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DERIVATION–HOMOMORPHISM FUNCTIONAL INEQUALITIES

CHOONKIL PARK

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Abstract. In this paper, we introduce and solve the following additive-additive (s,t)-functional inequality

$$\begin{aligned} &\|g(x+y)-g(x)-g(y)\|+\|h(x+y)+h(x-y)-2h(x)\| \\ &\leqslant \left\|s\left(2g\left(\frac{x+y}{2}\right)-g(x)-g(y)\right)\right\|+\left\|t\left(2h\left(\frac{x+y}{2}\right)+2h\left(\frac{x-y}{2}\right)-2h(x)\right)\right\|, \end{aligned} \tag{1}$$

where s and t are fixed nonzero complex numbers with $|s|<1$ and $|t|<1$. Using the direct method and the fixed point method, we prove the Hyers-Ulam stability of derivation-homomorphisms in complex Banach algebras, associated to the additive-additive (s,t)-functional inequality (1) and the following functional inequality

$$\|g(xy)-g(x)y-xg(y)\|+\|h(xy)-h(x)h(y)\| \leqslant \varphi(x,y). \tag{2}$$

1. Introduction and preliminaries

Let A be a complex Banach algebra. A $\mathbb{C}$-linear mapping $g:A\to A$ is a derivation if $g:A\to A$ satisfies

$$g(xy)=g(x)y+xg(y)$$

for all $x,y\in A$, and a $\mathbb{C}$-linear mapping $h:A\to A$ is a homomorphism if $h:A\to A$ satisfies

$$h(xy)=h(x)h(y)$$

for all $x,y\in A$.

The stability problem of functional equations originated from a question of Ulam [25] concerning the stability of group homomorphisms. Hyers [14] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [23] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [11] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

Gilányi [12] showed that if f satisfies the functional inequality

$$\|2f(x)+2f(y)-f(x-y)\| \leqslant \|f(x+y)\| \tag{3}$$

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then f satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)=f(x+y)+f(x-y).$$

See also [24]. Fechner [10] and Gilányi [13] proved the Hyers-Ulam stability of the functional inequality (3). Park [19, 20] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations and functional inequalities have been extensively investigated by a number of authors (see [7, 8, 17]).

We recall a fundamental result in fixed point theory.

THEOREM 1.1. [3, 6] *Let (X,d) be a complete generalized metric space and let $J:X\to X$ be a strictly contractive mapping with Lipschitz constant $\alpha<1$. Then for each given element $x\in X$, either*

$$d(J^nx,J^{n+1}x)=\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(1) $d(J^nx,J^{n+1}x)<\infty, \qquad \forall n\geqslant n_0$;
(2) *the sequence $\{J^nx\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y=\{y\in X \mid d(J^{n_0}x,y)<\infty\}$;*
(4) *$d(y,y^*)\leqslant\frac{1}{1-\alpha}d(y,Jy)$ for all $y\in Y$.*

In 1996, Isac and Rassias [15] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 5, 9, 21, 22]).

In this paper, we solve the additive-additive (s,t)-functional inequality (1). Furthermore, we investigate derivations and homomorphisms in complex Banach algebras associated to the additive-additive (s,t)-functional inequality (1) and the functional inequality (2) by using the direct method and by the fixed point method.

Throughout this paper, assume that A is a complex Banach algebra and that s and t are fixed nonzero complex numbers with $|s|<1$ and $|t|<1$.

2. Stability of additive-additive (s,t)-functional inequality (1): a direct method

In this section, we solve and investigate the additive-additive (s,t)-functional inequality (1) in complex Banach algebras.

LEMMA 2.1. *If mappings $g,h:A\to A$ satisfy $g(0)=h(0)=0$ and*

$$\begin{aligned}&\|g(x+y)-g(x)-g(y)\|+\|h(x+y)+h(x-y)-2h(x)\| \qquad (4)\\ &\leqslant\left\|s\left(2g\left(\frac{x+y}{2}\right)-g(x)-g(y)\right)\right\|+\left\|t\left(2h\left(\frac{x+y}{2}\right)+2h\left(\frac{x-y}{2}\right)-2h(x)\right)\right\|\end{aligned}$$

for all $x,y\in A$, then the mappings $g,h:A\to A$ are additive.

Proof. Letting $x = y$ in (4), we get

$$\|g(2x) - 2g(x)\| + \|h(2x) - 2h(x)\| \leqslant 0$$

for all $x \in A$. So $g(2x) = 2g(x)$ and $h(2x) = 2h(x)$ for all $x \in A$. It follows from (4) that

$$\begin{aligned}&\|g(x+y) - g(x) - g(y)\| + \|h(x+y) + h(x-y) - 2h(x)\|\\ &\leqslant \|s(g(x+y) - g(x) - g(y))\| + \|t(h(x+y) - h(x) - h(y))\|\end{aligned}$$

for all $x, y \in A$. Thus $g(x+y) - g(x) - g(y) = 0$ and $h(x+y) + h(x-y) - 2h(x) = 0$ for all $x \in A$, since $|s| < 1$ and $|t| < 1$. So the mappings $g, h : A \to A$ are additive. $\square$

LEMMA 2.2. [18, Theorem 2.1] *Let $f : A \to A$ be an additive mapping such that*

$$f(\lambda a) = \lambda f(a)$$

for all $\lambda \in \mathbb{T}^1 := \{\xi \in \mathbb{C} : |\xi| = 1\}$ and all $a \in A$. Then the mapping $f : A \to A$ is $\mathbb{C}$-linear.

Using the direct method, we prove the Hyers-Ulam stability of pairs of derivations and homomorphisms in complex Banach algebras associated to the additive-additive (s,t)-functional inequality (4).

THEOREM 2.3. *Let $\varphi : A^2 \to [0, \infty)$ be a function such that*

$$\sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{5}$$

for all $x, y \in A$. Let $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$ and

$$\begin{aligned}&\|g(\lambda(x+y)) - \lambda g(x) - \lambda g(y)\| + \|h(\lambda(x+y)) + h(\lambda(x-y)) - 2\lambda h(x)\|\\ &\leqslant \left\|s\left(2g\left(\lambda\frac{x+y}{2}\right) - \lambda g(x) - \lambda g(y)\right)\right\| \\ &\quad + \left\|t\left(2h\left(\lambda\frac{x+y}{2}\right) + 2h\left(\lambda\frac{x-y}{2}\right) - 2\lambda h(x)\right)\right\| + \varphi(x,y)\end{aligned} \tag{6}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y \in A$. If the mappings $g, h : A \to A$ satisfy

$$\|g(xy) - g(x)y - xg(y)\| + \|h(xy) - h(x)h(y)\| \leqslant \varphi(x,y) \tag{7}$$

for all $x, y \in A$, then there exist a unique derivation $D : A \to A$ and a unique homomorphism $H : A \to A$ such that

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \leqslant \sum_{j=1}^{\infty} 2^{j-1} \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) \tag{8}$$

for all $x \in A$.

Proof. Letting $\lambda = 1$ and $y = x$ in (6), we get

$$\|g(2x) - 2g(x)\| + \|h(2x) - 2h(x)\| \leqslant \varphi(x,x) \tag{9}$$

and so

$$\left\|g(x) - 2g\left(\frac{x}{2}\right)\right\| + \left\|h(x) - 2h\left(\frac{x}{2}\right)\right\| \leqslant \varphi\left(\frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in A$. Thus

$$\begin{aligned} &\left\|2^l g\left(\frac{x}{2^l}\right) - 2^m g\left(\frac{x}{2^m}\right)\right\| + \left\|2^l h\left(\frac{x}{2^l}\right) - 2^m h\left(\frac{x}{2^m}\right)\right\| \\ &\quad \leqslant \sum_{j=l}^{m-1} \left\|2^j g\left(\frac{x}{2^j}\right) - 2^{j+1} g\left(\frac{x}{2^{j+1}}\right)\right\| + \sum_{j=l}^{m-1} \left\|2^j h\left(\frac{x}{2^j}\right) - 2^{j+1} h\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\quad \leqslant \sum_{j=l+1}^{m} 2^{j-1} \varphi\left(\frac{x}{2^j}, \frac{x}{2^j}\right) \end{aligned} \tag{10}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (10) that the sequences $\{2^k g(\frac{x}{2^k})\}$ and $\{2^k h(\frac{x}{2^k})\}$ are Cauchy for all $x \in A$. Since Y is a Banach space, the sequences $\{2^k g(\frac{x}{2^k})\}$ and $\{2^k h(\frac{x}{2^k})\}$ converge. So one can define the mappings $D, H : A \to A$ by

$$D(x) := \lim_{k\to\infty} 2^k g\left(\frac{x}{2^k}\right),$$

$$H(x) := \lim_{k\to\infty} 2^k h\left(\frac{x}{2^k}\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing to the limit $m \to \infty$ in (10), we get (8).

It folllows from (6) that

$$\begin{aligned} &\|D(\lambda(x+y)) - \lambda D(x) - \lambda D(y)\| + \|H(\lambda(x+y)) + H(\lambda(x-y)) - 2\lambda H(x)\| \\ &= \lim_{n\to\infty} 2^n \left\|g\left(\lambda\frac{x+y}{2^n}\right) - \lambda g\left(\frac{x}{2^n}\right) - \lambda g\left(\frac{y}{2^n}\right)\right\| \\ &\quad + \lim_{n\to\infty} 2^n \left\|h\left(\lambda\frac{x+y}{2^n}\right) + h\left(\lambda\frac{x-y}{2^n}\right) - 2\lambda h\left(\frac{x}{2^n}\right)\right\| \\ &\leqslant \lim_{n\to\infty} 2^n \left\|s\left(2g\left(\lambda\frac{x+y}{2^{n+1}}\right) - \lambda g\left(\frac{x}{2^n}\right) - \lambda g\left(\frac{y}{2^n}\right)\right)\right\| \\ &\quad + \lim_{n\to\infty} 2^n \left\|t\left(2h\left(\lambda\frac{x+y}{2^{n+1}}\right) + 2h\left(\lambda\frac{x-y}{2^{n+1}}\right) - 2\lambda h\left(\frac{x}{2^n}\right)\right)\right\| + \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\ &= \left\|s\left(2D\left(\lambda\frac{x+y}{2}\right) - \lambda D(x) - \lambda D(y)\right)\right\| \\ &\quad + \left\|t\left(2H\left(\lambda\frac{x+y}{2}\right) + 2H\left(\lambda\frac{x-y}{2}\right) - 2\lambda H(x)\right)\right\| \end{aligned}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y \in A$. So

$$\begin{aligned} &\|D(\lambda(x+y)) - \lambda D(x) - \lambda D(y)\| + \|H(\lambda(x+y)) + H(\lambda(x-y)) - 2\lambda H(x)\| \quad (11) \\ &\leqslant \left\| s\left(2D\left(\lambda\frac{x+y}{2}\right) - \lambda D(x) - \lambda D(y)\right)\right\| \\ &\quad + \left\| t\left(2H\left(\lambda\frac{x+y}{2}\right) + 2H\left(\lambda\frac{x-y}{2}\right) - 2\lambda H(x)\right)\right\| \end{aligned}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y \in A$.

Let $\lambda = 1$ in (11). By Lemma 2.1, the mappings $D, H : A \to A$ are additive.

It follows from (11) and the additivity of D and H that

$$\begin{aligned} &\|D(\lambda(x+y)) - \lambda D(x) - \lambda D(y)\| + \|H(\lambda(x+y)) - H(\lambda(x-y)) - 2\lambda H(y)\| \\ &\leqslant \|s(D(\lambda(x+y)) - \lambda D(x) - \lambda D(y))\| + \|t(H(\lambda(x+y)) - H(\lambda(x-y)) - 2\lambda H(y))\| \end{aligned}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y \in A$. Since $|s| < 1$ and $|t| < 1$,

$$\begin{aligned} D(\lambda(x+y)) - \lambda D(x) - \lambda D(y) &= 0, \\ H(\lambda(x+y)) - H(\lambda(x-y)) - 2\lambda H(y) &= 0 \end{aligned}$$

and so $D(\lambda x) = \lambda D(x)$ and $H(\lambda x) = \lambda H(x)$ for all $\lambda \in \mathbb{T}^1$ and all $x, y \in A$. Thus by Lemma 2.2, the additive mappings $D, H : A \to A$ are $\mathbb{C}$-linear.

It follows from (7) and the additivity of D, H that

$$\begin{aligned} &\|D(xy) - D(x)y - xD(y)\| + \|H(xy) - H(x)H(y)\| \\ &= 4^n \left\| g\left(\frac{xy}{4^n}\right) - g\left(\frac{x}{2^n}\right)\frac{y}{2^n} - \frac{x}{2^n} g\left(\frac{y}{2^n}\right)\right\| + 4^n \left\| h\left(\frac{xy}{4^n}\right) - h\left(\frac{x}{2^n}\right) h\left(\frac{y}{2^n}\right)\right\| \\ &\leqslant 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right), \end{aligned}$$

which tends to zero as $n \to \infty$, by (5). So

$$\begin{aligned} D(xy) - D(x)y - xD(y) &= 0, \\ H(xy) - H(x)H(y) &= 0 \end{aligned}$$

for all $x, y \in A$. Hence the mapping $D : A \to A$ is a derivation and the mapping $H : A \to A$ is a homomorphism. $\square$

COROLLARY 2.4. *Let $r > 2$ and θ be nonnegative real numbers and $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$ and*

$$\begin{aligned} &\|g(\lambda(x+y)) - \lambda g(x) - \lambda g(y)\| + \|h(\lambda(x+y)) + h(\lambda(x-y)) - 2\lambda h(x)\| \\ &\quad \leqslant \left\| s\left(2g\left(\lambda\frac{x+y}{2}\right) - \lambda g(x) - \lambda g(y)\right)\right\| \quad (12) \\ &\quad + \left\| t\left(2h\left(\lambda\frac{x+y}{2}\right) + 2h\left(\lambda\frac{x-y}{2}\right) - 2\lambda h(x)\right)\right\| + \theta(\|x\|^r + \|y\|^r) \end{aligned}$$

for all $\lambda \in \mathbb{T}^1$ *and all* $x, y \in A$. *If the mappings* $g, h : A \to A$ *satisfy*

$$\|g(xy) - g(x)y - xg(y)\| + \|h(xy) - h(x)h(y)\| \leqslant \theta(\|x\|^r + \|y\|^r) \tag{13}$$

for all $x, y \in A$, *then there exist a unique derivation* $D : A \to A$ *and a unique homomorphism* $H : A \to A$ *such that*

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \leqslant \frac{2\theta}{2^r - 2}\|x\|^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.3 by $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in A$. □

THEOREM 2.5. *Let* $\varphi : A^2 \to [0, \infty)$ *be a function and* $g, h : A \to A$ *be mappings satisfying* $g(0) = h(0) = 0$, (6), (7) *and*

$$\Phi(x,y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y) < \infty \tag{14}$$

for all $x, y \in A$. *Then there exist a unique derivation* $D : A \to A$ *and a unique homomorphism* $H : A \to A$ *such that*

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \leqslant \frac{1}{2}\Phi(x,x) \tag{15}$$

for all $x \in A$.

Proof. It follows from (9) that

$$\left\|g(x) - \frac{1}{2}g(2x)\right\| + \left\|h(x) - \frac{1}{2}h(2x)\right\| \leqslant \frac{1}{2}\varphi(x,x) \tag{16}$$

for all $x \in A$. Thus

$$\begin{aligned} &\left\|\frac{1}{2^l}g\left(\frac{x}{2^l}\right) - \frac{1}{2^m}g\left(2^m x\right)\right\| + \left\|\frac{1}{2^l}h\left(2^l x\right) - \frac{1}{2^m}h\left(2^m x\right)\right\| \\ &\leqslant \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}g\left(2^j x\right) - \frac{1}{2^{j+1}}g\left(2^{j+1}x\right)\right\| + \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}h\left(2^j x\right) - \frac{1}{2^{j+1}}h\left(2^{j+1}x\right)\right\| \\ &\leqslant \frac{1}{2}\sum_{j=l}^{m-1}\frac{1}{2^j}\varphi\left(2^j x, 2^j x\right) \end{aligned} \tag{17}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (17) that the sequences $\{\frac{1}{2^k}g(2^k x)\}$ and $\{\frac{1}{2^k}h(2^k x)\}$ are Cauchy for all $x \in A$. Since Y is a

Banach space, the sequences $\{\frac{1}{2^k}g(2^kx)\}$ and $\{\frac{1}{2^k}h(2^kx)\}$ converge. So one can define the mappings $D,H : A \to A$ by

$$D(x) := \lim_{k\to\infty} \frac{1}{2^k} g\left(2^k x\right),$$

$$H(x) := \lim_{k\to\infty} \frac{1}{2^k} h\left(2^k x\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing to the limit $m \to \infty$ in (17), we get (15).

By the same reasoning as in the proof of Theorem 2.3, one can show that the mappings $D,H : A \to A$ are $\mathbb{C}$-linear.

It follows from (7) and the additivity of D,H that

$$\begin{aligned}
&\|D(xy) - D(x)y - xD(y)\| + \|H(xy) - H(x)H(y)\| \\
&= \frac{1}{4^n} \|g(4^n xy) - g(2^n x)(2^n y) - (2^n x)g(2^n y)\| + \frac{1}{4^n} \|h(4^n xy) - h(2^n x)h(2^n y)\| \\
&\leqslant \frac{1}{4^n} \varphi(2^n x, 2^n y),
\end{aligned}$$

which tends to zero as $n \to \infty$, by (14). So

$$\begin{aligned}
D(xy) - D(x)y - xD(y) &= 0, \\
H(xy) - H(x)H(y) &= 0
\end{aligned}$$

for all $x,y \in A$. Hence the mapping $D : A \to A$ is a derivation and the mapping $H : A \to A$ is a homomorphism. $\square$

COROLLARY 2.6. *Let $r < 1$ and θ be nonnegative real numbers and $g,h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$, (12) and (13). Then there exist a unique derivation $D : A \to A$ and a unique homomorphism $H : A \to A$ such that*

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \leqslant \frac{2\theta}{2-2^r}\|x\|^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.5 by $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x,y \in A$. $\square$

3. Stability of additive-additive (s,t)-functional inequality (1): a fixed point method

Using the fixed point method, we prove the Hyers-Ulam stability of pairs of derivations and homomorphisms in complex Banach algebras associated to the additive-additive (s,t)-functional inequality (1).

THEOREM 3.1. *Let $\varphi : A^2 \to [0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi\left(\frac{x}{2},\frac{y}{2}\right) \leqslant \frac{L}{4}\varphi(x,y) \leqslant \frac{L}{2}\varphi(x,y) \tag{18}$$

for all $x,y \in A$. Let $g,h : A \to A$ be mappings satisfying $g(0)=h(0)=0$, (6) and (7). Then there exist a unique derivation $D : A \to A$ and a unique homomorphism $H : A \to A$ such that

$$\|g(x)-D(x)\| + \|h(x)-H(x)\| \leqslant \frac{L}{2(1-L)}\varphi(x,x) \tag{19}$$

for all $x \in A$.

Proof. It follows from (18) that

$$\sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j},\frac{y}{2^j}\right) \leqslant \sum_{j=1}^{\infty} 4^j \frac{L^j}{4^j}\varphi(x,y) = \frac{L}{1-L}\varphi(x,y) < \infty$$

for all $x,y \in A$. By Theorem 2.3, there exist a unique derivation $D : A \to A$ and a unique homomorphism $H : A \to A$ satisfying (8).

Letting $\lambda = 1$ and $y = x$ in (6), we get

$$\|g(2x)-2g(x)\| + \|h(2x)-2h(x)\| \leqslant \varphi(x,x) \tag{20}$$

for all $x \in A$.

Consider the set

$$S := \{(g,h) : (A,A) \to (A,A),\ \ g(0)=h(0)=0\}$$

and introduce the generalized metric on S:

$$d((g,h),(g_1,h_1)) = \inf\{\mu \in \mathbb{R}_+ : \|g(x)-g_1(x)\| + \|h(x)-h_1(x)\| \leqslant \mu\varphi(x,x),\ \ \forall x \in A\},$$

where, as usual, $\inf\phi = +\infty$. It is easy to show that (S,d) is complete (see [16]).

Now we consider the linear mapping $J : S \to S$ such that

$$J(g,h)(x) := \left(2g\left(\frac{x}{2}\right), 2h\left(\frac{x}{2}\right)\right)$$

for all $x \in A$.

Let $(g,h),(g_1,h_1) \in S$ be given such that $d((g,h),(g_1,h_1)) = \varepsilon$. Then

$$\|g(x)-g_1(x)\| + \|h(x)-h_1(x)\| \leqslant \varepsilon\varphi(x,x)$$

for all $x \in A$. Since

$$\begin{aligned}&\leqslant \left\|2g\left(\frac{x}{2}\right)-2g_1\left(\frac{x}{2}\right)\right\| + \left\|2h\left(\frac{x}{2}\right)-2h_1\left(\frac{x}{2}\right)\right\| \\ &\leqslant 2\varepsilon\varphi\left(\frac{x}{2},\frac{x}{2}\right) \leqslant 2\varepsilon\frac{L}{2}\varphi(x,x) = L\varepsilon\varphi(x,x)\end{aligned}$$

for all $x \in A$, $d(J(g,h),J(g_1,h_1)) \leqslant L\varepsilon$. This means that

$$d(J(g,h),J(g_1,h_1)) \leqslant Ld((g,h),(g_1,h_1))$$

for all $(g,h),(g_1,h_1) \in S$.

It follows from (20) that

$$\left\|g(x) - 2g\left(\frac{x}{2}\right)\right\| + \left\|h(x) - 2h\left(\frac{x}{2}\right)\right\| \leqslant \varphi\left(\frac{x}{2},\frac{x}{2}\right) \leqslant \frac{L}{2}\varphi(x,x)$$

for all $x \in A$. So $d((g,h),(Jg,Jh)) \leqslant \frac{L}{2}$.

By Theorem 1.1, there exist mappings $D,H : A \to A$ satisfying the following:

(1) (D,H) is a fixed point of J, i.e.,

$$D(x) = 2D\left(\frac{x}{2}\right), \qquad H(x) = 2H\left(\frac{x}{2}\right) \tag{21}$$

for all $x \in A$. The mapping (D,H) is a unique fixed point of J. This implies that (D,H) is a unique mapping satisfying (21) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \leqslant \mu\varphi(x,x)$$

for all $x \in A$;

(2) $d(J^l(g,h),(D,H)) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^l g\left(\frac{x}{2^l}\right) = D(x), \quad \lim_{l\to\infty} 2^l h\left(\frac{x}{2^l}\right) = H(x)$$

for all $x \in A$;

(3) $d((g,h),(D,H)) \leqslant \frac{1}{1-L}d((g,h),J(g,h))$, which implies

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \leqslant \frac{L}{2(1-L)}\varphi(x,x)$$

for all $x \in A$. Thus we get the inequalitiy (19).

The rest of the proof is the same as in the proof of Theorem 2.3. □

COROLLARY 3.2. *Let $r > 2$ and θ be nonnegative real numbers and $g,h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$, (12) and (13). Then there exist a unique derivation $D : A \to A$ and a unique homomorphism $H : A \to A$ such that*

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \leqslant \frac{2\theta}{2^r - 2}\|x\|^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.1 by taking $L = 2^{1-r}$ and $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x,y \in A$. □

THEOREM 3.3. *Let* $\varphi : A^2 \to [0,\infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x,y) \leqslant 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{22}$$

for all $x, y \in A$. *Let* $g, h : A \to A$ *be mappings satisfying* $g(0) = h(0) = 0$, (6) *and* (7). *Then there exist a unique derivation* $D : A \to A$ *and a unique homomorphism* $H : A \to A$ *such that*

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \leqslant \frac{1}{2(1-L)}\varphi(x,x) \tag{23}$$

for all $x \in A$.

Proof. It follows from (22) that

$$\sum_{j=1}^{\infty} \frac{1}{4^j}\varphi\left(2^j x, 2^j y\right) \leqslant \sum_{j=1}^{\infty} \frac{1}{4^j}(4L)^j \varphi(x,y) = \frac{L}{1-L}\varphi(x,y) < \infty$$

for all $x, y \in A$. By Theorem 2.5, there exist a unique derivation $D : A \to A$ and a unique homomorphism $H : A \to A$ satisfying (15).

Let (S,d) be the generalized metric space defined in the proof of Theorem 3.1.

Now we consider the linear mapping $J : S \to S$ such that

$$J(g,h)(x) := \left(\frac{1}{2}g(2x), \frac{1}{2}h(2x)\right)$$

for all $x \in A$.

It follows from (20) that

$$\left\|g(x) - \frac{1}{2}g(2x)\right\| + \left\|h(x) - \frac{1}{2}h(2x)\right\| \leqslant \frac{1}{2}\varphi(x,x)$$

for all $x \in A$. Thus we get the inequalitiy (23).

The rest of the proof is similar to the proof of Theorem 3.1. $\square$

COROLLARY 3.4. *Let* $r < 1$ *and* θ *be nonnegative real numbers and* $g, h : A \to A$ *be mappings satisfying* $g(0) = h(0) = 0$, (12) *and* (13). *Then there exist a unique derivation* $D : A \to A$ *and a unique homomorphism* $H : A \to A$ *such that*

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \leqslant \frac{2\theta}{2-2^r}\|x\|^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.3 by taking $L = 2^{r-1}$ and $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in A$. $\square$

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Choonkil Park
Research Institute for Natural Sciences
Hanyang University
Seoul 04763, Korea
e-mail: baak@@hanyang.ac.kr