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Stability of a generalized n-variable mixed-type functional equation in fuzzy modular spaces

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Abstract
This research paper deals with general solution and the Hyers–Ulam stability of a new generalized n-variable mixed type of additive and quadratic functional equations in fuzzy modular spaces by using the fixed point method.

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1 Introduction and preliminaries

A mapping $f : U \to V$ is called additive if f satisfies the Cauchy functional equation

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

for all $x, y \in U$. It is easy to see that the function $f(x) = ax$ is a solution of functional equation (1.1) and every solution of functional equation (1.1) is said to be an additive mapping. A mapping $f : U \to V$ is called quadratic if f satisfies the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.2}$$

for all $x, y \in U$. It is easy to see that the quadratic function $f(x) = ax^2$ is a solution of functional equation (1.2), and every solution of functional equation (1.2) is said to be a quadratic mapping. Mixed-type functional equation is the advanced development in the field of functional equations. A single functional equation, which has more than one nature, is known as mixed-type functional equation. Further, one can refer to [1–23] for more information on functional equations and applications.

"Let G be a group and H be a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist $\delta > 0$ such that if a mapping $f : G \to H$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then there exists a homomorphism $a : G \to H$ with $d(f(x), a(x)) < \epsilon$ for all $x \in G$?" This problem for the stability of functional equations was raised by Ulam [24] and answered by Hyers [25]. Later, it was developed as Hyers–Ulam stability by Rassias [26], Rassias [27, 28], and Gavruta [29].

Definition 1.1 (Fuzzy modular space [30]) Let μ be a fuzzy set on $X \times \mathbb{R}^+$, V be a complex or real vector space, γ be a zero on V, and $*$ be a continuous triangular norm. The triple $(V, \mu, *)$ is said to be a fuzzy modular space and μ is said to be a fuzzy modular if it satisfies the following:

(i) $\mu(x,t) > 0$;
(ii) $\mu(x,t) = 1$ if and only if $x = \gamma$;
(iii) $\mu(x,t) = \mu(-x,t)$;
(iv) $\mu(ax + by, r + t) \geq \mu(x,r) * \mu(y,t)$, $a, b \geq 0$, $a + b = 1$;
(v) the function $\mu(x,\cdot) : (0,\infty) \to (0,1]$ is continuous.

Example 1.2 Let μ be a fuzzy set on $V \times \mathbb{R}^+$, V be a complex or real vector space, and $*$ be a continuous triangular norm such that $a * b = a *_M b = \min\{a,b\}$. Then

$$\mu(x,t) = \begin{cases} \frac{t}{t+\mu(x)}, & t > 0, x \in V, \\ 0, & t \leq 0, x \in V, \end{cases}$$

is a fuzzy modular space. This example holds even if we replace $a * b$ with $a *_P b$ and $a *_L b$.

Definition 1.3 Let $(V, \mu, *)$ be a fuzzy modular space. Let $\{z_n\}$ be a sequence in V.

(i) $\{z_n\}$ is said to be μ-convergent to z, denoted by $z_n \xrightarrow{\mu} z$, if there exists a positive integer m_0 such that $\mu(z_n - x, t) > 1 - \gamma$ for all $n \geq m_0$, $t > 0$, and $\gamma \in (0,1)$.
(ii) $\{z_n\}$ is said to be a Cauchy sequence if there exists a positive integer m_0 such that $\mu(z_n - z_m, t) > 1 - \gamma$ for all $n, m \geq m_0$, $t > 0$, and $\gamma \in (0,1)$.
(iii) Every μ-convergent sequence in an FM-space is a μ-Cauchy sequence. In $(V, \mu, *)$, if each μ-Cauchy sequence is μ-convergent sequence, then $(V, \mu, *)$ is called a μ-complete fuzzy modular space.

Definition 1.4 ([30]) If μ fulfills the property $\mu(\gamma z, t) = \mu(z, \frac{t}{|\gamma|^b})$ for some fixed $b \in (0,1]$ and a nonzero real number γ, then $(V, \mu, *)$ is said to be a b-homogeneous fuzzy modular space.

In 2002, J. M. Rassias [31] studied the Ulam stability of a mixed-type functional equation

$$g\left(\sum_{i=1}^{3} x_i\right) + \sum_{i=1}^{3} g(x_i) = \sum_{1 \leq i \leq j \leq 3} g(x_i + x_j).$$

Later, Nakmalachalasint [32] generalized the above functional equation and obtained an n-variable mixed-type functional equation of the form

$$g\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} g(x_i) = \sum_{1 \leq i \leq j \leq n} g(x_i + x_j)$$

for $n > 2$ and investigated its Ulam stability.

In 2005, Jun and Kim [33] introduced a generalized additive-quadratic functional equation of the form

$$g(x+ay)+ag(x-y)=g(x-ay)+ag(x+y) \tag{1.3}$$

for $a \neq 0, \pm 1$.

Shen and Chen [30] introduced the concept of fuzzy modular spaces in 2013. Further, Kumam [34, 35] and Wongkum *et al.* [36] introduced the fixed point concept in fuzzy modular spaces and obtained some properties. Wongkum and Kumam [37] investigated the Hyers–Ulam stability of sextic functional equation in fuzzy modular spaces.

Motivated by the notion of fuzzy modular spaces and by the mixed-type functional equations, we introduce a new generalized n-variable mixed-type functional equation of the form

$$\begin{aligned}&\sum_{i=1,j=i+1}^{n-1}\bigl(f(kx_i+x_j)\bigr)+f(kx_n+x_1)\\&\quad-k\Biggl[\sum_{i=1,j=i+1}^{n-1}\bigl(f(x_i+x_j)\bigr)+f(x_n+x_1)\Biggr]\\&=\frac{(1-k)^2}{2}\sum_{i=1}^{n}\bigl(f(x_i)+f(-x_i)\bigr)+\frac{1-k}{k^2-k}\sum_{i=1}^{n}\bigl(k^2f(x_i)-f(kx_i)\bigr)\end{aligned} \tag{1.4}$$

for positive integers $n, k \geq 2$ and investigate its Hyers–Ulam stability in fuzzy modular spaces.

This paper is structured as follows: In Sect. 1, we provide necessary introduction of this paper. In Sect. 2, we obtain the general solution of functional equation (1.4). In Sect. 3, we investigate the Hyers–Ulam stability of (1.4) in fuzzy modular spaces using the fixed point theory, and the conclusion is given in Sect. 4.

2 General solution of a mixed-type functional equation

Let U and V be real vector spaces. In this section we obtain the general solution of a generalized n-variable mixed-type functional equation (1.4).

Lemma 2.1 *Let a mapping $f: U \to V$ satisfy functional equation* (1.4). *If f is an even mapping, then f is quadratic.*

Proof Let a mapping $f: U \to V$ satisfy functional equation (1.4). Substituting $(x_1, x_2, \dots, x_n)$ by $(x, 0, \dots, 0)$ in (1.4), we have

$$\begin{aligned}&f(kx)+f(x)-2kf(x)\\&=\frac{(1-k)^2}{2}\bigl[f(x)+f(-x)\bigr]+\frac{1-k}{k^2-k}\bigl[k^2f(x)-f(kx)\bigr]\end{aligned} \tag{2.1}$$

for all $x \in U$. By the evenness of f, equation (2.1) leads to $f(kx) = k^2 f(x)$ for all $x \in U$, and so f is quadratic. Hence, by the evenness of f, the mixed-type functional equation (1.4) is reduced to the following quadratic functional equation of the form:

$$\sum_{i=1,j=i+1}^{n-1} \big(f(kx_i + x_j)\big) + f(kx_n + x_1) - k\left[\sum_{i=1,j=i+1}^{n-1} \big(f(x_i + x_j)\big) + f(x_n + x_1)\right] \\ = (1-k)^2 \sum_{i=1}^{n} \big(f(x_i)\big) + \frac{1-k}{k^2-k} \sum_{i=1}^{n} \big(k^2 f(x_i) - f(kx_i)\big) \tag{2.2}$$

for positive integers $n, k \geq 2$. □

Lemma 2.2 *Let a mapping $f : U \to V$ satisfy functional equation (1.4). If f is an odd mapping, then f is additive.*

Proof Let a mapping $f : U \to V$ satisfy functional equation (1.4). Substituting $(x_1, x_2, \dots, x_n)$ by $(x, 0, \dots, 0)$ in (1.4), we get (2.1). By the oddness of f, equation (2.1) leads to $f(kx) = kf(x)$ for all $x \in U$, and so f is additive. Hence, by the oddness of f, the mixed-type functional equation (1.4) is reduced to the following additive functional equation of the form:

$$\sum_{i=1,j=i+1}^{n-1} \big(f(kx_i + x_j)\big) + f(kx_n + x_1) - k\left[\sum_{i=1,j=i+1}^{n-1} \big(f(x_i + x_j)\big) + f(x_n + x_1)\right] \\ = \frac{1-k}{k^2-k} \sum_{i=1}^{n} \big(k^2 f(x_i) - f(kx_i)\big) \tag{2.3}$$

for $n \in \mathbb{N}$. □

Theorem 2.3 *Let an even mapping $f : U \to V$ satisfy functional equation (2.2), then f is quadratic.*

Proof Suppose that f is even and satisfies functional equation (2.2). Setting $x_1 = x_2 = \cdots = x_n = 0$ and replacing $(x_1, x_2, \dots, x_n)$ with $(x, 0, \dots, 0)$ in (2.2), we obtain $f(0) = 0$ and

$$f(kx) = k^2 f(x), \tag{2.4}$$

respectively, for all $x \in U$. Replacing $(x_1, x_2, x_3, \dots, x_n)$ with $(x_1, x_2, 0, \dots, 0)$ in (2.2) and using (2.4), we have

$$f(kx_1 + x_2) - kf(x_1 + x_2) = \big(k^2 - k\big) f(x_1) + (1-k) f(x_2) \tag{2.5}$$

for all $x_1, x_2 \in U$. Replacing x_2 with $-x_2$ in (2.5), using the evenness of f and again adding the resultant to (2.5), we get

$$f(kx_1 + x_2) + f(kx_1 - x_2) \\ = kf(x_1 + x_2) + kf(x_1 - x_2) + 2\big(k^2 - k\big) f(x_1) + 2(1-k) f(x_2) \tag{2.6}$$

for all $x_1, x_2 \in U$. Replacing (x_1, x_2) with $(x_1, x_1 + x_2)$ in (2.6), we get

$$\begin{aligned} &f\big((k+1)x_1 + x_2\big) + f\big((k-1)x_1 - x_2\big) \\ &\quad = kf(2x_1 + x_2) + f(-x_2) + 2\big(k^2 - k\big)f(x_1) + 2(1-k)f(x_1 + x_2) \end{aligned} \tag{2.7}$$

for all $x_1, x_2 \in U$. Replacing (x_1, x_2) with $(x_1, -x_2)$ in (2.7) and again adding the resultant to (2.7), we get

$$\begin{aligned} &f\big((k+1)x_1 + x_2\big) + f\big((k+1)x_1 - x_2\big) + f\big((k-1)x_1 + x_2\big) \\ &\qquad + f\big((k-1)x_1 - x_2\big) - k\big[f(2x_1 + x_2) + f(2x_1 - x_2) + 2f(x_2)\big] \\ &\quad = 4\big(k^2 - k\big)f(x_1) + 2(1-k)\big[f(x_1 + x_2) + f(x_1 - x_2)\big] \end{aligned} \tag{2.8}$$

for all $x_1, x_2 \in U$. Now, by (2.6) and (2.8) and by assuming different values of k as $k+1$, $k-1$, and 2, we obtain (1.2). Hence the mapping f is quadratic. □

Theorem 2.4 *Let an odd mapping $f : U \to V$ satisfy functional equation* (2.3). *Then f is additive.*

Proof Suppose that f is odd and satisfies functional equation(2.3). Replacing $(x_1, x_2, \ldots, x_n)$ with $(0, 0, \ldots, 0)$ and $(x, 0, \ldots, 0)$ in (2.3), we obtain $f(0) = 0$ and

$$f(kx) = kf(x), \quad \forall x \in U, \tag{2.9}$$

respectively. Replacing $(x_1, x_2, x_3, x_4, \ldots, x_n)$ with $(x_1, x_2, 0, 0, \ldots, 0)$ in (2.3), we obtain

$$f(kx_1 + x_2) - kf(x_1 + x_2) = (1-k)f(x_2) \tag{2.10}$$

for all $x_1, x_2 \in U$. Replacing x_2 with $-x_2$ in (2.10), using the oddness of f and again adding the resultant to (2.10), we get

$$f(kx_1 + x_2) + f(kx_1 - x_2) = kf(x_1 + x_2) + kf(x_1 - x_2) \tag{2.11}$$

for all $x_1, x_2 \in U$. Replacing (x_1, x_2) with (x_2, x_1) in (2.11), we get

$$f(x_1 + kx_2) - f(x_1 - kx_2) = kf(x_1 + x_2) - kf(x_1 - x_2) \tag{2.12}$$

for all $x_1, x_2 \in U$. Replacing x_2 with kx_2 in (2.11) and using (2.9), we get

$$f(x_1 + kx_2) + f(x_1 - kx_2) = f(x_1 + x_2) + f(x_1 - x_2) \tag{2.13}$$

for all $x_1, x_2 \in U$. Replacing x_1 with $x_1 + kx_2$ in (2.12), we get

$$f(x_1 + 2kx_2) - f(x_1) = kf\big((x_1 + x_2) + kx_2\big) - kf\big((x_1 - x_2) + kx_2\big) \tag{2.14}$$

for all $x_1, x_2 \in U$. Replacing x_2 with $-x_2$ in (2.14), adding the resultant to (2.14) and using (2.12), we obtain

$$f(x_1 + 2kx_2) + f(x_1 - 2kx_2) = k^2\bigl[f(x_1 + 2x_2) + f(x_1 - 2x_2)\bigr] - 2k^2 f(x_1) + 2f(x_1) \tag{2.15}$$

for all $x_1, x_2 \in U$. Replacing x_2 with $\frac{x_2}{2}$ in (2.15) and using (2.13), we get

$$f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1) \tag{2.16}$$

for all $x_1, x_2 \in U$. Replacing (x_1, x_2) with (x_2, x_1) in (2.16) and adding the resultant to (2.16), we obtain (1.1). Hence the mapping f is additive. □

Lemma 2.5 ([33]) *Let a mapping $f : U \to V$ satisfy functional equation* (1.3), *then f is additive-quadratic*.

Theorem 2.6 *Let an odd mapping $f : U \to V$ satisfy functional equation* (1.4). *Then f satisfies* (1.3).

Proof Suppose that an odd mapping f satisfies functional equation (1.4). Replacing $(x_1, x_2, \dots, x_n)$ with $(x_1, x_2, 0, \dots, 0)$ in (1.4), we obtain

$$f(kx_1 + x_2) - kf(x_1 + x_2) = (1 - k)f(x_2) \tag{2.17}$$

for all $x_1, x_2 \in U$. Replacing x_2 with $-x_2$ in (2.17), using the oddness of f, and again adding the resultant to (2.17), we get

$$f(kx_1 + x_2) + f(kx_1 - x_2) = kf(x_1 + x_2) + kf(x_1 - x_2) \tag{2.18}$$

for all $x_1, x_2 \in U$. Replacing (x_1, x_2) with (x_2, x_1) in (2.18), we get (1.3). □

3 Stability of a mixed-type functional equation

In this section, we obtain the Hyers–Ulam stability of a generalized n-variable mixed-type functional equation (1.4) in a fuzzy modular space by using the fixed point technique. For the mapping $f : M \to (V, \mu)$, consider

$$\begin{aligned} S(x_1, x_2, \dots, x_n) = {} & \sum_{i=1, j=i+1}^{n-1} \bigl(f(kx_i + x_j)\bigr) + f(kx_n + x_1) \\ & - k\Biggl[\sum_{i=1, j=i+1}^{n-1} \bigl(f(x_i + x_j)\bigr) + f(x_n + x_1)\Biggr] \\ & - \frac{(1-k)^2}{2} \sum_{i=1}^{n} \bigl(f(x_i) + f(-x_i)\bigr) - \frac{1-k}{k^2 - k} \sum_{i=1}^{n} \bigl(k^2 f(x_i) - f(kx_i)\bigr) \end{aligned}$$

for $n \in \mathbb{N}$, $k \geq 2$.

Theorem 3.1 *Let M be a linear space, V be a real vector space, $(V,\mu,*)$ be a μ-complete b-homogeneous fuzzy modular space, and $\alpha \in \{-1,1\}$ be fixed. Suppose that an even mapping $f: M \to (V,\rho,*)$ satisfies*

$$\mu\big(S(x_1,x_2,\dots,x_n),t\big) \geq \rho(x_1,x_2,\dots,x_n,t) \tag{3.1}$$

for all $x_1,x_2,\dots,x_n \in M$ and a given mapping $\rho: M\times M \to \Delta$ such that

$$\rho\big(k^a x_1,k^a x_2,\dots,k^a x_n,k^{2ba}Nt\big) \geq \rho(x_1,x_2,\dots,x_n,t) \tag{3.2}$$

for all $x_1,x_2,\dots,x_n \in M$ and

$$\lim_{m\to\infty} \rho\big(k^{am}x_1,k^{am}x_2,\dots,k^{am}x_2,k^{2bam}t\big) = 1 \tag{3.3}$$

for all $x_1,x_2,\dots,x_n \in M$ and a constant $0 < N < \frac{1}{(\frac{k^2-2k+1}{k^2-k})^b}$. Then there exists a unique quadratic mapping $Q: M \to (V,\mu)$ satisfying (1.4) and

$$\rho\left(Q(x)-f(x),\frac{t}{k^{2b}N^{\frac{\alpha-1}{2}}(1-(\frac{k^2-2k+1}{k^2-k})^b N)}\right) \geq \rho(x,0,\dots,0,t) \tag{3.4}$$

for all $x_1,x_2,\dots,x_n \in M$.

Proof Letting $(x_1,x_2,\dots,x_n)$ by $(x,0,\dots,0)$ in (3.1), we obtain

$$\mu\left(\frac{k^2-2k+1}{k^2-k}\big(f(kx)-k^2f(x)\big),t\right) \geq \rho(x,0,\dots,0,t) \tag{3.5}$$

for all $x \in M$, and so

$$\begin{aligned}\mu\left(\frac{f(kx)}{k^2}-f(x),t\right) &= \rho\left(\frac{k^2-2k+1}{k^2-k}\big(f(kx)-k^2f(x)\big),\left(\frac{k^2-2k+1}{k^2-k}\right)^b k^{2b}t\right)\\ &\geq \rho\left(x,0,\dots,0,\left(\frac{k^2-2k+1}{k^2-k}\right)^b k^{2b}t\right)\end{aligned} \tag{3.6}$$

for all $x \in M$. Replacing x with $k^{-1}x$ in (3.6), we obtain

$$\begin{aligned}\mu\left(\frac{f(k^{-1}x)}{k^{-2}}-f(x),t\right) &= \mu\left(\frac{f(x)}{k^2}-f\big(k^{-1}x\big),\frac{t}{k^{2b}}\right)\\ &\geq \rho\left(k^{-1}x,0,\dots,0,\left(\frac{k^2-2k+1}{k^2-k}\right)^b k^{2b}N^{-1}\frac{Nt}{k^{2b}}\right)\\ &\geq \rho\left(x,0,\dots,0,\left(\frac{k^2-2k+1}{k^2-k}\right)^b k^{2b}N^{-1}t\right).\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$\mu\left(\frac{f(k^a x)}{k^{2a}}-f(x),t\right) \geq \Psi(x,t) := \rho\left(x,0,\dots,0,\left(\frac{k^2-2k+1}{k^2-k}\right)^b k^{2b}N^{\frac{a-1}{2}}t\right) \tag{3.8}$$

for all $x \in M$. Consider $P := \{h : M \to (V, \mu) | h(0) = 0\}$ and define η on P as follows:

$$\eta(h) = \inf\{l > 0 : \rho(h(x), lt) \geq \Psi(x,t), \forall x \in M\}.$$

One can easily prove that η is modular on N and indulges the Δ_k-condition with $k^b = \kappa$ and the Fatou property. Additionally, N is η-complete (see [38]). Consider the mapping $R : P_\eta \to P_\eta$ as $RQ(x) := \frac{Q(k^a x)}{k^{2a}}$ for all $Q \in P_\eta$.

Let $h, j \in P_\eta$ and $l > 0$ be an arbitrary constant with $\eta(h - j) \leq l$. From the definition of η, we get

$$\mu(h(x) - j(x), lt) \geq \Psi(x,t)$$

for all $x \in M$, and so

$$\begin{aligned}
&\mu(Rh(x) - Rj(x), Nlt) \\
&\quad = \mu(k^{-2a}h(k^a x) - k^{-2a}j(k^a x), Nlt) \\
&\quad = \mu(h(k^a x) - j(k^a x), k^{2ba}Nlt) \\
&\quad \geq \Psi(k^a x, k^{2ba}Nt) \\
&\quad \geq \Psi(x,t)
\end{aligned}$$

for all $x \in M$. Hence $\eta(Rh - Rj) \leq N\eta(h - j)$ for all $h, j \in P_\eta$, which means that R is an η-strict contraction. Replacing x with $k^a x$ in (3.8), we get

$$\mu\left(\frac{f(k^{2a}x)}{k^{2a}} - f(k^a x), t\right) \geq \Psi(k^a x, t) \tag{3.9}$$

for all $x \in M$, and therefore

$$\begin{aligned}
&\mu(k^{-2(2a)}f(k^{2a}x) - k^{-2a}f(k^a x), Nt) &(3.10)\\
&\quad = \mu(k^{-2a}f(k^{2a}x) - f(k^a x), k^{2ba}Nt) \\
&\quad \geq \Psi(k^a x, k^{2ba}Nt) \\
&\quad \geq \Psi(x,t) &(3.11)
\end{aligned}$$

for all $x \in M$. Now

$$\begin{aligned}
&\mu\left(\frac{f(k^{2a}x)}{k^{2(2a)}} - f(x), \left(\frac{k^2 - 2k + 1}{k^2 - k}\right)^b (Nt + t)\right) &(3.12)\\
&\quad \geq \mu\left(\frac{f(k^{2a}x)}{k^{2(2a)}} - \frac{f(k^a x)}{k^{2a}}, Nt\right) \wedge \mu\left(\frac{f(k^a x)}{k^{2a}} - f(x), t\right) \\
&\quad \geq \Psi(x,t)
\end{aligned}$$

for all $x \in M$. In (3.12), replacing x with $k^a x$ and $(\frac{k^2-2k+1}{k^2-k})^b(Nt+t)$ with $(\frac{k^2-2k+1}{k^2-k})^b k^{2ba}(N^2t+Nt)$, we get

$$\mu\left(\frac{f(k^{3a}x)}{k^{2(2a)}}-f(k^a x), k^{2ba}\left(\frac{k^2-2k+1}{k^2-k}\right)^b(N^2t+Nt)\right) \geq \Psi(k^a x, k^{2ba}Nt) \geq \Psi(x,t) \tag{3.13}$$

for all $x \in E$. Therefore,

$$\mu\left(\frac{f(k^{3a}x)}{k^{3(2a)}}-\frac{f(k^a x)}{k^{2a}}, \left(\frac{k^2-2k+1}{k^2-k}\right)^b(N^2t+Nt)\right) \geq \Psi(x,t) \tag{3.14}$$

for all $x \in M$, and so

$$\begin{aligned}
&\mu\left(\frac{f(k^{3a}x)}{k^{3(2a)}}-f(x), \left(\frac{k^2-2k+1}{k^2-k}\right)^b\left(\left(\frac{k^2-2k+1}{k^2-k}\right)^b(N^2t+Nt)+t\right)\right) \\
&\quad\geq \mu\left(\frac{f(k^{3a}x)}{k^{3(2a)}}-\frac{f(k^a x)}{k^{2a}}, \left(\frac{k^2-2k+1}{k^2-k}\right)^b(N^2t+Nt)\right) \wedge \mu\left(\frac{f(k^a x)}{k^{2a}}-f(x), t\right) \\
&\quad\geq \Psi(x,t)
\end{aligned} \tag{3.15}$$

for all $x \in M$. Generalizing the above inequality, we obtain

$$\begin{aligned}
&\mu\Bigg(\frac{f(k^{am}x)}{k^{2(am)}}-f(x), \\
&\quad\left(\left(\left(\frac{k^2-2k+1}{k^2-k}\right)^b N\right)^{m-1}+\left(\frac{k^2-2k+1}{k^2-k}\right)^b\sum_{i=1}^{m-1}\left(\left(\frac{k^2-2k+1}{k^2-k}\right)^b N\right)^{i-1}\right)t\Bigg) \\
&\quad\geq \Psi(x,t)
\end{aligned} \tag{3.16}$$

for all $x \in M$ and a positive integer m. Hence we have

$$\begin{aligned}
&\eta(R^m f-f) \\
&\quad\leq \left(\left(\frac{k^2-2k+1}{k^2-k}\right)^b N\right)^{m-1}+\left(\frac{k^2-2k+1}{k^2-k}\right)^b\sum_{i=1}^{m-1}\left(\left(\frac{k^2-2k+1}{k^2-k}\right)^b N\right)^{i-1} \\
&\quad\leq \left(\frac{k^2-2k+1}{k^2-k}\right)^b\sum_{i=1}^{m}\left(\left(\frac{k^2-2k+1}{k^2-k}\right)^b N\right)^{i-1} \leq \frac{(\frac{k^2-2k+1}{k^2-k})^b}{1-(\frac{k^2-2k+1}{k^2-k})^b N}.
\end{aligned} \tag{3.17}$$

Now, one can easily prove that $\{R^m(f)\}$ is η-convergent to $Q \in P_\eta$ (see [37]). Therefore, (3.17) becomes

$$\eta(Q-f) \leq \frac{(\frac{k^2-2k+1}{k^2-k})^b}{1-(\frac{k^2-2k+1}{k^2-k})^b N}, \tag{3.18}$$

which implies

$$\mu\left(Q(x)-f(x),\frac{(\frac{k^2-2k+1}{k^2-k})^b}{1-(\frac{k^2-2k+1}{k^2-k})^bN}t\right) \geq \Psi(x,t)=\rho\left(x,0,\dots,0,\left(\frac{k^2-2k+1}{k^2-k}\right)^b k^{2b}N^{\frac{a-1}{2}}t\right) \tag{3.19}$$

for all $x\in M$, and hence we have

$$\mu\left(Q(x)-f(x),\frac{t}{k^{2b}N^{\frac{a-1}{2}}(1-(\frac{k^2-2k+1}{k^2-k})^bN)}\right)\geq \rho(x,0,\dots,0,t) \tag{3.20}$$

for all $x\in M$, and so inequality (3.4) holds. One can easily prove the uniqueness of Q (see [37]). □

Theorem 3.2 *Let M be a linear space, V be a real vector space, $(V,\mu,*)$ be a μ-complete b-homogeneous fuzzy modular space, and $\alpha\in\{-1,1\}$ be fixed. Suppose that an odd mapping $f:M\to(V,\rho,*)$ satisfies*

$$\mu\big(S(x_1,x_2,\dots,x_n),t\big)\geq\rho(x_1,x_2,\dots,x_n,t) \tag{3.21}$$

for all $x_1,x_2,\dots,x_n\in M$ and a given mapping $\rho:M\times M\to\Delta$ such that

$$\rho\big(k^ax_1,k^ax_2,\dots,k^ax_n,k^{ba}Nt\big)\geq\rho(x_1,x_2,\dots,x_n,t) \tag{3.22}$$

for all $x_1,x_2,\dots,x_n\in M$ and

$$\lim_{m\to\infty}\rho\big(k^{am}x_1,k^{am}x_2,\dots,k^{am}x_2,k^{bam}t\big)=1 \tag{3.23}$$

for all $x_1,x_2,\dots,x_n\in M$ and a constant $0<N<\frac{1}{(\frac{k^2-2k+1}{k^2-k})^b}$. Then there exists a unique additive mapping $A:M\to(V,\mu)$ satisfying (1.4) and

$$\rho\left(A(x)-f(x),\frac{t}{k^bN^{\frac{\alpha-1}{2}}(1-(\frac{k^2-2k+1}{k^2-k})^bN)}\right)\geq\rho(x,0,\dots,0,t) \tag{3.24}$$

for all $x_1,x_2,\dots,x_n\in M$.

Proof Replacing $(x_1,x_2,\dots,x_n)$ with $(x,0,\dots,0)$ in (3.21), we obtain

$$\mu\left(\frac{k^2-2k+1}{k^2-k}f(kx)-kf(x),t\right)\geq\rho(x,0,\dots,0,t) \tag{3.25}$$

for all $x\in M$, and so

$$\mu\left(\frac{f(kx)}{k}-f(x),t\right)=\rho\left(\frac{k^2-2k+1}{k^2-k}f(kx)-kf(x),\left(\frac{k(k^2-2k+1)}{k^2-k}\right)^bt\right) \geq\rho\left(x,0,\dots,0,\left(\frac{k(k^2-2k+1)}{k^2-k}\right)^bt\right) \tag{3.26}$$

for all $x \in M$. Replacing x with $k^{-1}x$ in (3.26), we obtain

$$\begin{aligned} \mu\left(\frac{f(k^{-1}x)}{k^{-1}} - f(x), t\right) &= \mu\left(\frac{f(x)}{k} - f\left(k^{-1}x\right), \frac{t}{(\frac{k(k^2-2k+1)}{k^2-k})^b}\right) \\ &\geq \rho\left(k^{-1}x, 0, \dots, 0, \left(\frac{k(k^2-2k+1)}{k^2-k}\right)^b N^{-1} \frac{Nt}{(\frac{k(k^2-2k+1)}{k^2-k})^b}\right) \\ &\geq \rho\left(x, 0, \dots, 0, \left(\frac{k(k^2-2k+1)}{k^2-k}\right)^b N^{-1} t\right). \end{aligned} \tag{3.27}$$

From (3.26) and (3.27), we obtain

$$\mu\left(\frac{f(k^a x)}{k^a} - f(x), t\right) \geq \Psi(x,t) := \rho\left(x, 0, \dots, 0, \left(\frac{k(k^2-2k+1)}{k^2-k}\right)^b N^{\frac{a-1}{2}} t\right) \tag{3.28}$$

for all $x \in M$. Consider $P := \{h : M \to (V, \mu) | h(0) = 0\}$ and define η on P as follows:

$$\eta(h) = \inf\bigl\{l > 0 : \rho\bigl(h(x), lt\bigr) \geq \Psi(x,t), \forall x \in M\bigr\}.$$

One can easily prove that η is modular on N and indulges the Δ_k-condition with $k^b = \kappa$ and the Fatou property. Additionally, N is η-complete (see [38]). Consider the mapping $R : P_\eta \to P_\eta$ as $RA(x) := \frac{A(k^a x)}{k^a}$ for all $A \in P_\eta$.

Let $h, j \in P_\eta$ and $l > 0$ be an arbitrary constant with $\eta(h - j) \leq l$. From the definition of η, we get

$$\mu\bigl(h(x) - j(x), lt\bigr) \geq \Psi(x,t)$$

for all $x \in M$, and so

$$\begin{aligned} &\mu\bigl(Rh(x) - Rj(x), Nlt\bigr) \\ &\quad = \mu\bigl(k^{-a}h\bigl(k^a x\bigr) - k^{-a}j\bigl(k^a x\bigr), Nlt\bigr) \\ &\quad = \mu\bigl(h\bigl(k^a x\bigr) - j\bigl(k^a x\bigr), k^{ba}Nlt\bigr) \\ &\quad \geq \Psi\bigl(k^a x, k^{ba}Nt\bigr) \\ &\quad \geq \Psi(x,t) \end{aligned}$$

for all $x \in M$. Hence $\eta(Rh - Rj) \leq N\eta(h - j)$ for all $h, j \in P_\eta$, which means that R is an η-strict contraction. Replacing x with $k^a x$ in (3.28), we have

$$\mu\left(\frac{f(k^{2a}x)}{k^a} - f\bigl(k^a x\bigr), t\right) \geq \Psi\bigl(k^a x, t\bigr) \tag{3.29}$$

for all $x \in M$, and therefore

$$\begin{aligned} &\mu\bigl(k^{-2a}f\bigl(k^{2a}x\bigr) - k^{-a}f\bigl(k^a x\bigr), Nt\bigr) \\ &\quad = \mu\bigl(k^{-a}f\bigl(k^{2a}x\bigr) - f\bigl(k^a x\bigr), k^{ba}Nt\bigr) \\ &\quad \geq \Psi\bigl(k^a x, k^{ba}Nt\bigr) \geq \Psi(x,t) \end{aligned} \tag{3.30}$$

for all $x \in M$. Now

$$\begin{aligned}
&\mu\left(\frac{f(k^{2a}x)}{k^{2a}} - f(x), \left(\frac{k^2-2k+1}{k^2-k}\right)^b (Nt+t)\right) \\
&\quad \ge \mu\left(\frac{f(k^{2a}x)}{k^{2a}} - \frac{f(k^a x)}{k^a}, Nt\right) \wedge \mu\left(\frac{f(k^a x)}{k^a} - f(x), t\right) \\
&\quad \ge \Psi(x,t)
\end{aligned} \tag{3.31}$$

for all $x \in M$. In (3.31), replacing x with $k^a x$ and $(\frac{k^2-2k+1}{k^2-k})^b(Nt+t)$ with $k^{ba}(\frac{k^2-2k+1}{k^2-k})^b(N^2t + Nt)$, we obtain

$$\begin{aligned}
&\mu\left(\frac{f(k^{3a}x)}{k^{2a}} - f\left(k^a x\right), k^{ba}\left(\frac{k^2-2k+1}{k^2-k}\right)^b \left(N^2 t + Nt\right)\right) \\
&\quad \ge \Psi\left(k^a x, k^{ba} Nt\right) \ge \Psi(x,t)
\end{aligned} \tag{3.32}$$

for all $x \in E$. Therefore,

$$\mu\left(\frac{f(k^{3a}x)}{k^{3a}} - \frac{f(k^a x)}{k^a}, \left(\frac{k^2-2k+1}{k^2-k}\right)^b \left(N^2 t + Nt\right)\right) \ge \Psi(x,t) \tag{3.33}$$

for all $x \in M$, and so

$$\begin{aligned}
&\mu\left(\frac{f(k^{3a}x)}{k^{3a}} - f(x), \left(\frac{k^2-2k+1}{k^2-k}\right)^b \left(\left(\frac{k^2-2k+1}{k^2-k}\right)^b \left(N^2 t + Nt\right) + t\right)\right) \\
&\quad \ge \mu\left(\frac{f(k^{3a}x)}{k^{3a}} - \frac{f(k^a x)}{k^a}, \left(\frac{k^2-2k+1}{k^2-k}\right)^b \left(N^2 t + Nt\right)\right) \wedge \mu\left(\frac{f(k^a x)}{k^a} - f(x), t\right) \\
&\quad \ge \Psi(x,t)
\end{aligned} \tag{3.34}$$

for all $x \in M$. Generalizing the above inequality, we get

$$\begin{aligned}
&\mu\left(\frac{f(k^{am}x)}{k^{am}} - f(x),\right. \\
&\qquad \left.\left(\left(\left(\frac{k^2-2k+1}{k^2-k}\right)^b N\right)^{m-1} + \left(\frac{k^2-2k+1}{k^2-k}\right)^b \sum_{i=1}^{m-1}\left(\left(\frac{k^2-2k+1}{k^2-k}\right)^b N\right)^{i-1}\right) t\right) \\
&\quad \ge \Psi(x,t)
\end{aligned} \tag{3.35}$$

for all $x \in M$ and a positive integer m. Hence we have

$$\begin{aligned}
&\eta\left(R^m f - f\right) \\
&\quad \le \left(\left(\frac{k^2-2k+1}{k^2-k}\right)^b N\right)^{m-1} + \left(\frac{k^2-2k+1}{k^2-k}\right)^b \sum_{i=1}^{m-1}\left(\left(\frac{k^2-2k+1}{k^2-k}\right)^b N\right)^{i-1} \\
&\quad \le \left(\frac{k^2-2k+1}{k^2-k}\right)^b \sum_{i=1}^{m}\left(\left(\frac{k^2-2k+1}{k^2-k}\right)^b N\right)^{i-1} \le \frac{(\frac{k^2-2k+1}{k^2-k})^b}{1-(\frac{k^2-2k+1}{k^2-k})^b N}.
\end{aligned} \tag{3.36}$$

Now, one can easily prove that $\{R^m(f)\}$ η-converges to $A \in P_\eta$ (see [37]). Therefore, (3.36) becomes

$$\eta(A-f) \le \frac{(\frac{k^2-2k+1}{k^2-k})^b}{1-(\frac{k^2-2k+1}{k^2-k})^b N}, \tag{3.37}$$

which implies

$$\begin{aligned} \mu\left(A(x)-f(x), \frac{(\frac{k^2-2k+1}{k^2-k})^b}{1-(\frac{k^2-2k+1}{k^2-k})^b N} t\right) &\ge \Psi(x,t) \\ &= \rho\left(x, 0, \ldots, 0, \left(\frac{k^2-2k+1}{k^2-k}\right)^b k^b N^{\frac{a-1}{2}} t\right) \end{aligned} \tag{3.38}$$

for all $x \in M$, and hence we have

$$\mu\left(A(x)-f(x), \frac{t}{k^b N^{\frac{a-1}{2}}(1-(\frac{k^2-2k+1}{k^2-k})^b N)}\right) \ge \rho(x, 0, \ldots, 0, t)$$

for all $x \in M$, and hence inequality (3.24) holds. One can easily prove the uniqueness of A (see [37]). □

4 Conclusion

In this paper, we introduced a new n-variable mixed-type functional equation which satisfies $f(x) = x + x^2$. Mainly, we obtained its general solution and investigated its Hyers–Ulam stability in fuzzy modular spaces by using the fixed point method, and we hope that this research work is a further improvement in the field of functional equations.

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