A GENERAL ADDITIVE FUNCTIONAL INEQUALITY
AND DERIVATION IN BANACH ALGEBRAS

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Abstract. Using the fixed point method, we prove the Hyers-Ulam stability of homomorphisms in complex Banach algebras and complex Banach Lie algebras and also of derivations on complex Banach algebras and complex Banach Lie algebras for the general additive functional inequality $\|f(\alpha x - \beta y) - \alpha f(x) - \beta f(-y)\| \leqslant \|r(f(\alpha x + \beta y) - \alpha f(x) - \beta f(y)\|$, where r is a fixed nonzero complex number with $|r| < 1$ and $\alpha, \beta \neq 0$.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [37] concerning the stability of group homomorphisms. Hyers [17] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [33] for linear mappings by considering an unbounded Cauchy difference. Those results have been recently complemented in [5]. A generalization of the Aoki and Rassias theorem was obtained by Găvruta [16], who used a more general function controlling the possibly unbounded Cauchy difference in the spirit of Rassias' approach. The stability problems for several functional equations or inequalities have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4, 12, 18, 19, 20, 21, 23, 25, 34, 35, 36]). In 2009, Kannappan produced a book, that has many new features in addition to the usual expected ones, introduce and cover as many important equations, areas, and methods of solution (see [22]).

The method provided by Hyers [17] which produces the additive function will be called a direct method. This method is the most important and powerful tool to study the stability of various functional equations. That is, the exact solution of the functional equation is explicitly constructed as a limit of a sequence, starting from the given approximate solution [27]. The other important method is fixed point theorem, that is , the exact solution of the functional equation is explicitly constructed as a fixed point of some certain map [24, 31, 28].

We recall a fundamental result in fixed point theorem.

THEOREM 1.1. (see [10, 13]) *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$, for all $n \geqslant n_0$;

(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(3) y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \leqslant \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

By using the fixed point method, the stability problems of several functional equations have been extensively investigated by a number of authors (see [6, 9, 10, 11, 29, 30, 38]).

To obtain a Jordan and von Neumann type characterization theorem for the quasi-inner-product spaces, Drygas [14] considered the functional equation

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y),$$

whose solution is called a *Drygas mapping*. The general solution of the above functional equation was given by Ebanks, Kannappan and Sahoo [15] as

$$f(x) = Q(x) + A(x)$$

where A is an additive mapping and Q is a quadratic mapping.

In this paper, we consider the following functional inequality

$$\|f(\alpha x - \beta y) - \alpha f(x) - \beta f(-y)\| \leqslant \|r(f(\alpha x + \beta y) - \alpha f(x) - \beta f(y))\| \quad (1.1)$$

for all $x, y \in X$ with $|r| < 1$, where α, β are nonzero real numbers. And we prove the Hyers-Ulam stability of homomorphisms in complex Banach algebras and of derivations on complex Banach algebras for the general Jensen-type functional equation. Moreover, we prove that the Hyers-Ulam stability of homomorphisms in complex Banach Lie algebras and of derivations on complex Banach Lie algebras.

2. Hyers-Ulam stability of (1.1) using a fixed point method

Throughout this section, assume that X is a complex algebra and that Y is a complex Banach algebra with norm $\| \cdot \|$.

LEMMA 2.1. *If a mapping $f : X \to Y$ satisfies*

$$\|f(\alpha x - \beta y) - \alpha f(x) - \beta f(-y)\| \leqslant \|r(f(\alpha x + \beta y) - \alpha f(x) - \beta f(y))\| \quad (2.1)$$

for all $x, y \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (2.1). Replacing y by $-y$ in (2.1), we get

$$\|f(\alpha x - \beta y) - \alpha f(x) - \beta f(y)\| \leqslant \|r(f(\alpha x - \beta y) - \alpha f(x) - \beta f(-y))\| \quad (2.2)$$

for all $x, y \in X$. It follows from (2.1) and (2.2) that

$$\|f(\alpha x - \beta y) - \alpha f(x) - \beta f(-y)\| \leqslant |r|^2 \|f(\alpha x - \beta y) - \alpha f(x) - \beta f(-y)\| \quad (2.3)$$

and so $f(\alpha x - \beta y) = \alpha f(x) + \beta f(-y)$ for all $x, y \in X$, since $|r| < 1$. And $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for all $x, y \in X$. It is easy to show that $f(x + y) = f(x) + f(y)$ for all $x, y \in X$. Thus, f is additive. $\square$

Using the fixed point method, we prove the Hyers-Ulam stability of the functional inequality (1.1) in complex Banach spaces.

THEOREM 2.2. *Let $\phi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\phi\left(\frac{x}{\alpha + \beta}, \frac{y}{\alpha + \beta}\right) \leqslant \frac{L}{|\alpha + \beta|}\phi(x, y) \quad (2.4)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\|f(\alpha x - \beta y) - \alpha f(x) - \beta f(-y)\| \leqslant \|r(f(\alpha x + \beta y) - \alpha f(x) - \beta f(y))\| + \phi(x, y) \quad (2.5)$$

for all $x, y \in X$. Then there exists a unique additive mapping $H : X \to Y$ such that

$$\|f(x) - H(x)\| \leqslant \frac{L}{|\alpha + \beta|(1 - L)}\left(\frac{|r|}{1 - |r|^2}\phi(x, x) + \frac{1}{1 - |r|^2}\phi(x, -x)\right) \quad (2.6)$$

for all $x \in X$.

Proof. Letting $y = -y$ in (2.5), we get

$$\|f(\alpha x + \beta y) - \alpha f(x) - \beta f(y)\| \leqslant |r|\|f(\alpha x - \beta y) - \alpha f(x) - \beta f(-y)\| + \phi(x, -y) \quad (2.7)$$

for all $x, y \in X$. Therefore, we get

$$\|f((\alpha + \beta)x) - (\alpha + \beta)f(x)\| \leqslant \frac{|r|}{1 - |r|^2}\phi(x, x) + \frac{1}{1 - |r|^2}\phi(x, -x) \quad (2.8)$$

for all $x \in X$.

Consider the set

$$A := \{h : X \to Y, h(0) = 0\}$$

and introduce the generalized metric on A:

$$d(g, h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\|\right.$$
$$\left.\leqslant \mu \left\{\frac{|r|}{1 - |r|^2}\phi(x, x) + \frac{1}{1 - |r|^2}\phi(x, -x)\right\}, \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [26]).

Now, we consider the linear mapping $J : A \to A$ such that

$$Jg(x) := |\alpha + \beta| g\left(\frac{x}{\alpha + \beta}\right)$$

for all $x \in X$.

Let $g, h \in A$ be given such that $d(g,h) = \varepsilon$, then

$$\|g(x) - h(x)\| \leqslant \varepsilon \phi(x,x)$$

for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\| = \left\| |\alpha + \beta| g\left(\frac{x}{\alpha + \beta}\right) - |\alpha + \beta| h\left(\frac{x}{\alpha + \beta}\right) \right\|$$

$$\leqslant |\alpha + \beta| \varepsilon \left\{ \frac{|r|}{1 - |r|^2} \phi\left(\frac{x}{\alpha + \beta}, \frac{x}{\alpha + \beta}\right) + \frac{1}{1 - |r|^2} \phi\left(\frac{x}{\alpha + \beta}, \frac{-x}{\alpha + \beta}\right) \right\}$$

$$\leqslant |\alpha + \beta| \varepsilon \frac{L}{|\alpha + \beta|} \left\{ \frac{|r|}{1 - |r|^2} \phi(x,x) + \frac{1}{1 - |r|^2} \phi(x,-x) \right\}$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \leqslant L\varepsilon$. This means that

$$d(Jg,Jh) \leqslant Ld(g,h)$$

for all $g, h \in A$.

It follows from (2.8) that

$$\left\| f(x) - (\alpha + \beta) f\left(\frac{x}{\alpha + \beta}\right) \right\|$$

$$\leqslant \frac{|r|}{1 - |r|^2} \phi\left(\frac{x}{\alpha + \beta}, \frac{x}{\alpha + \beta}\right) + \frac{1}{1 - |r|^2} \phi\left(\frac{x}{\alpha + \beta}, \frac{-x}{\alpha + \beta}\right)$$

$$\leqslant \frac{L}{|\alpha + \beta|} \left\{ \frac{|r|}{1 - |r|^2} \phi(x,x) + \frac{1}{1 - |r|^2} \phi(x,-x) \right\}$$

for all $x \in X$ and so $d(f, Jf) \leqslant \frac{L}{|\alpha + \beta|}$.

By Theorem (1.1), there exists a mapping $H : X \to Y$ satisfying the following:

(1) H is a fixed point of J, i.e.,

$$H(x) = |\alpha + \beta| H\left(\frac{x}{\alpha + \beta}\right) \tag{2.9}$$

for all $x \in X$. The mapping H is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g < \infty\}.$$

This implies that H is a unique mapping satisfying (2.9) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x) - H(x)\| \leqslant \mu \left\{ \frac{|r|}{1 - |r|^2} \phi(x,x) + \frac{1}{1 - |r|^2} \phi(x,-x) \right\}$$

for all $x \in X$;

(2) $d(J^l f, H) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l \to \infty} |\alpha + \beta|^n f\left(\frac{x}{(\alpha + \beta)^n}\right) = H(x)$$

for all $x \in X$;

(3) $d(f, H) \leqslant \frac{1}{1-L} d(f, Jf)$, which implies

$$\|f(x) - H(x)\| \leqslant \frac{L}{|\alpha + \beta|(1-L)} \left(\frac{|r|}{1 - |r|^2} \phi(x, x) + \frac{1}{1 - |r|^2} \phi(x, -x)\right)$$

for all $x \in X$. It follows from (2.4) and (2.5) that

$$\|H(\alpha x - \beta y) - \alpha H(x) - \beta H(-y)\|$$

$$\lim_{n \to \infty} |\alpha + \beta|^n \left\| f\left(\frac{\alpha x - \beta y}{(\alpha + \beta)^n}\right) - \alpha f\left(\frac{x}{(\alpha + \beta)^n}\right) - \beta f\left(-\frac{y}{(\alpha + \beta)^n}\right) \right\|$$

$$\leqslant \lim_{n \to \infty} |\alpha + \beta|^n |r| \left\| f\left(\frac{\alpha x + \beta y}{(\alpha + \beta)^n}\right) - \alpha f\left(\frac{x}{(\alpha + \beta)^n}\right) - \beta f\left(\frac{y}{(\alpha + \beta)^n}\right) \right\|$$

$$+ \lim_{n \to \infty} (\alpha + \beta)^n \phi\left(\frac{x}{(\alpha + \beta)^n}, \frac{y}{(\alpha + \beta)^n}\right)$$

$$= \|r(H(\alpha x + \beta y) - \alpha H(x) - \beta H(y))\|$$

for all $x, y \in X$. So

$$\|H(\alpha x - \beta y) - \alpha H(x) - \beta H(-y)\| \leqslant \|r(H(\alpha x + \beta y) - \alpha H(x) - \beta H(y))\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $H : X \to Y$ is additive. $\quad\square$

3. Hyers-Ulam stability of homomorphisms in complex Banach algebras

Note that a $\mathbb{C}$-linear mapping $H : X \to Y$ is called a *homomorphism* in complex Banach algebras if H satisfies $H(xy) = H(x)H(y)$ for all $x, y \in X$. Now we prove the Hyers-Ulam stability of homomorphism in complex Banach algebras.

We introduce a useful result that can be easily derived from [8, Lemma 1].

LEMMA 3.1. ([8]) *Let* $S \subset \mathbb{T}^1 := \{v \in \mathbb{C} : |v| = 1\}$ *be a connected set containing at least two points. Let* $f : X \to Y$ *be an additive mapping such that* $f(\lambda x) = \lambda f(x)$ *for all* x *and all* $\lambda \in S$. *Then* $f : X \to Y$ *is* $\mathbb{C}$-*linear.*

In the following parts of the paper, S stands for a connected subset of $\mathbb{T}_1$ such that $1 \in S$ and $S \setminus \{1\} \neq \emptyset$.

THEOREM 3.2. *Let* α, β *be fixed nonzero real numbers with* $|\alpha| < 1$. *Let* $f : X \to Y$ *be a mapping for which there exists a function* $\varphi : X \times X \to [0, \infty)$ *such that*

$$\|f(\lambda(\alpha x - \beta y)) - \lambda(\alpha f(x) + \beta f(-y))\|$$
$$\leqslant \|r(f(\lambda(\alpha x + \beta y)) - \lambda(\alpha f(x) + \beta f(y))\| + \varphi(x, y), \tag{3.1}$$

$$\|f(xy) - f(x)f(y)\| \leqslant \varphi(x,y) \tag{3.2}$$

for all $\lambda \in S$ and all $x, y \in X$. If there exists an $L < 1$ such that $\varphi(x,y) \leqslant \frac{L}{|\alpha|} \varphi(\alpha x, \alpha y)$ for all $x, y \in X$, then there exists a unique homomorphism $H : X \to Y$ such that

$$\|f(x) - H(x)\| \leqslant \frac{L}{|1 - |r|||\alpha|(1 - L)} \varphi(x,0) \tag{3.3}$$

for all $x \in X$.

　　Proof. Let $\lambda = 1$ and $y = 0$ in (3.1), we get

$$\left\|f(x) - \alpha f\left(\frac{x}{\alpha}\right)\right\| \leqslant \frac{\varphi(\frac{x}{\alpha},0)}{1 - |r|} \leqslant \frac{L}{|\alpha|} \varphi(x,0) \tag{3.4}$$

for all $x \in X$.

　　Consider the set

$$A := \{g : X \to X, g(0) = 0\}$$

and introduce the *generalized metric* on A:

$$d(g,h) = \inf\{C \in \mathbb{R}_+ : \|g(x) - h(x)\| \leqslant C\varphi(x,0), \forall x \in X\}.$$

It is easy to show that (A,d) is complete.

　　Now we consider the linear mapping $J : A \to A$ such that

$$Jg(x) := \alpha g\left(\frac{1}{\alpha}x\right)$$

for all $x \in X$.

　　By the similar reasoning as in the proof of Theorem 2.2, there is a unique additive mapping $H : X \to Y$ satifying (3.3) which are given by

$$H(x) = \lim_{n \to \infty} |\alpha|^n f\left(\frac{x}{\alpha^n}\right)$$

for all $x \in X$.

$$
\begin{aligned}
&\|H(\alpha x - \beta y) - \alpha H(x) - \beta H(-y)\| \\
&= \lim_{n \to \infty} |\alpha|^n \left\|f\left(\frac{\alpha x - \beta y}{\alpha^n}\right) - \alpha f\left(\frac{x}{\alpha^n}\right) - \beta f\left(\frac{-y}{\alpha^n}\right)\right\| \\
&\leqslant \lim_{n \to \infty} |\alpha|^n \left\|f\left(\frac{\alpha x + \beta y}{\alpha^n}\right) - \alpha f\left(\frac{x}{\alpha^n}\right) - \beta f\left(\frac{y}{\alpha^n}\right)\right\| + \lim_{n \to \infty} |\alpha|^n \varphi\left(\frac{x}{\alpha^n}, \frac{y}{\alpha^n}\right) \\
&= \|H(\alpha x + \beta y) - \alpha H(x) - \beta H(y)\|
\end{aligned}
$$

for all $x, y \in X$. Thus, H is additive.

Let $y = 0$ in (3.1), we get

$$\|f(\lambda \alpha x) - \lambda \alpha f(x)\| \leqslant \frac{\varphi(x,0)}{1 - |r|}$$

for all $\lambda \in \mathbb{T}^1$ and all $x \in X$. Thus

$$\|f(\lambda x) - \lambda f(x)\|$$
$$= \left\|f(\lambda x) - \lambda \alpha f\left(\frac{x}{\alpha}\right) + \lambda \alpha f\left(\frac{x}{\alpha}\right) - \lambda f(x)\right\|$$
$$\leqslant \left\|f(\lambda x) - \lambda \alpha f\left(\frac{x}{\alpha}\right)\right\| + |\lambda|\left\|f(x) - \alpha f\left(\frac{x}{\alpha}\right)\right\|$$
$$\leqslant \frac{2\varphi\left(\frac{x}{\alpha},0\right)}{1 - |r|}$$

for all $\lambda \in \mathbb{T}^1$ and all $x \in X$. So

$$\|H(\lambda x) - \lambda H(x)\| \leqslant |\alpha|^n \left\|f\left(\lambda \frac{x}{\alpha^n}\right) - \lambda f\left(\frac{x}{\alpha^n}\right)\right\|$$
$$\leqslant |\alpha|^n \varphi\left(\frac{x}{\alpha^{n+1}},0\right) \leqslant 2\frac{L^n}{1 - |r|}\varphi\left(\frac{x}{\alpha},0\right)$$

and thus $H(\lambda x) = \lambda H(x)$ for all $\lambda \in \mathbb{T}$ and all $x \in X$. By Lemma 3.1, the mapping $H : X \to Y$ is $\mathbb{C}$-linear.

It follows from (3.2) that

$$\|H(xy) - H(x)H(y)\| = \lim_{n \to \infty} |\alpha|^{2n} \left\|f\left(\frac{xy}{\alpha^{2n}}\right) - f\left(\frac{x}{\alpha^n}\right)f\left(\frac{y}{\alpha^n}\right)\right\|$$
$$\leqslant \lim_{n \to \infty} |\alpha|^{2n} \varphi\left(\frac{x}{\alpha^n}, \frac{y}{\alpha^n}\right) \leqslant \lim_{n \to \infty} |\alpha|^n \varphi\left(\frac{x}{\alpha^n}, \frac{y}{\alpha^n}\right) = 0$$

for all $x, y \in X$. So $H(xy) = H(x)H(y)$ for all $x, y \in X$. Thus, the $\mathbb{C}$-linear mapping $H : X \to Y$ is a homomorphism satisfying (3.3).   □

THEOREM 3.3. *Let* α, β *be fixed nonzero real numbers with* $|\alpha| > 1$. *Let* $f : X \to Y$ *be a mapping for which there exists a function* $\varphi : X^2 \to [0, \infty)$ *satisfying* (3.1) *and* (3.2). *If there exists an* $L < 1$ *such that* $\varphi(x,y) \leqslant L|\alpha|\varphi\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right)$ *for all* $x, y \in X$, *then there exists a unique homomorphism* $H : X \to Y$ *such that*

$$\|f(x) - H(x)\| \leqslant \frac{1}{(1 - |r|)|\alpha|(1 - L)}\varphi(x,0) \tag{3.5}$$

for all $x \in X$.

Proof. We consider the set A and the generalized metric d in the proof of Theorem 3.2.

We consider the linear mapping $J : A \to A$ such that

$$Jg(x) := \frac{1}{\alpha}g(\alpha x)$$

for all $x \in X$.

Letting $\lambda = 1$ and $y = 0$ in (3.1), we get that

$$\left\| f(x) - \frac{1}{\alpha} f(\alpha x) \right\|_Y \leqslant \frac{1}{|\alpha|} \frac{\varphi(x,0)}{1 - |r|}$$

for all $x \in X$. Hence $d(f, Jf) \leqslant \frac{1}{(1-|r|)|\alpha|}$.

The rest of the proof is similar to the proof of Theorem 3.2. □

4. Hyers-Ulam stability of derivations on complex Banach algebras

Throughout this section, assume that X is a complex Banach algebra with norm $\| \cdot \|$. Note that a $\mathbb{C}$-linear mapping $\delta : X \to X$ is called a *derivation* on X if δ satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in X$.

We prove the Hyers-Ulam stability of derivations on complex Banach algebras for the functional inequality (3.1).

THEOREM 4.1. *Let* α, β *be fixed nonzero real numbers with* $|\alpha| < 1$. *Let* $f : X \to X$ *be a mapping for which there exists a function* $\varphi : X^2 \to [0, \infty)$ *such that*

$$\begin{aligned} &\| f(\lambda(\alpha x - \beta y)) - \lambda(\alpha f(x) + \beta f(-y)) \| \\ &\leqslant \| r(f(\lambda(\alpha x + \beta y)) - \lambda(\alpha f(x) + \beta f(y)) \| + \varphi(x, y), \end{aligned} \tag{4.1}$$

$$\| f(xy) - f(x)y - xf(y) \| \leqslant \varphi(x, y) \tag{4.2}$$

for all $\lambda \in S$ *and all* $x, y \in X$. *If there exists an* $L < 1$ *such that* $\varphi(x, y) \leqslant \frac{L}{|\alpha|} \varphi(\alpha x, \alpha y)$ *for all* $x, y \in X$, *then there exists a unique derivation* $\delta : X \to X$ *such that*

$$\| f(x) - \delta(x) \| \leqslant \frac{L}{(1 - |r|)|\alpha|(1 - L)} \varphi(x, 0) \tag{4.3}$$

for all $x \in X$.

Proof. It follows from $\varphi(x, y) \leqslant \frac{L}{|\alpha|} \varphi(\alpha x, \alpha y)$ that

$$\lim_{j \to \infty} |\alpha|^j \varphi(\alpha^{-j} x, \alpha^{-j} y) = 0$$

for all $x, y \in X$.

Consider the set

$$A := \{ g : X \to X, g(0) = 0 \}$$

and introduce the *generalized metric* on A:

$$d(g, h) = \inf \{ C \in \mathbb{R}_+ : \| g(x) - h(x) \| \leqslant C\varphi(x, 0), \forall x \in X \}.$$

It is easy to show that (A,d) is complete.

Now we consider the linear mapping $J : A \to A$ such that

$$Jg(x) := \alpha g \left(\frac{1}{\alpha} x \right)$$

for all $x \in X$.

By [10, Theorem 3.1]

$$d(Jg, Jh) \leqslant Ld(g,h)$$

for all $g, h \in A$.

Letting $\lambda = 1$ and $y = 0$ in (4.1), we get

$$\left\| f(x) - \alpha f \left(\frac{x}{\alpha} \right) \right\| \leqslant \frac{1}{1 - |r|} \varphi \left(\frac{x}{\alpha}, 0 \right) \leqslant \frac{L}{(1 - |r|)|\alpha|} \varphi(x, 0)$$

for all $x \in X$.

Hence $d(f, Jf) \leqslant \frac{L}{(1 - |r|)|\alpha|}$.

By Theorem 1.1, there exists a mapping $\delta : X \to X$ such that

(1) δ is a fixed point of J, that is,

$$\alpha \delta \left(\frac{x}{\alpha} \right) = \delta(x) \tag{4.4}$$

for all $x \in X$. The mapping δ is a unique fixed point of J in the set

$$B = \{ g \in A : d(f,g) < \infty \}.$$

This implies that δ is a unique mapping satisfying (4.4) such that there exists $C \in (0, \infty)$ satisfying

$$\| \delta(x) - f(x) \| \leqslant C \varphi(x, 0)$$

for all $x \in X$.

(2) $d(J^n f, \delta) \to 0$ as $n \to \infty$. This implies the inequality

$$\lim_{n \to \infty} \alpha^n f \left(\frac{x}{\alpha^n} \right) = \delta(x)$$

for all $x \in X$.

(3) $d(f, \delta) \leqslant \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, \delta) \leqslant \frac{L}{(1 - |r|)|\alpha|(1 - L)} \varphi(x, 0).$$

This implies that the inequality (4.3) holds.

Next, we show that $\delta(x)$ is an additive mapping.

$$\|\delta(\alpha x - \beta y) - \alpha \delta(x) - \beta \delta(-y)\|$$

$$= \lim_{k \to \infty} \left\| \alpha^k f \left(\frac{\alpha x - \beta y}{\alpha^k} \right) - \alpha \alpha^k f \left(\frac{x}{\alpha^k} \right) - \beta \alpha^k f \left(\frac{-y}{\alpha^k} \right) \right\|$$

$$\leqslant \lim_{k \to \infty} |\alpha|^k \left\| f \left(\frac{\alpha x + \beta y}{\alpha^k} \right) - \alpha f \left(\frac{x}{\alpha^k} \right) - \beta f \left(\frac{y}{\alpha^k} \right) \right\| + \lim_{n \to \infty} |\alpha|^n \varphi \left(\frac{x}{\alpha^n}, \frac{y}{\alpha^n} \right)$$

$$= \|\delta(\alpha x + \beta y) - \alpha \delta(x) - \beta \delta(y)\|$$

Therefore, the mapping $\delta : X \to X$ is Cauchy additive.

Letting $y = 0$ in (4.1), we get

$$\left\| \lambda f(x) - \alpha f \left(\lambda \frac{x}{\alpha} \right) \right\| \leqslant \frac{1}{1 - |r|} \varphi \left(\frac{x}{\alpha}, 0 \right) \leqslant \frac{L}{(1 - |r|)|\alpha|} \varphi(x, 0)$$

for all $x \in X$. So

$$\|\lambda \delta(x) - \delta(\lambda x)\| = \lim_{n \to \infty} |\alpha|^n \left\| \lambda f \left(\frac{x}{\alpha^n} \right) - f \left(\frac{\lambda x}{\alpha^n} \right) \right\|$$

$$\leqslant \lim_{n \to \infty} |\alpha|^n \left\{ \left\| \lambda f \left(\frac{x}{\alpha^n} \right) - \lambda \alpha f \left(\frac{x}{\alpha \alpha^n} \right) \right\| + \left\| f \left(\frac{\lambda x}{\alpha^n} \right) - \lambda \alpha f \left(\frac{x}{\alpha \alpha^n} \right) \right\| \right\}$$

$$\leqslant \lim_{n \to \infty} \frac{2L|\alpha|^n}{|r|(1 - |r|)} \varphi \left(\frac{x}{\alpha^n}, 0 \right) = 0.$$

Thus

$$\lambda \delta(x) = \delta(\lambda x)$$

for all $\lambda \in S$ and all $x \in X$. By Lemma 3.1, the mapping $\delta : X \to X$ is $\mathbb{C}$-linear.

It follows from (4.2) that

$$\|\delta(xy) - \delta(x)y - x\delta(y)\|$$

$$= \lim_{n \to \infty} |\alpha|^{2n} \left\| f \left(\frac{x}{\alpha^n} \frac{y}{\alpha^n} \right) - f \left(\frac{x}{\alpha^n} \right) \frac{y}{\alpha^n} - \frac{x}{\alpha^n} f \left(\frac{y}{\alpha^n} \right) \right\|$$

$$\leqslant \lim_{n \to \infty} |\alpha|^{2n} \varphi \left(\frac{x}{\alpha^n}, \frac{y}{\alpha^n} \right) \leqslant \lim_{n \to \infty} |\alpha|^n \varphi \left(\frac{x}{\alpha^n}, \frac{y}{\alpha^n} \right) = 0$$

for all $x, y \in X$. So

$$\delta(xy) = \delta(x)y - x\delta(y)$$

for all $x, y \in X$. Thus $\delta : X \to X$ is a derivation satisfying (4.3), as desired. $\quad\square$

THEOREM 4.2. *Let* α, β *be fixed nonzero real numbers with* $|\alpha| > 1$. *Let* $f :$
$X \to X$ *be a mapping for which there exists a function* $\varphi : X^2 \to [0, \infty)$ *satisfying* (4.1)
and (4.2). *If there exists an* $L < 1$ *such that* $\varphi(x, y) \leqslant L|\alpha|\varphi \left(\frac{x}{\alpha}, \frac{y}{\alpha} \right)$ *for all* $x, y \in X$,
then there exists a unique derivation $\delta : X \to X$ *such that*

$$\|f(x) - \delta(x)\| \leqslant \frac{L}{(1 - |r|)|\alpha|(1 - L)} \varphi(x, 0) \tag{4.5}$$

for all $x \in X$.

Proof. The proof is similar to the proofs of Theorems 4.1. □

5. Hyers-Ulam stability of homomorphisms in complex Banach Lie algebras

A *k-Lie algebra or Lie algebra over k* (see [3]) consists of a vector space *a* over a field *k*, together with a *k*-bilinear map $[x,y] : a \times a \to a$ called the *Lie bracket*, such that for $x, y \in a$,

$$[x,y] := xy - yx.$$

Here *k*-bilinear means that for $x_1, x_2, x, y_1, y_2, y \in a$ and $r_1, r_2, r, s_1, s_2, s \in k$,

$$[r_1 x_1 + r_2 x_2, y] = r_1 [x_1, y] + r_2 [x_2, y],$$

$$[x, s_1 y_1 + s_2 y_2] = s_1 [x, y_1] + s_2 [x, y_2].$$

A complex Banach algebra $\mathscr{C}$, endowed with the Lie bracket on $\mathscr{C}$, is called a complex Banach Lie algebra.

DEFINITION 5.1. Let X be a complex Lie algebra and Y be complex Banach Lie algebra. A $\mathbb{C}$-linear mapping $H : X \to Y$ is called a *Lie homomorphism* if $H([x,y]) = [H(x), H(y)]$ for all $x, y \in X$.

Throughout this section, assume that X is a complex Lie algebra and that Y is a complex Banach Lie algebra with norm $\| \cdot \|$.

We prove the Hyers-Ulam stability of homomorphism in complex Banach Lie algebras for the functional inequality (4.1).

THEOREM 5.2. *Let* α, β *be fixed nonzero real numbers with* $|\alpha| < 1$. *Let* $f : X \to Y$ *be a mapping for which there exists a function* $\varphi : X^2 \to [0, \infty)$ *satisfying* (4.1)*such that*

$$\|f([x,y]) - [f(x), f(y)]\| \leqslant \varphi(x,y) \tag{5.1}$$

for all $x, y \in X$. *If there exists an* $L < 1$ *such that* $\varphi(x,y) \leqslant \frac{L}{|\alpha|} \varphi(\alpha x, \alpha y)$ *for all* $x, y \in X$, *then there exists a unique Lie homomorphism* $H : X \to Y$ *satisfying* (4.3).

Proof. It follows from $\varphi(x,y) \leqslant \frac{L}{|\alpha|} \varphi(\alpha x, \alpha y)$ that

$$\lim_{j \to \infty} |\alpha|^j \varphi(\alpha^{-j} x, \alpha^{-j} y) = 0$$

for all $x, y \in X$.

We consider the set A and the generalized metric d in the proof of Theorem 3.2. Now we consider the linear mapping $J : A \to A$ such that

$$Jg(x) := \alpha g \left(\frac{1}{\alpha} x \right)$$

for all $x \in X$.

By [10, Theorem 3.1]

$$d(Jg, Jh) \leqslant Ld(g, h)$$

for all $g, h \in A$.

Letting $\mu = 1$ and $y = 0$ in (4.1), we get

$$\left\| f(x) - \alpha f \left(\frac{x}{\alpha} \right) \right\| \leqslant \frac{1}{1 - |r|} \varphi \left(\frac{x}{\alpha}, 0 \right) \leqslant \frac{L}{(1 - |r|)|\alpha|} \varphi(x, 0)$$

for all $x \in X$.

Hence $d(f, Jf) \leqslant \frac{L}{(1 - |r|)|\alpha|}$.

By Theorem 1.1, there exists a mapping $H : X \to Y$ such that

$$\lim_{n \to \infty} \alpha^n f \left(\frac{x}{\alpha^n} \right) = H(x)$$

for all $x \in X$.

Next, the proof is similar to the proofs of Theorem 4.1. □

THEOREM 5.3. *Let* α, β *be fixed nonzero real numbers with* $|\alpha| > 1$. *Let* $f : X \to Y$ *be a mapping for which there exists a function* $\varphi : X^2 \to [0, \infty)$ *satisfying* (4.1) *and* (5.1). *If there exists an* $L < 1$ *such that* $\varphi(x, y) \leqslant L|\alpha| \varphi \left(\frac{x}{\alpha}, \frac{y}{\alpha} \right)$ *for all* $x, y \in X$, *then there exists a unique Lie homomorphism* $H : X \to Y$ *satisfying* (4.3).

Proof. The proof is similar to the proofs of Theorems 3.3 and 4.1. □

6. Hyers-Ulam stability of Lie derivations on complex Banach Lie algebras

DEFINITION 6.1. Let X be a complex Banach Lie algebra. A $\mathbb{C}$-linear mapping $\delta : X \to X$ is called *a Lie derivation* if $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in X$.

Throughout this section, assume that X is a complex Banach Lie algebra with norm $\| \cdot \|$.

We prove that the Hyers-Ulam stability of Lie derivations on complex Banach Lie algebras for the functional inequality (4.1).

THEOREM 6.2. *Let* α, β *be fixed nonzero real numbers with* $|\alpha| < 1$. *Let* $f : X \to X$ *be a mapping for which there exists a function* $\varphi : X^2 \to [0, \infty)$ *satisfying* (4.1) *such that*

$$\| f([x, y]) - [f(x), y] - [x, f(y)] \| \leqslant \varphi(x, y) \tag{6.1}$$

for all $x, y \in X$. *If there exists an* $L < 1$ *such that* $\varphi(x, y) \leqslant \frac{L}{|\alpha|} \varphi(\alpha x, \alpha y)$ *for all* $x, y \in X$, *then there exists a unique Lie derivation* $\delta : X \to X$ *satisfying* (4.3).

Proof. The proof is similar to the proof of Theorem 4.1. $\square$

THEOREM 6.3. *Let* α, β *be fixed nonzero real numbers with* $|\alpha| > 1$. *Let* $f :$ $X \rightarrow X$ *be a mapping for which there exists a function* $\varphi : X^2 \rightarrow [0, \infty)$ *satisfying* (4.1) *and* (6.1). *If there exists an* $L < 1$ *such that* $\varphi(x,y) \leqslant L|\alpha|\varphi\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right)$ *for all* $x, y \in X$, *then there exists a unique Lie derivation* $\delta : X \rightarrow X$ *satisfying* (4.5).

Proof. The proof is similar to the proofs of Theorems 6.2. $\square$

7. Hyperstability of homomorphisms in complex Banach algebras

For $\alpha = \beta = 1$ the functional inequality in Definition (1.1) is equivalent to the following inequality

$$\|f(x-y) - f(x) - f(-y)\| \leqslant \|r(f(x+y) - f(x) - f(y))\|.$$

Throughout this section, assume that X is a complex normed algebra with norm $\|\cdot\|$ and that Y is a complex Banach algebra with norm $\|\cdot\|$.

Bahyrycz and Piszczek [2] proved the hyperstability of the Jensen equation, for more information and further references concerning the issue of hyperstability we refer to [7, 5].

THEOREM 7.1. ([7, Theorems 2-4]) *Let* $\gamma, p, q \in \mathbb{R}, p + q \neq 0, 1, \gamma > 0$ *and let* $f : X \rightarrow Y$ *satisfy the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \gamma\|x\|^p\|y\|^q, \qquad x, y \in X \setminus \{0\}, x + y \neq 0.$$

Then

$$f(x+y) = f(x) + f(y), \qquad x, y \in X \setminus \{0\}, x + y \neq 0.$$

Now, we prove the hyperstability of homomorphisms in complex Banach algebras.

THEOREM 7.2. *Assume that there exist* $c, p, q \in \mathbb{R}, p > 0, p + q \neq 0, 1, c > 0$ *such that*

$$\begin{aligned} &\|f(\lambda(x-y)) - \lambda f(x) - \lambda f(-y)\| \\ &\leqslant \|r(f(\lambda(x+y)) - \lambda f(x) - \lambda f(y)\| + c\|x\|^p\|y\|^q, \end{aligned} \tag{7.1}$$

$$\|f(xy) - f(x)f(y)\| \leqslant c\|x\|^p\|y\|^q \tag{7.2}$$

for all $\lambda \in S$ *and all* $x, y \in X$ *with* $x \neq y$. *Then the mapping* $f : X \rightarrow Y$ *is a homomorphism.*

Proof. Taking $\lambda = 1$ in (7.1), we get

$$\|f(x+y) - f(x) - f(y)\| \leqslant c \frac{1+|r|}{1-|r|^2|} \|x\|^p \|y\|^q \tag{7.3}$$

for all $x, y \in X$ with $x \neq 0$. Letting $y = x$ in (7.3), we get

$$f(2x) = 2f(x), \qquad x \in X \tag{7.4}$$

and so $f(0) = 0$. By Theorem 7.1 and (7.4), $f(u+v) = f(u) + f(v)$ for all $u, v \in X \setminus \{0\}$ with $u+v \neq 0$. By (7.4),

$$f(u) = f(2u - u) = f(2u) + f(-u) = 2f(u) + f(-u)$$

and so $f(-u) = -f(u)$ all $u \in X$. Hence $f : X \to Y$ is Cauchy additive.

Letting $y = 0$ in (7.1), we get $\|\lambda f(x) - f(\lambda x)\| = 0$ for all $\lambda \in S$ and all $x \in X$. By Lemma 3.1, the mapping $f : X \to Y$ is $\mathbb{C}$-linear.

The rest of the proof is similar to the proof of Theorem 3.2. □

Similarly, Piszczek [32] proved the following result.

THEOREM 7.3. ([32, Theorem 2]) *Let* $\gamma, p \in \mathbb{R}, p < 0, \gamma > 0$ *and let* $f : X \to Y$ *satisfy the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \gamma(\|x\|^p + \|y\|^p), \qquad x, y \in X \setminus \{0\}.$$

Then

$$f(x+y) = f(x) + f(y), \qquad x, y \in X \setminus \{0\}. \tag{7.5}$$

Finally, we prove the hyperstability of homomorphisms in complex Banach algebras.

THEOREM 7.4. *Assume that there exist* $c, p \in \mathbb{R}, p < 0, c > 0$ *such that*

$$\begin{aligned}
&\|f(\lambda(x-y)) - \lambda f(x) - \lambda f(-y)\| \\
&\leqslant \|r(f(\lambda(x+y)) - \lambda f(x) - \lambda f(y)\| + c(\|x\|^p + \|y\|^q),
\end{aligned} \tag{7.6}$$

$$\|f(xy) - f(x)f(y)\| \leqslant c(\|x\|^p + \|y\|^p) \tag{7.7}$$

for all $\lambda \in S$ *and all* $x, y \in X$. *Then the mapping* $f : X \to Y$ *is a homomorphism.*

Proof. Analogously as in the proof of Theorem 7.2, we deduce that (7.5) holds. Without loss of generality, we can assume that $f(0) = 0$.

It is enough to show that (7.4) holds.

Clearly,

$$0 = f(0) = f\left(\frac{x-x}{2}\right) = \frac{f(x) + f(-x)}{2}$$

and so $f(-x) = -f(x)$ for all $x \in X$. Hence

$$f\left(\frac{x}{2}\right) = f\left(\frac{x+z-z}{2}\right) = \frac{f(x+z) - f(z)}{2} = \frac{1}{2}\left(\frac{1}{2}\left(f(2x) + f(2z)\right) - f(z)\right)$$

and so

$$2f\left(\frac{x}{2}\right) - \frac{1}{2}f(2x) = \frac{1}{2}f(2z) - f(z)$$

for all $x, z \in X \setminus \{0\}$. This means that $f\left(\frac{y}{2}\right) - \frac{1}{2}f(y) = d$ for all $y \in X \setminus \{0\}$ with some fixed number d, which implies that

$$3d = 2\left(f\left(\frac{x}{2}\right) - \frac{1}{2}f(x)\right) + f(x) - \frac{1}{2}f(2x)$$
$$= 2f\left(\frac{x}{2}\right) - \frac{1}{2}f(2x) = \frac{1}{2}f(z) - f(z) = -d$$

and finally $d = 0$. Thus we get (7.4).

The rest of the proof is similar to the proofs of Theorems 3.2 and 7.2.   □

REMARK 7.5. By the same method as in Theorems 7.2 and 7.4, we can prove the hyperstability of homomorphisms in complex Banach Lie algebras and derivations on complex Banach algebras and complex Banach Lie algebras.

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