Physics Letters B 749 (2015) 108–114

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Gauge invariance and holographic renormalization

Keun-Young Kim [a], Kyung Kiu Kim [a], Yunseok Seo [b], Sang-Jin Sin [c,d,*]

[a] School of Physics and Chemistry, Gwangju Institute of Science and Technology, Gwangju 500-712, Republic of Korea
[b] Research Institute for Natural Science, Hanyang University, Seoul 133-791, Republic of Korea
[c] Department of Physics, Hanyang University, Seoul 133-791, Republic of Korea
[d] School of Physics, Korea Institute for Advanced Study, Seoul 130-722, Republic of Korea

ARTICLE INFO

Article history:
Received 5 March 2015
Received in revised form 20 July 2015
Accepted 22 July 2015
Available online 28 July 2015
Editor: M. Cvetič

Keywords:
Gauge/gravity duality
Holographic renormalization
Gauge invariance

ABSTRACT

We study the gauge invariance of physical observables in holographic theories under the local diffeomorphism. We find that gauge invariance is intimately related to the holographic renormalization: the local counter terms defined in the boundary cancel most of gauge dependences of the on-shell action as well as the divergences. There is a mismatch in the degrees of freedom between the bulk theory and the boundary one. We resolve this problem by noticing that there is a residual gauge symmetry (RGS). By extending the RGS such that it satisfies infalling boundary condition at the horizon, we can understand the problem in the context of general holographic embedding of a global symmetry at the boundary into the local gauge symmetry in the bulk.

1. Introduction

According to AdS/CFT correspondence, any global symmetry at the boundary theory is lifted to a local symmetry in the bulk [1,2]. The gauge symmetry is essential to reduce the degree of freedom which is enlarged by going into one higher dimension. The physical goal in holography is the boundary quantities which do not know the presence of higher dimension or gauge degrees of freedom, while we use the tools in the bulk theory. Therefore the gauge invariance of a physical quantity is a critical issue for the validity of the AdS/CFT. Also tracing the gauge invariance gives much intuition on the way how holography actually works, especially how global symmetry is encoded in the local gauge symmetry.

One can find gauge invariant combinations of the fields, and express the physical quantities in terms of such master variables, however, it is not always easy to find such gauge invariant combination. Even in the case they are available, it is not very convenient to use such fields, especially if many fields are coupled, because the physical quantities are defined in terms of the field variables which are formally gauge dependent. For example [2], energy momentum tensor and chemical potential are defined in terms of metric/gauge field which is not gauge invariant. Similarly, heat currents can be related to the metric perturbation defined only in a specific gauge where time period has definite relation with temperature.

In recent works [3,4], based on [5,6], we developed a systematic method to numerically calculate the Green's functions and all AC transports quantities simultaneously for the case where many fields are coupled and there are *constraints* due to gauge symmetry. Although we have tested the validity of the procedure by showing the agreement of zero frequency limits of AC conductivities with the known analytic DC conductivities [7–9] we still think that we need to prove the gauge invariance of our procedure as a matter of principle. We found that the bulk gauge invariance is intimately related to the holographic renormalization. Although the local counter terms were introduced to kill the divergences, they also kill most of gauge dependence.

Furthermore, there is a residual gauge symmetry (RGS) even after we fix the axial gauge $g_{rx} = 0$. While equations of motion can be written in terms of the gauge invariant master fields $\mathcal{P}_h, \mathcal{P}_\chi$ (3.8), it turns out that the quadratic on-shell action, the generating function for two point retarded Green's functions, cannot be written as such. However, we prove that the Green's functions are still invariant under such a symmetry.

There is a mismatch in the degrees of freedom in the bulk and those at the boundary: there are only two independent bulk solutions satisfying the in-falling boundary conditions while we need three solutions at the boundary since there are three independent source fields. The RGS is the one that resolves the problem: since it cannot satisfy a proper boundary condition, it is not a proper gauge

* Corresponding author.
E-mail addresses: fortoe@gist.ac.kr (K.-Y. Kim), kimkyungkiu@gmail.com (K.K. Kim), yseo@hanyang.ac.kr (Y. Seo), sjsin@hanyang.ac.kr (S.-J. Sin).

http://dx.doi.org/10.1016/j.physletb.2015.07.058

symmetry but a 'solution generating symmetry'. It generate the desired solution at the boundary and therefore we should accept its bulk counter part as a new physical degree of freedom as well although it cannot satisfy the infalling boundary condition (BC). By extending the RGS such that it satisfies infalling boundary condition at the horizon, we can make the bulk solution more natural in the sense that it satisfies the infalling BC. With such solution we can also understand the problem in the context of general structure of holography, namely the correspondence between a global symmetry at the boundary and the local gauge symmetry in the bulk.

2. Action and background solution

Let us first briefly review the system we will discuss, which has been analysed in detail in [3,7,10]. The holographically renormalized action (S_{ren}) is given by

$$S_{\text{ren}} = S_{\text{EM}} + S_\psi + S_c\,, \tag{2.1}$$

where

$$S_{\text{EM}} = \int_M \mathrm{d}^4x\sqrt{-g}\left[R - 2\Lambda - \frac{1}{4}F^2\right] - 2\int_{\partial M}\mathrm{d}^3x\sqrt{-\gamma}K\,, \tag{2.2}$$

is the usual action for charged black hole in AdS space ($\Lambda < 0$) with the Gibbons–Hawking term and

$$S_\psi = \int_M \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}\sum_{I=1}^{2}(\partial\psi_I)^2\right], \tag{2.3}$$

is the action for two free massless scalars added for a momentum relaxation effect. S_c is the counter term

$$S_c = \eta_c\int_{\partial M}\mathrm{d}x^3\sqrt{-\gamma}\left(-4 - R[\gamma] + \frac{1}{2}\sum_{I=1}^{2}\gamma^{\mu\nu}\partial_\mu\psi_I\partial_\nu\psi_I\right), \tag{2.4}$$

which is included to cancel the divergence in $S_{\text{EM}} + S_\psi$. Here we introduced η_c to keep track of the effect of the counter term. At the end of the computation we will set $\eta_c = 1$.

The action (2.1) yields general equations of motion[1]

$$\begin{aligned} R_{MN} = {}& \frac{1}{2}g_{MN}\left(R - 2\Lambda - \frac{1}{4}F^2 - \frac{1}{2}\sum_{I=1}^{2}(\partial\psi_I)^2\right) \\ &+ \frac{1}{2}\sum_I \partial_M\psi_I\partial_N\psi_I + \frac{1}{2}F_M{}^P F_{NP}\,, \end{aligned} \tag{2.5}$$

$$\nabla_M F^{MN} = 0\,, \qquad \nabla^2\psi_I = 0\,, \tag{2.6}$$

which admit the following solutions

$$\mathrm{d}s^2 = G_{MN}\mathrm{d}x^M\mathrm{d}x^N = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,, \tag{2.7}$$

$$f(r) = r^2 - \frac{\beta^2}{2} - \frac{m_0}{r} + \frac{\mu^2}{4}\frac{r_0^2}{r^2}\,,$$

$$m_0 = r_0^3\left(1 + \frac{\mu^2}{4r_0^2} - \frac{\beta^2}{2r_0^2}\right), \tag{2.8}$$

$$A = \mu\left(1 - \frac{r_0}{r}\right)\mathrm{d}t\,, \tag{2.9}$$

$$\psi_I = \beta_{Ii}x^i = \beta\delta_{Ii}x^i\,. \tag{2.10}$$

[1] Index convention: $M, N, \cdots = 0, 1, 2, r$, and $\mu, \nu, \cdots = 0, 1, 2$, and $i, j, \cdots = 1, 2$.

These are reduced to AdS–Reissner–Nordstrom (AdS–RN) black brane solutions when $\beta = 0$. Here we have taken special β_{Ii}, which satisfies $\frac{1}{2}\sum_{I=1}^{2}\vec{\beta}_I\cdot\vec{\beta}_I = \beta^2$ for general cases.

The solutions (2.7)–(2.10) are characterized by three parameters: r_0, μ, and β. r_0 is the black brane horizon position ($f(r_0) = 0$) and can be replaced by temperature T for the dual field theory:

$$T = \frac{f'(r_0)}{4\pi} = \frac{1}{4\pi}\left(3r_0 - \frac{\mu^2 + 2\beta^2}{4r_0}\right). \tag{2.11}$$

Non-vanishing components of energy–momentum tensor and charge density read

$$\langle T^{tt}\rangle = 2m_0\,, \quad \langle T^{xx}\rangle = \langle T^{yy}\rangle = m_0\,, \quad \langle J^t\rangle = \mu r_0\,. \tag{2.12}$$

$\langle T^{tt}\rangle = 2\langle T^{xx}\rangle$ implies that charge carriers are still of massless character. From here we set $r_0 = 1$ not to clutter.

3. Gauge fixing and residual gauge transformation

To study electric, thermoelectric, and thermal conductivities we introduce small fluctuations around the background (2.7)–(2.10)

$$\delta A_x(t,r) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}e^{-i\omega t}a_x(\omega,r)\,, \tag{3.1}$$

$$\delta g_{tx}(t,r) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}e^{-i\omega t}r^2h_{tx}(\omega,r)\,, \tag{3.2}$$

$$\delta g_{rx}(t,r) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}e^{-i\omega t}r^2h_{rx}(\omega,r)\,, \tag{3.3}$$

$$\delta\psi_1(t,r) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}e^{-i\omega t}\chi(\omega,r)\,. \tag{3.4}$$

The fluctuations are chosen to be independent of x and y. This is allowed since all the background fields appearing in the equations of motion turn out to be independent of x and y. The gauge field fluctuation ($\delta A_x(t,r)$) sources metric ($\delta g_{tx}(t,r)$, $\delta g_{rx}(t,r)$) and scalar field ($\delta\psi_1(t,r)$) fluctuation and vice versa and all the other fluctuations are decoupled. We will work in momentum space and $h_{tx}(\omega,r)$ and $h_{rx}(\omega,r)$ is defined so that it goes to constant as r goes to infinity.

By linearizing the full equation of motion, we get four equations. However one of them can be obtained by the others. Thus we may consider following three equations:

$$(\chi' - \beta h_{rx}) - \frac{i\mu\omega a_x}{\beta r^2 f(r)} - \frac{ir^2\omega(h'_{tx} + i\omega h_{rx})}{\beta f(r)} = 0\,, \tag{3.5}$$

$$a''_x(r) + \frac{a'_x(r)f'(r)}{f(r)} + \frac{\omega^2 a_x(r)}{f(r)^2} + \frac{\mu(h'_{tx} + i\omega h_{rx})}{f(r)} = 0\,, \tag{3.6}$$

$$\begin{aligned} &f(r)f'(r)(\chi'(r) - \beta h_{rx}) + f(r)^2(\chi' - \beta h_{rx})' \\ &+ \frac{2f(r)^2(\chi' - \beta h_{rx})}{r} + \omega^2\chi(r) - i\beta\omega h_{tx}(r) = 0\,. \end{aligned} \tag{3.7}$$

If we differentiate the third equation with respect to r, all equations can be written in terms of three variables, $\mathcal{P}_\chi$, $\mathcal{P}_h$, and a_x, where

$$\mathcal{P}_\chi \equiv \chi' - \beta h_{rx}\,, \qquad \mathcal{P}_h \equiv h'_{tx} + i\omega h_{rx}\,. \tag{3.8}$$

Therefore, h_{rx} is a non-dynamical degree of freedom. Indeed, $\mathcal{P}_\chi$, $\mathcal{P}_h$, and a_x are invariant under a diffeomorphism generated by

$\xi^\mu = (0, \zeta(r)e^{-i\omega t}, 0, 0)$, under which the fields are transformed as follows:

$$\delta h_{rx} = \frac{1}{r^2}(\nabla_r \xi_x + \nabla_x \xi_r) = \zeta'(r)e^{-i\omega t}\,, \tag{3.9}$$

$$\delta h_{tx} = \frac{1}{r^2}(\nabla_t \xi_x + \nabla_x \xi_t) = -i\omega\zeta(r)e^{-i\omega t}\,, \tag{3.10}$$

$$\delta\chi = \beta\zeta(r)e^{-i\omega t}\,, \tag{3.11}$$

$$\delta a_x = 0\,. \tag{3.12}$$

Using this gauge degree of freedom, one may set $h_{rx} = 0$, which is so-called the axial gauge. The numerical calculation in [3] has been performed in this gauge. A question arises whether the resulting physical quantities are independent of such gauge fixing condition.

Furthermore, even after we fix $h_{rx} = 0$, one can still find a residual gauge transformation which is given by constant ζ [11]. This residual diffeomorphism doesn't change the gauge fixing condition $h_{rx} = 0$ and generates constant shift on h_{tx} and χ, because the equations of motion contain only derivatives of h_{tx} and χ and the linear combination of them, $\omega\chi(r) - i\beta h_{tx}(r)$, which is invariant under

$$h_{tx} \to h_{tx} + h_0, \text{ and } \chi \to \chi + i\frac{\beta}{\omega}h_0, \tag{3.13}$$

where h_0 is a constant. Thus there is one parameter constant solution given by

$$a_x = 0, \quad h_{tx} = h_0, \quad \chi = i\frac{\beta}{\omega}h_0\,, \tag{3.14}$$

which does not satisfy in-falling boundary condition so it is not a physical degree of freedom.[2] We call it the residual gauge symmetry (RGS) because it is generated by the zero mode of a diffeomorphism generator. This kind of solution was first introduced in [12].

Why should there be such a residual degree of freedom? It can be traced to the difference of the differential equation near horizon and those near boundary. Near the black hole horizon ($r \to 1$) the solutions are expanded as

$$h_{tx} = (r-1)^{\nu_\pm + 1}\left(h_{tx}^{(I)} + h_{tx}^{(II)}(r-1) + \cdots\right),$$
$$a_x = (r-1)^{\nu_\pm}\left(a_x^{(I)} + a_x^{(II)}(r-1) + \cdots\right),$$
$$\chi = (r-1)^{\nu_\pm}\left(\chi^{(I)} + \chi^{(II)}(r-1) + \cdots\right), \tag{3.15}$$

where $\nu_\pm = \pm i4\omega/(-12+2\beta^2+\mu^2) = \mp i\omega/(4\pi T)$ and the incoming boundary condition corresponds to $\nu = \nu_+$. By inserting these to the equations of motion, one can easily find a linear relations between the zero-th modes:

$$(\nu+1)h_{tx}^{(I)} + \mu a_x^{(I)} + \beta\chi^{(I)} = 0. \tag{3.16}$$

Notice that all other modes are generated by these. Thus there is a well defined constraint equation which reduces the degrees of freedom.

On the other hand, by inserting the expansion near the boundary ($r \to \infty$)

$$h_{tx} = h_{tx}^{(0)} + \frac{1}{r^2}h_{tx}^{(2)} + \frac{1}{r^3}h_{tx}^{(3)} + \cdots,$$
$$a_x = a_x^{(0)} + \frac{1}{r}a_x^{(1)} + \cdots,$$
$$\chi = \chi^{(0)} + \frac{1}{r^2}\chi^{(2)} + \frac{1}{r^3}\chi^{(3)} + \cdots, \tag{3.17}$$

to the equations of motion, we cannot get any relation between the zero-th modes $a_x^{(0)}$, $h_{tx}^{(0)}$, and $\chi^{(0)}$, all of which are related to the higher modes. More explicitly,

$$\omega(\omega\chi^{(0)} - i\beta h_{tx}^{(0)}) - 2\chi^{(2)} = 0,$$
$$i\beta(\omega\chi^{(0)} - i\beta h_{tx}^{(0)}) - 2h_{tx}^{(2)} = 0, \tag{3.18}$$

which are evolution equations in r-direction. Therefore, there is no constraint equation. Then there is a crisis of mismatch of degrees of freedom and this crisis is resolved by the effective residual degree of freedom described above. However, this residual gauge degree of freedom raises another issue of invariance of physics under this symmetry. We will address this issue at the end of Section 5.

4. Holographic renormalization and gauge invariance

Now we come back to the question whether physical quantities are independent of the choice of the gauge condition $h_{rx}(r) = 0$. We will show this by proving that the generating function of physical quantities, the on-shell action, is invariant even in the case with $h_{rx}(r) \neq 0$.

The on-shell renormalized action to quadratic order in fluctuation fields, $S_{\rm ren}^{(2)}$, is

$$\begin{aligned} S_{\rm ren}^{(2)} = \lim_{r\to\infty}\int d^3x\Bigg[&\delta\psi_1\left(\frac{1}{2}\beta f\delta g_{rx} - \frac{1}{2}fr^2\delta\psi_1'\right) \\ &+ \frac{2}{r}\delta g_{tx}^2 - \frac{1}{2}f\delta A_x\delta A_x' \\ &- \delta g_{tx}\left(\frac{1}{2}\delta\dot{g}_{rx} - \frac{1}{2}r^2\left(\frac{\delta g_{tx}}{r^2}\right)' + \frac{\mu}{2r^2}\delta A_x\right) \\ &+ \eta_c\left(\delta\psi_1\left(\frac{r^2\delta\ddot{\psi}_1}{2\sqrt{f}} - \frac{\beta\delta\dot{g}_{tx}}{2\sqrt{f}}\right)\right. \\ &\left.+ \frac{\beta\delta\dot{\psi}_1\delta g_{tx}}{2\sqrt{f}} - \left(\frac{2}{\sqrt{f}}\right)\delta g_{tx}^2\right)\Bigg], \end{aligned} \tag{4.1}$$

where $f(r) = r^2 - \frac{\beta^2}{2} - \frac{m_0}{r} + \frac{\mu^2}{4r^2}$. We dropped the boundary contribution from the horizon as a prescription for the retarded Green function [13].[3] Near boundary $r \to \infty$, the fluctuation fields in momentum space, (3.1)–(3.4), may be expanded as

$$h_{tx}(\omega, r) = \sum_{n=0}^{\infty}\frac{h_{tx}^{(n)}(\omega)}{r^n}\,, \qquad h_{rx}(\omega, r) = \sum_{n=0}^{\infty}\frac{h_{rx}^{(n)}(\omega)}{r^n}\,,$$
$$a_x(\omega, r) = \sum_{n=0}^{\infty}\frac{a_x^{(n)}(\omega)}{r^n}\,, \qquad \chi(\omega, r) = \sum_{n=0}^{\infty}\frac{\chi^{(n)}(\omega)}{r^n}\,, \tag{4.2}$$

and using the equations of motion, we can obtain a quadratic action as follows

[2] It is a regular solution at future horizon.

[3] In fact, the contribution of the incoming solution at the horizon is zero in (4.1), which is real. However, for a generating function of *retarded* Green's functions, we will take only part of (4.1) as explained below (4.3), which is complex. In this case, it turns out that the contribution from the horizon is pure imaginary. From this perspective, we should drop the contribution from the horizon.

$$S^{(2)}_{\text{ren}} = \frac{V_2}{2}\int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\Bigg[-\mu\bar{a}^{(0)}_x h^{(0)}_{tx} - \mu\bar{h}^{(0)}_{tx} a^{(0)}_x - 2m_0\bar{h}^{(0)}_{tx} h^{(0)}_{tx} + \bar{a}^{(0)}_x a^{(1)}_x$$
$$+\left(\bar{\chi}^{(0)} + \frac{i\beta}{\omega}\bar{h}^{(0)}_{tx}\right)\left(3\chi^{(3)} + \beta h^{(4)}_{rx}\right)$$
$$+(\eta_c - 1)\Big\{-\Lambda^3\left(4\bar{h}^{(0)}_{tx}h^{(0)}_{tx}\right)$$
$$-\Lambda^2\left(4\bar{h}^{(1)}_{tx}h^{(0)}_{tx} + 4i\bar{h}^{(0)}_{tx}h^{(2)}_{rx}\omega\right)$$
$$+\Lambda\left(i\beta\bar{h}^{(0)}_{tx}\chi^{(0)}\omega - 2i\bar{h}^{(0)}_{tx}h^{(3)}_{rx}\omega + \beta^2\bar{h}^{(0)}_{tx}h^{(0)}_{tx}\right)$$
$$+\Lambda\Big(-4i\bar{h}^{(1)}_{tx}h^{(2)}_{rx}\omega - 4\bar{h}^{(2)}_{tx}h^{(0)}_{tx} + i\beta\bar{\chi}^{(0)}h^{(0)}_{tx}\omega$$
$$-\bar{\chi}^{(0)}\chi^{(0)}\omega^2\Big) - 2m_0\bar{h}^{(0)}_{tx}h^{(0)}_{tx} - 4\bar{h}^{(0)}_{tx}h^{(3)}_{tx} - 2i\omega\bar{h}^{(1)}_{tx}h^{(3)}_{rx}$$
$$+\beta^2\bar{h}^{(1)}_{tx}h^{(0)}_{tx} + i\beta\omega\bar{h}^{(1)}_{tx}\chi^{(0)} - 4i\omega\bar{h}^{(2)}_{tx}h^{(2)}_{rx} - 4\bar{h}^{(3)}_{tx}h^{(0)}_{tx}$$
$$+i\beta\omega\bar{\chi}^{(1)}h^{(0)}_{tx} - \omega^2\bar{\chi}^{(1)}\chi^{(0)}\Big\}\Bigg] + [\text{c.c}], \tag{4.3}$$

where the argument of the fields[4] is ω. V_2 denotes volume in x–y space and [c.c] means the complex conjugated terms. From here, we will drop the [c.c] term since we want to compute *retarded* Green's functions [13].

The second line is proportional to a gauge invariant combination under (3.13). Furthermore, one of the equation of motion including $h^{(4)}_{rx}$ is

$$h^{(4)}_{rx} - \frac{1}{\beta^2-\omega^2}\left(3i\omega h^{(3)}_{tx} - i\mu\omega a^{(0)}_x - 3\beta\chi^{(3)}\right) = 0. \tag{4.4}$$

One can show that (4.4) is equivalent to a Ward identity

$$\nabla_\mu\langle T^{\mu\nu}\rangle + F_\lambda{}^\nu\langle J^\lambda\rangle - \langle \mathcal{O}^I\rangle\partial^\nu\psi_I = 0\,, \tag{4.5}$$

by using the boundary metric and the other fields in the linear approximation given as follows:

$$\begin{aligned}
&\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + 2h^{(0)}_{tx}e^{-i\omega t}\mathrm{d}t\mathrm{d}x\,,\\
&\left\langle T^{\mu\nu}\right\rangle = \left\langle T^{(0)\mu\nu}\right\rangle + \left\langle T^{(1)\mu\nu}\right\rangle\\
&F = -i\omega a^{(0)}_x e^{-i\omega t}\mathrm{d}t\wedge\mathrm{d}x\,,\\
&\left\langle J^\mu\right\rangle = \left\langle J^{(0)\mu}\right\rangle + \left\langle J^{(1)\mu}\right\rangle = (\mu,0,0) + \left(0, a^{(1)}_x - \mu h^{(0)}_{tx}\,, 0\right)e^{-i\omega t}\\
&\psi_I = (\beta x, \beta y)\,,\\
&\left\langle \mathcal{O}^I\right\rangle = \left\langle \mathcal{O}^{(1)I}\right\rangle = \left(3\chi^{(3)} + \beta h^{(4)}_{rx}, 0\right)e^{-i\omega t}\,,
\end{aligned} \tag{4.6}$$

where

$$\left\langle T^{(0)\mu\nu}\right\rangle = m_0\begin{pmatrix}2&0&0\\0&1&0\\0&0&1\end{pmatrix}\,,$$
$$\left\langle T^{(1)\mu\nu}\right\rangle = \left(-2m_0h^{(0)}_{tx} - 3h^{(3)}_{tx} + i\omega h^{(4)}_{rx}\right)\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}e^{-i\omega t}\,. \tag{4.7}$$

One may ask why Ward identity of the boundary theory is included in the bulk equation of motion. It is not accidental: The translation, $x\to x+\xi_0$ at the boundary theory is imbedded into the bulk diffeomorphism $x\to x+\xi(x)$, which induces the field transformation $\Phi\to\Phi+\delta_\xi\Phi$, which in turn is a special case of general variation, $\Phi\to\Phi+\delta\Phi$. Now the equation of motion is coming from the invariance of bulk action $\delta S_B = 0$ under the general variation, while the Ward identity is the requirement of the boundary action under the translation $\delta_{\xi_0}S_b = 0$. Because AdS/CFT request $S_B = S_b$ at the onshell, the latter is contained in the huge tower of equation of motion as a tiny piece.

The terms proportional to $(\eta_c - 1)$ in (4.3) include the divergent terms with Λ, a regularization parameter, and finite terms without Λ. A remarkable fact is that with the counter term of weight $\eta_c = 1$, not only the divergent terms are canceled, but also all the h_{rx} dependent finite terms disappears from the on-shell action, as we claimed in the beginning of this section.

5. Gauge invariance under the residual gauge transformation

Our starting point is the action[5]

$$S^{(2)}_{\text{ren}} = \frac{V_2}{2}\int_0^\infty\frac{\mathrm{d}\omega}{2\pi}\Bigg[-\mu\bar{a}^{(0)}_x h^{(0)}_{tx} - 2m_0\bar{h}^{(0)}_{tx}h^{(0)}_{tx} + \bar{a}^{(0)}_x a^{(1)}_x$$
$$-3\bar{h}^{(0)}_{tx}h^{(3)}_{tx} + 3\bar{\chi}^{(0)}\chi^{(3)} + \left(\beta\bar{\chi}^{(0)} + i\omega\bar{h}^{(0)}_{tx}\right)h^{(4)}_{rx}\Bigg]$$
$$+\text{c.c}, \tag{5.1}$$

which is still dependent on residual gauge (3.13) even after we set $h_{rx} = 0$. Since it is just a constant shift of the solution Φ, its effects are only shifts of zero-th modes and $\Phi'(r)$ and all of its modes, especially $(a^{(1)}_x, h^{(3)}_{tx}, \chi^{(3)}) := \Pi^a$ are intact. Notice that the recurrence relations derived from equations of motion relate higher modes with the zero-th modes $J^a = (a^{(0)}_x, h^{(0)}_{tx}, \chi^{(0)})$. However, all dependences of higher modes on zeroth modes is through the gauge invariant combination $\omega\chi^{(0)} - i\beta h^{(0)}_{tx}$. See, for example, (3.18). Thus all higher modes are gauge invariant, which makes the gauge invariance of the $\Phi'(r)$ intact in spite of the complicated dependence of higher modes on the zeroth modes.

The residual gauge dependence of (5.1) can be understood as follows. The full on shell action should be invariant under the residual gauge transformation. However, what we are looking at is the quadratic part of the action $S^{(2)}_{\text{ren}}$, which generates the 2-point function, in the expansion of

$$S_{\text{ren}}[\delta\Phi] = S^{(0)}_{\text{ren}} + S^{(1)}_{\text{ren}}[\delta\Phi] + S^{(2)}_{\text{ren}}[\delta\Phi] + \cdots, \tag{5.2}$$

where $\delta\Phi = (\delta\Phi_{\mu\nu}, \delta\Phi_\mu, \delta\Phi_I)$ collectively denotes the sources of the dual field theory, which are boundary values of $\frac{1}{r^2}\delta g_{\mu\nu}$, δA_μ and $\delta\psi_I$. $S^{(1)}_{\text{ren}}[\delta\Phi]$ and $S^{(2)}_{\text{ren}}[\delta\Phi]$ are given as follows:

$$S^{(1)}_{\text{ren}}[\delta\Phi] = \int\mathrm{d}^3x\left(\frac{1}{2}\delta\Phi_{\mu\nu}\left\langle T^{(0)\mu\nu}\right\rangle + \delta\Phi_\mu\left\langle J^{(0)\mu}\right\rangle + \delta\Phi_I\left\langle \mathcal{O}^{(0)I}\right\rangle\right), \tag{5.3}$$

$$S^{(2)}_{\text{ren}}[\delta\Phi] = \int\mathrm{d}^3x\left(\frac{1}{2}\delta\Phi_{\mu\nu}\left\langle T^{(1)\mu\nu}\right\rangle + \delta\Phi_\mu\left\langle J^{(1)\mu}\right\rangle + \delta\Phi_I\left\langle \mathcal{O}^{(1)I}\right\rangle\right). \tag{5.4}$$

[4] $\bar{a}^{(0)}_x(\omega) \equiv a^{(0)}_x(-\omega) = a^{(0)}_x(\omega)^*$ by the reality condition of δA_x. The same notation and reality condition apply to all the other fields.

[5] It comes from (4.1) before we get Eq. (4.3), for which we have to use the equations of motion.

Under the residual gauge transformation[6] with $h_0 = -i\omega\zeta_0$, the variations of these actions are

$$\delta S^{(1)}_{\rm ren}[\delta\Phi] = V_2 \int \frac{d\omega}{2\pi} \left\{ \bar{\zeta}_0 \left(i\omega\mu a_x^{(0)} + 2i\omega m_0 h_{tx}^{(0)} \right) + {\rm c.c} \right\}, \quad (5.5)$$

$$\delta S^{(2)}_{\rm ren}[\delta\Phi] = -\delta S^{(1)}_{\rm ren}[\delta\Phi] + V_2 \int \frac{d\omega}{2\pi} \left\{ \bar{\zeta}_0 \left(3\beta\chi^{(3)} - 3i\omega h_{tx}^{(3)} + i\omega\mu a_x^{(0)} + \left(\beta^2 - \omega^2\right) h_{rx}^{(4)} \right) + {\rm c.c} \right\}. \quad (5.6)$$

Thus the total variation is proportional to the Ward identity (4.4). Notice that $S_{\rm ren}$ is gauge invariant but $S^{(2)}_{\rm ren}$, which is starting point to derive the Green function, is not invariant by itself. Nevertheless physical observables derived from $S^{(2)}_{\rm ren}$ are invariant because the Green functions are second derivatives of the full on shell action at the zero source limit.

At this point one can discuss a puzzle in counting degrees of freedom. There are only two independent bulk solutions satisfying the in-falling boundary conditions,[7] while we need three solutions at the boundary since there are three independent source fields. Therefore, there is a crisis of mismatch of degrees of freedom between the bulk and boundary. What solves the problem is the RGS (3.14). We call it RGS because it is generated by the zero mode of a diffeomorphism generator. On the other hand, to be a proper gauge degree of freedom in the bulk, the diffeomorphism generator should satisfy the proper boundary conditions: infalling at horizon and Dirichlet at boundary. The residual gauge symmetry generator is a global shift and therefore it can satisfy neither of them. So such a shift by the diffeomorphism zero mode is not a true gauge symmetry, while it is a symmetry of the bulk equations of motion. In other words, the RGS is a "solution generating symmetry" rather than a gauge symmetry. Therefore, the gauge orbit of RGS *can* provide us the necessary degree of freedom (d.o.f) near boundary. To match the d.o.f, we need to accept its bulk orbit as physical configuration inspite of the fact that the resulting bulk solution does not satisfy the infalling BC.[8] One can give a more natural bulk solution by extending RGS to a diffeomorphism which satisfies the infalling boundary condition and it is reduced to our previous RGS near the boundary. It is generated by $\xi^\mu = (0, \zeta(r)e^{-i\omega t}, 0, 0)$, with[9]

$$\zeta(r) = \epsilon (f(r)/r^2)^{-i\omega/(4\pi T)}, \quad (5.7)$$

where f is the metric factor given in Eq. (2.8) and ϵ is a constant parameter. Notice that the RGS is the case where $\zeta(r)$ is constant. We will call this "boundary shifting diffeomorphism" (BSD). Now we can understand the degree of freedom mismatch as follows: Since it is not satisfying the Dirichlet bc, it is still not a proper gauge transformation. Notice also that under (5.7), the gauge slice is shifted and some of the gauge fields become singular. For the discussion on the treating these issues, we refer the reader to p. 24 of Ref. [9].[10] This is the reason why the BSD can generate a new solution in the boundary. It is precisely the same logic why RGS generate new solution.[11] Since RGS and BSD shift the boundary values of fields, they generate the Ward identity for the translation invariance. This is a typical example how a global symmetry is encoded in a local gauge transformation and how the apparent paradox of the degree of freedom can be resolved because of the holographic correspondence.[12]

6. Basis independence

In [3], we constructed a formalism to perform the AC conductivities for the case where multiple fields are coupled together. We had to choose a basis of initial conditions and one can ask whether different choices of basis give the same result. Answering this question will also provide an alternative reasoning of gauge invariance. To provide the setup, let us consider N fields $\Phi^a(x,r)$, $(a = 1, 2, \cdots, N)$,

$$\Phi^a(x,r) = \int \frac{d^d k}{(2\pi)^d} e^{-ikx} r^p \Phi^a(k,r), \quad (6.1)$$

where the index a may include components of higher spin fields. For convenience, r^p is multiplied such that the solution $\Phi^a(k,r)$ goes to constant at boundary. In our case, $(\Phi^1, \Phi^2, \Phi^3) = (a_x, h_{tx}, \chi)$ and $p = 0$ for Φ^1, Φ^3 and $p = 2$ for Φ^2.

Near horizon $(r = 1)$, solutions can be expanded as

$$\Phi^a_i(k,r) = (r-1)^{\nu_{a\pm}} \left(\varphi^a_i + \tilde{\varphi}^a_i (r-1) + \cdots \right), \quad (6.2)$$

where a new subscript i is introduced to denote the solutions corresponding to a specific independent set of initial conditions. For example, φ^a_i may be chosen as

$$\varphi^a_1 = \left(1, \ -(\tilde{\mu} + \tilde{\beta})/(1+\nu), \ \ 1\right)^{\rm T}, \qquad \varphi^a_2 = \left(1, \ -(\tilde{\mu} - \tilde{\beta})/(1+\nu), \ -1\right)^{\rm T}, \quad (6.3)$$

where we used (3.16) and $\nu = -i\omega/(4\pi T)$ as shown below (3.15) for incoming boundary condition to compute the retarded Green's function [13]. Due to incoming boundary condition, φ^a_i determines $\tilde{\varphi}^a_i$ through horizon-regularity condition so that we can determine the solution completely. Each initial value vector $\vec{\varphi}_i$ yields a solution, denoted by $\vec{\Phi}_i(r)$, which is expanded as

$$\Phi^a_i(k,r) \to \mathbb{S}^a_i + \cdots + \frac{\mathbb{O}^a_i}{r^{\delta_a}} + \cdots \qquad \text{(near boundary)}, \quad (6.4)$$

where $\mathbb{S}^a_i$ are the *sources* (leading terms) of i-th solution and $\mathbb{O}^a_i$ are the *operator* expectation values corresponding to sources $(\delta_a \geq 1)$.

Notice that we have only two solutions while we have a three dimensional vector space $\mathcal{J}$ of boundary values J^a, $a = 1, 2, 3$. To

[6] This transformation changes the sources of the action, $\delta\Phi_{\mu\nu}, \delta\Phi_\mu, \delta\Phi_I$. One should note that there are non-vanishing transformations for $\delta\Phi_{00}$ and $\delta\Phi_0$.

[7] We have two second order differential equations and one first order one in three variables: a_x, h_{tx}, χ. Therefore, there are 5 boundary conditions to fix. If we fix the in-falling boundary conditions for all three variables, we are left with two degrees of freedom. We recall equations (3.15) and (3.16).

[8] So far we discussed the degree of freedom mismatch using the RGS, since our formalism in [3] to calculate the conductivity is based on it.

[9] We thank the anonymous referee for suggesting to consider this.

[10] It is very tempting to consider BSD as a gauge transformation at least from bulk point of view. If we do it, we get to the problem: Its orbit in the boundary generate physical configuration while it does not in the bulk, so that crisis of d.o.f becomes real!

[11] This argument is further justified if we consider the numerical calculation starting from the boundary instead from horizon. After choosing 3 fields's values, we can adjust two "expectation values" such that we can get infalling boundary values at the horizon. It is easy to show that only when we start from a subspace of codimension 1, we get three infalling solution near the horizon. If we start from a point off this plane, we get one infalling and two fields which are mixture of infalling and a constant. In this calculation the gauge condition $h_{rx} = 0$ is intact. This demonstrates that we cannot impose infalling bc for all fields at hands. If we do the same numerical experiment for BSD, the picture is following. The BSD generate the orbit and it also move the gauge slice. Now in this case even in the case we start from the off the plane, we can get three infalling fields at the horizon. We need to calculate the r-evolution at each 'gauge fixing' plane which pass through the initial data.

[12] The apparent 'mismatch' is due to the difference in viewing the gauge orbit of BSD (or RGS) between the bulk and boundary. In the bulk, one could view it as gauge orbit. On the other hand, from the boundary theory point of view, there is no gauge structure and the orbit of translation symmetry is physical degree of freedom.

fix such mismatch of degree of freedom, we introduce a constant solution $\vec{\Phi}_0(r) = \vec{\mathbb{S}}_0 = (0, 1, i\beta/\omega)^\intercal$ along the gauge-orbit direction of the residual gauge transformation so that $\mathbb{S}_1^a, \mathbb{S}_2^a, \mathbb{S}_0^a$ form a basis of $\mathcal{J}$. Now $\mathbb{S}$ and $\mathbb{O}$ are generic regular matrices of order 3.

The general solution is a linear combination of them: let

$$\Phi^a(k,r) = \Phi_i^a(k,r)c^i\,, \tag{6.5}$$

with real constants c^i's. We can choose c^i such that the combined source term matches the boundary value J^a:

$$J^a = \mathbb{S}_i^a c^i\,, \tag{6.6}$$

which yields

$$\Phi^a(k,r) = \Phi_i^a(k,r)c^i \to J^a + \cdots + \frac{\Pi^a}{r^{\delta_a}} + \cdots\,, \quad \text{(near boundary)} \tag{6.7}$$

where, with (6.4) and (6.6),

$$\Pi^a = \mathbb{O}_i^a c^i = \mathbb{O}_i^a (\mathbb{S}^{-1})_b^i J^b =: C_b^a J^b\,. \tag{6.8}$$

Notice that both Π^a and C_b^a are invariant under the transformation $J^b \to J^b + \epsilon S_0^b$ because $C_b^a S_0^b = \mathbb{O}_i^a(\mathbb{S}^{-1})_b^i S_0^b = \mathbb{O}_0^a = 0$, where $\mathbb{O}_0^a = 0$ since it is the sub-leading term of the constant solutions.

A general on-shell quadratic action in momentum space has the form of

$$S_{\text{ren}}^{(2)} = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\left[\bar{J}^a \mathbb{A}_{ab}(k) J^b + \bar{J}^a \mathbb{B}_{ab}(k)\Pi^b\right], \tag{6.9}$$

where $\mathbb{A}$ and $\mathbb{B}$ are regular matrices of order N. $\bar{J}^a$ means $J^a(-k)$ and, in matrix notation, $\bar{J}^a$ can be understood as a row matrix. For example, in our case, the effective action (5.1) reads

$$S_{\text{ren}}^{(2)} = \frac{V_2}{2}\int \frac{d\omega}{2\pi}\left[\bar{J}^a \mathbb{A}_{ab}(\omega) J^b + \bar{J}^a \mathbb{B}_{ab}(\omega)\Pi^b\right], \tag{6.10}$$

where

$$J^a = \begin{pmatrix} a_x^{(0)} \\ h_{tx}^{(0)} \\ \chi^{(0)} \end{pmatrix}, \quad \Pi^a = \begin{pmatrix} a_x^{(1)} \\ h_{tx}^{(3)} \\ \chi^{(3)} \end{pmatrix},$$
$$\mathbb{A} = \begin{pmatrix} 0 & -\mu & 0 \\ -\mu & -2m_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{3i\beta}{\omega} \\ 0 & 0 & 3 \end{pmatrix}. \tag{6.11}$$

With (6.8) the action (6.9) becomes

$$\begin{aligned} S_{\text{ren}}^{(2)} &= \frac{1}{2}\int_{\omega\geq 0} \frac{d^dk}{(2\pi)^d}\left[\bar{J}^a\left[\mathbb{A}_{ab}(k) + \mathbb{B}_{ac}\mathbb{O}_i^c(\mathbb{S}^{-1})_b^i(k)\right]J^b\right] \\ &\equiv \frac{1}{2}\int_{\omega\geq 0} \frac{d^dk}{(2\pi)^d}\left[\bar{J}^a G_{ab}^R J^b\right], \end{aligned} \tag{6.12}$$

where the range of ω is chosen to be positive following the prescription in [13]. Notice that $\mathbb{O}_i^a(\mathbb{S}^{-1})_b^i$ is independent of J^a, because neither $\mathbb{S}_i^a$ nor $\mathbb{O}_i^a$ depends on J^a. Furthermore $\mathbb{O}_i^a(\mathbb{S}^{-1})_b^i$ is independent of the choice of the initial condition (6.3), because the different choice of initial value vectors is nothing but a linear transformation $\varphi_i^a \to \varphi_j^a R_i^j$, which induces right multiplications in the solutions: $\mathbb{S} \to \mathbb{S}R, \mathbb{O} \to \mathbb{O}R$. This proves the Green functions are independent of choice of basis for our initial conditions.

Notice that since $\mathbb{A}$ and $\mathbb{B}$ are also independent of J, G_{ab}^R is independent of J and manifestly gauge invariant, giving alternative reason for the invariance of the Green functions under the residual gauge symmetry.

7. Conclusion

We investigated the gauge invariance of physical observables in a holographic theory under the local diffeomorphism. We find that gauge invariance is closely related to the holographic renormalization. Apart from the zero-th mode residual gauge dependence, gauge dependence is canceled by the local counter terms defined in the boundary. However, due to the difference in the space–time structure between the near-horizon and near boundary regions, there are residual gauge structure near boundary. There is a subtle and deep connection between the degrees of freedom at the boundary and those at the bulk. There are three degrees of freedom at the boundary, out of which only two can be embedded into bulk fields such that they are the boundary values of the bulk fields satisfying the incoming boundary conditions. The residual gauge symmetry is not a proper gauge symmetry but a solution generator *near* the boundary. We proved the invariance of Green's functions under such a symmetry in the context of algorithm by which all AC transports are constructed simultaneously.

We can extend the RGS such that it satisfies the infalling boundary condition, which we call the boundary shifting diffeomorphism. Then we can view things more concisely and natural. RGS is not gauge symmetry but a solution generating transformation. Therefore it generate formally new solution both in boundary and bulk. By extending it to BSD, the bulk part of the solution can be accepted as a true bulk degree of freedom more naturally since the latter satisfies the in-falling boundary condition.

8. Note added in proof

After this work is almost finished, the paper [14] appeared where residual gauge invariance was discussed using a different method.

Acknowledgements

The work of KYK and KKK was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT & Future Planning (NRF-2014R1A1A1003220). The work of SS and YS was supported by Mid-career Researcher Program through the National Research Foundation of Korea (NRF) grant No. NRF-2013R1A2A2A05004846 and SS was also partially supported by the Korean–Eastern European cooperation in research and development through (NRF-2013K1A3A1A39073412). YS was also supported in part by Basic Science Research Program through NRF grant No. NRF-2012-R1A1A2040881.

References

[1] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. 2 (1998) 253–291, arXiv:hep-th/9802150.
[2] S.A. Hartnoll, Lectures on holographic methods for condensed matter physics, Class. Quantum Gravity 26 (2009) 224002, arXiv:0903.3246.
[3] K.-Y. Kim, K.K. Kim, Y. Seo, S.-J. Sin, Coherent/incoherent metal transition in a holographic model, arXiv:1409.8346.
[4] K.-Y. Kim, K.K. Kim, M. Park, A simple holographic superconductor with momentum relaxation, arXiv:1501.0044.
[5] I. Amado, M. Kaminski, K. Landsteiner, Hydrodynamics of holographic superconductors, J. High Energy Phys. 0905 (2009) 021, arXiv:0903.2209.
[6] M. Kaminski, K. Landsteiner, J. Mas, J.P. Shock, J. Tarrio, Holographic operator mixing and quasinormal modes on the brane, J. High Energy Phys. 1002 (2010) 021, arXiv:0911.3610.
[7] T. Andrade, B. Withers, A simple holographic model of momentum relaxation, J. High Energy Phys. 1405 (2014) 101, arXiv:1311.5157.
[8] A. Donos, J.P. Gauntlett, Thermoelectric DC conductivities from black hole horizons, arXiv:1406.4742.

[9] A. Donos, J.P. Gauntlett, Novel metals and insulators from holography, J. High Energy Phys. 1406 (2014) 007, arXiv:1401.5077.

[10] M. Bianchi, D.Z. Freedman, K. Skenderis, Holographic renormalization, Nucl. Phys. B 631 (2002) 159–194, arXiv:hep-th/0112119.

[11] A. Donos, J.P. Gauntlett, Holographic Q-lattices, J. High Energy Phys. 1404 (2014) 040, arXiv:1311.3292.

[12] S.A. Hartnoll, P. Kovtun, Hall conductivity from dyonic black holes, Phys. Rev. D 76 (2007) 066001, arXiv:0704.1160.

[13] D.T. Son, A.O. Starinets, Minkowski space correlators in AdS/CFT correspondence: recipe and applications, J. High Energy Phys. 0209 (2002) 042, arXiv:hep-th/0205051.

[14] A. Donos, J.P. Gauntlett, C. Pantelidou, Conformal field theories in $d = 4$ with a helical twist, arXiv:1412.3446.