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ISOMORPHISMS, DERIVATIONS AND ISOMETRIES IN PROPER CQ^*-ALGEBRAS*

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Abstract In this paper, we investigate homomorphisms in proper CQ^*-algebras, proper Lie CQ^*-algebras and proper Jordan CQ^*-algebras and derivations on proper CQ^*-algebras, proper Lie CQ^*-algebras and proper Jordan CQ^*-algebras associated with the Cauchy-Jensen functional equation

$$2f\left(\frac{x+y}{2}+z\right)=f(x)+f(y)+2f(z),$$

which was introduced and investigated in [3, 28].

Furthermore, Isometries and isometric isomorphisms in proper CQ^*-algebras are studied.

Keywords Cauchy-Jensen functional equation, proper CQ^*-algebra isomorphism, isometry, isometric isomorphism, proper Lie (Jordan) CQ^*-algebra homomorphism, derivation, Lie (Jordan) derivation.

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1. Introduction and preliminaries

Topological quasi $*$-algebras have been considered with a certain interest, first for their own mathematical structure and second for their possible applications in the mathematical description of a number of quantum models. A complete theory of topological quasi $*$-algebras is not yet at hand and, for several reasons, it has appeared more convenient to deal with relevant subclasses instead of considering the most general case. In this framework, Bagarello and Trapani have introduced and investigated the class of CQ^*-algebras ([6, 7]). Their interest relies mainly in the fact that they appear as the class of Banach partial $*$-algebras ([1]) that behaves more closely to C^*-algebras and share with these latter several structure properties. A CQ^*-algebra arises as the completion of a given C^*-algebra with respect to a weaker norm, with certain coupling properties of the two norms. In [4], Bagarello, Inoue and Trapani have considered the case where the CQ^*-algebra can be constructed from a given left Hilbert algebra, providing in this way the expected link with the Tomita-Takesaki theory ([40, 42]).

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The problem of the mathematical description of physical system has an ancient origin. Already for classical mechanics many different possibilities have been developed during the years. One of the most common is the phase space description, where the dynamics of the system is governed, for instance, by the Hamilton equations. In ordinary quantum mechanics the particles are described by vectors of a Hilbert space $\mathcal{H}$, while the operations performed over the system are described by self-adjoint operators acting on $\mathcal{H}$. As for the dynamics, again we can use several equivalent strategies: in the Heisenberg picture, for instance, the vectors (often called wave functions) are constant in time, while the operators evolve in the following fashion: $A \to A_t := e^{iHt}Ae^{-iHt}$. Here H is the hamiltonian operator, which describes the energy of the system. Opposite is the situation for the Schrödinger picture: here the operators are independent of time, while the wave function at time t is given by $\Psi_t := e^{iHt}\Psi$, Ψ being the initial condition.

In the so-called algebraic approach to quantum systems, one of the basic problems to solve consists in the rigorous definition of the algebraic dynamics, i.e., the time evolution of observables and states. For instance, in quantum statistical mechanics or in quantum field theory one tries to recover the dynamics by performing a certain limit of the strictly *local* dynamics. However, this can be successfully done only for few models and under quite strong topological assumptions (see [38] and references therein). The unbounded nature of the operators describing observables of a quantum mechanical system with a finite or infinite number of degrees of freedom is mathematically a *fact* which follows directly from the noncommutative nature of the quantum world in the sense that, as a consequence of the Wiener-von Neumann theorem, the commutation relation $[\hat{q}, \hat{p}] = iI$ for the position $\hat{q}$ and the momentum $\hat{p}$ is not compatible with the boundedness of both $\hat{q}$ and $\hat{p}$. Thus any operator representation of this commutation relation necessarily involves unbounded operators. The bosonic creation and annihilation operators $a^\dagger$ and a, $[a, a^\dagger] = I$, or the hamiltonian of the simple harmonic oscillator, $H = \frac{1}{2}(\hat{p}^2 + \hat{q}^2) = a^\dagger a + \frac{1}{2}I$, just to mention few examples, are all unbounded operators.

When an experiment is carried out, what is measured is an eigenvalue of an observable, which is surely a *finite* real number: for instance, if the physical system $\mathcal{S}$ on which measurements are performed is in a laboratory, then if we measure the position of a particle of $\mathcal{S}$ we must get a finite number as a result. If we measure the energy of a quantum particle in a, say, harmonic potential, we can only get a finite measure since the probability that the particle has infinite energy is zero. Moreover, in a true relativistic world, since the velocity of a particle cannot exceed the velocity of light c, any measurement of its momentum can only give a finite result. From the mathematical point of view this may correspond to restricting the operator to some *special subspaces* where the unboundedness is in fact removed. This procedure supports the practical point of view where it seems enough to deal with bounded operators only.

As it is extensively discussed in [39], the full description of a physical system $\mathcal{S}$ implies the knowledge of three basic ingredients: the sent of the observables, the set of the states and the dynamics that describes the time evolution of the system by means of the time dependence of the expectation value of a given observable on a given state. Originally the set of the observables was considered to be a C^*-algebra [18]. In many applications, however, this was shown not to be the most convenient choice and the C^*-algebra was replaced by a von Neumann algebra, because the role of the representation turns out to be crucial mainly when long range interactions are involved (see [5] and references therein). Here we use a different algebraic structure, similar to the one considered in [12], which is suggested by the considerations above: because of the relevance of the unbounded operators in the description of $\mathcal{S}$, we will assume that the observables of the system belong to a quasi $*$-algebra (A, A_0) (see [44] and references therein), while, in order to have a richer mathematical structure, we will use a slightly different algebraic structure: (A, A_0) will be assumed to be a proper CQ^*-algebra, which has nicer topological properties. In particular, for instance, A_0 is a C^*-algebra.

Let A be a linear space and A_0 is a $*$-algebra contained in A as a subspace. We say that A is a quasi $*$-algebra over A_0 if

(i) the right and left multiplications of an element of A and an element of A_0 are defined and linear;

(ii) $x_1(x_2a) = (x_1x_2)a$, $(ax_1)x_2 = a(x_1x_2)$ and $x_1(ax_2) = (x_1a)x_2$ for all $x_1, x_2 \in A_0$ and all $a \in A$;

(iii) an involution $*$, which extends the involution of A_0, is defined in A with the property $(ab)^* = b^*a^*$ whenever the multiplication is defined.

In this paper we will assume that the quasi $*$-algebra under consideration has a unit $e \in A_0$ such that $ae = ea = a$ for all $a \in A$.

The spatiality of derivations is a very classical problem when formulated in $*$-algebras and it as been extensively studied in the literature in a large variety of situations, mostly depending on the topological structure of the $*$-algebras under consideration (C^*-algebras, von Neumann algebras, etc. see [1,11,38]). We consider a more general set-up, turning our attention to derivations taking their values in a quasi $*$-algebra. This choice is motivated by possible applications to the physical situations described above. Indeed, if A_0 denotes the $*$-algebra of local observables of the system, in order to perform the so-called thermodynamical limits of certain local observables, one endows A_0 with a locally convex topology τ, conveniently chosen for this aim. The completion A of $A_0[\tau]$, where thermodynamical limits mostly live, may fail to be an algebra but it is in general *quasi $*$-algebra* [1, 44]. A quasi $*$-algebra (A, A_0) is said to be a locally convex quasi $*$-algebra if in A a locally convex topology τ is defined such that

(i) the involution is continuous and the multiplications are separately continuous;

(ii) A_0 is dense in $A[\tau]$.

Throughout this paper, we suppose that a locally convex quasi $*$-algebra $(A[\tau], A_0)$ is complete. For an overview on partial $*$-algebra and related topics we refer to [1].

In a series of papers [4,6–8], many authors have considered a special class of quasi $*$-algebras, called *proper CQ^*-algebras*, which arise as completions of C^*-algebras. They can be introduced in the following way:

Let A be a Banach bi-module over the C^*-algebra A_0 with involution $*$ and C^*-norm $\|\cdot\|_0$ such that $A_0 \subset A$. We say that (A, A_0) is a *proper CQ^*-algebra* if

(i) A_0 is dense in A with respect to its norm $\|\cdot\|$;

(ii) $\|y\|_0 = \sup_{a\in A, \|a\|\le 1} \|ay\|$ for all $y \in A_0$.

Several mathematician have contributed works on these subjects (see [13, 15, 20, 23, 24, 41, 43, 45–47]).

Definition 1.1. Let (A, A_0) and (B, B_0) be proper CQ^*-algebras.

(i) A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *proper CQ^*-algebra homomorphism* if $H(z) \in B_0$ and $H(zx) = H(z)H(x)$ for all $z \in A_0$ and all $x \in A$. If, in addition, the mapping $H : A \to B$ is bijective and the mapping $H|_{A_0} : A_0 \to B_0$ is a bijective involutive mapping, then the mapping $H : A \to B$ is called a *proper CQ^*-algebra isomorphism*;

(ii) A $\mathbb{C}$-linear mapping $\delta : A_0 \to A$ is called a *derivation* if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A_0$ (see [2]).

A C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x, y] := \frac{xy-yx}{2}$ on $\mathcal{C}$, is called a *Lie C^*-algebra*. (see [25, 27, 33]).

Definition 1.2. A proper CQ^*-algebra (A, A_0), endowed with the Lie product $[z, x] := \frac{zx-xz}{2}$ for all $z \in A_0$ and all $x \in A$, is called a *proper Lie CQ^*-algebra*.

Definition 1.3. Let (A, A_0) and (B, B_0) be proper Lie CQ^*-algebras.

(i) A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *proper Lie CQ^*-algebra homomorphism* if $H(z) \in B_0$ and $H([z, x]) = [H(z), H(x)]$ for all $z \in A_0$ and all $x \in A$;

(ii) A $\mathbb{C}$-linear mapping $\delta : A_0 \to A$ is called a *Lie derivation* if $\delta([x, y]) = [x, \delta(y)] + [\delta(x), y]$ for all $x, y \in A_0$.

A C^*-algebra $\mathcal{C}$, endowed with the Jordan product $x \circ y := \frac{xy+yx}{2}$ on $\mathcal{C}$, is called a *Jordan C^*-algebra* (see [26, 27, 33]).

Definition 1.4. A proper CQ^*-algebra (A, A_0), endowed with the Jordan product $z \circ x := \frac{zx+xz}{2}$ for all $z \in A_0$ and all $x \in A$, is called a *proper Jordan CQ^*-algebra*.

Definition 1.5. Let (A, A_0) and (B, B_0) be proper Jordan CQ^*-algebras.

(i) A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *proper Jordan CQ^*-algebra homomorphism* if $H(z) \in B_0$ and $H(z \circ x) = H(z) \circ H(x)$ for all $z \in A_0$ and all $x \in A$;

(ii) A $\mathbb{C}$-linear mapping $\delta : A_0 \to A$ is called a *Jordan derivation* if $\delta(x \circ y) = x \circ \delta(y) + \delta(x) \circ y$ for all $x, y \in A_0$.

In [16], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\|, \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [37]. Fechner [14] and Gilányi [17] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [31] proved the Hyers-Ulam stability of functional inequalities associated with Jordan-von Neumann type additive functional equations.

Lee et al. [21] proved the Hyers-Ulam stability of an additive functional inequality in proper CQ^*-algebras. Park and An [29] proved the Hyers-Ulam stability of isometric isomorphisms in proper CQ^*-algebras. Park and Boo [30] proved the

Hyers-Ulam stability of isomorphisms and derivations in proper CQ^*-algebras. Park et al. [32] proved the Hyers-Ulam stability of derivations on proper Jordan CQ^*-algebras.

In this paper, we will prove the superstability of isomorphisms and derivations in proper CQ^*-algebras, of homomorphisms and derivations in proper Lie CQ^*-algebras and of homomorphisms and derivations in proper Jordan CQ^*-algebras associated with the Cauchy-Jensen additive functional inequality

$$\|f(x)+f(y)+2f(z)\| \le \left\|2f\left(\frac{x+y}{2}+z\right)\right\|. \tag{1.2}$$

Moreover, we will prove the superstability of isometries and isometric isomorphisms in proper CQ^*-algebras associated with the Cauchy-Jensen additive functional inequality (1.2).

This paper is organized as follows: In Section 2, we investigate isomorphisms and derivations in proper CQ^*-algebras associated with the Cauchy-Jensen additive functional inequality (1.2).

In Section 3, we investigate homomorphisms and derivations in proper Lie CQ^*-algebras associated with the Cauchy-Jensen additive functional inequality (1.2).

In Section 4, we investigate homomorphisms and derivations in proper Jordan CQ^*-algebras associated with the Cauchy-Jensen additive functional inequality (1.2).

In Section 5, we investigate isometries and isometric isomorphisms in proper CQ^*-algebras associated with the Cauchy-Jensen additive functional inequality (1.2).

2. Isomorphisms and derivations in proper CQ^*-algebras

Throughout this section, assume that (A, A_0) is a proper CQ^*-algebra with C^*-norm $\|\cdot\|_{A_0}$ and norm $\|\cdot\|_A$, and that (B, B_0) is a proper CQ^*-algebra with C^*-norm $\|\cdot\|_{B_0}$ and norm $\|\cdot\|_B$.

We investigate isomorphisms in proper CQ^*-algebras associated with the Cauchy-Jensen functional equation.

Theorem 2.1. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to B$ a bijective mapping such that*

$$\|f(\mu x)+\mu f(y)+2f(z)\|_B \le \left\|2f\left(\frac{x+y}{2}+z\right)\right\|_B, \tag{2.1}$$

$$\|f(wx)-f(w)f(x)\|_B \le \theta(\|w\|_A^{2r}+\|x\|_A^{2r}), \tag{2.2}$$

$$\|f(w^*)-f(w)^*\|_B \le \theta\|w\|_A^r \tag{2.3}$$

for $\mu = 1, i$, all $w \in A_0$ and all $x, y, z \in A$. If $f|_{A_0} : A_0 \to B_0$ is bijective and if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $f : A \to B$ is a proper CQ^*-algebra isomorphism.

Proof. Let $\mu = 1$ in (2.1). By [31, Proposition 2.3], the mapping $f : A \to B$ is Cauchy additive. By Theorem of [34], the mapping $f : A \to B$ is $\mathbb{R}$-linear.

Letting $\mu = i$, $z = 0$ and $y = -x$ in (2.1), we get

$$f(ix) - if(x) = f(ix) + if(-x) = 0$$

for all $x \in A$. So $f(ix) = if(x)$ for all $x \in A$. For each $\lambda \in \mathbb{C}$, $\lambda = a + ib$ $(a, b \in \mathbb{R})$. Hence

$$f(\lambda x) = f(ax + ibx) = af(x) + bf(ix) = af(x) + ibf(x) = \lambda f(x)$$

for all $x \in A$. Thus $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (2.2),

$$\begin{aligned}\|f(wx) - f(w)f(x)\|_B &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n wx) - f(2^n w)f(2^n x)\|_B \\ &\leq \lim_{n\to\infty} \frac{4^{nr}}{4^n}\theta(\|w\|_A^{2r} + \|x\|_A^{2r}) = 0\end{aligned}$$

for all $w \in A_0$ and all $x \in A$. So

$$f(wx) = f(w)f(x)$$

for all $w \in A_0$ and all $x \in A$.
By (2.3),

$$\|f(w^*) - f(w)^*\|_B = \lim_{n\to\infty} \frac{1}{2^n}\|f(2^n w^*) - f(2^n w)^*\|_B \leq \lim_{n\to\infty} \frac{2^{nr}}{2^n}\theta\|w\|_A^r = 0$$

for all $w \in A_0$. So

$$f(w^*) = f(w)^*$$

for all $w \in A_0$;

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$\begin{aligned}f(wx) &= f(w)f(x), \\ f(w^*) &= f(w)^*\end{aligned}$$

for all $w \in A_0$ and all $x \in A$.

Since $f|_{A_0} : A_0 \to B_0$ is bijective, the mapping $f : A \to B$ is a proper CQ^*-algebra isomorphism, as desired. □

Theorem 2.2. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to B$ a bijective mapping satisfying* (2.1) *and* (2.3) *such that*

$$\|f(wx) - f(w)f(x)\|_B \leq \theta \cdot \|w\|_A^r \cdot \|x\|_A^r \tag{2.4}$$

for all $w \in A_0$ and all $x \in A$. If $f|_{A_0} : A_0 \to B_0$ is bijective and if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $f : A \to B$ is a proper CQ^*-algebra isomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.1, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (2.4),

$$\begin{aligned}\|f(wx) - f(w)f(x)\|_B &= \lim_{n\to\infty} \frac{1}{4^n} \|f(4^n wx) - f(2^n w)f(2^n x)\|_B \\ &\le \lim_{n\to\infty} \frac{4^{nr}}{4^n} \theta \cdot \|w\|_A^r \cdot \|x\|_A^r = 0\end{aligned}$$

for all $w \in A_0$ and all $x \in A$. So

$$f(wx) = f(w)f(x)$$

for all $w \in A_0$ and all $x \in A$;

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$f(wx) = f(w)f(x)$$

for all $w \in A_0$ and all $x \in A$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Now, we investigate derivations on proper CQ^*-algebras associated with the Cauchy-Jensen functional equation.

Theorem 2.3. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to A$ a mapping such that*

$$\|f(\mu x) + \mu f(y) + 2f(z)\|_A \le \left\| 2f\left(\frac{x+y}{2} + z\right)\right\|_A , \tag{2.5}$$

$$\|f(w_0 w_1) - f(w_0)w_1 - w_0 f(w_1)\|_A \le \theta(\|w_0\|_A^{2r} + \|w_1\|_A^{2r}) \tag{2.6}$$

for $\mu = 1, i$, all $w_0, w_1 \in A_0$ and all $x, y, z \in A$. Then the mapping $f : A \to A$ is a derivation on A.

Proof. By the same reasoning as in the proof of Theorem 2.1, the mapping $f : A \to A$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (2.6),

$$\begin{aligned}&\|f(w_0 w_1) - f(w_0)w_1 - w_0 f(w_1)\|_A \\ &= \lim_{n\to\infty} \frac{1}{4^n} \|f(4^n w_0 w_1) - f(2^n w_0) \cdot 2^n w_1 - 2^n w_0 f(2^n w_1)\|_A \\ &\le \lim_{n\to\infty} \frac{4^{nr}}{4^n} \theta(\|w_0\|_A^{2r} + \|w_1\|_A^{2r}) = 0\end{aligned}$$

for all $w_0, w_1 \in A_0$. So

$$f(w_0 w_1) = f(w_0)w_1 + w_0 f(w_1)$$

for all $w_0, w_1 \in A_0$;

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to A$ satisfies

$$f(w_0 w_1) = f(w_0)w_1 + w_0 f(w_1)$$

for all $w_0, w_1 \in A_0$.

Therefore, the mapping $f : A \to A$ is a derivation on A, as desired. □

Theorem 2.4. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to A$ a mapping satisfying* (2.5) *such that*

$$\|f(w_0 w_1) - f(w_0)w_1 - w_0 f(w_1)\|_A \leq \theta \cdot \|w_0\|_A^r \cdot \|w_1\|_A^r$$

for all $w_0, w_1 \in A_0$. Then the mapping $f : A \to A$ is a derivation on A.

Proof. The proof is similar to the proofs of Theorems 2.1 and 2.3. □

3. Homomorphisms and derivations in proper Lie CQ^*-algebras

Throughout this section, assume that (A, A_0) is a proper Lie CQ^*-algebra with C^*-norm $\|\cdot\|_{A_0}$ and norm $\|\cdot\|_A$, and that (B, B_0) is a proper Lie CQ^*-algebra with C^*-norm $\|\cdot\|_{B_0}$ and norm $\|\cdot\|_B$.

We investigate homomorphisms in proper Lie CQ^*-algebras associated with the Cauchy-Jensen functional equation.

Theorem 3.1. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to B$ a mapping satisfying* (2.1) *and $f(w) \in B_0$ for all $w \in A_0$ such that*

$$\|f([w, x]) - [f(w), f(x)]\|_B \leq \theta(\|w\|_A^{2r} + \|x\|_A^{2r}) \tag{3.1}$$

for all $w \in A_0$ and all $x \in A$. Then the mapping $f : A \to B$ is a proper Lie CQ^*-algebra homomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.1, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (3.1),

$$\begin{aligned}\|f([w, x]) - [f(w), f(x)]\|_B &= \lim_{n\to\infty} \frac{1}{4^n} \|f(4^n[w, x]) - [f(2^n w), f(2^n x)]\|_B] \\ &\leq \lim_{n\to\infty} \frac{4^{nr}}{4^n} \theta(\|w\|_A^{2r} + \|x\|_A^{2r}) = 0\end{aligned}$$

for all $w \in A_0$ and all $x \in A$. So

$$f([w, x]) = [f(w), f(x)]$$

for all $w \in A_0$ and all $x \in A$;

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$f([w, x]) = [f(w), f(x)]$$

for all $w \in A_0$ and all $x \in A$.

Therefore, the mapping $f : A \to B$ is a proper Lie CQ^*-algebra homomorphism. □

Theorem 3.2. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to B$ a mapping satisfying* (2.1) *and $f(w) \in B_0$ for all $w \in A_0$ such that*

$$\|f([w,x]) - [f(w), f(x)]\|_B \leq \theta \cdot \|w\|_A^r \cdot \|x\|_A^r \tag{3.2}$$

for all $w \in A_0$ and all $x \in A$. Then the mapping $f : A \to B$ is a proper Lie CQ^*-algebra homomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.1, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (3.2),

$$\begin{aligned}\|f([w,x]) - [f(w), f(x)]\|_B &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n[w,x]) - [f(2^n w), f(2^n x)]\|_B \\ &\leq \lim_{n\to\infty} \frac{4^{nr}}{4^n}\theta \cdot \|w\|_A^r \cdot \|x\|_A^r = 0\end{aligned}$$

for all $w \in A_0$ and all $x \in A$. So

$$f([w,x]) = [f(w), f(x)]$$

for all $w \in A_0$ and all $x \in A$;

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$f([w,x]) = [f(w), f(x)]$$

for all $w \in A_0$ and all $x \in A$.

Therefore, the mapping $f : A \to B$ is a proper Lie CQ^*-algebra homomorphism. □

Now we investigate derivations on proper Lie CQ^*-algebras associated with the Cauchy-Jensen functional equation.

Theorem 3.3. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to A$ a mapping satisfying* (2.5) *such that*

$$\|f([w_0, w_1]) - [f(w_0), w_1] - [w_0, f(w_1)]\|_A \leq \theta(\|w_0\|_A^{2r} + \|w_1\|_A^{2r}) \tag{3.3}$$

for all $w_0, w_1 \in A_0$. Then the mapping $f : A \to A$ is a Lie derivation on A.

Proof. By the same reasoning as in the proof of Theorem 2.1, the mapping $f : A \to A$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (3.3),

$$\begin{aligned}&\|f([w_0, w_1]) - [f(w_0), w_1] - [w_0, f(w_1)]\|_A \\ &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n[w_0, w_1]) - [f(2^n w_0), 2^n w_1] - [2^n w_0, f(2^n w_1)]\|_A \\ &\leq \lim_{n\to\infty} \frac{4^{nr}}{4^n}\theta(\|w_0\|_A^{2r} + \|w_1\|_A^{2r}) = 0\end{aligned}$$

for all $w_0, w_1 \in A_0$. So

$$f([w_0, w_1]) = [f(w_0), w_1] + [w_0, f(w_1)]$$

for all $w_0, w_1 \in A_0$;

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to A$ satisfies

$$f([w_0, w_1]) = [f(w_0), w_1] + [w_0, f(w_1)]$$

for all $w_0, w_1 \in A_0$.

Therefore, the mapping $f : A \to A$ is a Lie derivation on A, as desired. □

Theorem 3.4. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to A$ a mapping satisfying* (2.5) *such that*

$$\|f([w_0, w_1]) - [f(w_0), w_1] - [w_0, f(w_1)]\|_A \leq \theta \cdot \|w_0\|_A^r \cdot \|w_1\|_A^r$$

for all $w_0, w_1 \in A_0$. Then the mapping $f : A \to A$ is a Lie derivation on A.

Proof. The proof is similar to the proofs of Theorems 2.1 and 3.3. □

4. Homomorphisms and derivations in proper Jordan CQ^*-algebras

Throughout this section, assume that (A, A_0) is a proper Jordan CQ^*-algebra with C^*-norm $\|\cdot\|_{A_0}$ and norm $\|\cdot\|_A$, and that (B, B_0) is a proper Jordan CQ^*-algebra with C^*-norm $\|\cdot\|_{B_0}$ and norm $\|\cdot\|_B$.

We investigate homomorphisms in proper Jordan CQ^*-algebras associated with the Cauchy-Jensen functional equation.

Theorem 4.1. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to B$ a mapping satisfying* (2.1) *and $f(w) \in B_0$ for all $w \in A_0$ such that*

$$\|f(w \circ x) - f(w) \circ f(x)\|_B \leq \theta(\|w\|_A^{2r} + \|x\|_A^{2r}) \tag{4.1}$$

for all $w \in A_0$ and all $x \in A$. Then the mapping $f : A \to B$ is a proper Jordan CQ^*-algebra homomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.1, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (4.1),

$$\begin{aligned}\|f(w \circ x) - f(w) \circ f(x)\|_B &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n w \circ x) - f(2^n w) \circ f(2^n x)\|_B \\ &\leq \lim_{n\to\infty} \frac{4^{nr}}{4^n}\theta(\|w\|_A^{2r} + \|x\|_A^{2r}) = 0\end{aligned}$$

for all $w \in A_0$ and all $x \in A$. So

$$f(w \circ x) = f(w) \circ f(x)$$

for all $w \in A_0$ and all $x \in A$;

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$f(w \circ x) = f(w) \circ f(x)$$

for all $w \in A_0$ and all $x \in A$.

Therefore, the mapping $f : A \to B$ is a proper Jordan CQ^*-algebra homomorphism, as desired. $\square$

Theorem 4.2. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to B$ a mapping with respect to norms $\|\cdot\|_A$ and $\|\cdot\|_B$ satisfying* (2.1) *and $f(w) \in B_0$ for all $w \in A_0$ such that*

$$\|f(w \circ x) - f(w) \circ f(x)\|_B \leq \theta \cdot \|w\|_A^r \cdot \|x\|_A^r \tag{4.2}$$

for all $w \in A_0$ and all $x \in A$. Then the mapping $f : A \to B$ is a proper Jordan CQ^*-algebra homomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.1, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (4.2),

$$\begin{aligned}\|f(w \circ x) - f(w) \circ f(x)\|_B &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n w \circ x) - f(2^n w) \circ f(2^n x)\|_B \\ &\leq \lim_{n\to\infty} \frac{4^{nr}}{4^n}\theta \cdot \|w\|_A^r \cdot \|x\|_A^r = 0\end{aligned}$$

for all $w \in A_0$ and all $x \in A$. So

$$f(w \circ x) = f(w) \circ f(x)$$

for all $w \in A_0$ and all $x \in A$;

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$f(w \circ x) = f(w) \circ f(x)$$

for all $w \in A_0$ and all $x \in A$.

Therefore, the mapping $f : A \to B$ is a proper Jordan CQ^*-algebra homomorphism, as desired. $\square$

Now we investigate derivations on proper Jordan CQ^*-algebras associated with the Cauchy-Jensen functional equation.

Theorem 4.3. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to A$ a mapping satisfying* (2.5) *such that*

$$\|f(w_0 \circ w_1) - f(w_0) \circ w_1 - w_0 \circ f(w_1)\|_A \leq \theta(\|w_0\|_A^{2r} + \|w_1\|_A^{2r}) \tag{4.3}$$

for all $w_0, w_1 \in A_0$. Then the mapping $f : A \to A$ is a Jordan derivation on A.

Proof. By the same reasoning as in the proof of Theorem 2.1, the mapping $f : A \to A$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (4.3),

$$\begin{aligned}&\|f(w_0 \circ w_1) - f(w_0) \circ w_1 - w_0 \circ f(w_1)\|_A \\ &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n w_0 \circ w_1) - f(2^n w_0) \circ 2^n w_1 - 2^n w_0 \circ f(2^n w_1)\|_A \\ &\leq \lim_{n\to\infty} \frac{4^{nr}}{4^n}\theta(\|w_0\|_A^{2r} + \|w_1\|_A^{2r}) = 0\end{aligned}$$

for all $w_0, w_1 \in A_0$. So

$$f(w_0 \circ w_1) = f(w_0) \circ w_1 + w_0 \circ f(w_1)$$

for all $w_0, w_1 \in A_0$;

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to A$ satisfies

$$f(w_0 \circ w_1) = f(w_0) \circ w_1 + w_0 \circ f(w_1)$$

for all $w_0, w_1 \in A_0$.

Therefore, the mapping $f : A \to A$ is a Jordan derivation on A, as desired. □

Theorem 4.4. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to A$ a mapping satisfying* (2.5) *such that*

$$\|f(w_0 \circ w_1) - f(w_0) \circ w_1 - w_0 \circ f(w_1)\|_A \leq \theta \cdot \|w_0\|_A^r \cdot \|w_1\|_A^r$$

for all $w_0, w_1 \in A_0$. Then the mapping $f : A \to A$ is a Jordan derivation on A.

Proof. The proof is similar to the proofs of Theorems 2.1 and 4.3. □

5. Isometries and isometric isomorphisms in proper CQ^*-algebras

Throughout this section, assume that (A, A_0) is a proper CQ^*-algebra with C^*-norm $\|\cdot\|_{A_0}$ and norm $\|\cdot\|_A$, and that (B, B_0) is a proper CQ^*-algebra with C^*-norm $\|\cdot\|_{B_0}$ and norm $\|\cdot\|_B$.

Surjective isometries between normed vector spaces have been investigated by several authors ([9, 10, 19, 22, 35, 36]).

Definition 5.1. A mapping $I : A \to B$, which satisfies $I(w) \in B_0$ for all $w \in A_0$, is called an *isometry in proper CQ^*-algebras* if

$$\|I(x) - I(y)\|_B = \|x - y\|_A,$$
$$\|I(z) - I(w)\|_{B_0} = \|z - w\|_{A_0}$$

for all $z, w \in A_0$ and all $x, y \in A$.

We investigate isometries in proper CQ^*-algebras associated to the Cauchy-Jensen functional equation.

Theorem 5.1. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to B$ a mapping satisfying* (2.1) *and $f(w) \in B_0$ for all $w \in A_0$ such that*

$$|\ \|f(w)\|_{B_0} + \|f(x)\|_B - \|w\|_{A_0} - \|x\|_A\ | \leq \theta(\|w\|_A^r + \|x\|_A^r) \tag{5.1}$$

for all $w \in A_0$ and all $x \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $f : A \to B$ is an isometry in proper CQ^*-algebras.

Proof. By the same reasoning as in the proof of Theorem 2.1, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. Letting $w = 0$ in (5.1), we get

$$| \ \|f(x)\|_B - \|x\|_A \ | \le \theta \|x\|_A^r$$

for all $x \in A$. Hence

$$\begin{aligned} | \ \|f(x)\|_B - \|x\|_A \ | &= \lim_{n\to\infty} \frac{1}{2^n} | \ \|f(2^n x)\|_B - \|2^n x\|_A \ | \\ &\le \lim_{n\to\infty} \frac{2^{nr}}{2^n} \theta \|x\|_A^r = 0 \end{aligned}$$

for all $x \in A$. So $\|f(x)\|_B = \|x\|_A$ for all $x \in A$. Since $f : A \to B$ is $\mathbb{C}$-linear,

$$\|f(x) - f(y)\|_B = \|f(x-y)\|_B = \|x - y\|_A$$

for all $x, y \in A$.
Letting $x = 0$ in (5.1), we get

$$| \ \|f(w)\|_{B_0} - \|w\|_{A_0} \ | \le \theta \|w\|_A^r$$

for all $w \in A_0$. Hence

$$\begin{aligned} | \ \|f(w)\|_{B_0} - \|w\|_{A_0} \ | &= \lim_{n\to\infty} \frac{1}{2^n} | \ \|f(2^n w)\|_{B_0} - \|2^n w\|_{A_0} \ | \\ &\le \lim_{n\to\infty} \frac{2^{nr}}{2^n} \theta \|w\|_A^r = 0 \end{aligned}$$

for all $w \in A_0$. So $\|f(w)\|_{B_0} = \|w\|_{A_0}$ for all $w \in A_0$. Since $f|_{A_0} : A_0 \to B_0$ is $\mathbb{C}$-linear,

$$\|f(z) - f(w)\|_{B_0} = \|f(z - w)\|_{B_0} = \|z - w\|_{A_0}$$

for all $z, w \in A_0$;

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$\begin{aligned} \|f(x) - f(y)\|_B &= \|x - y\|_A, \\ \|f(z) - f(w)\|_{B_0} &= \|z - w\|_{A_0} \end{aligned}$$

for all $z, w \in A_0$ and all $x, y \in A$.

Therefore, the mapping $f : A \to B$ is an isometry in proper CQ^*-algebras. □

Definition 5.2. A proper CQ^*-algebra isomorphism $H : A \to B$ is called an *isometric isomorphism in proper CQ^*-algebras* if H is an isometry in proper CQ^*-algebras.

We investigate isometric isomorphisms in proper CQ^*-algebras associated to the Cauchy-Jensen functional equation.

Theorem 5.2. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to B$ a bijective mapping satisfying* (2.1)*,* (2.2)*,* (2.3) *and* (5.1)*. If $f|_{A_0} : A_0 \to B_0$ is bijective and if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $f : A \to B$ is an isometric isomorphism in proper CQ^*-algebras.*

Proof. The proof is similar to the proofs of Theorems 2.1 and 5.2. □

Conclusions

We have proved the superstability of isomorphisms and derivations in proper CQ^*-algebras, of homomorphisms and derivations in proper Lie CQ^*-algebras and of homomorphisms and derivations in proper Jordan CQ^*-algebras associated with the Cauchy-Jensen additive functional inequality (1.2).

Moreover, we have proved the superstability of isometries and isometric isomorphisms in proper CQ^*-algebras associated with the Cauchy-Jensen additive functional inequality (1.2). Our results generalize the previous results given in [21, 29, 30, 32].

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