frac{1}{7}A(w-u)\right)\right) \\
&\leqslant \frac{1}{7}\rho\left(A\left(\frac{u+v}{2} - w\right) - \frac{1}{2^n} f_o\left(\frac{2^n(u+v)}{2} - 2^n w\right)\right) \\
&\quad + \frac{1}{7}\rho\left(A\left(\frac{v+w}{2} - u\right) - \frac{1}{2^n} f_o\left(\frac{2^n(v+w)}{2} - 2^n u\right)\right) \\
&\quad + \frac{1}{7}\rho\left(A\left(\frac{w+u}{2} - v\right) - \frac{1}{2^n} f_o\left(\frac{2^n(w+u)}{2} - 2^n v\right)\right) + \frac{1}{7}\rho\left(-A(u-v) + \frac{1}{2^n} f_o\left(2^n u - 2^n v\right)\right) \\
&\quad + \frac{1}{7}\rho\left(-A(v-w) + \frac{1}{2^n} f_o\left(2^n v - 2^n w\right)\right) + \frac{1}{7}\rho\left(-A(w-u) + \frac{1}{2^n} f_o\left(2^n w - 2^n u\right)\right) \\
&\quad + \frac{1}{7}\rho\left(\frac{1}{2^n}\left(f_o\left(\frac{2^n(u+v)}{2} - w\right) + f_o\left(\frac{2^n(v+w)}{2} - u\right) + f_o\left(\frac{2^n(w+u)}{2} - v\right)\right.\right. \\
&\quad \left.\left. - \left(f_o\left(2^n u - 2^n v\right) + f_o\left(2^n v - 2^n w\right) + f_o\left(2^n w - 2^n u\right)\right)\right)\right)
\end{aligned} \tag{4.13}$$

for all $u, v, w \in V$. Taking $n \to \infty$ in (4.13), we get

$$A\left(\frac{u+v}{2} - w\right) + A\left(\frac{v+w}{2} - u\right) + A\left(\frac{w+u}{2} - v\right) - \left(A(u-v) + A(v-w) + A(w-v)\right) = 0$$

for all $u, v, w \in V$. This gives that A is additive.

In order to prove A is unique, let A' be another additive mapping satisfying (4.3) and (1.3). Since A and A' are additive mappings, $A(2^n u) = 2^n A(u)$ and $A'(2^n u) = 2^n A'(u)$ hold. So

$$\begin{aligned}
\rho\left(\frac{1}{2}A(u) - \frac{1}{2}A'(u)\right) &= \frac{1}{2}\rho\left(\frac{A(2^n u)}{2^n} - \frac{f_o(2^n u)}{2^n}\right) + \frac{1}{2}\rho\left(\frac{f_o(2^n u)}{2^n} - \frac{A'(2^n u)}{2^n}\right) \\
&\leqslant \frac{1}{2^{n+1}}\rho\left(A(2^n u) - f_o(2^n u)\right) + \frac{1}{2^{n+1}}\rho\left(f_o(2^n u) - A'(2^n u)\right) \\
&\leqslant \frac{1}{2^n}\sum_{i=0}^{\infty}\frac{1}{2^{i+1}}\Gamma\left(2^{i+n}u, 2^{i+n}u, -2^{i+n}u\right) \\
&\leqslant \sum_{i=n}^{\infty}\frac{1}{2^{i+1}}\Gamma\left(2^i u, 2^i u, -2^i u\right) \to 0 \text{ as } n \to \infty
\end{aligned}$$

for all $u \in V$. Hence A is unique.

For $j = -1$, we can prove the similar stability result. Hence we complete the proof. □

Corollary 4.2. *Let Θ and τ be nonnegative real numbers. If an odd mapping $f_o : V \to X_\rho$ satisfies the inequality*

$$\rho\left(Df_o(u, v, w)\right) \leqslant \Theta$$

for all $u, v, w \in V$, then there exists a unique additive mapping $A : V \to X_\rho$ such that

$$\rho\left(f_o(u) - A(u)\right) \leqslant \frac{\Theta}{|1|}$$

for all $u \in V$.

Corollary 4.3. *Let Θ and τ be nonnegative real numbers with $\tau \neq 1$. If an odd mapping $f_o : V \to X_\rho$ satisfies the inequality*

$$\rho\left(Df_o(u, v, w)\right) \leqslant \Theta\left(\|u\|^\tau + \|v\|^\tau + \|w\|^\tau\right)$$

for all $u, v, w \in V$, then there exists a unique additive mapping $A : V \to X_\rho$ such that

$$\rho\left(f_o(u) - A(u)\right) \leqslant \frac{3\Theta\|u\|^\tau}{|2 - 2^\tau|}$$

for all $u \in V$.

Corollary 4.4. *Let Θ and τ be nonnegative real numbers with $\tau \neq \frac{1}{3}$. If an odd mapping $f_o : V \to X_\rho$ satisfies the inequality*

$$\rho\left(Df_o(u, v, w)\right) \leqslant \Theta\left(\|u\|^\tau \|v\|^\tau \|w\|^\tau\right)$$

for all $u, v, w \in V$, then there exists a unique additive mapping $A : V \to X_\rho$ such that

$$\rho\left(f_o(u) - A(u)\right) \leqslant \frac{\Theta\|u\|^{3\tau}}{|2 - 2^{3\tau}|}$$

for all $u \in V$.

Corollary 4.5. *Let Θ and τ be nonnegative real numbers with $\tau \neq \frac{1}{3}$. If an odd mapping $f_o : V \to X_\rho$ satisfies the inequality*

$$\rho\left(Df_o(u, v, w)\right) \leqslant \Theta\left(\|u\|^\tau \|v\|^\tau \|w\|^\tau + \|u\|^{3\tau} + \|v\|^{3\tau} + \|w\|^{3\tau}\right)$$

for all $u, v, w \in V$, then there exists a unique additive mapping $A : V \to X_\rho$ such that

$$\rho\left(f_o(u) - A(u)\right) \leqslant \frac{4\Theta\|u\|^{3\tau}}{|2 - 2^{3\tau}|}$$

for all $u \in V$.

4.2. Additive stability of (1.3) in modular space with Δ_2-conditions

In this subsection, we present the Ulam-Hyers stability of the additive-quartic functional equation (1.3) using Hyers' direct method in modular space with Δ_2-conditions.

Theorem 4.6. *Let $j \in \{-1, 1\}$. Let $\Gamma : V^3 \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \left(\frac{k^2}{2}\right)^n \Gamma\left(\frac{u}{2^{nj}}, \frac{u}{2^{nj}}, \frac{-u}{2^{nj}}\right) < +\infty \quad \text{and} \quad \lim_{n\to\infty} k^{nj}\Gamma\left(\frac{u}{2^{nj}}, \frac{v}{2^{nj}}, \frac{w}{2^{nj}}\right) = 0 \tag{4.14}$$

for all $u, v, w \in V$ *and* $f_o : V \to X_\rho$ *be an odd mapping satisfying the inequality*

$$\rho\left(D\, f_o(u,v,w)\right) \leqslant \Gamma\left(u,v,w\right) \tag{4.15}$$

for all $u, v, w \in V$. *Then there exists a unique additive mapping* $A : X \to H$ *such that*

$$\rho\left(f_o(u) - A(u)\right) \leqslant \frac{1}{2}\sum_{i=\frac{1-j}{2}}^{\infty}\left(\frac{k^2}{2}\right)^i \Gamma\left(\frac{u}{2^{ij}}, \frac{u}{2^{ij}}, \frac{-u}{2^{ij}}\right) \tag{4.16}$$

for all $u \in V$. *The mapping* $A(u)$ *is defined by*

$$A(u) = \rho - \lim_{n\to\infty} 2^{nj} f_o\left(\frac{u}{2^{nj}}\right) \tag{4.17}$$

for all $u \in V$.

Proof. Assume $j = 1$. Replacing (u, v, w) by $\left(\frac{u}{2}, \frac{u}{2}, \frac{-u}{2}\right)$ in (4.15), we get

$$\rho\left(f_o(u) - 2f_o\left(\frac{u}{2}\right)\right) \leqslant \Gamma\left(\frac{u}{2}, \frac{u}{2}, \frac{-u}{2}\right)$$

for all $u \in V$, since f is an odd mapping. Then it follows from Δ_2-condition and the convexity of the modular ρ that

$$\rho\left(f_o(u) - 2^n f_o\left(\frac{u}{2^n}\right)\right) = \rho\left(\sum_{i=1}^{n}\frac{1}{2^i}\left(f_o\left(\frac{u}{2^{i-1}}\right) - 2^{2i} f_o\left(\frac{u}{2^i}\right)\right)\right) \leqslant \frac{1}{k}\sum_{i=1}^{n}\left(\frac{k^2}{2}\right)^i \Gamma\left(\frac{u}{2^i}, \frac{u}{2^i}, \frac{-u}{2^i}\right)$$

for all $u \in V$. So, for all $n, m \in \mathbb{N}$ with $n \geqslant m$, we have

$$\begin{aligned}
\rho\left(2^n f_o\left(\frac{u}{2^n}\right) - 2^m f_o\left(\frac{u}{2^m}\right)\right) &= \rho\left(2^m\left(2^{n-m} f_o\left(\frac{u}{2^{n-m}}\right) - f_o\left(\frac{u}{2^m}\right)\right)\right)\\
&\leqslant k^m \sum_{i=1}^{n-m}\left(\frac{k^2}{2}\right)^i \Gamma\left(\frac{u}{2^i.2^m}, \frac{u}{2^i.2^m}, \frac{-u}{2^i.2^m}\right)\\
&\leqslant \frac{2^m}{k^{m+1}} \sum_{i=m+1}^{n}\left(\frac{k^2}{2}\right)^2 \Gamma\left(\frac{u}{2^i}, \frac{u}{2^i}, \frac{-u}{2^i}\right)
\end{aligned}$$

for all $u \in V$. Since the right hand side of the above inequality tends to zero as n goes to infinity, the sequence $\left\{2^n f_o\left(\frac{u}{2^n}\right)\right\}$ is a ρ-Cauchy sequence in X_ρ and so the sequence $\left\{2^n f_o\left(\frac{u}{2^n}\right)\right\}$ is a ρ-convergent sequence on X_ρ. Thus, we may define a mapping $A : V \to X_\rho$ as

$$A(u) = \rho - \lim_{n\to\infty} 2^n f_o\left(\frac{u}{2^n}\right), \ \text{ i.e., } \lim_{n\to\infty} \rho\left(2^n f_o\left(\frac{u}{2^n}\right) - A(u)\right) = 0$$

for all $u \in V$. According to the Δ_2-condition, we obtain the following inequality

$$\begin{aligned}
\rho\left(A(u) - f(u)\right) &\leqslant \frac{1}{2}\rho\left(2A(u) - 2^{n+1} f_o\left(\frac{u}{2^n}\right)\right) + \frac{1}{2}\rho\left(2^{n+1} f_o\left(\frac{u}{2^n}\right) - 2f(u)\right)\\
&\leqslant \frac{k}{2}\rho\left(A(u) - 2^n f_o\left(\frac{u}{2^n}\right)\right) + \frac{k}{2}\rho\left(2^n f_o\left(\frac{u}{2^n}\right) - f(u)\right)\\
&\leqslant \frac{1}{2}\sum_{i=1}^{\infty}\left(\frac{k^2}{2}\right)^i \Gamma\left(\frac{u}{2^i}, \frac{u}{2^i}, \frac{-u}{2^i}\right) + \frac{k}{2}\rho\left(2^n f_o\left(\frac{u}{2^n}\right) - f(u)\right)
\end{aligned}$$

for all $u \in V$. Taking n tends to ∞, we conclude that the estimation (4.16) of f by A holds for all $u \in V$. Now, we claim that the mapping A is additive. Replacing (u,v,w) by $\left(\frac{u}{2^n},\frac{v}{2^n},\frac{w}{2^n}\right)$ in (4.15) with Δ_2-condition, we obtain

$$\rho\left(2^n Df_o\left(\frac{u}{2^n},\frac{v}{2^n},\frac{w}{2^n}\right)\right) \leqslant k^n \Gamma\left(\frac{u}{2^n},\frac{v}{2^n},\frac{w}{2^n}\right)$$

for all $u,v,w \in V$. Thus, it follows from the Δ_2-condition such that

$$\begin{aligned}
&\rho\left(A\left(\frac{u+v}{2}-w\right)+A\left(\frac{v+w}{2}-u\right)+A\left(\frac{w+u}{2}-v\right)-(A(u-v)+A(v-w)+A(w-u))\right)\\
&\leqslant \frac{1}{7}\rho\left(7\left(A\left(\frac{u+v}{2}-w\right)-2^n f_o\left(\frac{u+v}{2^{n+1}}-\frac{w}{2^n}\right)\right)\right)\\
&\quad+\frac{1}{7}\rho\left(7\left(A\left(\frac{v+w}{2}-u\right)-2^n f_o\left(\frac{v+w}{2^{n+1}}-\frac{u}{2^n}\right)\right)\right)\\
&\quad+\frac{1}{7}\rho\left(7\left(A\left(\frac{w+u}{2}-v\right)-2^n f_o\left(\frac{w+u}{2^{n+1}}-\frac{v}{2^n}\right)\right)\right)+\frac{1}{7}\rho\left(7\left(-A(u-v)+2^n f_o\left(\frac{u-v}{2^n}\right)\right)\right)\\
&\quad+\frac{1}{7}\rho\left(7\left(-A(v-w)+2^n f_o\left(\frac{v-w}{2^n}\right)\right)\right)+\frac{1}{7}\rho\left(7\left(-A(w-u)+2^n f_o\left(\frac{w-u}{2^n}\right)\right)\right)\\
&\quad+\frac{1}{7}\rho\left(7\left(2^n\left(f_o\left(\frac{u+v}{2^{n+1}}-\frac{w}{2^n}\right)+f_o\left(\frac{v+w}{2^{n+1}}-\frac{u}{2^n}\right)+f_o\left(\frac{w+u}{2^{n+1}}-\frac{v}{2^n}\right)\right.\right.\right.\\
&\quad\left.\left.\left.-\left(f_o\left(\frac{u-v}{2^n}\right)+f_o\left(\frac{v-w}{2^n}\right)+f_o\left(\frac{w-u}{2^n}\right)\right)\right)\right)\right)\\
&\leqslant \frac{k}{7}\rho\left(A\left(\frac{u+v}{2}-w\right)-2^n f_o\left(\frac{u+v}{2^{n+1}}-\frac{w}{2^n}\right)\right)+\frac{k}{7}\rho\left(A\left(\frac{v+w}{2}-u\right)-2^n f_o\left(\frac{v+w}{2^{n+1}}-\frac{u}{2^n}\right)\right)\\
&\quad+\frac{k}{7}\rho\left(A\left(\frac{w+u}{2}-v\right)-2^n f_o\left(\frac{w+u}{2^{n+1}}-\frac{v}{2^n}\right)\right)+\frac{k}{7}\rho\left(-A(u-v)+2^n f_o\left(f_o\left(\frac{u-v}{2^n}\right)\right)\right)\\
&\quad+\frac{k}{7}\rho\left(-A(v-w)+2^n f_o\left(f_o\left(\frac{v-w}{2^n}\right)\right)\right)+\frac{k}{7}\rho\left(-A(w-u)+2^n f_o\left(f_o\left(\frac{w-u}{2^n}\right)\right)\right)\\
&\quad+\frac{k}{7}\rho\left(2^n\left(f_o\left(\frac{u+v}{2^{n+1}}-\frac{w}{2^n}\right)+f_o\left(\frac{v+w}{2^{n+1}}-\frac{u}{2^n}\right)+f_o\left(\frac{w+u}{2^{n+1}}-\frac{v}{2^n}\right)\right.\right.\\
&\quad\left.\left.-\left(f_o\left(\frac{u-v}{2^n}\right)+f_o\left(\frac{v-w}{2^n}\right)+f_o\left(\frac{w-u}{2^n}\right)\right)\right)\right)
\end{aligned}$$

for all $u,v,w \in V$ and all positive integers n. Let us take the limit as n tends to ∞, one see that A is additive. In order to prove A is unique, let $A'(u)$ be another additive mapping satisfying (4.16) and (1.3). Since A and A' are additive mappings, $A\left(\frac{u}{2^n}\right)=\frac{1}{2^n}A(u)$ and $A'\left(\frac{u}{2^n}\right)=\frac{1}{2^n}A'(u)$ hold. Thus

$$\begin{aligned}
\rho\left(A(u)-A'(u)\right) &= \frac{1}{2}\rho\left(2^{n+1}A\left(\frac{u}{2^n}\right)-2^{n+1}f_o\left(\frac{u}{2^n}\right)\right)+\frac{1}{2}\rho\left(2^{n+1}f_o\left(\frac{u}{2^n}\right)-2^{n+1}A'\left(\frac{u}{2^n}\right)\right)\\
&\leqslant \frac{k^{n+1}}{2}\rho\left(A\left(\frac{u}{2^n}\right)-f_o\left(\frac{u}{2^n}\right)\right)+\frac{k^{n+1}}{2}\rho\left(f_o\left(\frac{u}{2^n}\right)-A'\left(\frac{u}{2^n}\right)\right)\\
&\leqslant \left(\frac{2}{k}\right)^{n-1}\sum_{i=n+1}^{\infty}\left(\frac{k^2}{2}\right)^i \Gamma\left(\frac{u}{2^i},\frac{u}{2^i},\frac{-u}{2^i}\right) \to 0 \text{ as } n\to\infty
\end{aligned}$$

for all $u \in V$. Hence A is unique.

For $j=-1$, we can prove the similar stability result. Hence we complete the proof. □

Corollary 4.7. *Let Θ and τ be nonnegative real numbers and X_ρ satisfies Δ_2 condition. If an odd mapping $f_o : V \to X_\rho$ satisfies the inequality*

$$\rho\left(Df_o(u,v,w)\right) \leqslant \Theta$$

for all $u, v, w \in V$, *hen there exists a unique additive mapping* $A : V \to X_\rho$ *such that*

$$\rho\left(f_o(u) - A(u)\right) \leqslant \frac{k^2\Theta}{|1|}$$

for all $u \in V$.

Corollary 4.8. *Let* Θ *and* τ *be nonnegative real numbers and* X_ρ *satisfies* Δ_2 *condition with* $2^\tau \neq k^2$. *If an odd mapping* $f_o : V \to X_\rho$ *satisfies the inequality*

$$\rho\left(Df_o(u, v, w)\right) \leqslant \Theta\left(\|u\|^\tau + \|v\|^\tau + \|w\|^\tau\right)$$

for all $u, v, w \in V$, *then there exists a unique additive mapping* $A : V \to X_\rho$ *such that*

$$\rho\left(f_o(u) - A(u)\right) \leqslant \frac{3k^2\Theta\|u\|^\tau}{|k^2 - 2^\tau|}$$

for all $u \in V$.

Corollary 4.9. *Let* Θ *and* τ *be nonnegative real numbers and* X_ρ *satisfies* Δ_2 *condition with* $2^{3\tau} \neq k^2$. *If an odd mapping* $f_o : V \to X_\rho$ *satisfies the inequality*

$$\rho\left(Df_o(u, v, w)\right) \leqslant \Theta\left(\|u\|^\tau \|v\|^\tau \|w\|^\tau\right)$$

for all $u, v, w \in V$, *then there exists a unique additive mapping* $A : V \to X_\rho$ *such that*

$$\rho\left(f_o(u) - A(u)\right) \leqslant \frac{k^2\Theta\|u\|^{3\tau}}{|k^2 - 2^{3\tau}|}$$

for all $u \in V$.

Corollary 4.10. *Let* Θ *and* τ *be nonnegative real numbers and* X_ρ *satisfies* Δ_2 *condition with* $2^{3\tau} \neq k^2$. *If an odd mapping* $f_o : V \to X_\rho$ *satisfies the inequality*

$$\rho\left(Df_o(u, v, w)\right) \leqslant \Theta\left(\|u\|^\tau \|v\|^\tau \|w\|^\tau + \|u\|^{3\tau} + \|v\|^{3\tau} + \|w\|^{3\tau}\right)$$

for all $u, v, w \in V$, *then there exists a unique additive mapping* $A : V \to X_\rho$ *such that*

$$\rho\left(f_o(u) - A(u)\right) \leqslant \frac{4k^2\Theta\|u\|^{3\tau}}{|k^2 - 2^{3\tau}|}$$

for all $u \in V$.

4.3. Quartic stability of (1.3) *in modular space without* Δ_2*-conditions*

In this subsection, we present the Ulam-Hyers stability of the additive-quartic functional equation (1.3) using Hyers' direct method in modular space without Δ_2-conditions.

Theorem 4.11. *Let* $j \in \{-1, 1\}$. *Let* $\Gamma : V^3 \to [0, \infty)$ *be a function such that*

$$\sum_{k=0}^{\infty} \frac{\Gamma\left(2^{kj}u, 2^{kj}v, 2^{kj}w\right)}{2^{4kj}} < \infty \tag{4.18}$$

for all $u, v, w \in V$ *and* $f_{q_2} : V \to X_\rho$ *be an even mapping satisfying the inequality*

$$\rho\left(D\, f_{q_2}(u, v, w)\right) \leqslant \Gamma\left(u, v, w\right)$$

for all $u, v, w \in V$. *Then there exists a unique quartic mapping* $Q_2 : V \to X_\rho$ *such that*

$$\rho\left(f_{q_2}(u) - Q_2(u)\right) \leqslant \sum_{i=\frac{1-j}{2}}^{\infty} \frac{\Gamma(2^{ij}u, 2^{ij}u, -2^{ij}u)}{2^{4ij}}$$

for all $u \in V$. *The mapping* $Q_2(u)$ *is defined by*

$$Q_2(u) = \rho - \lim_{k\to\infty} \frac{f_{q_2}(2^{kj}u)}{2^{4kj}} \tag{4.19}$$

for all $u \in V$.

Proof. Replacing (u, v, w) by $(u, u, -u)$ and dividing by 2^4, we get

$$\rho\left(\frac{f_{q_2}(2u)}{2^4} - f_{q_2}(u)\right) \leqslant \frac{1}{2^4}\Gamma(u, u, -u)$$

for all $u \in V$, since f is an even mapping. Then the conclusion is a direct consequence of Theorem 4.1. □

Corollary 4.12. *Let* Θ *and* τ *be nonnegative real numbers. If an even mapping* $f_{q_2} : V \to X_\rho$ *satisfies the inequality*

$$\rho\left(Df_{q_2}(u, v, w)\right) \leqslant \Theta$$

for all $u, v, w \in V$, *then there exists a unique quartic mapping* $Q_2 : V \to X_\rho$ *such that*

$$\rho\left(f_{q_2}(u) - Q_2(u)\right) \leqslant \frac{\Theta}{|15|}$$

for all $u \in V$.

Corollary 4.13. *Let* Θ *and* τ *be nonnegative real numbers with* $\tau \neq 4$. *If an even mapping* $f_{q_2} : V \to X_\rho$ *satisfies the inequality*

$$\rho\left(Df_{q_2}(u, v, w)\right) \leqslant \Theta\left(\|u\|^\tau + \|v\|^\tau + \|w\|^\tau\right)$$

for all $u, v, w \in V$, *then there exists a unique quartic mapping* $Q_2 : V \to X_\rho$ *such that*

$$\rho\left(f_{q_2}(u) - Q_2(u)\right) \leqslant \frac{3\Theta\|u\|^\tau}{|2^4 - 2^\tau|}$$

for all $u \in V$.

Corollary 4.14. *Let* Θ *and* τ *be nonnegative real numbers with* $\tau \neq \frac{4}{3}$. *If an odd mapping* $f_{q_2} : V \to X_\rho$ *satisfies the inequality*

$$\rho\left(Df_{q_2}(u, v, w)\right) \leqslant \Theta\left(\|u\|^\tau \|v\|^\tau \|w\|^\tau\right)$$

for all $u, v, w \in V$, *then there exists a unique quartic mapping* $Q_2 : V \to X_\rho$ *such that*

$$\rho\left(f_{q_2}(u) - Q_2(u)\right) \leqslant \frac{\Theta\|u\|^{3\tau}}{|2^4 - 2^{3\tau}|}$$

for all $u \in V$.

Corollary 4.15. *Let* Θ *and* τ *be nonnegative real numbers with* $\tau \neq \frac{4}{3}$. *If an even mapping* $f_{q_2} : V \to X_\rho$ *satisfies the inequality*

$$\rho\left(Df_{q_2}(u, v, w)\right) \leqslant \Theta\left(\|u\|^\tau \|v\|^\tau \|w\|^\tau + \|u\|^{3\tau} + \|v\|^{3\tau} + \|w\|^{3\tau}\right)$$

for all $u, v, w \in V$, *then there exists a unique quartic mapping* $Q_2 : V \to X_\rho$ *such that*

$$\rho\left(f_{q_2}(u) - Q_2(u)\right) \leqslant \frac{4\Theta\|u\|^{3\tau}}{|2^4 - 2^{3\tau}|}$$

for all $u \in V$.

4.4. Quartic stability of (1.3) *in modular space with* Δ_2*-conditions*

In this subsection, we present the Ulam-Hyers stability of the additive-quartic functional equation (1.3) using Hyers' direct method in modular space with Δ_2-conditions.

Theorem 4.16. *Let* $j \in \{-1, 1\}$. *Let* $\Gamma : V^3 \to [0, \infty)$ *be a function such that*

$$\sum_{n=0}^{\infty} \left(\frac{k^8}{2^4}\right)^n \Gamma\left(\frac{u}{2^{nj}}, \frac{u}{2^{nj}}, \frac{-u}{2^{nj}}\right) < \infty \quad \text{and} \quad \lim_{n\to\infty} k^{4nj}\Gamma\left(\frac{u}{2^{nj}}, \frac{v}{2^{nj}}, \frac{w}{2^{nj}}\right) = 0 \tag{4.20}$$

for all $u, v, w \in V$ *and* $f_{q_2} : V \to X_\rho$ *be an even mapping satisfying the inequality*

$$\rho\left(D\ f_{q_2}(u, v, w)\right) \leqslant \Gamma\left(u, v, w\right) \tag{4.21}$$

for all $u, v, w \in V$. *Then there exists a unique quartic mapping* $Q_2 : V \to X_\rho$ *such that*

$$\rho\left(f_{q_2}(u) - Q_2(u)\right) \leqslant \frac{1}{2^4} \sum_{i=\frac{1-j}{2}}^{\infty} \left(\frac{k^8}{16}\right)^i \Gamma\left(\frac{u}{2^{ij}}, \frac{u}{2^{ij}}, \frac{-u}{2^{ij}}\right)$$

for all $u \in V$. *The mapping* $Q_2(u)$ *is defined by*

$$Q_2(u) = \rho - \lim_{n\to\infty} 2^{4nj} f_{q_2}\left(\frac{u}{2^{nj}}\right) \tag{4.22}$$

for all $u \in V$.

Proof. Assume $j = 1$. Replacing (u, v, w) by $\left(\frac{u}{2}, \frac{u}{2}, \frac{-u}{2}\right)$ in (4.21), we get

$$\rho\left(f_{q_2}(u) - 2^4 f_{q_2}\left(\frac{u}{2}\right)\right) \leqslant \Gamma\left(\frac{u}{2}, \frac{u}{2}, \frac{-u}{2}\right)$$

for all $u \in V$, since f is an even mapping. Then the conclusion is a direct consequence of Theorem 4.6. □

Corollary 4.17. *Let* Θ *and* τ *be nonnegative real numbers and* X_ρ *satisfies* Δ_2 *condition. If an even mapping* $f_{q_2} : V \to X_\rho$ *satisfies the inequality*

$$\rho\left(Df_{q_2}(u, v, w)\right) \leqslant \Theta$$

for all $u, v, w \in V$, *then there exists a unique quartic mapping* $Q_2 : V \to X_\rho$ *such that*

$$\rho\left(f_{q_2}(u) - Q_2(u)\right) \leqslant \frac{k^8\Theta}{|15|}$$

for all $u \in V$.

Corollary 4.18. *Let* Θ *and* τ *be nonnegative real numbers and* X_ρ *satisfies* Δ_2 *condition with* $2^\tau \neq k^8$. *If an even mapping* $f_{q_2} : V \to X_\rho$ *satisfies the inequality*

$$\rho\left(Df_{q_2}(u, v, w)\right) \leqslant \Theta\left(\|u\|^\tau + \|v\|^\tau + \|w\|^\tau\right)$$

for all $u, v, w \in V$, *then there exists a unique quartic mapping* $Q_2 : V \to X_\rho$ *such that*

$$\rho\left(f_{q_2}(u) - Q_2(u)\right) \leqslant \frac{3k^8\Theta\|u\|^\tau}{|k^8 - 2^\tau|}$$

for all $u \in V$.

Corollary 4.19. *Let Θ and τ be nonnegative real numbers and X_ρ satisfies Δ_2 condition with $2^{3\tau} \neq k^8$. If an even mapping $f_{q_2} : V \to X_\rho$ satisfies the inequality*

$$\rho\left(Df_{q_2}(u,v,w)\right) \leqslant \Theta\left(\|u\|^{\tau}\|v\|^{\tau}\|w\|^{\tau}\right)$$

for all $u,v,w \in V$, then there exists a unique quartic mapping $Q_2 : V \to X_\rho$ such that

$$\rho\left(f_{q_2}(u) - Q_2(u)\right) \leqslant \frac{k^8\Theta\|u\|^{3\tau}}{|k^8 - 2^{3\tau}|}$$

for all $u \in V$.

Corollary 4.20. *Let Θ and τ be nonnegative real numbers and X_ρ satisfies Δ_2 condition with $2^{3\tau} \neq k^8$. If an even mapping $f_{q_2} : V \to X_\rho$ satisfies the inequality*

$$\rho\left(Df_{q_2}(u,v,w)\right) \leqslant \Theta\left(\|u\|^{\tau}\|v\|^{\tau}\|w\|^{\tau} + \|u\|^{3\tau} + \|v\|^{3\tau} + \|w\|^{3\tau}\right)$$

for all $u,v,w \in V$, then there exists a unique quartic mapping $Q_2 : V \to X_\rho$ such that

$$\rho\left(f_{q_2}(u) - Q_2(u)\right) \leqslant \frac{4k^8\Theta\|u\|^{3\tau}}{|k^8 - 2^{3\tau}|}$$

for all $u \in V$.

4.5. Additive-quartic stability of (1.3) in modular space without Δ_2-conditions

In this subsection, we present the Ulam -Hyers stability of the additive-quartic mixed type functional equation (1.3) using Hyers' direct method in modular space without Δ_2-conditions.

Theorem 4.21. *Let $j \in \{-1, 1\}$. Let $\Gamma : V \to [0,\infty)$ be a function satisfying (4.1) and (4.18) for all $u,v,w \in V$. Let $f : V \to X_\rho$ be a mapping satisfying the inequality*

$$\rho\left(Df(u,v,w)\right) \leqslant \Gamma\left(u,v,w\right)$$

for all $u,v,w \in V$. Then there exists a unique additive mapping $A : V \to X_\rho$ and a unique quartic mapping $Q_2 : V \to X_\rho$ such that

$$\rho\left(f(u) - A(u) - Q_2(u)\right) \leqslant \frac{1}{2}\left\{\sum_{i=\frac{1-j}{2}}^{\infty}\frac{\Gamma(2^{ij}u, 2^{ij}u, -2^{ij}u)}{2^{ij}} + \sum_{i=\frac{1-j}{2}}^{\infty}\frac{\Gamma(2^{ij}u, 2^{ij}u, -2^{ij}u)}{2^{4ij}}\right\}$$

for all $u \in V$. The mappings $A(u)$ and $Q_2(u)$ respectively are defined in (4.4) and (4.19).

Proof. Let $f_{q_2}(u) = \frac{1}{2}\{f(u) + f(-u)\}$ for all $u \in V$. Then $f_{q_2}(0) = 0, f_{q_2}(u) = f_{q_2}(-u)$. Hence

$$\begin{aligned}\rho\left(Df_{q_2}(u,v,w)\right) &= \frac{1}{2}\{\rho\left(Df(u,v,w) + Df(-u,-v,-w)\right)\}\\ &\leqslant \frac{1}{2}\{\rho\left(Df(u,v,w)\right) + \rho\left(Df(-u,-v,-w)\right)\} \leqslant \frac{1}{2}\{\Gamma(u,v,w) + \Gamma(-u,-v,-w)\}\end{aligned}$$

for all $u \in V$. Hence by Theorem 4.11, there exists a unique quartic mapping $Q_2 : V \to X_\rho$ such that

$$\rho\left(f_{q_2}(u) - Q_2(u)\right) \leqslant \frac{1}{2}\left\{\sum_{i=0}^{\infty}\left(\frac{\Gamma\left(2^i u, 2^i u, -2^i u\right)}{2^{4i}} + \frac{\Gamma\left(-2^i u, -2^i u, 2^i u\right)}{2^{4i}}\right)\right\} \tag{4.23}$$

for all $u \in V$. Again $f_o(u) = \frac{1}{2}\{f(u) - f(-u)\}$ for all $u \in V$. Then $f_c(0) = 0, f_o(u) = -f_o(-u)$. Hence

$$\begin{aligned}\rho(Df_o(u,v,w)) &= \frac{1}{2}\{\rho(Df(u,v,w) + Df(u,v,w))\}\\ &\leqslant \frac{1}{2}\{\rho(Df(u,v,w)) + \rho(Df(-u,-v,-w))\} \leqslant \frac{1}{2}\{\Gamma(u,v,w) + \Gamma(-u,-v,-w)\}\end{aligned}$$

for all $u \in V$. Hence by Theorem 4.1, there exists a unique additive mapping $A : V \to X_\rho$ such that

$$\rho(f_o(u) - A(u)) \leqslant \frac{1}{2}\left\{\sum_{i=0}^{\infty}\left(\frac{\Gamma(2^i u, 2^i u, -2^i u)}{2^i} + \frac{\Gamma(-2^i u, -2^i u, 2^i u)}{2^i}\right)\right\} \tag{4.24}$$

for all $u \in V$. Since $f(u) = f_{q_2}(u) + f_o(u)$, it follows from (4.23) and (4.24) that

$$\begin{aligned}\rho(f(u) - A(u) - Q_2(u)) &= \rho(f_o(u) + f_{q_2}(u) - A(u) - Q_2(u))\\ &\leqslant \rho(f_{q_2}(u) - Q_2(u)) + \rho(f_o(u) - A(u))\\ &\leqslant \frac{1}{2}\left\{\sum_{i=0}^{\infty}\left(\frac{\Gamma(2^i u, 2^i u, -2^i u)}{2^i} + \frac{\Gamma(-2^i u, -2^i u, -2^i u)}{2^i}\right)\right.\\ &\quad\left. + \sum_{i=0}^{\infty}\left(\frac{\Gamma(2^i u, 2^i u, -2^i u)}{2^{4i}} + \frac{\Gamma(-2^i u, -2^i u, 2^i u)}{2^{4i}}\right)\right\}\end{aligned}$$

for all $u \in V$. Hence we complete the proof. □

Corollary 4.22. *Let Θ and τ be nonnegative real numbers. If a mapping $f : V \to X_\rho$ satisfies the inequality*

$$\rho(Df(u,v,w)) \leqslant \Theta$$

for all $u, v, w \in V$, then there exists a unique additive mapping $A : V \to X_\rho$ and a unique quartic mapping $Q_2 : V \to X_\rho$ such that

$$\rho(f(u) - A(u) - Q_2(u)) \leqslant \frac{16\Theta}{15}$$

for all $u \in V$.

Corollary 4.23. *Let Θ and τ be nonnegative real numbers with $\tau \neq 1, 4$. If a mapping $f : V \to X_\rho$ satisfies the inequality*

$$\rho(Df(u,v,w)) \leqslant \Theta(\|u\|^\tau + \|v\|^\tau + \|w\|^\tau)$$

for all $u, v, w \in V$, then there exists a unique additive mapping $A : V \to X_\rho$ and a unique quartic mapping $Q_2 : V \to X_\rho$ such that

$$\rho(f(u) - A(u) - Q_2(u)) \leqslant 3\Theta\left(\frac{1}{|2 - 2^\tau|} + \frac{1}{|2^4 - 2^\tau|}\right)\|u\|^\tau$$

for all $u \in V$.

Corollary 4.24. *Let Θ and τ be nonnegative real numbers with $\tau \neq \frac{1}{3}, \frac{4}{3}$. If a mapping $f : V \to X_\rho$ satisfies the inequality*

$$\rho(Df(u,v,w)) \leqslant \Theta(\|u\|^\tau \|v\|^\tau \|w\|^\tau)$$

for all $u, v, w \in V$, then there exists a unique additive mapping $A : V \to X_\rho$ and a unique quartic mapping $Q_2 : V \to X_\rho$ such that

$$\rho(f(u) - A(u) - Q_2(u)) \leqslant \Theta\left(\frac{1}{|2 - 2^{3\tau}|} + \frac{1}{|2^4 - 2^{3\tau}|}\right)\|u\|^{3\tau}$$

for all $u \in V$.

Corollary 4.25. *Let Θ and τ be nonnegative real numbers with $\tau \neq \frac{1}{3}, \frac{4}{3}$. If a mapping $f : V \to X_\rho$ satisfies the inequality*

$$\rho\left(Df(u,v,w)\right) \leqslant \Theta\left(\|u\|^{\tau}\|v\|^{\tau}\|w\|^{\tau} + \|u\|^{3\tau} + \|v\|^{3\tau} + \|w\|^{3\tau}\right)$$

for all $u,v,w \in V$, then there exists a unique additive mapping $A : V \to X_\rho$ and a unique quartic mapping $Q_2 : V \to X_\rho$ such that

$$\rho\left(f(u) - A(u) - Q_2(u)\right) \leqslant 4\Theta\left(\frac{1}{|2-2^{3\tau}|} + \frac{1}{|2^4-2^{3\tau}|}\right)\|u\|^{3\tau}$$

for all $u \in V$.

4.6. Additive-quartic stability of (1.3) in modular space with Δ_2-conditions

In this subsection, we present the Ulam-Hyers stability of the additive-quartic mixed type functional equation (1.3) using Hyers' direct method in modular space with Δ_2-conditions.

Theorem 4.26. *Let $j \in \{-1, 1\}$. Let $\Gamma : V \to [0,\infty)$ be a function satisfying (4.14) and (4.20) for all $u,v,w \in V$. Let $f : V \to X_\rho$ be a mapping satisfying the inequality*

$$\rho\left(D\ f(u,v,w)\right) \leqslant \Gamma\left(u,v,w\right)$$

for all $u,v,w \in V$. Then there exists a unique additive mapping $A : V \to X_\rho$ and a unique quartic mapping $Q_2 : V \to X_\rho$ such that

$$\rho\left(f(u) - A(u) - Q_2(u)\right) \leqslant \frac{1}{2}\left\{\sum_{i=\frac{1-j}{2}}^{\infty}\left(\frac{k^2}{2}\right)^i\Gamma\left(\frac{u}{2^{ij}},\frac{u}{2^{ij}},\frac{-u}{2^{ij}}\right) + \sum_{i=\frac{1-j}{2}}^{\infty}\left(\frac{k^8}{2^4}\right)^i\Gamma\left(\frac{u}{2^{ij}},\frac{u}{2^{ij}},\frac{-u}{2^{ij}}\right)\right\}$$

for all $u \in V$. The mappings $A(u)$ and $Q_2(u)$ respectively are defined in (4.17) and (4.22).

Proof. Let $f_{q_2}(u) = \frac{1}{2}\{f(u) + f(-u)\}$ for all $u \in V$. Then $f_{q_2}(0) = 0, f_{q_2}(u) = f_{q_2}(-u)$. Hence

$$\begin{aligned}\rho\left(Df_{q_2}(u,v,w)\right) &= \frac{1}{2}\{\rho\left(Df(u,v,w) + Df(-u,-v,-w)\right)\}\\ &\leqslant \frac{1}{2}\{\rho\left(Df(u,v,w)\right) + \rho\left(Df(-u,-v,-w)\right)\} \leqslant \frac{1}{2}\{\Gamma(u,v,w) + \Gamma(-u,-v,-w)\}\end{aligned}$$

for all $u \in V$. Hence by Theorem 4.16, there exist sa unique quartic mapping $Q_2 : V \to X_\rho$ such that

$$\rho\left(f_{q_2}(u) - Q_2(u)\right) \leqslant \frac{1}{2}\left\{\sum_{i=0}^{\infty}\left(\frac{k^8}{16}\right)^i\left(\Gamma\left(\frac{u}{2^i},\frac{u}{2^i},\frac{-u}{2^i}\right) + \Gamma\left(\frac{-u}{2^i},\frac{-u}{2^i},\frac{u}{2^i}\right)\right)\right\} \tag{4.25}$$

for all $u \in V$. Again $f_o(u) = \frac{1}{2}\{f(u) - f(-u)\}$ for all $u \in V$. Then $f_c(0) = 0, f_o(u) = -f_o(-u)$. Hence

$$\begin{aligned}\rho\left(Df_o(u,v,w)\right) &= \frac{1}{2}\{\rho\left(Df(u,v,w) + Df(u,v,w)\right)\}\\ &\leqslant \frac{1}{2}\{\rho\left(Df(u,v,w)\right) + \rho\left(Df(-u,-v,-w)\right)\} \leqslant \frac{1}{2}\{\Gamma(u,v,w) + \Gamma(-u,-v,-w)\}\end{aligned}$$

for all $u \in V$. Hence by Theorem 4.6, there exists a unique additive mapping $A : V \to X_\rho$ such that

$$\rho\left(f_o(u) - A(u)\right) \leqslant \frac{1}{2}\left\{\sum_{i=0}^{\infty}\left(\frac{k^2}{2}\right)^i\left(\Gamma\left(\frac{u}{2^i},\frac{u}{2^i},\frac{-u}{2^i}\right) + \Gamma\left(\frac{-u}{2^i},\frac{-u}{2^i},\frac{u}{2^i}\right)\right)\right\} \tag{4.26}$$

for all $u \in V$. Since $f(u) = f_{q_2}(u) + f_o(u)$, it follows from (4.25) and (4.26) that

$$\begin{aligned}
\rho\left(f(u) - A(u) - Q_2(u)\right) &= \rho\left(f_o(u) + f_{q_2}(u) - A(u) - Q_2(u)\right) \\
&\leqslant \rho\left(f_{q_2}(u) - Q_2(u)\right) + \rho\left(f_o(u) - A(u)\right) \\
&\leqslant \frac{1}{2}\left\{\sum_{i=0}^{\infty}\left(\frac{k^8}{16}\right)^i\left(\Gamma\left(\frac{u}{2^i}, \frac{u}{2^i}, \frac{-u}{2^i}\right) + \Gamma\left(\frac{-u}{2^i}, \frac{-u}{2^i}, \frac{u}{2^i}\right)\right)\right. \\
&\quad \left. + \sum_{i=0}^{\infty}\left(\frac{k^2}{2}\right)^i\left(\Gamma\left(\frac{u}{2^i}, \frac{u}{2^i}, \frac{-u}{2^i}\right) + \Gamma\left(\frac{-u}{2^i}, \frac{-u}{2^i}, \frac{u}{2^i}\right)\right)\right\}
\end{aligned}$$

for all $u \in V$. Hence we complete the proof. □

Corollary 4.27. *Let Θ and τ be nonnegative real numbers and X_ρ satisfies Δ_2 condition. If a mapping $f : V \to X_\rho$ satisfies the inequality*

$$\rho\left(Df(u, v, w)\right) \leqslant \Theta$$

for all $u, v, w \in V$, then there exists a unique additive mapping $A : V \to X_\rho$ and a unique quartic mapping $Q_2 : V \to X_\rho$ such that

$$\rho\left(f(u) - A(u) - Q_2(u)\right) \leqslant k^2\Theta\left(1 + \frac{k^6}{15}\right)$$

for all $u \in V$.

Corollary 4.28. *Let Θ and τ be nonnegative real numbers and X_ρ satisfies Δ_2 condition with $2^\tau \neq k^2, k^8$. If a mapping $f : V \to X_\rho$ satisfies the inequality*

$$\rho\left(Df(u, v, w)\right) \leqslant \Theta\left(\|u\|^\tau + \|v\|^\tau + \|w\|^\tau\right)$$

for all $u, v, w \in V$, then there exists a unique additive mapping $A : V \to X_\rho$ and a unique quartic mapping $Q_2 : V \to X_\rho$ such that

$$\rho\left(f(u) - A(u) - Q_2(u)\right) \leqslant 3\Theta\left(\frac{k^2}{|k^2 - 2^\tau|} + \frac{k^8}{|k^8 - 2^\tau|}\right)\|u\|^\tau$$

for all $u \in V$.

Corollary 4.29. *Let Θ and τ be nonnegative real numbers and X_ρ satisfies Δ_2 condition with $2^{3\tau} \neq k^2, k^8$. If a mapping $f : V \to X_\rho$ satisfies the inequality*

$$\rho\left(Df(u, v, w)\right) \leqslant \Theta\left(\|u\|^\tau \|v\|^\tau \|w\|^\tau\right)$$

for all $u, v, w \in V$, then there exists a unique additive mapping $A : V \to X_\rho$ and a unique quartic mapping $Q_2 : V \to X_\rho$ such that

$$\rho\left(f(u) - A(u) - Q_2(u)\right) \leqslant \Theta\left(\frac{k^2}{|k^2 - 2^{3\tau}|} + \frac{k^8}{|k^8 - 2^{3\tau}|}\right)\|u\|^{3\tau}$$

for all $u \in V$.

Corollary 4.30. *Let Θ and τ be nonnegative real numbers and X_ρ satisfies Δ_2 condition with $2^{3\tau} \neq k^2, k^8$. If a mapping $f : V \to X_\rho$ satisfies the inequality*

$$\rho\left(Df(u, v, w)\right) \leqslant \Theta\left(\|u\|^\tau \|v\|^\tau \|w\|^\tau + \|u\|^{3\tau} + \|v\|^{3\tau} + \|w\|^{3\tau}\right)$$

for all $u, v, w \in V$, then there exists a unique additive mapping $A : V \to X_\rho$ and a unique quartic mapping $Q_2 : V \to X_\rho$ such that

$$\rho\left(f(u) - A(u) - Q_2(u)\right) \leqslant 4\Theta\left(\frac{k^2}{|k^2 - 2^{3\tau}|} + \frac{k^8}{|k^8 - 2^{3\tau}|}\right)\|u\|^{3\tau}$$

for all $u \in V$.

5. Conclusion

In this article, we have proved the stability results of additive functional equation, quartic functional equation, and additive-quartic mixed type functional equations in modular spaces with and without using the Δ_2-condition by the direct method.

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