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ON AN EQUATION CHARACTERIZING MULTI–JENSEN–QUARTIC MAPPINGS AND ITS STABILITY

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Abstract. In this paper, we introduce a new form of the multi-quartic mappings and then unify the system of functional equations defining a multi-Jensen-quartic mapping to a single equation. Applying a fixed point theorem, we study the generalized Hyers-Ulam stability of multi-Jensen-quartic mappings. We present a few corollaries corresponding to some known stability outcomes on the multi-quartic and the multi-Jensen-quartic functional equations.

1. Introduction

The first stability problem concerning of group homomorphisms was introduced by Ulam [35] in 1940. The famous Ulam stability problem was partially solved by Hyers [22] for linear functional equation of Banach spaces. Hyers' theorem was generalized in 1950 by Aoki [1] for additive mappings and in 1978 by Th. M. Rassias [32] for linear mappings by considering an unbounded Cauchy difference. Subsequently, in 1982, J. M. Rassias [29] following the spirit of the approach of [28] and by replacing the sum of two p-norms with the product of two p-norms obtained a result similar to that of [32] for the stability of the linear mappings. A generalization of the Rassias theorem was obtained by Găvruţa [21] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. The terminology Hyers-Ulam-Rassias stability originates from these historical backgrounds and this terminology is also applied to the cases of other functional equations.

Let V be a commutative group, W be a linear space, and $n \geqslant 2$ be an integer. Recall from [16] that a mapping $f : V^n \longrightarrow W$ is called *multi-additive* if it is additive (satisfies Cauchy's functional equation $A(x+y) = A(x)+A(y)$) in each variable. Furthermore, f is said to be *multi-quadratic* if it is quadratic (satisfies the quadratic functional equation $Q(x+y)+Q(x-y) = 2Q(x)+2Q(y)$) in each variable [15]. In [38], Zhao et al. showed that such mappings can be unified as an equation. Various versions of multi-quadratic mappings which are recently studied can be found in [8] and [33]. In [16] and [15], Ciepliński studied the generalized Hyers-Ulam stability of multi-additive and multi-quadratic mappings in Banach spaces, respectively (see also

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[38]). The mentioned mapping f is also called a *multi-cubic* if it is cubic (satisfies the equation $C(2x+y)+C(2x-y)=2C(x+y)+2C(x-y)+12C(x)$) in each variable [24]). This notion of mappings was introduced by Bodaghi and Shojaee in [11] for the first time. They also studied Hyers-Ulam stability and hyperstability of such mappings in that paper. For other forms of multi-cubic mappings and functional equations which are recently studied, we refer to [19] and [26]. For other forms of cubic functional equations and their stabilities refer to [5], [9], [23] and [31].

The quartic functional equation

$$\mathfrak{Q}(x+2y)+\mathfrak{Q}(x-2y)=4\mathfrak{Q}(x+y)+4\mathfrak{Q}(x-y)-6\mathfrak{Q}(x)+24\mathfrak{Q}(y). \tag{1.1}$$

was introduced for the first time by Rassias [30]. The functional equation (1.1) was generalized by Bodaghi and Kang in [6] and [25], respectively. Motivated by equation (1.1), Bodaghi et al. [7] defined the multi-quartic mappings and provided a characterization of such mappings. In other words, they showed that every multi-quartic mapping can be shown a single functional equation and vice versa. Moreover, they established the generalized Hyers-Ulam stability for the multi-quartic functional equations [7].

Prager and Schwaiger [27] introduced the notion of multi-Jensen mappings $f: V^n \longrightarrow W$ (V and W being vector spaces over the rational numbers) with the connection with generalized polynomials and obtained their general form. The aim of this note was to study the stability of the multi-Jensen equation. After that, the stability of multi-Jensen mappings in various normed spaces have been investigated by a number of mathematicians (see [17], [18], [28], [36] and [37]).

In this paper, we firstly define new multi-quartic mappings and characterize them as an equation. Then, we introduce the multi-Jensen-quartic mappings which are Jensen in each of some k variables and is quartic in each of the other variables and then present a characterization of such mappings. In other words, we reduce the system of n equations defining the multi-Jensen-quartic mappings to obtain a single functional equation. We also prove the generalized Hyers-Ulam stability for multi-Jensen-quartic functional equations by using the fixed point method which was introduced and used for the first time by Brzdęk in [12] (see also [13]). For more applications of this approach for the stability of multi-Cauchy-Jensen, multi-additive-quadratic, multi-Jensen-quadratic and multi-mixed additive-quadratic mappings in Banach spaces, we refer to [2, 3, 4, 10, 20, 34].

2. Characterization of multi-quartic mappings

It is shown in [30] that if the mapping $\mathfrak{Q}: V \longrightarrow W$ satisfies (1.1), then it is even and $\mathfrak{Q}(x)=2^{-4}\mathfrak{Q}(2x)$ for all $x \in V$. This result lead us to the following elementary consequence.

PROPOSITION 2.1. *The mapping $\mathfrak{Q}: V \longrightarrow W$ satisfies (1.1) if and only if it is fulfilling in*

$$16\left[\mathfrak{Q}\left(\frac{x+2y}{2}\right)+\mathfrak{Q}\left(\frac{x-2y}{2}\right)\right]=4\left[\mathfrak{Q}(x+y)+\mathfrak{Q}(x-y)\right]-6\mathfrak{Q}(x)+24\mathfrak{Q}(y). \tag{2.1}$$

for all $x,y\in V$.

Throughout this paper, $\mathbb{N}$ and $\mathbb{Q}$ stand for the set of all positive integers and rational numbers, respectively, and also $\mathbb{N}_0:=\mathbb{N}\cup\{0\},\mathbb{R}_+:=[0,\infty),n\in\mathbb{N}$. For any $l\in\mathbb{N}_0,m\in\mathbb{N}$, $t=(t_1,\dots,t_m)\in\{-2,2\}^m$ and $x=(x_1,\dots,x_m)\in V^m$, we write $lx:=(lx_1,\dots,lx_m)$ and $tx:=(t_1x_1,\dots,t_mx_m)$, where lx stands, as usual, for the scalar product of l on x in the linear space V.

Let $n\in\mathbb{N}$ with $n\geqslant 2$ and $x_i^n=(x_{i1},x_{i2},\dots,x_{in})\in V^n$, where $i\in\{1,2\}$. We shall denote x_i^n by x_i if there is no risk of ambiguity. For $x_1,x_2\in V^n$ and $p_i\in\mathbb{N}_0$ with $0\leqslant p_i\leqslant n$, put $\mathscr{N}_s^n=\left\{(N_{s+1},N_{s+2},\dots,N_n)|\,N_j\in\{x_{1j}\pm x_{2j},x_{1j},x_{2j}\}\right\}$, where $j\in\{1,\dots,n\}$ and $i\in\{1,2\}$. We denote $\mathscr{N}_0^n$ by $\mathscr{N}^n$. Consider the subset $\mathscr{N}_{(p_1,p_2)}^n$ of $\mathscr{N}^n$ as follows:

$$\mathscr{N}_{(p_1,p_2)}^n:=\left\{\mathfrak{N}_n=(N_1,N_2,\dots,N_n)\in\mathscr{N}^n|\,\mathrm{Card}\{N_j:N_j=x_{ij}\}=p_i\ \ (i\in\{1,2\})\right\}.$$

Let V and W be vector spaces over $\mathbb{Q}$. We say the mapping $f:V^n\longrightarrow W$ is n*-multi-quartic* or *multi-quartic* if f is quartic in each variable (see equation (2.1)). In this section, for such mappings, we use the following notations:

$$f\left(\mathscr{N}_{(p_1,p_2)}^n\right):=\sum_{\mathfrak{N}_n\in\mathscr{N}_{(p_1,p_2)}^n}f(\mathfrak{N}_n),\tag{2.2}$$

$$f\left(\mathscr{N}_{(p_1,p_2)}^n,z\right):=\sum_{\mathfrak{N}_n\in\mathscr{N}_{(p_1,p_2)}^n}f(\mathfrak{N}_n,z)\qquad(z\in V).$$

For each $x_1,x_2\in V^n$, we consider the equation

$$16^n\sum_{t\in\{-2,2\}^n}f\left(\frac{x_1+tx_2}{2}\right)=\sum_{p_2=0}^{n}\sum_{p_1=0}^{n-p_2}4^{n-p_1-p_2}(-6)^{p_1}24^{p_2}f\left(\mathscr{N}_{(p_1,p_2)}^n\right).\tag{2.3}$$

By a mathematical computation, one can check that the mapping $f(z_1,\dots,z_n)=a\prod_{j=1}^n z_j^4$ satisfies (2.3) and so this equation is said to be *multi-quartic functional equation*. In this section, we show that the mapping $f:V^n\longrightarrow W$ is multi-quartic if and only if it satisfying the multi-quartic functional equation (2.3).

In the sequel, $\binom{n}{k}$ is the binomial coefficient defined for all $n,k\in\mathbb{N}$ with $n\geqslant k$ by $n!/(k!(n-k)!)$.

We say the mapping $f:V^n\longrightarrow W$ satisfies (has) *the m-power condition* in the jth variable if

$$f(z_1,\dots,z_{j-1},2z_j,z_{j+1},\dots,z_n)=2^m f(z_1,\dots,z_{j-1},z_j,z_{j+1},\dots,z_n),\quad(z_1,\dots,z_n)\in V^n.$$

REMARK 2.2. It is easily verified that if f is a multi-quartic mapping, then it satisfies 4-power condition in all variables. But the converse is not true. Here, by means of an example we show that 4-power condition in all variables for a mapping f does not imply that it is multi-quartic. Let $(\mathscr{A},\|\cdot\|)$ be a Banach algebra. Fix the vector

a_0 in $\mathscr{A}$ (not necessarily unit). Define the mapping $h:\mathscr{A}^n\longrightarrow\mathscr{A}$ by $h(a_1,\ldots,a_n)=\prod_{j=1}^n\|a_j\|^4a_0$ for $(a_1,\ldots,a_n)\in\mathscr{A}^n$. It is easily verified that the mapping h satisfies 4-power condition in all variables but h is not multi-quartic even for $n=1$, that is h does not satisfy in equation (2.1).

Let $0\leqslant q\leqslant n-1$. Put $\mathscr{K}_q=\{{}_qx:=(0,\ldots,0,x_{j_1},0,\ldots,0,x_{j_q},0,\ldots,0)\in V^n\}$, where $1\leqslant j_1<\ldots<j_q\leqslant n$. In other words, $\mathscr{K}_q$ is the set of all vectors in V^n that exactly their q components are non-zero.

We wish to show that the mapping $f:V^n\longrightarrow W$ satisfies equation (2.3) if and only if it is multi-quartic. In order to do this, we need the next lemma.

LEMMA 2.3. *If the mapping $f:V^n\longrightarrow W$ satisfies equation (2.3) and has 4-power condition in all variables, then $f(x)=0$ for any $x\in V^n$ with at least one component which is equal to zero.*

Proof. We argue by induction on q that for each ${}_qx\in\mathscr{K}_q$, $f({}_qx)=0$ for $0\leqslant q\leqslant n-1$. For $q=0$, by putting $x_1=x_2=(0,\ldots,0)$ in (2.3), we have

$$\begin{aligned}&16^n\times 2^nf(0,\ldots,0)\\&=\sum_{p_2=0}^{n}\sum_{p_1=0}^{n-p_2}4^{n-p_1-p_2}(-6)^{p_1}24^{p_2}\binom{n}{n-p_1-p_2}\binom{p_1+p_2}{p_1}2^{n-p_1-p_2}f(0,\ldots,0).\end{aligned}\tag{2.4}$$

One can easy to check that

$$\binom{n-q}{n-q-p_1-p_2}\binom{p_1+p_2}{p_1}=\binom{n-q}{p_2}\binom{n-q-p_2}{p_1}\tag{2.5}$$

for $0\leqslant q\leqslant n-1$. Using (2.5) for $q=0$, we compute the right side of (2.4) as follows:

$$\begin{aligned}&\sum_{p_2=0}^{n}\sum_{p_1=0}^{n-p_2}4^{n-p_1-p_2}(-6)^{p_1}24^{p_2}\binom{n}{n-p_1-p_2}\binom{p_1+p_2}{p_1}2^{n-p_1-p_2}f(0,\ldots,0)\\&=\left[\sum_{p_2=0}^{n}\binom{n}{p_2}24^{p_2}\sum_{p_1=0}^{n-p_2}\binom{n-p_2}{p_1}8^{n-p_1-p_2}(-6)^{p_1}\right]f(0,\ldots,0)\\&=\left[\sum_{p_2=0}^{n}\binom{n}{p_2}24^{p_2}(8-6)^{n-p_2}\right]f(0,\ldots,0)\\&=(24+2)^nf(0,\ldots,0)=26^nf(0,\ldots,0).\end{aligned}\tag{2.6}$$

It follows from relations (2.4) an (2.6) that $f(0,\ldots,0)=0$. Assume that for each ${}_{q-1}x\in\mathscr{K}_{q-1}$, $f({}_{q-1}x)=0$. We show that if ${}_qx\in\mathscr{K}_q$, then $f({}_qx)=0$. By some suitable replacements in (3.1), we obtain

$$
\begin{aligned}
&16^n \times 2^n f({}_q x)\\
&= \sum_{p_2=0}^{n-q} \sum_{p_1=0}^{n-q-p_2} 4^{n-p_1-p_2}(-6)^{p_1} 24^{p_2} \binom{n-q}{n-q-p_1-p_2} \binom{p_1+p_2}{p_1} 2^{n-p_1-p_2} f(2_q x)\\
&= 2^{4q} \sum_{p_2=0}^{n-q} \sum_{p_1=0}^{n-q-p_2} 4^{n-p_1-p_2}(-6)^{p_1} 24^{p_2} \binom{n-q}{n-q-p_1-p_2} \binom{p_1+p_2}{p_1} 2^{n-p_1-p_2} f({}_q x)\\
&= 2^{7q} \left[\sum_{p_2=0}^{n-q} \binom{n-q}{p_2} 24^{p_2} \sum_{p_1=0}^{n-q-p_2} \binom{n-q-p_2}{p_1} 8^{n-q-p_1-p_2}(-6)^{p_1} \right] f({}_q x)\\
&= 2^{7q} \left[\sum_{p_2=0}^{n-q} \binom{n-q}{p_2} 24^{p_2}(8-6)^{n-q-p_2} \right] f({}_q x)\\
&= 2^{7q}(24+2)^{n-q} f({}_q x) = 2^{7q} \times 26^{n-q} f({}_q x)
\end{aligned}
\tag{2.7}
$$

Hence, $f({}_q x) = 0$. This shows that $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero. □

We now prove the main result of this section.

THEOREM 2.4. *If the mapping $f : V^n \longrightarrow W$ is multi-quartic, then f satisfies equation (2.3). The converse is true provided that f has 4-power condition in each variable.*

Proof. Assume that f is a multi-quartic. We prove that f satisfies equation (2.3) by induction on n. For $n = 1$, it is trivial that f satisfies equation (2.1). If (2.3) is valid for some positive integer $n > 1$, then

$$
\begin{aligned}
&16^{n+1} \sum_{t\in\{-2,2\}^{n+1}} f\left(\frac{x_1^{n+1}+tx_2^{n+1}}{2}\right)\\
&= 4\times 16^n \sum_{t\in\{-2,2\}^n} f\left(\frac{x_1^n+tx_2^n}{2}, x_{1,n+1}+x_{2,n+1}\right)\\
&\quad + 4\times 16^n \sum_{t\in\{-2,2\}^n} f\left(\frac{x_1^n+tx_2^n}{2}, x_{1,n+1}-x_{2,n+1}\right)\\
&\quad - 6\times 16^n \sum_{t\in\{-2,2\}^n} f\left(\frac{x_1^n+tx_2^n}{2}, x_{1,n+1}\right) + 24\times 16^n \sum_{t\in\{-2,2\}^n} f\left(\frac{x_1^n+tx_2^n}{2}, x_{2,n+1}\right)\\
&= 4 \sum_{p_2=0}^{n} \sum_{p_1=0}^{n-p_2} \sum_{s\in\{-1,1\}} 4^{n-p_1-p_2}(-6)^{p_1} 24^{p_2} f\left(\mathscr{N}^n_{(p_1,p_2)}, x_{1,n+1}+sx_{2,n+1}\right)\\
&\quad - 6 \sum_{p_2=0}^{n} \sum_{p_1=0}^{n-p_2} 4^{n-p_1-p_2}(-6)^{p_1} 24^{p_2} f\left(\mathscr{N}^n_{(p_1,p_2)}, x_{1,n+1}\right)
\end{aligned}
$$

$$+24\sum_{p_2=0}^{n}\sum_{p_1=0}^{n-p_2}4^{n-p_1-p_2}(-6)^{p_1}24^{p_2}f\left(\mathcal{N}^{n}_{(p_1,p_2)},x_{2,n+1}\right)$$
$$=\sum_{p_2=0}^{n+1}\sum_{p_1=0}^{n+1-p_2}4^{n+1-p_1-p_2}(-6)^{p_1}24^{p_2}f\left(\mathcal{N}^{n+1}_{(p_1,p_2)}\right).$$

This means that (2.3) holds for $n+1$.

Conversely, suppose that f satisfies equation (2.3). Fix $j\in\{1,\dots,n\}$. Set

$$\begin{aligned}f^*(x_{1j},x_{2j}):=&f\left(x_{11},\dots,x_{1j-1},x_{1j}+x_{2j},x_{1j+1},\dots,x_{1n}\right)\\&+f\left(x_{11},\dots,x_{1j-1},x_{1j}-x_{2j},x_{1j+1},\dots,x_{1n}\right),\end{aligned}$$

and

$$f^*(x_{2j}):=f\left(x_{11},\dots,x_{1j-1},x_{2j},x_{1j+1},\dots,x_{1n}\right).$$

Putting $x_{2k}=0$ for all $k\in\{1,\dots,n\}\setminus\{j\}$ in (2.3) and using Lemma 2.3, we get

$$\begin{aligned}&16^n\times 2^{n-1}f\left(x_{11},\dots,x_{1j-1},\frac{x_{1j}+2x_{2j}}{2},x_{1j+1},\dots,x_{1n}\right)\\
&\quad+16^n\times 2^{n-1}f\left(x_{11},\dots,x_{1j-1},\frac{x_{1j}-2x_{2j}}{2},x_{1j+1},\dots,x_{1n}\right)\\
&=16^{n-1}\sum_{p_1=0}^{n-1}\binom{n-1}{p_1}4^{n-p_1}(-6)^{p_1}2^{n-p_1-1}f^*(x_{1j},x_{2j})\\
&\quad+16^{n-1}\sum_{p_1=1}^{n}\binom{n-1}{p_1-1}4^{n-p_1}(-6)^{p_1}2^{n-p_1}f(x_{11},\dots,x_{1n})\\
&\quad+16^{n-1}\sum_{p_1=1}^{n}\binom{n-1}{p_1-1}4^{n-p_1}(-6)^{p_1-1}2^{n-p_1}f^*(x_{2j})\\
&=4\times 32^{n-1}\sum_{p_1=0}^{n-1}\binom{n-1}{p_1}4^{n-1-p_1}(-3)^{p_1}f^*(x_{1j},x_{2j})\\
&\quad-6\times 32^{n-1}\sum_{p_1=0}^{n-1}\binom{n-1}{p_1}4^{n-1-p_1}(-3)^{p_1}f(x_{11},\dots,x_{1n})\\
&\quad+24\times 32^{n-1}\sum_{p_1=0}^{n-1}\binom{n-1}{p_1}4^{n-1-p_1}(-3)^{p_1}f^*(x_{2j})\\
&=4\times 32^{n-1}f^*(x_{1j},x_{2j})-6\times 32^{n-1}f(x_{11},\dots,x_{1n})+24\times 32^{n-1}f^*(x_{2j}).\end{aligned}$$

Note that we have used the following relation in the above computations.

$$\sum_{p_1=0}^{n-1}\binom{n-1}{p_1}4^{n-1-p_1}(-3)^{p_1}=(4-3)^{n-1}=1.$$

Therefore, the above relation implies that f is quartic in the jth variable. Since j is arbitrary, we obtain the desired result. $\square$

3. Characterization of multi-Jensen-quartic mappings

Let V and W be linear spaces, $n\in\mathbb{N}$ and $k\in\{0,\dots,n\}$. A mapping $f: V^n \longrightarrow W$ is called k-Jensen and $n-k$-quartic (briefly, multi-Jensen-quartic) if f is Jensen in each of some k variables and is quartic in each of the other variables (see equation (2.1)). In this note, we suppose for simplicity that f is Jensen in each of the first k variables, but one can obtain analogous results without this assumption. Let us note that for $k=n$ $(k=0)$, the above definition leads to the so-called multi-Jensen (multi-quartic) mappings; some basic facts on Jensen mappings can be found for instance in [27].

From now on, we assume that V and W are vector spaces over $\mathbb{Q}$. Moreover, we identify $x=(x_1,\dots,x_n)\in V^n$ with $(x^k,x^{n-k})\in V^k\times V^{n-k}$, where $x^k:=(x_1,\dots,x_k)$ and $x^{n-k}:=(x_{k+1},\dots,x_n)$, and we adopt the convention that $(x^n,x^0):=x^n:=(x^0,x^n)$. Put $x_i^k=(x_{i1},\dots,x_{ik})\in V^k$ and $x_i^{n-k}=(x_{i,k+1},\dots,x_{in})\in V^{n-k}$, where $i\in\{1,2\}$. Recall that

$$\mathscr{N}^{n-k}_{(p_1,p_2)}:=\left\{\mathfrak{N}_n=(N_{k+1},N_{k+2},\dots,N_n)\in \mathscr{N}_k^n \,|\, \mathrm{Card}\{N_j: N_j=x_{ij}\}=p_i \ (i\in\{1,2\})\right\}.$$

We also use the following notation:

$$f\left(x_i^k,\mathscr{N}^{n-k}_{(p_1,p_2)}\right):=\sum_{\mathfrak{N}_n\in\mathscr{N}^{n-k}_{(p_1,p_2)}} f(x_i^k,\mathfrak{N}_n) \qquad (i\in\{1,2\}).$$

In this section, we wish to show that the mapping $f: V^n\longrightarrow W$ is multi-Jensen-quartic if and only if it satisfies the equation

$$\begin{aligned}&2^{4n-3k}\sum_{t\in\{-2,2\}^{n-k}} f\left(\frac{x_1^k+x_2^k}{2},\frac{x_1^{n-k}+tx_2^{n-k}}{2}\right)\\&=\sum_{j_1,\dots,j_k\in\{1,2\}}\sum_{p_2=0}^{n-k}\sum_{p_1=0}^{n-k-p_2}4^{n-k-p_1-p_2}(-6)^{p_1}24^{p_2} f\left(x_{j_1 1},\dots,x_{l_k k},\mathscr{N}^{n-k}_{(p_1,p_2)}\right)\end{aligned} \tag{3.1}$$

for all $x_i^k=(x_{i1},\dots,x_{ik})\in V^k$ and $x_i^{n-k}=(x_{i,k+1}\dots,x_{in})\in V^{n-k}$, where $i\in\{1,2\}$.

Here, we reduce the system of n equations defining the multi Jensen-quartic mapping to obtain a single functional equation.

THEOREM 3.1. *Let $n\in\mathbb{N}$ and $k\in\{0,\dots,n\}$. If the mapping $f: V^n\longrightarrow W$ is multi-Jensen-quartic mapping, then it satisfies equation (3.1). The converse holds provided that f has 4-power condition in the last $n-k$ variables.*

Proof. (Necessity) Suppose that f is a multi-Jensen-quartic mapping. Since for $k\in\{0,n\}$ our assertion follows from [28, Lemma 1.1] and Theorem 2.4, we can assume that $k\in\{1,\dots,n-1\}$. For any $x^{n-k}\in V^{n-k}$, define the mapping $g_{x^{n-k}}: V^k\longrightarrow W$ by

$g_{x^{n-k}}(x^k) := f(x^k, x^{n-k})$ for $x^k \in V^k$. By assumption, $g_{x^{n-k}}$ is k-Jensen, and hence Lemma 1.1 from [28] implies that

$$2^k g_{x^{n-k}}\left(\frac{x_1^k + x_2^k}{2}\right) = \sum_{j_1, j_2, \ldots, j_k \in \{1,2\}} g_{x^{n-k}}(x_{j_1 1}, x_{j_2 2}, \ldots, x_{j_k k}), \qquad x_1^k, x_2^k \in V^k.$$

It now follows from the above equality that

$$2^k f\left(\frac{x_1^k + x_2^k}{2}, x^{n-k}\right) = \sum_{j_1, j_2, \ldots, j_k \in \{1,2\}} f(x_{j_1 1}, x_{j_2 2}, \ldots, x_{j_k k}, x^{n-k}) \tag{3.2}$$

for all $x_1^k, x_2^k \in V^k$ and $x^{n-k} \in V^{n-k}$. Similar to the above, for any $x^k \in V^k$, consider the mapping $h_{x^k} : V^{n-k} \longrightarrow W$ defined via $h_{x^k}(x^{n-k}) := f(x^k, x^{n-k})$, $x^{n-k} \in V^{n-k}$ which is $n-k$-quartic. It now Theorem 2.4 implies that

$$\begin{aligned} &16^{n-k} \sum_{t \in \{-2,2\}^{n-k}} h_{x^k}\left(\frac{x_1^{n-k} + t x_2^{n-k}}{2}\right) \\ &= \sum_{p_2=0}^{n-k} \sum_{p_1=0}^{n-k-p_2} 4^{n-k-p_1-p_2} (-6)^{p_1} 24^{p_2} h_{x^k}\left(\mathcal{N}_{(p_1,p_2)}^{n-k}\right) \end{aligned} \tag{3.3}$$

for all $x_1^{n-k}, x_2^{n-k} \in V^{n-k}$. By the definition of h_{x^k}, relation (3.3) is equivalent to

$$\begin{aligned} &16^{n-k} \sum_{t \in \{-2,2\}^{n-k}} f\left(x^k, \frac{x_1^{n-k} + t x_2^{n-k}}{2}\right) \\ &= \sum_{p_2=0}^{n-k} \sum_{p_1=0}^{n-k-p_2} 4^{n-k-p_1-p_2} (-6)^{p_1} 24^{p_2} f\left(x^k, \mathcal{N}_{(p_1,p_2)}^{n-k}\right) \end{aligned} \tag{3.4}$$

for all $x_1^{n-k}, x_2^{n-k} \in V^{n-k}$ and $x^k \in V^k$. Plugging equality (3.2) into (3.4), we get

$$\begin{aligned} &2^{4n-3k} \sum_{t \in \{-2,2\}^{n-k}} f\left(\frac{x_1^k + x_2^k}{2}, \frac{x_1^{n-k} + t x_2^{n-k}}{2}\right) \\ &= \sum_{t \in \{-2,2\}^{n-k}} \sum_{j_1, j_2, \ldots, j_k \in \{1,2\}} f\left(x_{j_1 1}, x_{j_2 2}, \ldots, x_{j_k k}, \frac{x_1^{n-k} + t x_2^{n-k}}{2}\right) \\ &= \sum_{j_1, \ldots, j_k \in \{1,2\}} \sum_{p_2=0}^{n-k} \sum_{p_1=0}^{n-k-p_2} 4^{n-k-p_1-p_2} (-6)^{p_1} 24^{p_2} f\left(x_{j_1 1}, \ldots, x_{j_k k}, \mathcal{N}_{(p_1,p_2)}^{n-k}\right) \end{aligned}$$

for all $x_i^k = (x_{i1}, \ldots, x_{ik}) \in V^k, x_i^{n-k} = (x_{ik+1} \ldots, x_{in}) \in V^{n-k}$ and $i \in \{1,2\}$, which proves that f satisfies equation (3.1).

(Sufficiency) Assume that f satisfies equation (3.1). Putting $x_2^{n-k}=0$ in (3.1) and using the assumption, we obtain

$$\begin{aligned}
&2^{4n-3k}\times 2^{n-k} f\left(\frac{x_1^k+x_2^k}{2}, x_1^{n-k}\right)\\
&=\sum_{j_1,j_2,\dots,j_k\in\{1,2\}}\sum_{p_1=0}^{n-k}\binom{n-k}{p_1}4^{n-k-p_1}\times(-6)^{p_1}\times 2^{n-k-p_1}\\
&\quad\times 16^{n-k} f(x_{j_1 1},x_{j_2 2},\dots,x_{j_k k},2x_1^{n-k})\\
&=\sum_{j_1,j_2,\dots,j_k\in\{1,2\}}(8-6)^{n-k}\times 16^{n-k} f(x_{j_1 1},x_{j_2 2},\dots,x_{j_k k},x_1^{n-k})\\
&=2^{5n-5k}\sum_{j_1,j_2,\dots,j_k\in\{1,2\}} f(x_{j_1 1},x_{j_2 2},\dots,x_{j_k k},x_1^{n-k}).
\end{aligned}$$

Thus,

$$2^k f\left(\frac{x_1^k+x_2^k}{2}, x_1^{n-k}\right)=\sum_{j_1,j_2,\dots,j_k\in\{1,2\}} f(x_{j_1 1},x_{j_2 2},\dots,x_{j_k k},x_1^{n-k}) \tag{3.5}$$

for all $x_1^k, x_2^k\in V^n$ and $x_1^{n-k}\in V^{n-k}$. In view of [28, Lemma 1.1], we see that f is Jensen in each of the k first variables. Furthermore, by putting $x_1^k=x_2^k$ in (2.3), we have

$$\begin{aligned}
&2^{4n-3k}\sum_{t\in\{-2,2\}^{n-k}} f\left(x_1^k,\frac{x_1^{n-k}+tx_2^{n-k}}{2}\right)\\
&=2^k\sum_{p_2=0}^{n-k}\sum_{p_1=0}^{n-k-p_2}4^{n-k-p_1-p_2}(-6)^{p_1}24^{p_2} f\left(x_1^k,\mathcal{N}_{(p_1,p_2)}^{n-k}\right)
\end{aligned}$$

and so

$$\begin{aligned}
&16^{n-k}\sum_{t\in\{-2,2\}^{n-k}} f\left(x_1^k,\frac{x_1^{n-k}+tx_2^{n-k}}{2}\right)\\
&=\sum_{p_2=0}^{n-k}\sum_{p_1=0}^{n-k-p_2}4^{n-k-p_1-p_2}(-6)^{p_1}24^{p_2} f\left(x_1^k,\mathcal{N}_{(p_1,p_2)}^{n-k}\right)
\end{aligned}$$

for all $x_1^k\in V^k$ and $x_1^{n-k}, x_2^{n-k}\in V^{n-k}$. In light of Theorem 2.4, we see that f is a multi-Jensen-quartic mapping. $\square$

4. Stability results of (3.1)

In this section, we prove the generalized Hyers-Ulam stability of equation (3.1) by a fixed point result (Theorem 4.1) in Banach spaces. Throughout, for two sets X and Y, the set of all mappings from X to Y is denoted by Y^X. Here, we introduce the oncoming three hypotheses:

(A1) Y is a Banach space, $\mathscr{S}$ is a nonempty set, $j \in \mathbb{N}$, $g_1, \ldots, g_j : \mathscr{S} \longrightarrow \mathscr{S}$ and $L_1, \ldots, L_j : \mathscr{S} \longrightarrow \mathbb{R}_+$,

(A2) $\mathscr{T} : Y^{\mathscr{S}} \longrightarrow Y^{\mathscr{S}}$ is an operator satisfying the inequality

$$\|\mathscr{T}\lambda(x) - \mathscr{T}\mu(x)\| \leqslant \sum_{i=1}^{j} L_i(x)\,\|\lambda(g_i(x)) - \mu(g_i(x))\|, \quad \lambda, \mu \in Y^{\mathscr{S}}, x \in \mathscr{S},$$

(A3) $\Lambda : \mathbb{R}_+^{\mathscr{S}} \longrightarrow \mathbb{R}_+^{\mathscr{S}}$ is an operator defined through

$$\Lambda\delta(x) := \sum_{i=1}^{j} L_i(x)\delta(g_i(x)), \qquad \delta \in \mathbb{R}_+^{\mathscr{S}}, x \in \mathscr{S}.$$

In the following, we present a result in fixed point theory [13, Theorem 1] which plays a key tool in obtaining our aim in this paper.

THEOREM 4.1. *Let hypotheses* (A1)-(A3) *hold and the function* $\theta : \mathscr{S} \longrightarrow \mathbb{R}_+$ *and the mapping* $\phi : \mathscr{S} \longrightarrow Y$ *fulfill the following two conditions:*

$$\|\mathscr{T}\phi(x) - \phi(x)\| \leqslant \theta(x), \quad \theta^*(x) := \sum_{l=0}^{\infty} \Lambda^l \theta(x) < \infty \qquad (x \in \mathscr{S}).$$

Then, there exists a unique fixed point ψ *of* $\mathscr{T}$ *such that*

$$\|\phi(x) - \psi(x)\| \leqslant \theta^*(x) \qquad (x \in \mathscr{S}).$$

Moreover, $\psi(x) = \lim_{l\to\infty} \mathscr{T}^l \phi(x)$ *for all* $x \in \mathscr{S}$.

Here and subsequently, for the mapping $f : V^n \longrightarrow W$, we consider the difference operator $\mathscr{D}_{(J,q)}f : V^n \times V^n \longrightarrow W$ by

$$\begin{aligned}\mathscr{D}_{(J,q)}f(x_1,x_2) := 2^{4n-3k} &\sum_{t\in\{-2,2\}^{n-k}} f\left(\frac{x_1^k + x_2^k}{2}, \frac{x_1^{n-k} + t x_2^{n-k}}{2}\right) \\ &- \sum_{j_1,\ldots,j_k\in\{1,2\}} \sum_{p_2=0}^{n-k} \sum_{p_1=0}^{n-k-p_2} 4^{n-k-p_1-p_2}(-6)^{p_1}24^{p_2} \\ &\times f\left(x_{j_1 1}, \ldots, x_{j_k k}, \mathscr{N}_{(p_1,p_2)}^{n-k}\right)\end{aligned}$$

for all $x_i^k = (x_{i1}, \ldots, x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1}, \ldots, x_{in}) \in V^{n-k}$.

We have the next stability result for the functional equation (3.1) which is our main result in this section.

THEOREM 4.2. *Let* $\beta \in \{-1,1\}$ *be fixed,* V *be a linear space and* W *be a Banach space. Suppose that* $\psi : V^n \times V^n \longrightarrow \mathbb{R}_+$ *is a mapping satisfying the equality*

$$\lim_{l\to\infty} \left(\frac{1}{2^{(4n-3k)\beta}}\right)^l \psi(2^{\beta l}x_1, 2^{\beta l}x_2) = 0 \tag{4.1}$$

for all $x_1 = (x_1^k, x_1^{n-k}), x_2 = (x_2^k, x_2^{n-k}) \in V^n$ *and*

$$\Psi(x) =: \frac{1}{2^{n-k+(4n-3k)\frac{\beta+1}{2}}} \sum_{l=0}^{\infty} \left(\frac{1}{2^{(4n-3k)\beta}}\right)^l \psi\left(2^{\beta l+\frac{\beta+1}{2}}x, 0\right) < \infty \tag{4.2}$$

for all $x \in V^n$. *Assume also* $f : V^n \longrightarrow W$ *is a mapping satisfying the inequality*

$$\|\mathscr{D}_{(J,q)} f(x_1, x_2)\| \leqslant \psi(x_1, x_2) \tag{4.3}$$

for all $x_1, x_2 \in V^n$ *and* $f(x) = 0$ *for any* $x \in V^n$ *with at least one component which is equal to zero, then there exists a unique solution* $\mathscr{F} : V^n \longrightarrow W$ *of* (3.1) *such that*

$$\|f(x) - \mathscr{F}(x)\| \leqslant \Psi(x) \tag{4.4}$$

for all $x = (x_1^k, x_1^{n-k}) \in V^n$.

Proof. Replacing $x_1 = (x_1^k, x_1^{n-k})$ and $x_2 = (x_2^k, x_2^{n-k})$ by $2x_1 = 2(x_1^k, x_1^{n-k})$ and $(0,0)$ in (4.3) respectively, and using the assumptions, we have

$$\left\| 2^{4n-3k} \times 2^{n-k} f(x) - \sum_{p_1=0}^{n-k} \binom{n-k}{p_1} 4^{n-k-p_1} \times (-6)^{p_1} \times 2^{n-k-p_1} f(2x) \right\| \leqslant \psi(2x, 0), \tag{4.5}$$

where $x = x_1 = (x_1^k, x_1^{n-k}) \in V^n$. Since

$$\sum_{p_1=0}^{n-k} \binom{n-k}{p_1} 8^{n-k-p_1} (-6)^{p_1} = (8-6)^{n-k} = 2^{n-k},$$

relation (4.5) shows that

$$\left\| f(2x) - 2^{4n-3k} f(x) \right\| \leqslant \frac{1}{2^{n-k}} \psi(2x, 0) \tag{4.6}$$

for all $x \in V^n$. Set

$$\theta(x) := \frac{1}{2^{n-k+(4n-3k)\frac{\beta+1}{2}}} \psi\left(2^{\frac{\beta+1}{2}} x, 0\right), \text{ and } \mathscr{T}\theta(x) := \frac{1}{2^{(4n-3k)\beta}} \theta(2^\beta x) \qquad (\theta \in W^{V^n}).$$

Then, relation (4.6) can be modified as

$$\|f(x) - \mathscr{T} f(x)\| \leqslant \theta(x) \qquad (x \in V^n). \tag{4.7}$$

Define $\Lambda\eta(x) := \frac{1}{2^{(4n-3k)\beta}} \eta(2^\beta x)$ for all $\eta \in \mathbb{R}_+^{V^n}, x \in V^n$. We now see that Λ has the form described in (A3) with $\mathscr{S} = V^n$, $g_1(x) = 2^\beta x$ and $L_1(x) = \frac{1}{2^{(4n-3k)\beta}}$ for all $x \in V^n$. Furthermore, for each $\lambda, \mu \in W^{V^n}$ and $x \in V^n$, we get

$$\|\mathscr{T}\lambda(x) - \mathscr{T}\mu(x)\| = \left\| \frac{1}{2^{(4n-3k)\beta}} \left[\lambda(2^\beta x) - \mu(2^\beta x)\right] \right\| \leqslant L_1(x) \|\lambda(g_1(x)) - \mu(g_1(x))\|.$$

The above relation shows that the hypothesis (A2) holds. By induction on l, one can check that for any $l \in \mathbb{N}_0$ and $x \in V^n$, we have

$$\Lambda^l \theta(x) := \left(\frac{1}{2^{(4n-3k)\beta}}\right)^l \theta(2^{\beta l}x) = \frac{1}{2^{n-k+(4n-3k)\frac{\beta+1}{2}}}\left(\frac{1}{2^{(4n-3k)\beta}}\right)^l \psi\left(2^{\beta l+\frac{\beta+1}{2}}x, 0\right) \tag{4.8}$$

for all $x \in V^n$. The relations (4.2) and (4.8) necessitate that all assumptions of Theorem 4.1 are satisfied. Hence, there exists a unique mapping $\mathscr{F} : V^n \longrightarrow W$ such that

$$\mathscr{F}(x) = \lim_{l\to\infty}(\mathscr{T}^l f)(x) = \frac{1}{2^{(4n-3k)\beta}}\mathscr{F}(2^\beta x) \qquad (x \in V^n),$$

and (4.4) holds. We shall to show that

$$\|\mathscr{D}_{(J,q)}(\mathscr{T}^l f)(x_1,x_2)\| \leqslant \left(\frac{1}{2^{(4n-3k)\beta}}\right)^l \psi(2^{\beta l}x_1, 2^{\beta l}x_2) \tag{4.9}$$

for all $x_1, x_2 \in V^n$ and $l \in \mathbb{N}_0$. We argue by induction on l. The inequality (4.9) is valid for $l = 0$ by (4.3). Assume that (4.9) is true for an $l \in \mathbb{N}_0$. Then

$$\begin{aligned}
&\|\mathscr{D}_{(J,q)}(\mathscr{T}^{l+1} f)(x_1,x_2)\| \\
&= \Bigg\| 2^{4n-3k}\sum_{t\in\{-2,2\}^{n-k}} (\mathscr{T}^{l+1}f)\left(\frac{x_1^k + x_2^k}{2}, \frac{x_1^{n-k} + tx_2^{n-k}}{2}\right) \\
&\quad -\sum_{j_1,\dots,j_k\in\{1,2\}}\sum_{p_2=0}^{n-k}\sum_{p_1=0}^{n-k-p_2} 4^{n-k-p_1-p_2}(-6)^{p_1}24^{p_2}(\mathscr{T}^{l+1}f)\left(x_{j_1 1},\dots,x_{l_k k},\mathscr{N}^{n-k}_{(p_1,p_2)}\right)\Bigg\| \\
&= \frac{1}{2^{(4n-3k)\beta}}\Bigg\| 2^{4n-3k}\sum_{t\in\{-2,2\}^{n-k}} (\mathscr{T}^{l}f)\left(2^\beta\frac{x_1^k + x_2^k}{2}, 2^\beta\frac{x_1^{n-k} + tx_2^{n-k}}{2}\right) \\
&\quad -\sum_{j_1,\dots,j_k\in\{1,2\}}\sum_{p_2=0}^{n-k}\sum_{p_1=0}^{n-k-p_2} 4^{n-k-p_1-p_2}(-6)^{p_1}24^{p_2}(\mathscr{T}^{l}f)\left(2^\beta x_{j_1 1},\dots,2^\beta x_{l_k k},2^\beta\mathscr{N}^{n-k}_{(p_1,p_2)}\right)\Bigg\| \\
&= \frac{1}{2^{(4n-3k)\beta}}\left\|\mathscr{D}_{(J,q)}(\mathscr{T}^l f)(2^\beta x_1, 2^\beta x_2)\right\| \leqslant \left(\frac{1}{2^{(4n-3k)\beta}}\right)^{l+1}\psi(2^{\beta(l+1)}x_1, 2^{\beta(l+1)}x_2)
\end{aligned}$$

for all $x_1, x_2 \in V^n$. Letting $l \to \infty$ in (4.9) and applying (4.1), we arrive at $\mathscr{D}_{(J,q)}\mathscr{F}(x_1,x_2) = 0$ for all $x_1, x_2 \in V^n$. This means that the mapping $\mathscr{F}$ satisfies (3.1), and hence the proof is now complete. $\square$

Let A be a nonempty set, (X,d) a metric space, $\psi \in \mathbb{R}_+^{A^n}$, and $\mathscr{F}_1, \mathscr{F}_2$ operators mapping a nonempty set $D \subset X^A$ into X^{A^n}. We say that operator equation

$$\mathscr{F}_1\varphi(a_1,\dots,a_n) = \mathscr{F}_2\varphi(a_1,\dots,a_n) \tag{4.10}$$

is ψ-hyperstable provided every $\varphi_0 \in D$ satisfying inequality

$$d(\mathscr{F}_1\varphi_0(a_1,\dots,a_n), \mathscr{F}_2\varphi_0(a_1,\dots,a_n)) \leqslant \psi(a_1,\dots,a_n), \qquad a_1,\dots,a_n \in A,$$

fulfils (4.10); this definition is introduced in [14]. In other words, a functional equation $\mathscr{F}$ is *hyperstable* if any mapping f satisfying the equation $\mathscr{F}$ approximately is a true solution of $\mathscr{F}$. Under some conditions the functional equation (3.1) can be hyperstable as follows.

COROLLARY 4.3. *Let $\delta > 0$. Suppose that $p_{ij} > 0$ for $i \in \{1,2\}$ and $j \in \{1,\dots,n\}$ fulfill $\sum_{i=1}^{2}\sum_{j=1}^{n} p_{ij} \neq 4n-3k$. Let V be a normed space and W be a Banach space. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\mathscr{D}_{(J,q)} f(x_1,x_2)\| \leqslant \delta \prod_{i=1}^{2}\prod_{j=1}^{n} \|x_{ij}\|^{p_{ij}}$$

for all $x_1,x_2 \in V^n$ and $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero, then f is a unique solution of (3.1).

In the following corollary, we show that the functional equation (3.1) is stable. Since the proof is routine, we include it without proof.

COROLLARY 4.4. *Let $\delta > 0$ and $\alpha \in \mathbb{R}$ with $\alpha \neq 4n-3k$. Let also V be a normed space and W be a Banach space. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\mathscr{D}_{(J,q)} f(x_1,x_2)\| \leqslant \delta \sum_{i=1}^{2}\sum_{j=1}^{n} \|x_{ij}\|^{\alpha}$$

for all $x_1,x_2 \in V^n$ and $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero, then there exists a unique solution $\mathscr{F} : V^n \longrightarrow W$ of (3.1) *such that*

$$\|f(x) - \mathscr{F}(x)\| \leqslant \frac{2^{\alpha}}{2^{n-k}|2^{\alpha} - 2^{4n-3k}|}\,\delta \sum_{j=1}^{n} \|x_{1j}\|^{\alpha}$$

for all $x \in V^n$.

Putting $k = 0$ in Theorem 4.2, we obtain the upcoming result on the stability of multi-quartic mappings.

COROLLARY 4.5. *Let $\delta > 0$. Let also V be a normed space and W be a Banach space. If $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\left\| 16^n \sum_{t \in \{-2,2\}^n} f\left(\frac{x_1 + t x_2}{2}\right) - \sum_{p_2=0}^{n}\sum_{p_1=0}^{n-p_2} 4^{n-p_1-p_2}(-6)^{p_1} 24^{p_2} f\left(\mathscr{N}^n_{(p_1,p_2)}\right) \right\| \leqslant \delta$$

for all $x_1,x_2 \in V^n$ and $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero, then there exists a unique solution $\mathscr{Q} : V^n \longrightarrow W$ of (2.3) *such that*

$$\|f(x) - \mathscr{Q}(x)\| \leqslant \frac{\delta}{2^n(2^{4n}-1)}$$

for all $x \in V^n$.

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