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New oscillation criteria for second-order neutral differential equations with distributed deviating arguments

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Abstract

In this work, we create new oscillation conditions for solutions of second-order differential equations with continuous delay. The new criteria were created based on Riccati transformation technique and comparison principles. Furthermore, we obtain iterative criteria that can be applied even when the other criteria fail. The results obtained in this paper improve and extend the relevant previous results as illustrated by examples.

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1 Introduction

The importance of studying delay differential equations DDEs is not limited to the theoretical side only, but the applications of this type of equations extend to many branches of applied science. In fact, the neutral DDEs arise in the examination of vibrating masses attached to an elastic bar, in the solution of variational problems with time delays, and in problems concerning electric networks containing lossless transmission lines (as in high speed computers), see [1, 2].

The great development in the study of asymptotic behavior of DDEs is easily noted in many works in recent times. Some of these works that are concerned with improving the oscillation criteria of DDEs are [3–6]. In addition, many improved methods and interesting results can be found in studies [7–23], which study the oscillatory behavior of the NDDEs of different order.

In this work, we discuss the oscillation properties of the second-order NDDE with distributed deviating arguments

$$\left(r(t)\left((x+p\cdot x\circ\tau)'(t)\right)^{\alpha}\right)' + \Lambda[q\cdot f\circ x\circ g;a,b](t) = 0, \tag{1.1}$$

where $t \geq t_0$ and

$$\Lambda[F;a,b](t) \stackrel{\text{def}}{=} \int_a^b F(t,s)\,ds.$$

Throughout this work, we assume that

$$\aleph(t) \stackrel{\text{def}}{=} (x + p \cdot x \circ \tau)(t)$$

and the following hypotheses hold:

(H$_1$) α is a ratio of odd natural numbers, $r \in C([t_0,\infty),(0,\infty))$, and

$$\int_{t_0}^{\infty} r^{-1/\alpha}(s)\,ds = \infty. \tag{1.2}$$

(H$_2$) $p \in C([t_0,\infty))$, $q \in C([t_0,\infty) \times [a,b])$, $0 \leq p(t) < 1$, and $q(t,s) > 0$ is not zero on any half line $[t_*,\infty) \times [a,b]$ for all $t_* \geq t_0$.

(H$_3$) $\tau \in C([t_0,\infty),\mathbb{R})$, $g \in C([t_0,\infty) \times [a,b],\mathbb{R})$, $\tau(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \infty$, $g(t,s) \leq t$, $\lim_{t\to\infty} g(t,s) = \infty$ for $s \in [a,b]$, and g is strictly increasing with respect to t and s for all $s \in (a,b)$.

(H$_4$) $f \in C((-\infty,\infty))$ and $f(x)/x^{\alpha} \geq \kappa$ for $x \neq 0$, where κ is a positive constant.

For a solution of (1.1), we mean a function $x \in C([t_x,\infty))$, $t_x \geq t_0$, which has the property $\aleph(t)$ and $r(t)(\aleph'(t))^{\alpha}$ are continuously differentiable for $t \in [t_x,\infty)$ and satisfies (1.1) on $[t_x,\infty)$. We focus only on the solutions of (1.1) which satisfy $\sup\{|x(t)| : t_x \leq t\} > 0$ for $t \geq t_x$. A solution x of (1.1) is called nonoscillatory if it is either eventually positive or eventually negative; otherwise it is called oscillatory.

In the next part of the introduction, we provide some related work that contributed to the development of the study of oscillatory behavior of NDDEs.

In 1985, Grammatikopoulos et al. [7] studied the asymptotic behavior of NDDE

$$\bigl(x(t) + p(t)x(t-\tau_0)\bigr)'' + q(t)x(t-g_0) = 0. \tag{1.3}$$

They proved that all solutions of (1.3) are oscillatory if $p(t) \in [0,1]$ and

$$\int_{t_0}^{\infty} q(v)\bigl(1 - p(v-g_0)\bigr)\,dv = \infty.$$

However, Erbe et al. [8] established the oscillation condition when $q(t) \geq q_0 > 0$, $p(t) \in [p_1,p_2]$ and $p(t)$ is not eventually negative. Posteriorly, Grace and Lalli [9] studied the oscillation of the NDDE

$$\bigl(r(t)\bigl(x(t) + p(t)x(t-\tau_0)'\bigr)\bigr)' + q(t)f\bigl(x(t-g_0)\bigr) = 0,$$

under the condition

$$\int_{t_0}^{\infty} \left(\rho(s)q(s)\bigl(1 - p(s-g_0)\bigr) - \frac{r(s-g_0)(\rho'(s))^2}{4\kappa\rho(s)}\right) ds = \infty,$$

where $\rho \in C^1([t_0,\infty),(0,\infty))$.

In the previous decade, under the hypothesis $\tau \circ g = g \circ \tau$, Han et al. [10] presented the oscillation criteria for the NDDE

$$\left(r(t)\left(x(t) + p(t)x\left(\tau(t)\right)\right)'\right)' + q(t)x\left(g(t)\right)) = 0.$$

In 2012, by using the Riccati transformation technique, Liu et al. [11] and Wu et al. [12] obtained the oscillation conditions for the NDDE

$$\left(r(t)\left|w'(t)\right|^{\alpha-1}w'(t)\right)' + q(t)\left|x\left(g(t)\right)\right|^{\beta-1}x\left(g(t)\right) = 0,$$

where $w(t) := x(t) + p(t)x(\tau(t))$, $\alpha \geq \beta$, $r'(t) > 0$, and $g'(t) > 0$. Based on establishing new comparison theorems that compare the second-order equation with a first-order DDE, Baculikova and Dzurina [13] studied the NDDE

$$\left(r(t)\left(\left(x(t) + p(t)x\left(\tau(t)\right)\right)'\right)^{\alpha}\right)' + q(t)x^{\beta}\left(g(t)\right) = 0, \tag{1.4}$$

under the conditions

$$0 \leq p(t) \leq p_0 < \infty, \qquad \tau'(t) \geq \tau_0, \quad \text{and} \quad \tau \circ g = g \circ \tau.$$

Of interesting works recently, Moaaz et al. in [14, 15] studied the oscillatory properties of (1.4) and improved the results in [13].

For NDDE with distributed deviating arguments (1.1), Candan [16] studied the sufficient conditions for the oscillation of solutions.

In this work, we are creating an improved relationship between the corresponding function $\aleph$ and its first derivative. This new relationship helps us to get sharp criteria for testing the oscillation. Based on the Riccati transformation and comparison principles, we obtain new and different criteria for the oscillation of solutions of (1.1). The results obtained in this paper improve and extend the relevant previous results as illustrated by examples.

To prove our main results, we need the following auxiliary lemmas. The proof of the first lemma is similar to that of [13, Lemma 3] and hence we omit it.

Lemma 1.1 *If x is a positive solution of* (1.1) *on $[t_0, \infty)$, then there exists $t_1 \geq t_0$ such that*

$$\aleph(t) > 0, \qquad \aleph'(t) > 0, \qquad \left(r(t)\left(\aleph'(t)\right)^{\alpha}\right)' \leq 0 \tag{1.5}$$

for $t \geq t_1$.

Lemma 1.2 ([14, Lemma 1.2]) *Suppose that $F(s) = As - Bs^{(\alpha+1)/\alpha}$, where $A, B > 0$ are constants. Then F attains its maximum value on $\mathbb{R}$ at $s^* = (\alpha A/((\alpha+1)B))^{\alpha}$ and*

$$\max_{x \in r} F = F\left(s^*\right) = \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{A^{\alpha+1}}{B^{\alpha}}.$$

Remark 1.1 All functional inequalities are assumed to hold eventually, that is, they are satisfied for all $t > t_1$, where t_1 is large enough.

2 Main results

For convenience, we denote the class of all eventually positive solutions by S^+. Moreover, we assume the following notations: $\beta \stackrel{\text{def}}{=} (\alpha+1)^{\alpha+1}$, $g_a(t) \stackrel{\text{def}}{=} g(t,a)$,

$$\Theta(t) \stackrel{\text{def}}{=} \Lambda\bigl[q \cdot (1-p \circ g)^\alpha; a, b\bigr](t),$$
$$\mu(t) \stackrel{\text{def}}{=} \int_{t_1}^t r^{-1/\alpha}(\theta)\,\mathrm{d}\theta,$$
$$\widetilde{\mu}(t) \stackrel{\text{def}}{=} \mu(t) + \frac{\kappa}{\alpha}\int_{t_1}^t \mu(\theta)\mu^\alpha\bigl(g(\theta,a)\bigr)\Theta(\theta)\,\mathrm{d}\theta,$$
$$\widehat{\mu}(t) \stackrel{\text{def}}{=} \exp\biggl(-\alpha\int_{g_a(t)}^t \frac{\mathrm{d}\theta}{\widetilde{\mu}(\theta) r^{1/\alpha}(\theta)}\biggr),$$

and $F_+(t) \stackrel{\text{def}}{=} \max\{0, F(t)\}$, where $t_1 \geq t_0$.

The following theorem gives a criterion for the oscillation of (1.1), depending on the comparison with a first-order DDE.

Theorem 2.1 *Every solution of* (1.1) *is oscillatory if the first-order DDE*

$$\phi'(t) + \kappa\Theta(t)\widetilde{\mu}^\alpha\bigl(g_a(t)\bigr)\phi\bigl(g_a(t)\bigr) = 0 \tag{2.1}$$

is oscillatory.

Proof Assume the contrary that there is a nonoscillatory solution x of (1.1). Then we can assume $x \in S^+$, and so $x(t)$, $x(\tau(t))$, and $x(g(t,s))$ are positive for $t \geq t_1 \geq t_0$ and $s \in [a,b]$. It follows from Lemma 1.1 that (1.5) holds. Using the fact that $r(t)(\aleph'(t))^\alpha$ is a nonincreasing function, we get

$$\int_{t_1}^t \biggl(\frac{r(\theta)(\aleph'(\theta))^\alpha}{r(\theta)}\biggr)^{1/\alpha}\,\mathrm{d}\theta \geq \mu(t)\bigl(r(t)\bigl(\aleph'(t)\bigr)^\alpha\bigr)^{1/\alpha},$$

and hence

$$\aleph(t) \geq \mu(t)\bigl(r(t)\bigl(\aleph'(t)\bigr)^\alpha\bigr)^{1/\alpha}. \tag{2.2}$$

It follows from (1.1) and (H_1) that

$$\bigl(r(t)\bigl(\aleph'(t)\bigr)^\alpha\bigr)' \leq -\kappa\int_a^b q(t,s)x^\alpha\bigl(g(t,s)\bigr)\,\mathrm{d}s. \tag{2.3}$$

From the definition of $\aleph$, we have

$$\begin{aligned} x\bigl(g(t,s)\bigr) &= \aleph\bigl(g(t,s)\bigr) - p\bigl(g(t,s)\bigr)x\bigl(\tau\bigl(g(t,s)\bigr)\bigr) \\ &\geq \aleph\bigl(g(t,s)\bigr) - p\bigl(g(t,s)\bigr)\aleph\bigl(\tau\bigl(g(t,s)\bigr)\bigr) \\ &\geq \aleph\bigl(g(t,s)\bigr)\bigl(1-p\bigl(g(t,s)\bigr)\bigr), \end{aligned}$$

which with (2.3) gives

$$\begin{aligned}\left(r(t)\left(\aleph'(t)\right)^{\alpha}\right)' &\leq -\kappa\int_a^b q(t,s)\aleph^{\alpha}\left(g(t,s)\right)\left[1-p\left(g(t,s)\right)\right]^{\alpha}\,\mathrm{d}s \\ &\leq -\kappa\aleph^{\alpha}\left(g_a(t)\right)\int_a^b q(t,s)\left[1-p\left(g(t,s)\right)\right]^{\alpha}\,\mathrm{d}s \\ &\leq -\kappa\Theta(t)\aleph^{\alpha}\left(g_a(t)\right),\end{aligned} \tag{2.4}$$

which is a direct result of the facts that $\aleph'(t) > 0$ and $\partial_s g(t,s) > 0$. Combining

$$\mu\frac{\mathrm{d}}{\mathrm{d}t}\left(r^{1/\alpha}\aleph'\right)^{\alpha} = \mu\alpha\left(r^{1/\alpha}\aleph'\right)^{\alpha-1}\left(r^{1/\alpha}\aleph'\right)'$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\aleph-\mu r^{1/\alpha}\aleph'\right) = \aleph' - \mu' r^{1/\alpha}\aleph' - \mu\left(r^{1/\alpha}\aleph'\right)' = -\mu\left(r^{1/\alpha}\aleph'\right)',$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\aleph-\mu r^{1/\alpha}\aleph'\right) = -\frac{1}{\alpha}\mu\left(r^{1/\alpha}\aleph'\right)^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^{1/\alpha}\aleph'\right)^{\alpha}.$$

Thus, from (2.4), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\aleph(t)-\mu(t)r^{1/\alpha}(t)\aleph'(t)\right) \geq \frac{\kappa}{\alpha}\mu(t)\left(r^{1/\alpha}(t)\aleph'(t)\right)^{1-\alpha}\Theta(t)\aleph^{\alpha}\left(g_a(t)\right). \tag{2.5}$$

Integrating (2.5) from $t_1 \to t$, we have

$$\aleph(t) \geq \mu(t)r^{1/\alpha}(t)\aleph'(t) + \frac{\kappa}{\alpha}\int_{t_1}^t \mu(\theta)\Theta(\theta)\left(r^{1/\alpha}(\theta)\aleph'(\theta)\right)^{1-\alpha}\aleph^{\alpha}\left(g(\theta,a)\right)\,\mathrm{d}\theta. \tag{2.6}$$

Now, we set $\phi(t) := r(t)(\aleph'(t))^{\alpha}$. Then, from (2.2) and (2.6), we obtain

$$\begin{aligned}\aleph(t) &\geq \mu(t)\phi^{1/\alpha}(t) + \frac{\kappa}{\alpha}\int_{t_1}^t \mu(\theta)\Theta(\theta)\phi^{(1-\alpha)/\alpha}(\theta)\mu^{\alpha}\left(g(\theta,a)\right)\phi\left(g(\theta,a)\right)\,\mathrm{d}\theta \\ &\geq \mu(t)\phi^{1/\alpha}(t) + \frac{\kappa}{\alpha}\int_{t_1}^t \mu(\theta)\Theta(\theta)\phi^{(1-\alpha)/\alpha}(\theta)\mu^{\alpha}\left(g(\theta,a)\right)\phi(\theta)\,\mathrm{d}\theta \\ &\geq \phi^{1/\alpha}(t)\left(\mu(t) + \frac{\kappa}{\alpha}\int_{t_1}^t \mu(\theta)\mu^{\alpha}\left(g(\theta,a)\right)\Theta(\theta)\,\mathrm{d}\theta\right) \\ &\geq \widetilde{\mu}(t)\phi^{1/\alpha}(t).\end{aligned} \tag{2.7}$$

Combining (2.4) and (2.7), we have that ϕ is a positive solution of the first-order DD inequality

$$\phi'(t) + \kappa\Theta(t)\widetilde{\mu}^{\alpha}\left(g_a(t)\right)\phi\left(g_a(t)\right) \leq 0.$$

From [24, Theorem 1], DDE (2.1) also has a positive solution, which is a contradiction. This contradiction completes the proof. □

Applying a well-known condition [25, Theorem 2.1.1] for oscillation of first-order DDE (2.1), we get immediately the following criteria for oscillation of (1.1).

Corollary 2.1 *Every solution of* (1.1) *is oscillatory if one of the following conditions is satisfied*:

$$\liminf_{t\to\infty}\int_{g_a(t)}^{t}\widetilde{\mu}^{\alpha}\big(g_a(\theta)\big)\Theta(\theta)\,\mathrm{d}\theta>\frac{1}{\kappa\,\mathrm{e}} \tag{2.8}$$

or

$$\limsup_{t\to\infty}\int_{g_a(t)}^{t}\widetilde{\mu}^{\alpha}\big(g_a(\theta)\big)\Theta(\theta)\,\mathrm{d}\theta>\frac{1}{\kappa}\quad \textit{and}\quad \partial_t g(t,s)\geq 0. \tag{2.9}$$

The next theorem gives another criterion for the oscillation of (1.1), depending on the Riccati transformation technique.

Theorem 2.2 *Every solution of* (1.1) *is oscillatory if there is a positive function* $\rho\in C^1([t_0,\infty))$ *satisfying*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(\kappa\rho(\theta)\widehat{\mu}(\theta)\Theta(\theta)-\frac{r(\theta)(\rho_+'(\theta))^{\alpha+1}}{\beta\rho^{\alpha}(\theta)}\right)\mathrm{d}\theta=\infty, \tag{2.10}$$

where t_1 *is sufficiently large*.

Proof Assume the contrary that there is a nonoscillatory solution x of (1.1). Then we can assume $x\in S^+$, and so $x(t)$, $x(\tau(t))$, and $x(g(t,s))$ are positive for $t\geq t_1\geq t_0$ and $s\in[a,b]$. It follows from Lemma 1.1 that (1.5) holds. Now, we set

$$w\stackrel{\text{def}}{=}\rho r\left(\frac{\aleph'}{\aleph}\right)^{\alpha}. \tag{2.11}$$

Thus, we note that $w(t)>0$ for $t\geq t_1$. By differentiating w, we get

$$\begin{aligned} w'&=\rho' r\left(\frac{\aleph'}{\aleph}\right)^{\alpha}+\rho\frac{(r\aleph')'}{\aleph^{\alpha}}-\alpha\rho r\left(\frac{\aleph'}{\aleph}\right)^{\alpha+1}\\ &=\frac{\rho'}{\rho}w+\rho\frac{(r\aleph')'}{\aleph^{\alpha}}-\alpha\rho^{-1/\alpha}r^{-1/\alpha}w^{1+1/\alpha}. \end{aligned} \tag{2.12}$$

Next, as in the proof of Theorem 2.1, we obtain (2.4) and (2.7). Then, from (2.7), we obtain $\aleph(t)\geq\widetilde{\mu}(t)r^{1/\alpha}(t)\aleph'(t)$ for $t\geq t_1$. Then, applying the Grönwall inequality, we find

$$\aleph\big(g_a(t)\big)\geq\aleph(t)\exp\left[-\int_{g_a(t)}^{t}\widetilde{\mu}^{-1}(\theta)r^{-1/\alpha}(\theta)\,\mathrm{d}\theta\right]=\widehat{\mu}^{1/\alpha}(t)\aleph(t),$$

which with (2.4) gives

$$\big(r(t)\big(\aleph'(t)\big)^{\alpha}\big)'\leq-\kappa\Theta(t)\left(\frac{\aleph(g_a(t))}{\aleph(t)}\right)^{\alpha}\aleph^{\alpha}(t)\leq-\kappa\Theta(t)\widehat{\mu}(t)\aleph^{\alpha}(t). \tag{2.13}$$

Combining (2.12) and (2.13), we arrive at

$$w' \leq \frac{\rho'_+}{\rho} w - \kappa \rho \Theta \widehat{\mu}(t) - \alpha \rho^{-1/\alpha} r^{-1/\alpha} w^{1+1/\alpha}. \tag{2.14}$$

Using Lemma 1.2 with $A = \rho'_+/\rho$ and $B = \alpha(\rho r)^{-1/\alpha}$, we get

$$w' \leq -\kappa \rho \Theta \widehat{\mu} + \frac{1}{\beta} r\left(\rho'_+\right)^{\alpha+1} \rho^{-\alpha}. \tag{2.15}$$

Integrating (2.15) from $t_1 \to t$, we find

$$w(t_1) \geq \int_{t_1}^{t} \left(\kappa \rho(\theta) \Theta(\theta) \widehat{\mu}(\theta) - \frac{1}{\beta} r(\theta) \left(\rho'_+(\theta)^{\alpha+1}\right) \rho^{-\alpha}(\theta) \right) \mathrm{d}\theta,$$

which contradicts (2.10). This contradiction completes the proof. □

It is easy to see that Corollary 2.1 cannot be applied in the case where

$$\int_{g_a(t)}^{t} \widetilde{\mu}^{\alpha}\left(g_a(\theta)\right) \Theta(\theta) \,\mathrm{d}\theta \leq \frac{1}{\kappa \mathrm{e}}. \tag{2.16}$$

However, if $x \in S^+$ and (2.16) holds, then we can get a sharp estimate of $z(g(t))/z(t)$. Thus, we can obtain a sharp criteria for the oscillation of (1.1).

Lemma 2.1 *Assume that $x \in S^+$ and*

$$\liminf_{t \to \infty} \int_{g_a(t)}^{t} \widetilde{\mu}^{\alpha}\left(g_a(\theta)\right) \Theta(\theta) \,\mathrm{d}\theta \geq \delta \tag{2.17}$$

for some $\delta > 0$. Then

$$\frac{r(g_a(t))}{r(t)} \left(\frac{\aleph'(g_a(t))}{\aleph'(t)} \right)^{\alpha} \geq \vartheta_n(\delta) \tag{2.18}$$

for every $n \geq 0$, where

$$\vartheta_m(\delta) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } m = 0; \\ \exp\left(\delta \vartheta_{m-1}(\delta)\right) & \text{if } m > 0. \end{cases} \tag{2.19}$$

Proof The proof of the first lemma is similar to that of [26, Lemma 1], and hence we omit it. □

Theorem 2.3 *Assume that (2.17) holds for some $\delta < 0$. Every solution of (1.1) is oscillatory if there is a positive function $\rho \in C^1([t_0, \infty))$ satisfying*

$$\limsup_{t \to \infty} \int_{t_1}^{t} \left(\kappa \rho(\theta) \Theta(\theta) - \frac{(\rho'_+(\theta))^{\alpha+1} r(g_a(\theta))}{\beta \vartheta_m(\delta) \rho^{\alpha}(\theta) (g'_a(\theta))^{\alpha}} \right) = \infty, \tag{2.20}$$

for some $m \geq 0$, where $\vartheta_m(\delta)$ is defined as (2.19).

Proof Assume the contrary that there is a nonoscillatory solution x of (1.1). Then we can assume $x \in S^+$, and so $x(t)$, $x(\tau(t))$, and $x(g(t,s))$ are positive for $t \geq t_1 \geq t_0$ and $s \in [a,b]$. It follows from Lemma 1.1 that (1.5) holds. As in the proof of Theorem 2.1, we obtain (2.4). Now, we set

$$\omega \stackrel{\text{def}}{=} \rho r \left(\frac{\aleph'}{\aleph(g_a)} \right)^{\alpha}.$$

Thus, we note that $\omega(t) > 0$ for $t \geq t_1$. By differentiating ω and using (2.4), we obtain

$$\begin{aligned} \omega' &= \rho' r \left(\frac{\aleph'}{\aleph(g_a)} \right)^{\alpha} + \rho \frac{(r(\aleph')^{\alpha})'}{\aleph^{\alpha}(g_a)} - \alpha \rho r \frac{(\aleph')^{\alpha}}{\aleph^{\alpha+1}(g_a)} \aleph'(g_a) g_a' \\ &\leq \frac{\rho'}{\rho} \omega - \kappa \rho \Theta - \alpha \rho r \frac{(\aleph')^{\alpha}}{\aleph^{\alpha+1}(g_a)} \aleph'(g_a) g_a'. \end{aligned}$$

Thus, it follows from Lemma 2.1 that

$$\begin{aligned} \omega' &\leq \frac{\rho'}{\rho} \omega - \kappa \rho \Theta - \alpha \rho \frac{r^{1+1/\alpha} \vartheta_n^{1/\alpha}(\delta)}{r^{1/\alpha}(g_a)} \left(\frac{\aleph'}{\aleph^{\alpha+1}(g_a)} \right)^{\alpha+1} g_a' \\ &\leq \frac{\rho_+'}{\rho} \omega - \kappa \rho \Theta - \alpha \frac{\vartheta_n^{1/\alpha}(\delta) g_a'}{\rho^{1/\alpha} r^{1/\alpha}(g_a)} \omega^{1+1/\alpha}. \end{aligned} \tag{2.21}$$

Using Lemma 1.2 with $A = \rho_+'/\rho$ and $B = \alpha \vartheta_m^{1/\alpha}(\delta)/(\rho r(g))^{1/\alpha}$, we get

$$\omega' \leq -\kappa \rho \Theta + \frac{(\rho_+')^{\alpha+1} r(g_a)}{\beta \vartheta_m(\delta) \rho^{\alpha} (g_a')^{\alpha}}. \tag{2.22}$$

Integrating (2.22) from $t_1 \to t$, we find

$$\omega(t_1) \geq \int_{t_1}^{t} \left(\kappa \rho(\theta) \Theta(\theta) - \frac{(\rho_+'(\theta))^{\alpha+1} r(g_a(\theta))}{\beta \vartheta_m(\delta) \rho^{\alpha}(\theta) (g_a'(\theta))^{\alpha}} \right) \mathrm{d}\theta,$$

which contradicts (2.20). This contradiction completes the proof. □

3 Further results

It is easy to notice that (2.7) is a sharper estimate than (2.2) for the relationship between $\aleph$ and $\aleph'$. By repeating the same steps that improved (2.2), we obtain iterative criteria that can be applied even when the other criteria fail.

Lemma 3.1 *Assume that $x \in S^+$. Then*

$$\aleph(t) \geq U_k(t) r^{1/\alpha}(t) \aleph'(t) \tag{3.1}$$

for $k = 0, 1, \ldots$, where $U_0(t) := \widetilde{\mu}(t)$ and

$$U_{k+1}(t) := \int_{t_1}^{t} \left(\frac{1}{r(s)} \exp\left(\int_s^t \kappa \Theta(\theta) U_n^{\alpha}\big(g_a(\theta)\big) \, \mathrm{d}\theta \right) \right)^{1/\alpha} \mathrm{d}s. \tag{3.2}$$

Proof Assume that $x \in S^+$. Then $x(t)$, $x(\tau(t))$, and $x(g(t,s))$ are positive for $t \geq t_1 \geq t_0$ and $s \in [a,b]$. It follows from Lemma 1.1 that (1.5) holds. By induction, we will prove (3.1).

Now, as in the proof of Theorem 2.1, we obtain (2.4) and (2.7). From (2.7), we obtain

$$\aleph \geq \widetilde{\mu}(t) r^{1/\alpha} \aleph' = U_0 r^{1/\alpha} \aleph'.$$

Next, for $k = n$, we suppose that $\aleph \geq U_n r^{1/\alpha} \aleph'$. Hence, we get

$$\aleph(g_a) \geq U_n(g_a) r^{1/\alpha}(g_a) \aleph'(g_a) \geq U_n(g_a) r^{1/\alpha} \aleph',$$

which with (2.4) gives

$$\kappa \Theta(t) \aleph^\alpha \big(g_a(t)\big)\big(r(t)\big(\aleph'(t)\big)^\alpha\big)' + \kappa \Theta(t) U_n^\alpha \big(g_a(t)\big) r(t)\big(\aleph'(t)\big)^\alpha \leq 0. \tag{3.3}$$

Letting $H := r(\aleph')^\alpha$, (3.3) reduces to

$$H'(t) + \kappa \Theta(t) U_n^\alpha \big(g_a(t)\big) H(t) \leq 0. \tag{3.4}$$

Applying the Grönwall inequality in (3.4), we find

$$H(s) \geq H(t) \exp\left(\int_s^t \kappa \Theta(\theta) U_n^\alpha \big(g_a(\theta)\big)\, \mathrm{d}\theta\right)$$

for $t \geq s \geq t_1$, and so

$$\aleph'(s) \geq r^{1/\alpha}(t) \aleph'(t) \left(\frac{1}{r(s)} \exp\left(\int_s^t \kappa \Theta(\theta) U_n^\alpha \big(g_a(\theta)\big)\, \mathrm{d}\theta\right)\right)^{1/\alpha}. \tag{3.5}$$

Integrating (3.5) from $t_1 \to t$, we see that

$$\begin{aligned} \aleph(t) &\geq r^{1/\alpha}(t) \aleph'(t) \int_{t_1}^t \left(\frac{1}{r(s)} \exp\left(\int_s^t \kappa \Theta(\theta) U_n^\alpha \big(g_a(\theta)\big)\, \mathrm{d}\theta\right)\right)^{1/\alpha} \mathrm{d}s \\ &= U_{n+1}(t) r^{1/\alpha}(t) \aleph'(t). \end{aligned}$$

This completes the proof. □

Theorem 3.1 *Assume that U_k are defined as in Lemma 3.1. Every solution of (1.1) is oscillatory if*

$$\int_{t_0}^\infty \Theta(\theta) U_k(\theta)\, \mathrm{d}\theta = \infty \tag{3.6}$$

for some $k = 0, 1, \ldots$.

Proof Assume the contrary that there is a nonoscillatory solution x of (1.1). Then we can assume $x \in S^+$, and so $x(t)$, $x(\tau(t))$, and $x(g(t,s))$ are positive for $t \geq t_1 \geq t_0$ and $s \in [a,b]$. It follows from Lemma 1.1 and 3.1 that (1.5) and (3.1) hold. As in the proof of Theorem 2.1, we obtain (2.4).

Now, we define w as in (2.11) with $\rho \equiv 1$. Proceeding as in the proof of Theorem 2.2 and substituting (2.7) with (3.1), we arrive at

$$w'(t) + \kappa \Theta(t) U_k(t) + \alpha r^{-1/\alpha}(t) w^{1+1/\alpha}(t) \leq 0 \tag{3.7}$$

or

$$w'(t) + \kappa \Theta(t) U_k(t) \leq 0. \tag{3.8}$$

Integrating (3.8) from $t_1 \to t$, we get

$$w(t) \leq w(t_1) - \kappa \int_{t_1}^{t} \Theta(\theta) U_k(\theta)\,\mathrm{d}\theta.$$

Therefore, $w(t) \to -\infty$ as $t \to \infty$, which is a contradiction. This contradiction completes the proof. □

Theorem 3.2 *Assume that U_k are defined as in Lemma 3.1. Every solution of (1.1) is oscillatory if*

$$\liminf_{t\to\infty} \frac{1}{G(t)} \int_t^{\infty} r^{-1/\alpha}(\theta) G^{1+1/\alpha}(\theta)\,\mathrm{d}\theta > \frac{1}{\beta^{1/\alpha}}, \tag{3.9}$$

where

$$G(t) \stackrel{\text{def}}{=} \int_t^{\infty} \Theta(\theta) U_k(\theta)\,\mathrm{d}\theta.$$

Proof Assume the contrary that there is a nonoscillatory solution x of (1.1). Then we can assume $x \in S^+$, and so $x(t)$, $x(\tau(t))$, and $x(g(t,s))$ are positive for $t \geq t_1 \geq t_0$ and $s \in [a,b]$. By the same procedure as in the proof of Theorem 3.1, we arrive at (3.7). Then, integrating (3.7) from $t \to v$, we find

$$\int_t^{v} \Theta(\theta) U_k(\theta)\,\mathrm{d}\theta + \alpha \int_t^{v} r^{-1/\alpha}(\theta) w^{1+1/\alpha}(\theta)\,\mathrm{d}\theta \leq w(t) - w(v).$$

Letting $v \to \infty$, we get

$$G(t) + \alpha \int_t^{\infty} r^{-1/\alpha}(\theta) w^{1+1/\alpha}(\theta)\,\mathrm{d}\theta \leq w(t), \tag{3.10}$$

or equivalently,

$$1 + \alpha \int_t^{\infty} r^{-1/\alpha}(\theta) G^{1+1/\alpha}(\theta) \left(\frac{w(\theta)}{G(\theta)} \right)^{1+1/\alpha} \mathrm{d}\theta \leq \frac{w(t)}{G(t)}. \tag{3.11}$$

If we set $\varrho = \inf_{t \geq t_1}(w(t)/G(t))$, then we note that $\varrho \geq 1$. However, from (3.9) and (3.11), we get

$$\varrho \geq 1 + \alpha \left(\frac{\varrho}{\alpha+1} \right)^{\frac{\alpha+1}{\alpha}}. \tag{3.12}$$

Therefore, relationship (3.12) can be modeled on the form

$$\frac{\varrho}{\alpha+1} \geq \frac{1}{\alpha+1} + \frac{\alpha}{\alpha+1}\left(\frac{\varrho}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}},$$

which contradicts the possible values of ϱ and α. This contradiction completes the proof. □

For the following theorem, we need to define the sequence $\{\phi_n(t)\}_{n=0}^{\infty}$ as

$$\phi_n(t) \stackrel{\text{def}}{=} \phi_0(t) + \alpha \int_t^{\infty} r^{-1/\alpha}(\theta)\phi_{n-1}^{1+1/\alpha}(\theta)\,\mathrm{d}\theta, \quad n = 1,2,3,\ldots, \tag{3.13}$$

and

$$\phi_0(t) \stackrel{\text{def}}{=} G(t)$$

for all $t \geq t_1 \geq t_0$, where G is defined as in Theorem 3.2.

Lemma 3.2 *Assume that $x \in S^+$ and w is defined as in (2.11) with $\rho \equiv 1$. Then $w(t) \geq \phi_n(t)$. In addition, there is a function $\phi \in C([t_1,\infty),(0,\infty))$ such that $\lim_{n\to\infty}\phi_n(t) = \phi(t)$ and*

$$\phi(t) = \phi_0(t) + \alpha \int_t^{\infty} r^{-1/\alpha}(\theta)\phi^{1+1/\alpha}(\theta)\,\mathrm{d}\theta. \tag{3.14}$$

Proof Assume that $x \in S^+$. Then $x(t)$, $x(\tau(t))$, and $x(g(t,s))$ are positive for $t \geq t_1 \geq t_0$ and $s \in [a,b]$. By the same procedure as in the proof of Theorem 3.2, we arrive at (3.10), and hence

$$\phi_0(t) = G(t) \leq w(t).$$

Thus, from the definition of $\phi_n(t)$, we note that $w(t) \geq \phi_n(t)$ for all $n > 1$ and $t \geq t_1$. Since $\{\phi_n(t)\}_{n=0}^{\infty}$ is an increasing sequence and bounded from above, $\phi_n(t)$ converges to $\phi(t)$. By using Lebesgue's monotone convergence theorem, if we take the limit of (3.13) as $n \to \infty$, then we obtain that (3.14) hold. The proof is complete. □

Theorem 3.3 *Assume that U_k are defined as in Lemma 3.1. Every solution of (1.1) is oscillatory if*

$$\limsup_{t\to\infty} \phi_n(t)\left(\int_{t_0}^{t} r^{-\frac{1}{\alpha}}(s)\,\mathrm{d}s\right)^{\alpha} > 1 \tag{3.15}$$

for some positive integers n.

Proof Assume the contrary that there is a nonoscillatory solution x of (1.1). Then we can assume $x \in S^+$, and so $x(t)$, $x(\tau(t))$, and $x(g(t,s))$ are positive for $t \geq t_1 \geq t_0$ and $s \in [a,b]$. Now, we define w as in (2.11) with $\rho \equiv 1$. From the fact that $(r(s)(\aleph'(s))^{\alpha})' \leq 0$, we have

$$\aleph(t) = \aleph(t_1) + \int_{t_1}^{t} \frac{1}{r^{1/\alpha}(\theta)}\left(r(\theta)\left(\aleph'(\theta)\right)^{\alpha}\right)^{1/\alpha}\mathrm{d}\theta$$

$$\geq \left(r(t)\left(\aleph'(t)\right)^{\alpha}\right)^{1/\alpha} \int_{t_1}^{t} \frac{1}{r^{1/\alpha}(\theta)}\,\mathrm{d}\theta,$$

which with (2.11) gives

$$\begin{aligned} w(t) &= r(t)\left(\aleph'(t)\right)^{\alpha}\aleph^{-\alpha}(t) \\ &\leq r(t)\left(\aleph'(t)\right)^{\alpha}\left(\left(r(t)\left(\aleph'(t)\right)^{\alpha}\right)^{1/\alpha}\int_{t_1}^{t}\frac{1}{r^{1/\alpha}(\theta)}\,\mathrm{d}\theta\right)^{-\alpha} \\ &= \left(\int_{t_1}^{t}\frac{1}{r^{1/\alpha}(\theta)}\,\mathrm{d}\theta\right)^{-\alpha}. \end{aligned}$$

Thus,

$$w(t)\left(\int_{t_0}^{t}\frac{1}{r^{1/\alpha}(\theta)}\,\mathrm{d}\theta\right)^{\alpha} \leq \left(\frac{\int_{t_0}^{t} r^{-1/\alpha}(\theta)\,\mathrm{d}\theta}{\int_{t_1}^{t} r^{-1/\alpha}(\theta)\,\mathrm{d}\theta}\right)^{\alpha} \leq 1$$

for $t \geq t_1$. Taking into account Lemma 3.2, we get a contradiction with (3.15). This contradiction completes the proof. □

4 Examples

In this section, we apply our main results to some special cases of (1.1) and also compare our results with the previous related results.

Example 4.1 Consider the second-order NDDE

$$\left(\left(\left(x(t)+p_0 x\circ\tau\right)'\right)^{\alpha}\right)' + \Lambda\left[\frac{q_0}{t^{\alpha+1}}\cdot(x\circ g)^{\alpha};a,1\right] = 0, \tag{4.1}$$

where $p_0 \in [0,1)$, $q_0 > 0$, $\delta \in [0,1)$, $\tau(t) = \eta t$, $\eta \in (0,1)$, and $g(t,s) = st$ for $s \in [a,1]$. Obviously, we see that

$$\begin{aligned} &b = 1, \qquad r(t) \equiv 1, \qquad p(t) \equiv p_0, \qquad q(t,s) \equiv q_0/t^{\alpha+1} \quad \text{and} \\ &f(x) \equiv x^{\alpha}, \quad \text{with constant } \kappa = 1. \end{aligned}$$

Therefore, it is easy to verify that

$$\begin{aligned} &\Theta(t) = \frac{q_0}{t^{\alpha+1}}(1-a)(1-p_0)^{\alpha}, \qquad \mu(t) = t, \qquad \widetilde{\mu}(t) = (1+\lambda)t, \\ &\widehat{\mu}(t) = a^{1/(1+\lambda)}, \quad \text{and} \quad G(t) = \frac{\lambda a^{\alpha/(1+\lambda)}}{a^{\alpha}}\frac{1}{t^{\alpha}}, \end{aligned}$$

where

$$\lambda \stackrel{\text{def}}{=} \frac{1}{\alpha}a^{\alpha}(1-a)(1-p_0)^{\alpha}q_0.$$

Next, to apply Corollary 2.1, we must first check either condition (2.9) or (2.8). By substitution and a simple computation, we obtain

$$\begin{aligned}\liminf_{t\to\infty}\int_{g_a(t)}^{t}\widetilde{\mu}^{\alpha}\big(g_a(\theta)\big)\Theta(\theta)\,\mathrm{d}\theta &= \alpha\lambda(1+\lambda)^{\alpha}\liminf_{t\to\infty}\int_{at}^{t}\frac{1}{\theta}\,\mathrm{d}\theta\\ &= \alpha\lambda(1+\lambda)^{\alpha}\ln\frac{1}{a}.\end{aligned}$$

Thus, by using Corollary 2.1, (4.1) is oscillatory if

$$\alpha\lambda(1+\lambda)^{\alpha}\ln\frac{1}{a} > \frac{1}{\mathrm{e}}.$$

On the other hand, condition (3.9) with $k = 1$ reduces to

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{G(t)}\int_{t}^{\infty}r^{-1/\alpha}(\theta)G^{1+1/\alpha}(\theta)\,\mathrm{d}\theta &= \left(\frac{\lambda}{\alpha}\frac{a^{\alpha+\alpha/(1+\lambda)}}{\alpha}\right)^{1/\alpha}\liminf_{t\to\infty}t^{\alpha}\int_{t}^{\infty}\frac{1}{\theta^{\alpha+1}}\,\mathrm{d}\theta\\ &= \frac{1}{\alpha}\left(\frac{\lambda}{\alpha}\frac{a^{\alpha+\alpha/(1+\lambda)}}{\alpha}\right)^{1/\alpha} > \frac{1}{\beta^{1/\alpha}}.\end{aligned}$$

From Theorem 3.2, equation (4.1) is oscillatory if

$$\lambda a^{\alpha+\alpha/(1+\lambda)} > \frac{\alpha^{\alpha+2}}{\beta}.$$

Example 4.2 Consider the second-order NDDE

$$\Big(\Big(\big(x(t)+(1-\delta)x(\eta t)\big)'\Big)^{\alpha}\Big)' + \frac{q_0}{t^{\alpha+1}}x^{\alpha}(\lambda t) = 0, \tag{4.2}$$

where $\delta\in(0,1]$, $q_0 > 0$, and $\eta,\lambda\in(0,1)$. Obviously, we see that

$$\begin{aligned}&a = 0,\qquad b = 1,\qquad r(t)\equiv 1,\qquad p(t)\equiv 1-\gamma,\qquad q(t,s)\equiv q_0/t^{\alpha+1},\qquad \tau(t)\equiv\eta t,\\ &g(t,s)\equiv\lambda t\quad\text{and}\quad f(x)\equiv x^{\alpha},\quad\text{with constant }\kappa = 1.\end{aligned}$$

Therefore, it is easy to verify that

$$\Theta(t) = \delta^{\alpha}q_0\frac{1}{t^{\alpha+1}},\qquad \mu(t) = t,\qquad U_0(t) = \left(1+\frac{1}{\alpha}\lambda^{\alpha}\delta^{\alpha}q_0\right)t$$

and

$$U_1(t) = \lambda^{\alpha/(1+\frac{1}{\alpha}\lambda^{\alpha}\delta^{\alpha}q_0)}.$$

From Theorem 3.2, equation (4.1) is oscillatory if

$$q_0\delta^{\alpha}\lambda^{\alpha/(1+\frac{1}{\alpha}\lambda^{\alpha}\delta^{\alpha}q_0)} > \frac{\alpha^{\alpha+1}}{\beta}. \tag{4.3}$$

Remark 4.1 Consider the special case of (4.2) where $\delta = 1$, $\alpha = 1/3$, and $\lambda = 0.9$. The oscillation criteria in [13, Corollary 2] and [17, Corollary 2.1] reduce to $q_0 > 3.61643$ and $q_0 > 1.92916$, respectively. However, (4.3) reduces to $q_0 > 0.16131$. So, our results improve and extend some of the previous results.

5 Conclusion

The oscillation theory of DDEs has many applications in applied sciences. Thus, studying the oscillation of the solutions of these equations has practical importance besides the theoretical importance. In this study, we obtained different oscillation criteria with different techniques. These new criteria enable us to test the oscillation of a class of NDDEs with continuous delay. Our results extended to recently published works [14, 15], and also improved [13, 17].

Modeling by fractional-order differential equations has more advantages than by classical integer-order ones as it considers the effects of existence of time memory or long-range space interactions. So, it would be interesting to extend the results of this paper to the fractional delay differential equations. Moreover, it is interesting to study the periodicity behavior of solutions of the studied equation as an extension of the works [19, 23].

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