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Aboodh transform and the stability of second order linear differential equations

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Abstract

In this paper, we introduce a new integral transform, namely Aboodh transform, and we apply the transform to investigate the Hyers–Ulam stability, Hyers–Ulam–Rassias stability, Mittag-Leffler–Hyers–Ulam stability, and Mittag-Leffler–Hyers–Ulam–Rassias stability of second order linear differential equations.

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1 Introduction

In [1], Ulam proposed the universal Ulam stability problem in metric groups. In [2], Hyers gave the first affirmative answer to the question of Ulam for additive functional equations in Banach spaces. Since then the Hyers result has produced many significant generalizations [3–8]. Furthermore, useful non-stability results for various functional equations have been given by Gajda [9], Bodaghi, Senthil Kumar, and Rassias [10], Alessa *et al.* [11] and Karthikeyan, Park, Rassias, and Lee [12].

The theory of stability is an important branch of the qualitative theory of differential equations. During the last decades many interesting results have been investigated on different types differential equations (for more details, see [13–18]).

A generalization of Ulam's problem was proposed by replacing functional equations with differential equations: The differential equation $\phi(f,x,x',x'',\dots,x^{(n)}) = 0$ has the Hyers–Ulam stability if, for given $\epsilon > 0$ and a function x such that

$$\left|\phi\left(f,x,x',x'',\dots,x^{(n)}\right)\right| \leq \epsilon,$$

there exists a solution x_a of the differential equation such that $|x(t) - x_a(t)| \leq K(\epsilon)$ and

$$\lim_{\epsilon\to 0} K(\epsilon) = 0.$$

If the preceding statement is also true when we replace ϵ and $K(\epsilon)$ with $\phi(t)$ and $\varphi(t)$, where ϕ, φ are appropriate functions not depending on x and x_a explicitly, then we say that the corresponding differential equation has the generalized Hyers–Ulam stability or Hyers–Ulam–Rassias stability.

Alsina and Ger [19] investigated the stability of the differential equation $x'(t) - x(t)$. They proved the following celebrated theorem.

Theorem 1.1 ([19]) *Let $f : I \to \mathbf{R}$ be a differentiable function, which is a solution of the following differential inequality $\|x'(t) - x(t)\| \le \epsilon$, where I is an open interval of $\mathbf{R}$. Then there is a solution $g : I \to \mathbf{R}$ of $x'(t) = x(t)$ such that, for any $t \in I$, we have $\|f(t) - g(t)\| \le 3\epsilon$.*

This result was generalized by Takahasi *et al.* [20], who proved the Hyers–Ulam stability for the Banach space-valued differential equation $y'(t) = \lambda y(t)$. Furthermore, the Hyers–Ulam stability has been proved for the first order linear differential equations in more general settings [21–25].

In 2007, Wang, Zhou, and Sun [26] established the Hyers–Ulam stability of a class of first order linear differential equations.

Many different methods for solving differential equations have been used to study the Hyers–Ulam stability problem for various differential equations. But some initial conditions have more significant advantage for solving differential equations. In 2011, Gavruta, Jung, and Li [27] studied the Hyers–Ulam stability for the second order linear differential equation $y'' + \beta(x)y = 0$ with initial and boundary conditions using Taylor's formula.

In 2014, Alqifiary and Jung [28] investigated the generalized Hyers–Ulam stability of

$$x^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k x^{(k)}(t) = f(t)$$

by using the Laplace transform method. In 2020, Murali and Selvan [29] established the Mittag-Leffler–Hyers–Ulam stability of the first order linear differential equation for both homogeneous and non-homogeneous cases by using Laplace transformation. The Hyers–Ulam stability of differential equations has been given attention, and it was established by many authors (see [30–34]).

Recently, Murali, Selvan, and Park [35] investigated the Hyers–Ulam stability of various differential equations by using the Fourier transform method (also [36, 37]).

In this paper, our main intention is to establish the Hyers–Ulam stability and the Mittag-Leffler–Hyers–Ulam stability of the following second order linear differential equations:

$$u''(t) + \mu^2 u = 0 \tag{1.1}$$

and

$$u''(t) + \mu^2 u = q(t) \tag{1.2}$$

for all $t \in I$, $u(t) \in C^2(I)$ and $q(t) \in C(I)$, $I = [a, b]$, $-\infty < a < b < \infty$, by using a new integral transform method, i.e., Aboodh transform method.

2 Preliminaries

In this section, we introduce some standard notations and definitions which will be very useful to obtain our main results.

Throughout this paper, $\mathbf{F}$ denotes the real field $\mathbf{R}$ or the complex field $\mathbf{C}$. A function $f:(0,\infty)\to\mathbf{F}$ is said to be of exponential order if there exist constants $A,B\in\mathbf{R}$ such that $|f(t)|\leq Ae^{tB}$ for all $t>0$.

Definition 2.1 ([38, 39]) The Aboodh integral transform is defined, for a function of exponential order $f(t)$, by

$$\mathcal{A}\{f(t)\}=\frac{1}{\xi}\int_0^\infty f(t)e^{-\xi t}\,dt=F(\xi),\quad t\geq 0,$$

provided that the integral exists for some ξ, where $\xi\in(k_1,k_2)$. $\mathcal{A}$ is called the Aboodh integral transform operator.

Let f and g be Lebesgue integrable functions on $(-\infty,+\infty)$. Let S denote the set of x for which the Lebesgue integral

$$h(x)=\int_{-\infty}^{\infty}f(t)g(x-t)\,dt$$

exists. This integral defines a function h on S called the *convolution* of f and g. We also write $h=f*g$ to denote this function.

Definition 2.2 ([40]) The Mittag-Leffler function of one parameter, denoted by $E_\alpha(z)$, is defined as

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+1)}z^k,$$

where $z,\alpha\in\mathbf{C}$ and $\mathrm{Re}(\alpha)>0$. If we put $\alpha=1$, then the above equation becomes

$$E_1(z)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k+1)}z^k=\sum_{k=0}^{\infty}\frac{z^k}{k}=e^z.$$

Definition 2.3 ([40]) A generalization of $E_\alpha(z)$ is defined as a function

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+\beta)}z^k,$$

where $z,\alpha,\beta\in\mathbf{C}$, $\mathrm{Re}(\alpha)>0$ and $\mathrm{Re}(\beta)>0$.

Let $I,J\subseteq\mathbf{R}$ be intervals. Throughout this paper, we denote the space of k continuously differentiable functions from I to J by $C^k(I,J)$ and denote $C^k(I,I)$ by $C^k(I)$. Furthermore, $C(I,J)=C^0(I,J)$ denotes the space of continuous functions from I to J. In addition, $\mathbf{R}_+:=[0,\infty)$. From now on, we assume that $I=[a,b]$, where $-\infty<a<b<\infty$.

Here, we give some definitions of various forms of Hyers–Ulam stability and Mittag-Leffler–Hyers–Ulam stability of differential equations (1.1) and (1.2).

Definition 2.4 We say that differential equation (1.1) has the Hyers–Ulam stability if there exists a constant $L>0$ satisfying the following condition: For every $\epsilon>0$ and some $u(t)\in$

$C^2(I)$ satisfying the inequality

$$\left|u''(t) + \mu^2 u\right| \le \epsilon$$

for all $t \in I$, there exists a solution $v \in C^2(I)$ satisfying the differential equation $v''(t) + \mu^2 v = 0$ and $|u(t) - v(t)| \le L\epsilon$ for all $t \in I$. We call such L the Hyers–Ulam stability constant for (1.1).

Definition 2.5 We say that differential equation (1.1) has the Hyers–Ulam–Rassias stability with respect to $\phi \in C(\mathbf{R}_+, \mathbf{R}_+)$ if there exists a constant $L_\phi > 0$ with the following property: For every $\epsilon > 0$ and some $u(t) \in C^2(I)$ satisfying the inequality

$$\left|u''(t) + \mu^2 u\right| \le \epsilon\phi(t)$$

for all $t \in I$, there exists a solution $v \in C^2(I)$ satisfying the differential equation $v''(t) + \mu^2 v = 0$ and

$$\left|u(t) - v(t)\right| \le L_\phi \epsilon\phi(t)$$

for all $t \in I$. We call such L the Hyers–Ulam–Rassias stability constant for (1.1).

Definition 2.6 We say that differential equation (1.2) has the Hyers–Ulam stability if there exists a constant $L > 0$ satisfying the following condition: For every $\epsilon > 0$ and some $u(t) \in C^2(I)$ satisfying the inequality

$$\left|u''(t) + \mu^2 u - q(t)\right| \le \epsilon$$

for all $t \in I$, there exists some $v \in C^2(I)$ satisfying $v''(t) + \mu^2 v = q(t)$ and

$$\left|u(t) - v(t)\right| \le L\epsilon$$

for all $t \in I$. We call such L the Hyers–Ulam stability constant for (1.2).

Definition 2.7 We say that differential equation (1.2) has the Hyers–Ulam–Rassias stability with respect to $\phi \in C(\mathbf{R}_+, \mathbf{R}_+)$ if there exists a constant $L_\phi > 0$ such that, for every $\epsilon > 0$ and some $u(t) \in C^2(I)$ satisfying the inequality

$$\left|u''(t) + \mu^2 u - q(t)\right| \le \epsilon\phi(t)$$

for all $t \in I$, there exists some $v \in C^2(I)$ satisfying the differential equation $v''(t) + \mu^2 v = q(t)$ and

$$\left|u(t) - v(t)\right| \le L_\phi \epsilon\phi(t)$$

for all $t \in I$. We call such L the Hyers–Ulam–Rassias stability constant for (1.2).

Definition 2.8 We say that differential equation (1.1) has the Mittag-Leffler–Hyers–Ulam stability if there exists a positive constant L satisfying the following condition: For every $\epsilon > 0$ and some $u(t) \in C^2(I)$ satisfying the inequality

$$\left|u''(t) + \mu^2 u\right| \leq \epsilon E_\alpha(t)$$

for all $t \in I$, there exists a solution $v \in C^2(I)$ satisfying $v''(t) + \mu^2 v = 0$ and

$$\left|u(t) - v(t)\right| \leq L\epsilon E_\alpha(t)$$

for all $t \in I$. We call such L the Mittag-Leffler–Hyers–Ulam stability constant for (1.1).

Definition 2.9 We say that differential equation (1.1) has the Mittag-Leffler–Hyers–Ulam–Rassias stability with respect to $\phi : (0,\infty) \to (0,\infty)$ if there exists a positive constant L_ϕ satisfying the following condition: For every $\epsilon > 0$ and some $u(t) \in C^2(I)$ satisfying the inequality

$$\left|u''(t) + \mu^2 u\right| \leq \phi(t)\epsilon E_\alpha(t)$$

for all $t \in I$, there exists a solution $v \in C^2(I)$ satisfying $v''(t) + \mu^2 v = 0$ and

$$\left|u(t) - v(t)\right| \leq L_\phi \phi(t)\epsilon E_\alpha(t)$$

for all $t \in I$. We call such L_ϕ the Mittag-Leffler–Hyers–Ulam–Rassias stability constant for (1.1).

Definition 2.10 We say that differential equation (1.2) has the Mittag-Leffler–Hyers–Ulam stability if there exists a positive constant L satisfying the following condition: For every $\epsilon > 0$ and some $u(t) \in C^2(I)$ satisfying the inequality

$$\left|u''(t) + \mu^2 u - q(t)\right| \leq \epsilon E_\alpha(t)$$

for all $t \in I$, there exists a solution $v \in C^2(I)$ satisfying the linear differential equation $v''(t) + \mu^2 v = q(t)$ and

$$\left|u(t) - v(t)\right| \leq L\epsilon E_\alpha(t)$$

for all $t \in I$. We call such L the Mittag-Leffler–Hyers–Ulam stability constant for (1.2).

Definition 2.11 We say that differential equation (1.2) has the Mittag-Leffler–Hyers–Ulam–Rassias stability with respect to $\phi : (0,\infty) \to (0,\infty)$ if there exists a positive constant L_ϕ satisfying the following condition: For every $\epsilon > 0$ and some $u(t) \in C^2(I)$ satisfying the inequality

$$\left|u''(t) + \mu^2 u - q(t)\right| \leq \phi(t)\epsilon E_\alpha(t)$$

for all $t \in I$, there exists a solution $v \in C^2(I)$ satisfying the linear differential equation $v''(t) + \mu^2 v = q(t)$ and $|u(t) - v(t)| \leq L_\phi \phi(t)\epsilon E_\alpha(t)$ for all $t \in I$. We call such L_ϕ the Mittag-Leffler–Hyers–Ulam–Rassias stability constant for (1.2).

3 Hyers–Ulam stability for (1.1)

In this section, we prove the Hyers–Ulam stability, Hyers–Ulam–Rassias stability, Mittag-Leffler–Hyers–Ulam stability, and Mittag-Leffler–Hyers–Ulam–Rassias stability of differential equation (1.1) by using the Aboodh transform.

Theorem 3.1 *Differential equation* (1.1) *is Hyers–Ulam stable.*

Proof Let $\epsilon > 0$. Suppose that $u(t) \in C^2(I)$ satisfies

$$\left|u''(t) + \mu^2 u\right| \leq \epsilon \tag{3.1}$$

for all $t \in I$. We prove that there exists a real number $L > 0$ which is independent of ϵ and u such that $|u(t) - v(t)| \leq L\epsilon$ for some $v \in C^2(I)$ satisfying $v''(t) + \mu^2 v = 0$ for all $t \in I$. Define a function $p : (0,\infty) \to \mathbf{F}$ such that $p(t) =: u''(t) + \mu^2 u(t)$ for all $t > 0$. In view of (3.1), we have $|p(t)| \leq \epsilon$. Taking the Aboodh transform to $p(t)$, we have

$$\mathcal{A}\{p\} = \left(\xi^2 + \mu^2\right)\mathcal{A}\{u\} - u(0) - \frac{u'(0)}{\xi}, \tag{3.2}$$

and thus

$$\mathcal{A}\{u\} = \frac{\mathcal{A}\{p\} + u(0) + \frac{u'(0)}{\xi}}{\xi^2 + \mu^2}.$$

In view of (3.2), a function $u_0 : (0,\infty) \longrightarrow \mathbf{F}$ is a solution of (1.1) if and only if

$$\left(\xi^2 + \mu^2\right)\mathcal{A}\{u_0\} - u_0(0) - \frac{u_0'(0)}{\xi} = 0.$$

If there exist constants l and m in $\mathbf{F}$ such that $\xi^2 + \mu^2 = (\xi - l)(\xi - m)$ with $l + m = 0$ and $lm = \mu^2$, then (3.2) becomes

$$\mathcal{A}\{u\} = \frac{\mathcal{A}\{p\} + u(0) + \frac{u'(0)}{\xi}}{(\xi - l)(\xi - m)}. \tag{3.3}$$

Set

$$v(t) = u(0)\left(\frac{le^{lt} - me^{mt}}{l - m}\right) + u'(0)\left(\frac{e^{lt} - e^{mt}}{l - m}\right).$$

We have $v(0) = u(0)$ and $u'(0) = v'(0)$. Taking the Aboodh transform to $v(t)$, we obtain

$$\mathcal{A}\{v\} = \frac{u(0)}{(\xi - l)(\xi - m)} + \frac{u'(0)}{\xi}\frac{1}{(\xi - l)(\xi - m)}. \tag{3.4}$$

On the other hand, $\mathcal{A}\{v''(t) + \mu^2 v\} = (\xi^2 + \mu^2)\mathcal{A}\{v\} - v(0) - \frac{v'(0)}{\xi}$. Using (3.4), we get $\mathcal{A}\{v''(t) + \mu^2 v\} = 0$. Since $\mathcal{A}$ is one-to-one and linear, $v''(t) + \mu^2 v = 0$. This means that $v(t)$ is a solution of (1.1). It follows from (3.3) and (3.4) that

$$\mathcal{A}\{u\} - \mathcal{A}\{v\} = \frac{\mathcal{A}\{p\} + u(0) + \frac{u'(0)}{\xi}}{(\xi - l)(\xi - m)} - \frac{u(0) + \frac{u'(0)}{\xi}}{(\xi - l)(\xi - m)} = \frac{\mathcal{A}\{p\}}{(\xi - l)(\xi - m)},$$

$$\mathcal{A}\{u(t)-v(t)\} = \mathcal{A}\left\{p(t) * \left(\frac{e^{lt}-e^{mt}}{l-m}\right)\right\}.$$

The above equalities show that

$$u(t)-v(t) = p(t) * \left(\frac{e^{lt}-e^{mt}}{l-m}\right).$$

Taking modulus on both sides and using $|p(t)| \leq \epsilon$, we get

$$\begin{aligned}
\left|u(t)-v(t)\right| &= \left|p(t) * \left(\frac{e^{lt}-e^{mt}}{l-m}\right)\right| \\
&\leq \left|\int_0^t p(x)\left(\frac{e^{l(t-x)}-e^{m(t-x)}}{l-m}\right)dx\right| \\
&\leq \epsilon\left|\int_0^t \left(\frac{e^{l(t-x)}-e^{m(t-x)}}{l-m}\right)dx\right|
\end{aligned}$$

for all $t > 0$, where

$$\begin{aligned}
L &= \left|\int_0^t \left(\frac{e^{l(t-x)}-e^{m(t-x)}}{l-m}\right)dx\right| \\
&\leq \frac{1}{|l-m|}\left\{e^{\mathcal{R}(l)t}\int_0^t e^{-\mathcal{R}(l)x}\,dx + e^{\mathcal{R}(m)t}\int_0^t e^{-\mathcal{R}(m)x}\,dx\right\} \leq \frac{\mathcal{K}}{|l-m|},
\end{aligned}$$

where $\int_0^t e^{-\mathcal{R}(l)x}\,dx$ and $\int_0^t e^{-\mathcal{R}(m)x}\,dx$ exist. Hence $|u(t)-v(t)| \leq \frac{\mathcal{K}}{|l-m|}\epsilon = L\epsilon$. By Definition 2.4, linear differential equation (1.1) has the Hyers–Ulam stability. This finishes the proof. □

By using the same technique as in Theorem 3.1, we can also prove the Hyers–Ulam–Rassias stability of differential equation (1.1). The method of the proof is similar, but we include it for the sake of completeness.

Theorem 3.2 *Differential equation* (1.1) *is Hyers–Ulam–Rassias stable.*

Proof Assume that $u(t) \in C^2(I)$ satisfies

$$\left|u''(t) + \mu^2 u\right| \leq \epsilon\phi(t) \tag{3.5}$$

for all $t \in I$, $\epsilon > 0$ and an integrable function $\phi \in C(\mathbf{R}_+, \mathbf{R}_+)$. We show that there exists a real number $L_\phi > 0$ such that $|u(t)-v(t)| \leq L_\phi\epsilon\phi(t)$ for some $v \in C^2(I)$ satisfying $v''(t) + \mu^2 v = 0$ for all $t \in I$.

Define a function $p : (0,\infty) \to \mathbf{F}$ such that $p(t) =: u''(t) + \mu^2 u(t)$ for all $t > 0$. By (3.5), we have $|p(t)| \leq \epsilon\phi(t)$. Now, taking the Aboodh transform to $p(t)$, we have

$$\mathcal{A}\{p\} = \left(\xi^2 + \mu^2\right)\mathcal{A}\{u\} - u(0) - \frac{u'(0)}{\xi}. \tag{3.6}$$

We know the function $u_0 : (0,\infty) \longrightarrow \mathbf{F}$ is a solution of (1.1) if and only if

$$\left(\xi^2 + \mu^2\right)\mathcal{A}\{u_0\} - u_0(0) - \frac{u_0'(0)}{\xi} = 0.$$

If there exist two constants l and m in $\mathbf{F}$ such that $\xi^2+\mu^2=(\xi-l)(\xi-m)$ with $l+m=0$ and $lm=\mu^2$, then (3.6) becomes

$$\mathcal{A}\{u\}=\frac{\mathcal{A}\{p\}+u(0)+\frac{u'(0)}{\xi}}{(\xi-l)(\xi-m)}. \tag{3.7}$$

Let $v(t)=u(0)(\frac{le^{lt}-me^{mt}}{l-m})+u'(0)(\frac{e^{lt}-e^{mt}}{l-m})$. Then $v(0)=u(0)$ and $u'(0)=v'(0)$. Taking again the Aboodh transform to $v(t)$, we have

$$\mathcal{A}\{v\}=\frac{u(0)+\frac{u'(0)}{\xi}}{(\xi-l)(\xi-m)}. \tag{3.8}$$

Furthermore, $\mathcal{A}\{v''(t)+\mu^2v\}=(\xi^2+\mu^2)\mathcal{A}\{v\}-v(0)-\frac{v'(0)}{\xi}$. Thus, using (3.8), we get $\mathcal{A}\{v''(t)+\mu^2v\}=0$, and so $v''(t)+\mu^2v=0$. Applying (3.7) and (3.8), we get

$$\begin{aligned}\mathcal{A}\{u\}-\mathcal{A}\{v\}&=\frac{\mathcal{A}\{p\}+u(0)+\frac{u'(0)}{\xi}}{(\xi-l)(\xi-m)}-\frac{u(0)+\frac{u'(0)}{\xi}}{(\xi-l)(\xi-m)}=\frac{\mathcal{A}\{p\}}{(\xi-l)(\xi-m)},\\ \mathcal{A}\bigl\{u(t)-v(t)\bigr\}&=\mathcal{A}\biggl\{p(t)*\biggl(\frac{e^{lt}-e^{mt}}{l-m}\biggr)\biggr\}.\end{aligned}$$

Therefore, $u(t)-v(t)=p(t)*(\frac{e^{lt}-e^{mt}}{l-m})$. Using $|p(t)|\le\epsilon\phi(t)$, we get

$$\bigl|u(t)-v(t)\bigr|\le\epsilon\biggl|\int_0^t\biggl(\frac{e^{l(t-x)}-e^{m(t-x)}}{l-m}\biggr)\phi(t)\,dx\biggr|$$

for all $t>0$, where

$$\begin{aligned}L_\phi&=\biggl|\int_0^t\biggl(\frac{e^{l(t-x)}-e^{m(t-x)}}{l-m}\biggr)\phi(x)\,dx\biggr|\\ &\le\frac{1}{|l-m|}\biggl\{e^{\mathcal{R}(l)t}\int_0^t e^{-\mathcal{R}(l)x}\phi(x)\,dx+e^{\mathcal{R}(m)t}\int_0^t e^{-\mathcal{R}(m)x}\phi(x)\,dx\biggr\}\le\frac{\mathcal{K}_\phi\phi(t)}{|l-m|},\end{aligned}$$

where $\int_0^t e^{-\mathcal{R}(l)x}\phi(x)\,dx$ and $\int_0^t e^{-\mathcal{R}(m)x}\phi(x)\,dx$ exist for all $t>0$ and an integrable function ϕ. Hence $|u(t)-v(t)|\le\frac{\mathcal{K}_\phi\phi(t)}{|l-m|}\epsilon=L_\phi\epsilon\phi(t)$. □

Theorem 3.3 *Differential equation* (1.1) *has Mittag-Leffler–Hyers–Ulam stability.*

Proof Let $\epsilon>0$. Suppose that $u(t)\in C^2(I)$ satisfies

$$\bigl|u''(t)+\mu^2u\bigr|\le\epsilon E_\alpha(t) \tag{3.9}$$

for all $t\in I$. We prove that there exists a real number $L>0$ which is independent of ϵ and u such that $|u(t)-v(t)|\le L\epsilon E_\alpha(t)$ for some $v\in C^2(I)$ satisfying $v''(t)+\mu^2v=0$ for all $t\in I$. Define a function $p:(0,\infty)\to\mathbf{F}$ such that $p(t)=:u''(t)+\mu^2u(t)$ for all $t>0$. In view of (3.9), we have $|p(t)|\le\epsilon E_\alpha(t)$. Taking the Aboodh transform to $p(t)$, we have

$$\mathcal{A}\{p\}=\bigl(\xi^2+\mu^2\bigr)\mathcal{A}\{u\}-u(0)-\frac{u'(0)}{\xi}, \tag{3.10}$$

and thus

$$\mathcal{A}\{u\} = \frac{\mathcal{A}\{p\} + u(0) + \frac{u'(0)}{\xi}}{\xi^2 + \mu^2}.$$

By (3.10), a function $u_0 : (0, \infty) \longrightarrow \mathbf{F}$ is a solution of (1.1) if and only if

$$\left(\xi^2 + \mu^2\right)\mathcal{A}\{u_0\} - u_0(0) - \frac{u_0'(0)}{\xi} = 0.$$

If there exist constants l and m in $\mathbf{F}$ such that $\xi^2 + \mu^2 = (\xi - l)(\xi - m)$ with $l + m = 0$ and $lm = \mu^2$, then (3.10) becomes

$$\mathcal{A}\{u\} = \frac{\mathcal{A}\{p\} + u(0) + \frac{u'(0)}{\xi}}{(\xi - l)(\xi - m)}. \tag{3.11}$$

Set

$$v(t) = u(0)\left(\frac{le^{lt} - me^{mt}}{l - m}\right) + u'(0)\left(\frac{e^{lt} - e^{mt}}{l - m}\right).$$

We have $v(0) = u(0)$ and $v'(0) = u'(0)$. Taking the Aboodh transform to $v(t)$, we obtain

$$\mathcal{A}\{v\} = \frac{u(0) + \frac{u'(0)}{\xi}}{(\xi - l)(\xi - m)}. \tag{3.12}$$

On the other hand, $\mathcal{A}\{v''(t) + \mu^2 v\} = (\xi^2 + \mu^2)\mathcal{A}\{v\} - v(0) - \frac{v'(0)}{\xi}$. Using (3.12), we get $\mathcal{A}\{v''(t) + \mu^2 v\} = 0$. Since $\mathcal{A}$ is one-to-one and linear, $v''(t) + \mu^2 v = 0$. This means that $v(t)$ is a solution of (1.1). It follows from (3.11) and (3.12) that

$$\begin{aligned} \mathcal{A}\{u\} - \mathcal{A}\{v\} &= \frac{\mathcal{A}\{p\} + u(0) + \frac{u'(0)}{\xi}}{(\xi - l)(\xi - m)} - \frac{u(0) + \frac{u'(0)}{\xi}}{(\xi - l)(\xi - m)} = \frac{\mathcal{A}\{p\}}{(\xi - l)(\xi - m)}, \\ \mathcal{A}\bigl\{u(t) - v(t)\bigr\} &= \mathcal{A}\left\{p(t) * \left(\frac{e^{lt} - e^{mt}}{l - m}\right)\right\}. \end{aligned}$$

The above equalities show that

$$u(t) - v(t) = p(t) * \left(\frac{e^{lt} - e^{mt}}{l - m}\right),$$

and by using $|p(t)| \leq \epsilon E_\alpha(t)$, we get

$$\bigl|u(t) - v(t)\bigr| \leq \epsilon E_\alpha(t)\left|\int_0^t \left(\frac{e^{l(t-x)} - e^{m(t-x)}}{l - m}\right) dx\right|$$

for all $t > 0$, where

$$\begin{aligned} L &= \left|\int_0^t \left(\frac{e^{l(t-x)} - e^{m(t-x)}}{l - m}\right) dx\right| \leq \frac{1}{|l - m|}\left\{e^{\mathcal{R}(l)t}\int_0^t e^{-\mathcal{R}(l)x}\, dx + e^{\mathcal{R}(m)t}\int_0^t e^{-\mathcal{R}(m)x}\, dx\right\} \\ &\leq \frac{\mathcal{K}}{|l - m|}, \end{aligned}$$

where $\int_0^t e^{-\mathcal{R}(l)x}\,dx$ and $\int_0^t e^{-\mathcal{R}(m)x}\,dx$ exist. Hence $|u(t)-v(t)| \le L\epsilon E_\alpha(t)$. By Definition 2.8, linear differential equation (1.1) has the Hyers–Ulam stability. This finishes the proof. □

The following corollary proves the Mittag-Leffler–Hyers–Ulam–Rassias stability of differential equation (1.1). The method of proof is similar to the proof of Theorem 3.3.

Corollary 3.4 *For every $\epsilon > 0$, let $u(t)$ be a twice continuously differentiable function on I which satisfies the inequality*

$$\left|u''(t) + \mu^2 u\right| \le \epsilon\phi(t)E_\alpha(t)$$

for all $t \in I$. Then there exists a real number $L_\phi > 0$ which is independent of ϵ and u such that

$$\left|u(t) - v(t)\right| \le L_\phi \epsilon\phi(t)E_\alpha(t)$$

for some $v \in C^2(I)$ satisfying $v''(t) + \mu^2 v = 0$ for all $t \in I$.

4 Hyers–Ulam stability for (1.2)

In this section, we investigate the Hyers–Ulam stability, the Hyers–Ulam–Rassias stability, the Mittag-Leffler–Hyers–Ulam stability, and the Mittag-Leffler–Hyers–Ulam–Rassias stability of non-homogeneous differential equation (1.2).

Firstly, we prove the Hyers–Ulam stability of linear differential equation (1.2).

Theorem 4.1 *Non-homogeneous linear differential equation* (1.2) *has Hyers–Ulam stability.*

Proof For every $\epsilon > 0$ and for each solution $u(t) \in C^2(I)$ satisfying

$$\left|u''(t) + \mu^2 u - q(t)\right| \le \epsilon \tag{4.1}$$

for all $t \in I$, we prove that there exists a real number $L > 0$ which is independent of ϵ and u such that $|u(t) - v(t)| \le L\epsilon$ for some $v \in C^2(I)$ satisfying $v''(t) + \mu^2 v = q(t)$ for all $t \in I$. Define a function $p : (0,\infty) \to \mathbf{F}$ such that $p(t) =: u''(t) + \mu^2 u(t) - q(t)$ satisfies $|p(t)| \le \epsilon$. Taking the Aboodh transform to $p(t)$, we have

$$\mathcal{A}\{u\} = \frac{\mathcal{A}\{p\} + u(0) + \frac{u'(0)}{\xi} + \mathcal{A}\{q\}}{\xi^2 + \mu^2}. \tag{4.2}$$

Equality (4.2) shows that a function $u_0 : (0,\infty) \longrightarrow \mathbf{F}$ is a solution of (1.2) if and only if

$$\left(\xi^2 + \mu^2\right)\mathcal{A}\{u_0\} - u_0(0) - \frac{u_0'(0)}{\xi} = \mathcal{A}\{q\}.$$

If there exist constants l and m in $\mathbf{F}$ such that $\xi^2 + \mu^2 = (\xi - l)(\xi - m)$ with $l + m = 0$ and $lm = \mu^2$, then (4.2) becomes

$$\mathcal{A}\{u\} = \frac{\mathcal{A}\{p\} + u(0) + \frac{u'(0)}{\xi} + \mathcal{A}\{q\}}{(\xi - l)(\xi - m)}. \tag{4.3}$$

Set $r(t) = \frac{e^{lt}-e^{mt}}{l-m}$ and

$$v(t) = u(0)\left(\frac{le^{lt} - me^{mt}}{l-m}\right) + u'(0)r(t) + \big[(r * q)(t)\big].$$

Then $v(0) = u(0)$ and $u'(0) = v'(0)$. Once more, taking the Aboodh transform to $v(t)$, we have

$$\mathcal{A}\{v\} = \frac{u(0) + \frac{u'(0)}{\xi} + \mathcal{A}\{q\}}{(\xi - l)(\xi - m)}. \tag{4.4}$$

On the other hand,

$$\mathcal{A}\big\{v''(t) + \mu^2 v\big\} = \big(\xi^2 + \mu^2\big)\mathcal{A}\{v\} - v(0) - \frac{v'(0)}{\xi}.$$

By (4.4), the last equality becomes $\mathcal{A}\{v''(t) + \mu^2 v\} = \mathcal{A}\{q\}$. Since $\mathcal{A}$ is one-to-one and linear, $v''(t) + \mu^2 v = q(t)$, which shows that $v(t)$ is a solution of (1.2). Now, relations (4.3) and (4.4) necessitate that

$$\mathcal{A}\big\{u(t) - v(t)\big\} = \mathcal{A}\{u\} - \mathcal{A}\{v\} = \frac{\mathcal{A}\{p\}}{(\xi - l)(\xi - m)} = \mathcal{A}\big\{p(t) * r(t)\big\},$$

and hence $u(t) - v(t) = p(t) * r(t)$. Taking modulus on both sides of the last equality and using $|p(t)| \le \epsilon$, we get

$$\big|u(t) - v(t)\big| = \big|p(t) * r(t)\big| \le \epsilon \left|\int_0^t \left(\frac{e^{l(t-x)} - e^{m(t-x)}}{l-m}\right) dt\right| \le L\epsilon$$

for all $t > 0$, where

$$\begin{aligned} L &= \left|\int_0^t \left(\frac{e^{l(t-x)} - e^{m(t-x)}}{l-m}\right) dx\right| \\ &\le \frac{1}{|l-m|}\left\{e^{\mathcal{R}(l)t}\int_0^t e^{-\mathcal{R}(l)x}\,dx + e^{\mathcal{R}(m)t}\int_0^t e^{-\mathcal{R}(m)x}\,dx\right\} \le \frac{\mathcal{K}}{|l-m|}, \end{aligned}$$

where the integrals $\int_0^t e^{-\mathcal{R}(l)x}\,dx$ and $\int_0^t e^{-\mathcal{R}(m)x}\,dx$ exist. Hence $|u(t) - v(t)| \le \frac{\mathcal{K}}{|l-m|}\epsilon = L\epsilon$. Therefore, linear differential equation (1.2) has the Hyers–Ulam stability. □

In analogous way to Theorem 4.1, we have the following result which proves the Hyers–Ulam-Rassias stability of differential equation (1.2).

Theorem 4.2 *Differential equation* (1.2) *has Hyers–Ulam–Rassias stability.*

Proof Let $\epsilon > 0$ and $\phi \in C(\mathbf{R}_+, \mathbf{R}_+)$. Suppose that $u(t) \in C^2(I)$ satisfies

$$\big|u''(t) + \mu^2 u - q(t)\big| \le \epsilon\phi(t) \tag{4.5}$$

for all $t \in I$. We prove that there exists a real number $L_\phi > 0$ such that $|u(t) - v(t)| \le L_\phi\epsilon\phi(t)$ for some $v \in C^2(I)$ satisfying $v''(t) + \mu^2 v = q(t)$ for all $t \in I$. Define a function $p : (0, \infty) \to \mathbf{F}$

by $p(t) =: u''(t) + \mu^2 u(t) - q(t)$ for all $t > 0$. In view of (4.5), we have $|p(t)| \leq \epsilon\phi(t)$. Now, taking the Aboodh transform to $p(t)$, we get

$$\mathcal{A}\{u\} = \frac{\mathcal{A}\{p\} + u(0) + \frac{u'(0)}{\xi} + \mathcal{A}\{q\}}{\xi^2 + \mu^2}. \tag{4.6}$$

In addition, by (4.6), a function $u_0 : (0, \infty) \to \mathbf{F}$ is a solution of (1.2) if and only if

$$(\xi^2 + \mu^2)\mathcal{A}\{u_0\} - u_0(0) - \frac{u_0'(0)}{\xi} = \mathcal{A}\{q\}.$$

However, (4.6) becomes

$$\mathcal{A}\{u\} = \frac{\mathcal{A}\{p\} + u(0) + \frac{u'(0)}{\xi} + \mathcal{A}\{q\}}{(\xi - l)(\xi - m)}. \tag{4.7}$$

Assume that there exist constants l and m in $\mathbf{F}$ such that $\xi^2 + \mu^2 = (\xi - l)(\xi - m)$ with $l + m = 0$ and $lm = \mu^2$. Putting $r(t) = \frac{e^{lt} - e^{mt}}{l-m}$ and

$$v(t) = u(0)\left(\frac{le^{lt} - me^{mt}}{l - m}\right) + u'(0)r(t) + \big[(r * q)(t)\big],$$

one can easily obtain $v(0) = u(0)$ and $u'(0) = v'(0)$. Taking the Aboodh transform to $v(t)$, we have

$$\mathcal{A}\{v\} = \frac{u(0) + \frac{u'(0)}{\xi} + \mathcal{A}\{q\}}{(\xi - l)(\xi - m)}. \tag{4.8}$$

Furthermore, $\mathcal{A}\{v'' + \mu^2 v\} = (\xi^2 + \mu^2)\mathcal{A}\{v\} - v(0) - \frac{v'(0)}{\xi}$. By (4.8), we obtain $\mathcal{A}\{v''(t) + \mu^2 v\} = \mathcal{A}\{q\}$. The last equality implies that $v''(t) + \mu^2 v(t) = q(t)$. This means that $v(t)$ is a solution of (1.2). Hence, by (4.7) and (4.8), we obtain

$$\mathcal{A}\big\{u(t) - v(t)\big\} = \mathcal{A}\{u\} - \mathcal{A}\{v\} = \frac{\mathcal{A}\{p\}}{(\xi - l)(\xi - m)} = \mathcal{A}\big\{p(t) * r(t)\big\}.$$

Thus $u(t) - v(t) = p(t) * r(t)$. Then, by using $|p(t)| \leq \epsilon\phi(t)$, we get

$$\big|u(t) - v(t)\big| \leq \epsilon \left| \int_0^t \left(\frac{e^{l(t-x)} - e^{m(t-x)}}{l - m}\right)\phi(t)\,dx \right| \leq L_\phi \epsilon \phi(t)$$

for all $t > 0$, where

$$\begin{aligned} L_\phi &= \left| \int_0^t \left(\frac{e^{l(t-x)} - e^{m(t-x)}}{l - m}\right)\phi(x)\,dx \right| \\ &\leq \frac{1}{|l - m|}\left\{ e^{\mathcal{R}(l)t}\int_0^t e^{-\mathcal{R}(l)x}\phi(x)\,dx + e^{\mathcal{R}(m)t}\int_0^t e^{-\mathcal{R}(m)x}\phi(x)\,dx \right\} \leq \frac{\mathcal{K}_\phi \phi(t)}{|l - m|}, \end{aligned}$$

where the integrals $\int_0^t e^{-\mathcal{R}(l)x}\phi(x)\,dx$ and $\int_0^t e^{-\mathcal{R}(m)x}\phi(x)\,dx$ exist for all $t > 0$ and an integrable function ϕ. Hence

$$\left|u(t) - v(t)\right| \leq \frac{\mathcal{K}_\phi \phi(t)}{|l - m|}\epsilon = L_\phi \epsilon \phi(t).$$

This finishes the proof. □

By using the same technique as in Theorem 3.1, we can also prove the Mittag-Leffler–Hyers–Ulam stability of differential equation (1.2). The method of the proof is similar, but we include it for the sake of completeness.

Theorem 4.3 *Differential equation* (1.2) *is Mittag-Leffler–Hyers–Ulam stable.*

Proof For every $\epsilon > 0$ and for each solution $u(t) \in C^2(I)$ satisfying

$$\left|u''(t) + \mu^2 u - q(t)\right| \leq \epsilon E_\alpha(t) \tag{4.9}$$

for all $t \in I$, we prove that there exists a real number $L > 0$ which is independent of ϵ and u such that $|u(t) - v(t)| \leq L\epsilon E_\alpha(t)$ for some $v \in C^2(I)$ satisfying $v''(t) + \mu^2 v = q(t)$ for all $t \in I$. Then the function $p : (0,\infty) \to \mathbf{F}$ defined by $p(t) =: u''(t) + \mu^2 u(t) - q(t)$ satisfies $|p(t)| \leq \epsilon E_\alpha(t)$. Taking the Aboodh transform to $p(t)$, we have

$$\mathcal{A}\{u\} = \frac{\mathcal{A}\{p\} + u(0) + \frac{u'(0)}{\xi} + \mathcal{A}\{q\}}{\xi^2 + \mu^2}. \tag{4.10}$$

Equality (4.10) shows that a function $u_0 : (0,\infty) \to \mathbf{F}$ is a solution of (1.2) if and only if

$$\left(\xi^2 + \mu^2\right)\mathcal{A}\{u_0\} - u_0(0) - \frac{u_0'(0)}{\xi} = \mathcal{A}\{q\}.$$

If there exist constants l and m in $\mathbf{F}$ such that $\xi^2 + \mu^2 = (\xi - l)(\xi - m)$ with $l + m = 0$ and $lm = \mu^2$, then (4.10) becomes

$$\mathcal{A}\{u\} = \frac{\mathcal{A}\{p\} + u(0) + \frac{u'(0)}{\xi} + \mathcal{A}\{q\}}{(\xi - l)(\xi - m)}. \tag{4.11}$$

Set $r(t) = \frac{e^{lt} - e^{mt}}{l-m}$ and

$$v(t) = u(0)\left(\frac{le^{lt} - me^{mt}}{l - m}\right) + u'(0)r(t) + \left[(r * q)(t)\right].$$

Then $v(0) = u(0)$ and $v'(0) = u'(0)$. Once more, taking the Aboodh transform to $v(t)$, we obtain

$$\mathcal{A}\{v\} = \frac{u(0) + \frac{u'(0)}{\xi} + \mathcal{A}\{q\}}{(\xi - l)(\xi - m)}. \tag{4.12}$$

On the other hand, $\mathcal{A}\{v''(t)+\mu^2 v\} = (\xi^2+\mu^2)\mathcal{A}\{v\} - v(0) - \frac{v'(0)}{\xi}$. By (4.12), the last equality becomes $\mathcal{A}\{v''(t)+\mu^2 v\} = \mathcal{A}\{q\}$. Since $\mathcal{A}$ is one-to-one and linear, $v''(t)+\mu^2 v = q(t)$, which shows that $v(t)$ is a solution of (1.2). Now, relations (4.11) and (4.12) necessitate that

$$\mathcal{A}\{u(t)-v(t)\} = \mathcal{A}\{u\} - \mathcal{A}\{v\} = \frac{\mathcal{A}\{p\}}{(\xi-l)(\xi-m)} = \mathcal{A}\{p(t)*r(t)\},$$

and hence $u(t)-v(t) = p(t)*r(t)$. Taking modulus on both sides of the last equality and using $|p(t)| \leq \epsilon E_\alpha(t)$, we get

$$|u(t)-v(t)| \leq L\epsilon E_\alpha(t),$$

where $L = |\int_0^t (\frac{e^{l(t-x)}-e^{m(t-x)}}{l-m})\,dt|$ and the integral exists for all $t>0$. Hence linear differential equation (1.2) has the Mittag-Leffler–Hyers–Ulam stability. □

In analogous way to Theorem 4.3, we have the following corollary which proves the Mittag-Leffler–Hyers–Ulam–Rassias stability of differential equation (1.2).

Corollary 4.4 *For every $\epsilon > 0$, let $u(t)$ be a twice continuously differentiable function on I which satisfies the inequality*

$$|u''(t)+\mu^2 u - q(t)| \leq \epsilon\phi(t)E_\alpha(t)$$

for all $t \in I$. Then there exists a real number $L_\phi > 0$ which is independent of ϵ and u such that

$$|u(t)-v(t)| \leq L_\phi\epsilon\phi(t)E_\alpha(t)$$

for some $v \in C^2(I)$ satisfying $v''(t)+\mu^2 v = q(t)$ for all $t \in I$.

5 Examples and remarks

In this section, we provide some examples to make it easier to understand the main results of this paper.

Example 5.1 We consider the following homogeneous linear differential equation of second order:

$$u''(t)+u(t) = 0, \tag{5.1}$$

where $\mu^2 = 1$, with the initial conditions $u(0) = u'(0) = 1$. Letting $p(t) = u''(t)+u(t)$ in Theorem 3.1 and taking the Aboodh transform, we get

$$P(\xi) = \xi^2 U(\xi) - \frac{u'(0)}{\xi} - u(0) + U(\xi).$$

By the initial conditions, we have $\mathcal{A}\{u\} = \frac{\xi P(\xi)+\xi+1}{\xi(\xi^2+1)}$. If a continuously differentiable function $u : [0,\infty) \to \mathbf{F}$ of exponential order satisfies

$$|u''(t)+u(t)| \leq \epsilon$$

for all $t \geq 0$ and for some $\epsilon > 0$, then, by Theorem 3.1, there exists a solution $v : [0, \infty) \to \mathbf{F}$ of differential equation (5.1) such that

$$\left|u(t) - v(t)\right| \leq L\epsilon$$

for all $t \geq 0$. In fact, $v(t) = c_1 \cos t + c_2 \sin t$ for some constants $c_1, c_2 \in \mathbf{F}$.

Example 5.2 Let us take the non-homogeneous linear differential equation

$$u''(t) + 3u(t) = t, \tag{5.2}$$

with the initial conditions $u(0) = u'(0) = -1$. Here $q(t) = t$ is a function of exponential order and $\mu^2 = 3$.

If a continuously differentiable function $u : [0, \infty) \to \mathbf{F}$ of exponential order satisfies

$$\left|u''(t) + 3u(t) - t\right| \leq \varepsilon$$

for all $t \geq 0$ and some $\varepsilon > 0$, then, by Theorem 4.1, there exists a solution $v : [0, \infty) \to \mathbf{F}$ of differential equation (5.2) such that $v(t)$ is of exponential order and

$$\left|u(t) - v(t)\right| \leq L\varepsilon$$

for all $t \geq 0$. In fact, $v(t) = c_1 \cos \sqrt{3}t + c_2 \sin \sqrt{3}t + \frac{1}{3}t$ for some constants $c_1, c_2 \in \mathbf{F}$.

Example 5.3 Consider the non-homogeneous linear differential equation

$$u''(t) + 2u(t) = 4e^{3t}, \tag{5.3}$$

with the initial conditions

$$u(0) = -3 \quad \text{and} \quad u'(0) = 5,$$

where $q(t) = 4e^{3t}$ is a function of exponential order with $\mu^2 = 2$.

Letting $p(t) = u''(t) + 2u(t) - 4e^{3t}$ in Theorem 4.1 and taking the Aboodh transform, we get

$$P(\xi) = \xi^2 U(\xi) - \frac{u'(0)}{\xi} - u(0) + 2U(\xi) - \frac{4}{\xi(\xi - 3)}.$$

By the initial conditions, we have

$$U(\xi) = \mathcal{A}\{u\} = \frac{1}{\xi^2 + 2}\left[P(\xi) + \frac{4}{\xi(\xi - 3)} + \frac{5}{\xi} - 3\right].$$

If a continuously differentiable function $u : [0, \infty) \to \mathbf{F}$ of exponential order satisfies

$$\left|u''(t) + 2u(t) - 4e^{3t}\right| \leq \epsilon$$

for all $t \geq 0$ and some $\epsilon > 0$, then, by Theorem 4.1, there exists a solution $v : [0, \infty) \to \mathbf{F}$ of differential equation (5.3) such that

$$\left|u(t) - v(t)\right| \leq L\epsilon$$

for all $t \geq 0$. In fact, $v(t) = c_1 \cos \sqrt{2}t + c_2 \sin \sqrt{2}t + 4e^{3t}$ for some constants $c_1, c_2 \in \mathbf{F}$.

Example 5.4 Consider the linear differential equation

$$u''(t) + 9u(t) = 2\cos t \tag{5.4}$$

with the initial conditions

$$u(0) = 3 \quad \text{and} \quad u'(0) = 4,$$

where $q(t) = 2\cos t$ is a function of exponential order with $\mu^2 = 2$.

Letting $p(t) = u''(t) + 9u(t) - 2\cos t$ in Theorem 4.1 and taking the Aboodh transform, we get

$$P(\xi) = \xi^2 U(\xi) - \frac{u'(0)}{\xi} - u(0) + 9U(\xi) - \frac{2}{\xi^2 + 1}.$$

By the initial conditions, we have

$$U(\xi) = \mathcal{A}\{u\} = \frac{1}{\xi^2 + 9}\left[P(\xi) + \frac{2}{\xi^2 + 1} + \frac{4}{\xi} + 3\right].$$

If a continuously differentiable function $u : [0, \infty) \to \mathbf{F}$ of exponential order satisfies

$$\left|u''(t) + 9u(t) - 2\cos t\right| \leq \epsilon$$

for all $t \geq 0$ and some $\epsilon > 0$, then, by Theorem 4.1, there exists a solution $v : [0, \infty) \to \mathbf{F}$ of differential equation (5.4) such that

$$\left|u(t) - v(t)\right| \leq L\epsilon$$

for all $t \geq 0$. In fact, $v(t) = c_1 \cos 3t + c_2 \sin 3t + \frac{2}{5}\cos t$ for some constants $c_1, c_2 \in \mathbf{F}$.

Remark 5.5 The above examples are also true when we replace ϵ and $K\varepsilon$ with $\phi(t)\varepsilon$ and $K\phi(t)\epsilon$, respectively, where $\phi(t)$ is an increasing function. In this case, we see that the corresponding differential equations have the Hyers–Ulam–Rassias stability.

Remark 5.6 Differential equations (5.1), (5.2), (5.3), and (5.4) have the Mittag-Leffler–Hyers–Ulam stability when $\alpha > 0$. Moreover, they also have the Mittag-Leffler–Hyers–Ulam–Rassias stability when $\phi(t)$ is an increasing function and $\alpha > 0$.

6 Conclusion

In this paper, we introduced a new integral transform, namely Aboodh transform, and we applied the transform to investigate the Hyers–Ulam stability, the Hyers–Ulam–Rassias stability, the Mittag-Leffler–Hyers–Ulam stability, and the Mittag-Leffler–Hyers–Ulam–Rassias stability of second order linear differential equations with constant coefficients.

In other words, we established sufficient criteria for the Hyers–Ulam stability of second order linear differential equations with constant coefficients by using the Aboodh transform method. Moreover, this paper provides a new method to investigate the Hyers–Ulam stability of differential equations. This is the first attempt to use the Aboodh transformation to prove the Hyers–Ulam stability for linear differential equations of second order. Furthermore, this paper shows that the Aboodh transform method is more convenient for investigating the stability problems for linear differential equations with constant coefficients. Readers can also apply this terminology to various problems on differential equations.

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