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The tangential Thom class of a Poincaré duality group

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ABSTRACT

For each Poincaré duality group Γ there exists a class, which we call the tangential Thom class of Γ, in the group cohomology of $\Gamma \times \Gamma$ with a right choice of the coefficient module. The class has the crucial properties, even if stated in a purely algebraic language, which correspond to those of Thom class of the tangent bundle of a closed manifold. In particular the Thom isomorphism has been proved to exist by observing that certain two sequences of homological functors, one being the homology of Γ and the other that of $\Gamma \times \Gamma$, being regarded as functors defined on the category of $\mathbb{Z}\Gamma$-modules are homological and effaceable.

1. Introduction

Let Γ be a Poincaré duality group of dimension n in the sense of K.S. Brown (pp. 220-1, [1]). This means that Γ is of type FP and there is a homomorphism $w : \Gamma \to \{\pm 1\}$ and a class $[\Gamma]$ in the group homology $H_n(\Gamma; \mathbb{Z}^w)$ so that there is the Poincaré duality isomorphism

$$\cdot \cap [\Gamma] : H^i(\Gamma; M) \to H_{n-i}(\Gamma; M^w)$$

for any $i \in \mathbb{Z}$ and for any left $\mathbb{Z}\Gamma$-module M. Here $\mathbb{Z}\Gamma$ denotes the integral group ring and M^w denotes the $\mathbb{Z}\Gamma$-module whose underlying abelian group is M itself while its $\mathbb{Z}\Gamma$-module structure is given by linearly expanding the rule

$$g \cdot m = w(g)gm$$

for any $g \in \Gamma$ and any $m \in M$. In fact, '$\cdot \cap [\Gamma]$' in the above maps into $H_{n-i}(\Gamma; M \otimes \mathbb{Z}^w)$. Note that the integer group $\mathbb{Z}$ can be understood as a $\mathbb{Z}\Gamma$-module by letting Γ act trivially. Then we may identify

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the homology groups by the natural isomorphism $M \otimes \mathbb{Z}^w \equiv M^w$ of coefficients, where we understand the module structure of $M \otimes \mathbb{Z}^w$ is given by the diagonal action of Γ.

It is an interesting question whether the Eilenberg–MacLane space $X = K(\Gamma, 1)$ is the homotopy type of a closed manifold for any Poincaré duality group Γ. However Poincaré duality does not guarantee that Γ is finitely presented according to M.W. Davis [4]. Even when this condition is imposed on Γ, the question is still open (cf. [5]). Also there is a Poincaré duality group Γ for which X can be chosen as a closed topological manifold but not as a smooth or PL manifold (see [6] and also [7]).

If X is a smooth n-manifold, then the diagonal submanifold $\Delta(X) \subset X \times X$ has a neighborhood $N \subset X \times X$ which is diffeomorphic to the disk bundle of the tangent bundle of X. The decomposition $X \times X = (X \times X \smallsetminus \operatorname{int} N) \cup N$ provides a standard model for the so-called *Poincaré embedding structure on the diagonal* for general Poincaré duality spaces. Of course here the manifold X should be closed for the notion of Poincaré embedding to be relevant. An obstruction to the existence of a Poincaré embedding structure on the diagonal for general Poincaré complexes, which is exact in dimensions ≥ 4, has been defined by J.R. Klein [8]. If there is a finitely presented Poincaré duality group for which the obstruction does not vanish, the example will be truly interesting. However it appears that there is no known example of Poincaré duality group Γ for which the Klein obstruction for $X = K(\Gamma, 1)$ does not vanish.

In general for any Poincaré complex of dimension n, its tangent spherical fibration may be defined as an $(n-1)$-spherical fibration which is a stable inverse to the Spivak fibration with the right Euler characteristic if n is even or with the right b-invariant if n is odd, which always exists and unique up to fiberwise homotopy equivalence [2].

In this paper we will show that the Poincaré duality of the group Γ implies automatically an algebraic tangential property of Γ in the sense that there is a cohomology class U defined by 2.1 below. It seems appropriate for one to call the class U the *tangential Thom class*. The terminology is partially justified by 2.2, 2.3 and 3.1 below and also supported by 2.4 below since they make the class U appear to be a group cohomology version of the Thom class of the tangent bundle of a manifold. In fact U has been interpreted as the Thom class of a 'tangential' sphere fibration over $K(\Gamma, 1)$ by 5.1 and 6.2, 3 in [3] in case $K(\Gamma, 1)$ is the type of a finite complex whose universal cover is forward tame and simply connected at infinity.

We note that each of 2.2, 2.3 and 2.4 has a corresponding statement in [3, §6] even if the contexts are not the same. On the other hand the Thom isomorphism with U, which is established by 3.1 below, did not require any other argument in [3] than to show that U corresponds to the real Thom class of a spherical fibration. In this paper we prove 3.1 by showing that certain two sequences of homological functors are homological and effaceable. In fact one is the homology of Γ and the other is that of $\Gamma \times \Gamma$, which are regarded as sequences of functors defined on the category of $\mathbb{Z}\Gamma$-modules. Eventually this will imply that the two sequences are isomorphic to each other and prove that there is the Thom isomorphism.

2. The tangential Thom class

In this section we define the tangential Thom class of Γ. Then we will show that it has some properties which one expects from the Thom class of the tangent bundle of a manifold. The existence of Thom isomorphism demands more sophisticated notions for its justification and is postponed until the next section. The notations of the introduction are kept.

Every module in this paper is a left module with the only exception being implicit in the definition of the group homology for which we take the tensor $P \otimes_\Gamma M$ between left $\mathbb{Z}\Gamma$-modules understanding that the needed right module structure of P comes from the involution of $\mathbb{Z}\Gamma$ which maps g to g^{-1} for any $g \in \Gamma$. Every tensor product in this paper means one between abelian groups if not otherwise specified by a subscript.

Note that $\Gamma \times \Gamma$ is also a Poincaré duality group with the homomorphism $w \times w : \Gamma \times \Gamma \to \{\pm 1\}$ defined by $(w \times w)(g, h) = w(g)w(h)$ for any $(g, h) \in \Gamma \times \Gamma$ and the class $[\Gamma \times \Gamma] = [\Gamma] \times [\Gamma] \in H_{2n}(\Gamma \times \Gamma; \mathbb{Z}^{w \times w})$, where we understand the identification $\mathbb{Z}^w \otimes \mathbb{Z}^w \equiv \mathbb{Z}^{w \times w}$.

Let $\Delta : \Gamma \to \Gamma \times \Gamma$ denote the diagonal homomorphism and let $\Gamma\Delta$ denote the set of left cosets, $(\Gamma \times \Gamma)/\Delta\Gamma$. Then the free abelian group $\mathbb{Z}\Gamma\Delta$ is a $\mathbb{Z}(\Gamma \times \Gamma)$-module.

Also we introduce the homomorphisms $w_l, w_r : \Gamma \times \Gamma \to \{\pm 1\}$ defined respectively by $w_l(g, h) = w(g)$ and $w_r(g, h) = w(h)$ for any $(g, h) \in \Gamma \times \Gamma$.

Then we consider the homomorphism between $\mathbb{Z}\Gamma$-modules

$$\alpha : \mathbb{Z}^w \to \mathbb{Z}\Gamma\Delta^{w_r} \tag{2-1}$$

defined by $\alpha(k) = k\Delta\Gamma$ for any $k \in \mathbb{Z}^w$, where $\mathbb{Z}\Gamma\Delta^{w_r}$ is understood as a $\mathbb{Z}\Gamma$-module by means of the ring homomorphism $\mathbb{Z}\Gamma \to \mathbb{Z}(\Gamma \times \Gamma)$ induced by the diagonal homomorphism Δ. Note that

$$g \cdot \Delta\Gamma = (g, g) \cdot \Delta\Gamma = w_r(g, g)\Delta\Gamma = w(g)\alpha(1) = \alpha(g \cdot 1)$$

and therefore α is indeed a homomorphism between $\mathbb{Z}\Gamma$-modules. Since α is a homomorphism between $\mathbb{Z}\Gamma$-modules when $\mathbb{Z}\Gamma\Delta^{w_r}$ is regarded as a $\mathbb{Z}\Gamma$-module by Δ, it follows that there is a homomorphism (cf. p. 79, [1]),

$$(\Delta, \alpha)_* : H_n(\Gamma; \mathbb{Z}^w) \to H_n(\Gamma \times \Gamma; \mathbb{Z}\Gamma\Delta^{w_r}).$$

In what follows we understand the identification $M^{w_l} \otimes \mathbb{Z}^{w \times w} \equiv M^{w_r}$, where $M^{w_l} \otimes \mathbb{Z}^{w \times w}$ is given the $\mathbb{Z}(\Gamma \times \Gamma)$-module structure by the diagonal action of $\Gamma \times \Gamma$. Similar identifications will be often implicit throughout the paper when we take the cap or the cup products.

Definition 2.1. The *tangential Thom class U* is the class in $H^n(\Gamma \times \Gamma; \mathbb{Z}\Gamma\Delta^{w_l})$ such that

$$U \cap [\Gamma \times \Gamma] = (\Delta, \alpha)_*[\Gamma] \in H_n(\Gamma \times \Gamma; \mathbb{Z}\Gamma\Delta^{w_r}).$$

Here we would like to note again that U above has in fact an interpretation as the Thom class of a spherical fibration when the situation is favorable as noted in the introduction, in the paragraph just above the last.

Let us regard $\mathbb{Z}\Gamma$ as a module over $\mathbb{Z}\Gamma$ itself in the obvious way. Then we have $H^i(\Gamma; \mathbb{Z}\Gamma) \cong \tilde{H}^i(S^n; \mathbb{Z})$ for any $i \in \mathbb{Z}$, where $\tilde{H}^i(S^n; \mathbb{Z})$ denotes the reduced singular cohomology of the n-sphere (cf. pp. 220-1, [1]). We also have $H^i(\Gamma; \mathbb{Z}\Gamma^w) \cong H_{n-i}(\Gamma; \mathbb{Z}\Gamma) \cong \tilde{H}^i(S^n; \mathbb{Z})$, which can be seen by combining the Poincaré duality with (iii), (6.1), p. 72 in [1]. Note the natural isomorphisms

$$H_0(\Gamma; \mathbb{Z}\Gamma) \equiv \mathbb{Z} \otimes_\Gamma (\mathbb{Z}\Gamma) \equiv \mathbb{Z}.$$

Then each of $H^n(\Gamma; \mathbb{Z}\Gamma^w)$ and $H_0(\Gamma; \mathbb{Z}\Gamma)$ has a *preferred generator* which corresponds to $1 \in \mathbb{Z}$ by the isomorphisms, $H^n(\Gamma; \mathbb{Z}\Gamma^w) \cong H_0(\Gamma; \mathbb{Z}\Gamma) \equiv \mathbb{Z} \otimes_\Gamma (\mathbb{Z}\Gamma) \equiv \mathbb{Z}$. In particular, the preferred generator of $H^n(\Gamma; \mathbb{Z}\Gamma^w)$ depends on the choice of the fundamental class $[\Gamma] \in H_n(\Gamma; \mathbb{Z}^w)$. Likewise we note that there are natural isomorphisms $H^n(\Gamma; \mathbb{Z}^w) \cong H_0(\Gamma; \mathbb{Z}) \equiv \mathbb{Z} \otimes_\Gamma \mathbb{Z} \equiv \mathbb{Z}$ and also call the generators of $H^n(\Gamma; \mathbb{Z}^w)$ and $H_0(\Gamma; \mathbb{Z})$ corresponding to $1 \in \mathbb{Z}$ the preferred generators.

Let e denote the identity of Γ and consider the homomorphism $\iota_l : \Gamma \to \Gamma \times \Gamma$ defined by $\iota_l(g) = (g, e)$ for any $g \in \Gamma$. Also we define the map

$$\beta : \mathbb{Z}\Gamma\Delta^{w_l} \to \mathbb{Z}\Gamma^w$$

defined by $\beta[g,h] = gh^{-1}$ for any $[g,h] = (g,h)\Delta\Gamma \in \Gamma\Delta$. Then β is in fact a homomorphism between $\mathbb{Z}\Gamma$-modules if we regard $\mathbb{Z}\Gamma\Delta^{w_l}$ as a $\mathbb{Z}\Gamma$-module by means of ι_l. Therefore for each $i \in \mathbb{Z}$ there is a well-defined homomorphism (cf. p. 79, [1]),

$$(\iota_l, \beta)^* : H^i(\Gamma \times \Gamma; \mathbb{Z}\Gamma\Delta^{w_l}) \to H^i(\Gamma; \mathbb{Z}\Gamma^w).$$

Now we may state a property of the tangential Thom class which corresponds to the fact that the pull-back of the Thom class of a spherical fibration to each fiber is a generator. The proof will be postponed until at the end of the section.

Theorem 2.2. *The pull-back $(\iota_l, \beta)^* U \in H^n(\Gamma; \mathbb{Z}\Gamma^w) \equiv \mathbb{Z}$ is the preferred generator.*

On the other hand note that the identity map $1 : \mathbb{Z}^w \to \mathbb{Z}^{w_r}$ is a homomorphism between $\mathbb{Z}\Gamma$-modules if we provide $\mathbb{Z}^{w_r}$ with a $\mathbb{Z}\Gamma$-module structure by means of the diagonal $\Delta : \Gamma \to \Gamma \times \Gamma$. Therefore for each $i \in \mathbb{Z}$ there is a homomorphism,

$$\Delta_* : H_i(\Gamma; \mathbb{Z}^w) \to H_i(\Gamma \times \Gamma; \mathbb{Z}^{w_r}).$$

Consider the class $\Delta_*[\Gamma] \in H_n(\Gamma \times \Gamma; \mathbb{Z}^{w_r})$ and let $u \in H^n(\Gamma \times \Gamma; \mathbb{Z}^{w_l})$ be the class satisfying

$$u \cap [\Gamma \times \Gamma] = \Delta_*[\Gamma]$$

which we call the *diagonal cohomology class* (cf. p. 125, [10] and [2]).

Also there is a homomorphism between $\mathbb{Z}(\Gamma \times \Gamma)$-modules

$$\varepsilon : \mathbb{Z}\Gamma\Delta^{w_l} \to \mathbb{Z}^{w_l} \tag{2-2}$$

defined by $\varepsilon(\sum_{g\in\Gamma} n_g[g,e]) = \sum_{g\in\Gamma} n_g$ where n_g are integers such that $n_g = 0$ except for finitely many g's. It follows that for each $i \in \mathbb{Z}$ there is a homomorphism

$$\varepsilon_* : H^i(\Gamma \times \Gamma; \mathbb{Z}\Gamma\Delta^{w_l}) \to H^i(\Gamma \times \Gamma; \mathbb{Z}^{w_l}).$$

The following provides the main reason why U is referred to as being 'tangential', the proof of which is provided at the later part of the section.

Theorem 2.3. *We have: $\varepsilon_* U = u \in H^n(\Gamma \times \Gamma; \mathbb{Z}^{w_l})$.*

Note that $\varepsilon : \mathbb{Z}\Gamma\Delta^{w_l} \to \mathbb{Z}^w$ can be regarded as a homomorphism between $\mathbb{Z}\Gamma$-modules when $\mathbb{Z}\Gamma\Delta^{w_l}$ is given a $\mathbb{Z}\Gamma$-module structure by means of Δ. Then there exists a homomorphism $(\Delta, \varepsilon)^* : H^n(\Gamma \times \Gamma; \mathbb{Z}\Gamma\Delta^{w_l}) \to H^n(\Gamma; \mathbb{Z}^w)$. On the other hand we also note that $X = K(\Gamma, 1)$ is the homotopy type of a finitely dominated complex since Γ is assumed to be of type FP. As an immediate consequence of 2.3 we have:

Corollary 2.4. *We have for the Kronecker index*

$$\langle (\Delta, \varepsilon)^* U, [\Gamma] \rangle = \chi(X),$$

where $\chi(X)$ denotes the Euler–Poincaré number of X.

Proof. Consider the homomorphism $\Delta^* : H^n(\Gamma \times \Gamma; \mathbb{Z}^{w_l}) \to H^n(\Gamma; \mathbb{Z}^w)$. Note that $(\Delta, \varepsilon)^* U = \Delta^* \varepsilon_* U = \Delta^* u$, where the last equality comes from 2.3 above. Now the assertion follows by the same argument as 3.4 in [2] (see also p. 130, [10]). □

Remark. We note that $(\Delta, \varepsilon)^* U$ in the above can be referred to as the *Euler class* of Γ. However we would like to note that there is another occasion in which the terminology is used to refer to an obstruction to a finiteness condition in a group cohomology theory [9].

In the rest of the section we prove 2.2 and 2.3 above. We would like to mention that they are in essence the contents of 6.2, [3]. However the two are written in a different context. In addition there is a slight improvement in 2.2 above in the sense that we do not assume anymore that w is trivial. We start from the following observation (cf. p. 254, [11]).

Lemma 2.5. *Let $f : G \to H$ be a homomorphism between groups. Let N be a $\mathbb{Z}G$-module and M, N' be $\mathbb{Z}H$-modules, which are regarded also as $\mathbb{Z}G$-modules by means of f. Let $\rho : N \to N'$ be a homomorphism between $\mathbb{Z}G$-modules. Then for any $a \in H^i(H; M)$ and any $x \in H_j(G; N)$ we have that*

$$(f, 1 \otimes \rho)_*((f^* a) \cap x) = a \cap (f, \rho)_* x \in H_{j-i}(H; M \otimes N').$$

Proof. Let $\Delta : G \to G \times G$ and $\Delta' : H \to H \times H$ be the diagonal homomorphisms. Choose projective resolutions $P \to \mathbb{Z}$ and $Q \to \mathbb{Z}$ respectively for G and for H and chain maps $\Delta_\sharp : P \to P \otimes P$ and $\Delta'_\sharp : Q \to Q \otimes Q$, where we understand $P \otimes P$ and $Q \otimes Q$ are given module structures by the diagonal action. We need to choose them together with $f_\sharp : P \to Q$ so that the following diagram commutes

$$\begin{array}{ccc} P & \xrightarrow{\Delta_\sharp} & P \otimes P \\ {\scriptstyle f_\sharp} \downarrow & & {\scriptstyle f_\sharp \otimes f_\sharp} \downarrow \\ Q & \xrightarrow{\Delta'_\sharp} & Q \otimes Q. \end{array}$$

This can be done for instance by taking as the resolutions the singular simplicial chain complexes of the universal cover of the Eilenberg–MacLane spaces, as $\Delta_\sharp$ and $\Delta'_\sharp$ the Alexander–Whitney diagonal approximations (cf. p. 250, [11]) and as $f_\sharp$ the chain map induced by the lifting of a continuous map between the Eilenberg–MacLane spaces realizing f.

Now let $\alpha \in \mathrm{Hom}_H(Q_i, M)$ and $\xi = p \otimes n \in P_j \otimes_G N$. Let $\Delta_\sharp p = \sum_k p_k \otimes p'_k \in (P \otimes P)_j$. Then we have $\Delta'_\sharp f_\sharp p = \sum_k f_\sharp p_k \otimes f_\sharp p'_k$ by the commutativity of the above diagram. Therefore in $Q_{j-i} \otimes_H (M \otimes N')$ we have:

$$(f, 1 \otimes \rho)_\sharp (f^\sharp(\alpha) \cap \xi) = \sum_k f_\sharp(p'_k) \otimes (\alpha(f_\sharp p_k) \otimes \rho(n)) = \alpha \cap (f, \rho)_\sharp \xi.$$

This completes the proof. □

We also need the following, which is obvious. We omit the proof.

Lemma 2.6. *Let G be a group and $\phi : M_1 \to M_2$, $\psi : N_1 \to N_2$ be homomorphisms between $\mathbb{Z}G$-modules. Then for any $a \in H^i(G; M_1)$ and any $x \in H_j(G; N_1)$ we have:*

$$(\phi \otimes \psi)_*(a \cap x) = \phi_*(a) \cap \psi_*(x) \in H_{j-i}(G; M_2 \otimes N_2).$$

To prove 2.2 above we use 2.3 above and therefore we prove 2.3 first. Even if 2.3 is proved in 6.2 in [3] in essence, we provide a proof for the convenience of the reader.

Proof of 2.3. Recall the homomorphism $\varepsilon : \mathbb{Z}\Gamma\Delta^{w_l} \to \mathbb{Z}^{w_l}$ from (2-2) above. By applying 2.6 we have:

$$(\varepsilon \otimes 1)_*(U \cap [\Gamma \times \Gamma]) = \varepsilon_*(U) \cap [\Gamma \times \Gamma] \in H_n(\Gamma \times \Gamma; \mathbb{Z}^{w_l} \otimes \mathbb{Z}^{w\times w}) \equiv H_n(\Gamma \times \Gamma; \mathbb{Z}^{w_r}).$$

By definition it is enough to prove $\varepsilon_*(U) \cap [\Gamma \times \Gamma] = \Delta_*[\Gamma]$, where $\Delta_* : H_n(\Gamma; \mathbb{Z}^w) \to H_n(\Gamma \times \Gamma; \mathbb{Z}^{w_r})$ is as in the above.

It seems at this point necessary for us to provide notations for some natural identifications of coefficients: μ for $\mathbb{Z}\Gamma\Delta^{w_l} \otimes \mathbb{Z}^{w\times w} \overset{\equiv}{\to} \mathbb{Z}\Gamma\Delta^{w_r}$ and μ' for $\mathbb{Z}^{w_l} \otimes \mathbb{Z}^{w\times w} \overset{\equiv}{\to} \mathbb{Z}^{w_r}$. Note that the map $\varepsilon : \mathbb{Z}\Gamma\Delta^{w_r} \to \mathbb{Z}^{w_r}$ is a homomorphism between the $\mathbb{Z}(\Gamma \times \Gamma)$-modules. Then it is obvious that $\mu'(\varepsilon \otimes 1) = \varepsilon\mu$. Furthermore rewrite the definition of U in 2.1 above as follows:

$$\mu_*(U \cap [\Gamma \times \Gamma]) = (\Delta, \alpha)_*[\Gamma] \in H_n(\Gamma \times \Gamma; \mathbb{Z}\Gamma\Delta^{w_r}).$$

Also note the identity $\varepsilon\alpha = 1$. Then we have in $H_n(\Gamma \times \Gamma; \mathbb{Z}^{w_r})$,

$$\mu'_*(\varepsilon \otimes 1)_*(U \cap [\Gamma \times \Gamma]) = \varepsilon_*\mu_*(U \cap [\Gamma \times \Gamma]) = \varepsilon_*(\Delta, \alpha)_*[\Gamma] = (\Delta, \varepsilon\alpha)_*[\Gamma] = \Delta_*[\Gamma].$$

This proves 2.3. □

The proof of 2.2 is much less straightforward which proceeds as follows:

Proof of 2.2. Let $\epsilon : \mathbb{Z}\Gamma^w \to \mathbb{Z}^w$ denote the homomorphism satisfying $\epsilon(g) = 1$ for any $g \in \Gamma$, which is a homomorphism between $\mathbb{Z}\Gamma$-modules.

We begin by considering the commutative diagram:

$$\begin{array}{ccc} H^n(\Gamma; \mathbb{Z}\Gamma^w) & \xrightarrow{\epsilon_*} & H^n(\Gamma; \mathbb{Z}^w) \\ {\scriptstyle \cap[\Gamma]}\downarrow & & {\scriptstyle \cap[\Gamma]}\downarrow \\ H_0(\Gamma; \mathbb{Z}\Gamma) & \xrightarrow{\epsilon_*} & H_0(\Gamma; \mathbb{Z}). \end{array}$$

Since it is obvious that the preferred generator of $H_0(\Gamma; \mathbb{Z}\Gamma) \equiv \mathbb{Z} \otimes_\Gamma \mathbb{Z}\Gamma \equiv \mathbb{Z}$ is mapped to that of $H_0(\Gamma; \mathbb{Z}) \equiv \mathbb{Z} \otimes_\Gamma \mathbb{Z} \equiv \mathbb{Z}$ by ϵ_*, we conclude from this commutative diagram that the preferred generator of $H^n(\Gamma; \mathbb{Z}\Gamma^w)$ is mapped to that of $H^n(\Gamma; \mathbb{Z}^w)$ by ϵ_*.

Furthermore it is also clear that the diagram commutes:

$$\begin{array}{ccc} H^n(\Gamma \times \Gamma; \mathbb{Z}\Gamma\Delta^{w_l}) & \xrightarrow{\varepsilon_*} & H^n(\Gamma \times \Gamma; \mathbb{Z}^{w_l}) \\ {\scriptstyle (\iota_l,\beta)^*}\downarrow & & {\scriptstyle \iota_l^*}\downarrow \\ H^n(\Gamma; \mathbb{Z}\Gamma^w) & \xrightarrow{\epsilon_*} & H^n(\Gamma; \mathbb{Z}^w). \end{array}$$

Since $\varepsilon_*U = u$ by 2.3 above, it is enough to see that $\iota_l^* u \in H^n(\Gamma; \mathbb{Z}^w)$ is the preferred generator.

We will show that $(\iota_l^* u) \cap [\Gamma] \in H_0(\Gamma; \mathbb{Z}^w \otimes \mathbb{Z}^w) \equiv H_0(\Gamma; \mathbb{Z}) \equiv \mathbb{Z}$ is the preferred generator.

By 2.5 above we have

$$(\iota_l, 1 \otimes 1)_*((\iota_l^* u) \cap [\Gamma]) = u \cap \iota_{l*}[\Gamma],$$

where $(\iota_l, 1 \otimes 1)_*$ is the map $H_0(\Gamma; \mathbb{Z}^w \otimes \mathbb{Z}^w) \to H_0(\Gamma \times \Gamma; \mathbb{Z}^{w_l} \otimes \mathbb{Z}^{w_l})$. Now let $1^* \in H^0(\Gamma; \mathbb{Z})$ and $[\Gamma]^* \in H^n(\Gamma; \mathbb{Z}^w)$ respectively be such that $1^* \cap [\Gamma]$ is the fundamental class $[\Gamma] \in H_n(\Gamma; \mathbb{Z}^w)$ and $[\Gamma]^* \cap [\Gamma]$ is the preferred generator of $H_0(\Gamma; \mathbb{Z})$. It is easy to see that the homomorphism $\iota_{l*} : H_n(\Gamma; \mathbb{Z}^w) \to H_n(\Gamma \times \Gamma; \mathbb{Z}^{w_l})$ maps $[\Gamma]$ to $[\Gamma] \times 1 \in H_n(\Gamma \times \Gamma; \mathbb{Z}^w \otimes \mathbb{Z}) \equiv H_n(\Gamma \times \Gamma; \mathbb{Z}^{w_l})$, where 1 denotes the preferred generator of $H_0(\Gamma; \mathbb{Z})$. On the other hand consider the projection $p_r : \Gamma \times \Gamma \to \Gamma$ to the second component. Then the homomorphism

$p_r^* : H^n(\Gamma;\mathbb{Z}^w) \to H^n(\Gamma\times\Gamma;\mathbb{Z}^{w_r})$ maps $[\Gamma]^*$ to $1^*\times[\Gamma]^* \in H^n(\Gamma\times\Gamma;\mathbb{Z}\otimes\mathbb{Z}^w) \equiv H^n(\Gamma\times\Gamma;\mathbb{Z}^{w_r})$. Now we have in $H_0(\Gamma\times\Gamma;\mathbb{Z})$:

$$\begin{aligned} u\cap \iota_{l*}[\Gamma] &= u\cap([\Gamma]\times 1) = (-1)^n u\cap((1^*\times[\Gamma]^*)\cap([\Gamma]\times[\Gamma])) \\ &= (1^*\times[\Gamma]^*)\cap(u\cap[\Gamma\times\Gamma]) = p_r^*([\Gamma]^*)\cap\Delta_*[\Gamma]. \end{aligned}$$

By applying 2.5 above again we have that

$$p_r^*([\Gamma]^*)\cap\Delta_*[\Gamma] = \Delta_*((\Delta^* p_r^*[\Gamma]^*)\cap[\Gamma]) = \Delta_*([\Gamma]^*\cap[\Gamma]) = \Delta_*(1).$$

To summarize, we have that

$$(\iota_l, 1\otimes 1)_*((\iota_l^* u)\cap[\Gamma]) = \Delta_*(1) \in H_0(\Gamma\times\Gamma;\mathbb{Z}),$$

which is the preferred generator. This is possible only if $(\iota_l^* u)\cap[\Gamma] = 1 \in H_0(\Gamma;\mathbb{Z})$, which means that $\iota_l^* u$ is the preferred generator. □

3. The Thom isomorphism

Let M be a $\mathbb{Z}\Gamma$-module which we regard also as a $\mathbb{Z}(\Gamma\times\Gamma)$-module by means of the projection $p_r : \Gamma\times\Gamma\to\Gamma$ to the second component. We will show that, with the tangential Thom class, $U \in H^n(\Gamma\times\Gamma;\mathbb{Z}\Gamma\Delta^{w_l})$, there exists a Thom isomorphism in the sense of 3.1 below. This completes the list of properties of the class U, the first two being 2.2 and 2.3 above, which make U deserve its name. Moreover this last property appears to demand the most nontrivial arguments of the three for the justification.

Theorem 3.1. *For each $i\in\mathbb{Z}$ we have an isomorphism:*

$$p_r^*(\,\cdot\,)\cup U : H^i(\Gamma;M) \to H^{i+n}(\Gamma\times\Gamma;M\otimes(\mathbb{Z}\Gamma\Delta^{w_l})).$$

We note again that $M\otimes(\mathbb{Z}\Gamma\Delta^{w_l})$ is given the $\mathbb{Z}(\Gamma\times\Gamma)$-module structure by the diagonal action of $\Gamma\times\Gamma$. Such diagonal actions must be understood in similar situations below. The statement of 3.1 above is motivated by 5.1, [3] which states $H^*(\Gamma\times\Gamma;\mathbb{Z}\Gamma\Delta^{w_l})$ is naturally isomorphic to the integral cohomology of the Thom space of a spherical fibration. However it is under the assumption that $K(\Gamma,1)$ is the type of a finite complex whose universal cover is forward tame and simply connected at infinity. Also note that it appears far fetched to expect that a cohomology with coefficient $M\otimes(\mathbb{Z}\Gamma\Delta^{w_l})$ might be isomorphic to the one with coefficient M. However 3.2 below mitigates this first impression to some extent.

Consider the diagonal homomorphism $\Delta:\Gamma\to\Gamma\times\Gamma$. Also recall the homomorphism $\alpha:\mathbb{Z}^w\to\mathbb{Z}\Gamma\Delta^{w_r}$ from (2-1) above. Then for each $i\in\mathbb{Z}$ we have a well-defined homomorphism

$$(\Delta, 1\otimes\alpha)_* : H_i(\Gamma;M\otimes\mathbb{Z}^w) \to H_i(\Gamma\otimes\Gamma;M\otimes(\mathbb{Z}\Gamma\Delta^{w_r})).$$

Then the following is the most technically demanding statement for one to justify in this paper.

Lemma 3.2. *For each $i\in\mathbb{Z}$, $(\Delta,1\otimes\alpha)_*$ is an isomorphism.*

Proof. We consider the two sequences of functors on the category of $\mathbb{Z}\Gamma$-modules,

$$(H_i(\Gamma;\,\cdot\otimes\mathbb{Z}^w))_{i\in\mathbb{Z}} \text{ and } (H_i(\Gamma\times\Gamma;\,\cdot\otimes(\mathbb{Z}\Gamma\Delta^{w_r})))_{i\in\mathbb{Z}}.$$

We will show that both sequences of functors are *homological* and *effaceable* if $i > 0$ in the terminology of K. Brown (p. 73 and p. 75, [1]). Furthermore we will show that $(\Delta, 1 \otimes \alpha)_*$ is an isomorphism if $i = 0$. Then by (7.3) on p. 75, [1], we may conclude that $(\Delta, 1 \otimes \alpha)_*$ is an isomorphism for any $i \in \mathbb{Z}$.

Firstly we show that both are homological. Given any short exact sequence of $\mathbb{Z}\Gamma$-modules

$$0 \to M' \to M \to M'' \to 0$$

both of the sequences formed by taking tensor products over $\mathbb{Z}$ respectively with $\mathbb{Z}^w$ and with $\mathbb{Z}\Gamma\Delta^{w_r}$ are in fact short exact sequences since both $\mathbb{Z}^w$ and $\mathbb{Z}\Gamma\Delta^{w_r}$ are free abelian groups. Therefore both sequences of functors give rise to long exact sequences. Furthermore it is clear that both define functors from the category of short exact sequences of $\mathbb{Z}\Gamma$-modules to the category of long exact sequence of abelian groups.

Secondly we show that both are effaceable if $i > 0$. Let F be a free $\mathbb{Z}\Gamma$-module. Then $F \otimes \mathbb{Z}^w$ is a free $\mathbb{Z}\Gamma$-module since $\mathbb{Z}^w$ is a free abelian group (see p. 69, [1]). We also assert that $F \otimes (\mathbb{Z}\Gamma\Delta^{w_r})$ is a free $\mathbb{Z}(\Gamma \times \Gamma)$-module: Since $F \otimes (\mathbb{Z}\Gamma\Delta^{w_r}) \cong (F \otimes (\mathbb{Z}\Gamma\Delta)) \otimes \mathbb{Z}^{w_r}$, it is enough to see that $F \otimes (\mathbb{Z}\Gamma\Delta)$ is free. Furthermore it suffices to see that $(\mathbb{Z}\Gamma) \otimes (\mathbb{Z}\Gamma\Delta)$ is free. Note that we regard $\mathbb{Z}\Gamma$ as a $\mathbb{Z}(\Gamma \times \Gamma)$-module by means of $p_r : \Gamma \times \Gamma \to \Gamma$, which is in fact the rule kept throughout the section. Consider the homomorphism between abelian groups

$$\varphi : (\mathbb{Z}\Gamma) \otimes (\mathbb{Z}\Gamma\Delta) \to \mathbb{Z}(\Gamma \times \Gamma)$$

defined by $\varphi(g \otimes [h, k]) = (hk^{-1}g, g)$. Clearly φ is well-defined and bijective. Furthermore φ is a homomorphism between $\mathbb{Z}(\Gamma \times \Gamma)$-modules since we have that

$$\begin{aligned} \varphi((l, m)(g \otimes [h, k])) &= \varphi((mg) \otimes [lh, mk]) = (lhk^{-1}m^{-1}mg, mg) \\ &= (l, m)(hk^{-1}g, g) = (l, m)\varphi(g \otimes [h, k]) \end{aligned}$$

for any $g, h, k, l, m \in \Gamma$. Thus φ is an isomorphism between $\mathbb{Z}(\Gamma \times \Gamma)$-modules. Therefore we conclude that if $i > 0$ the i-th functor vanishes on F for each of the two sequences of functors (see p. 72, [1]).

Lastly we show that $(\Delta, 1 \otimes \alpha)_* : H_0(\Gamma; M \otimes \mathbb{Z}^w) \to H_0(\Gamma \times \Gamma; M \otimes (\mathbb{Z}\Gamma\Delta^{w_r}))$ is an isomorphism. Note that there is a commutative diagram

$$\begin{array}{ccc} H_0(\Gamma; M \otimes \mathbb{Z}^w) & \xrightarrow{(\Delta, 1\otimes\alpha)_*} & H_0(\Gamma \times \Gamma; M \otimes (\mathbb{Z}\Gamma\Delta^{w_r})) \\ \equiv\downarrow & & \equiv\downarrow \\ \mathbb{Z} \otimes_\Gamma (M \otimes \mathbb{Z}^w) & \xrightarrow{1\otimes(1\otimes\alpha)} & \mathbb{Z} \otimes_{\Gamma\times\Gamma} (M \otimes (\mathbb{Z}\Gamma\Delta^{w_r})). \end{array}$$

Since we have $1 \otimes (m \otimes \Delta\Gamma) = 1 \otimes (m \otimes ((g, e)\Delta\Gamma))$ in $\mathbb{Z} \otimes_{\Gamma\times\Gamma} (M \otimes (\mathbb{Z}\Gamma\Delta^{w_r}))$ for any $m \in M$ ad any $g \in \Gamma$, we conclude that $(\Delta, 1 \otimes \alpha)_*$ is surjective at dimension 0. On the other hand $(\Delta, 1 \otimes \alpha)_*$ is injective, in fact, at every dimension. Recall the homomorphism $\varepsilon : \mathbb{Z}\Gamma\Delta^{w_l} \to \mathbb{Z}^{w_l}$ from (2-2) above. Then note that the following homomorphism is well-defined:

$$(p_r, 1 \otimes \varepsilon)_* : H_i(\Gamma \times \Gamma; M \otimes (\mathbb{Z}\Gamma\Delta^{w_r})) \to H_i(\Gamma; M \otimes \mathbb{Z}^w),$$

for each $i \in \mathbb{Z}$. Then the composite $(p_r, 1 \otimes \varepsilon)_*(\Delta, 1 \otimes \alpha)_*$ is the identity on $H_i(\Gamma; M \otimes \mathbb{Z}^w)$. This completes the proof. $\square$

Now we provide:

Proof of 3.1. Let $a \in H^i(\Gamma; M)$. Then up to the natural identifications of the homology groups by natural isomorphisms between the coefficients we have the following identities in $H_{n-i}(\Gamma \times \Gamma; M \otimes (\mathbb{Z}\Gamma\Delta^{w_r}))$:

$$\begin{aligned}(p_r^*(a) \cup U) \cap [\Gamma \times \Gamma] &= p_r^*(a) \cap (U \cap [\Gamma \times \Gamma]) \\ &= p_r^*(a) \cap (\Delta, \alpha)_*[\Gamma] \\ &= (\Delta, 1 \otimes \alpha)_*(\Delta^* p_r^*(a) \cap [\Gamma]) \\ &= (\Delta, 1 \otimes \alpha)_*(a \cap [\Gamma]).\end{aligned}$$

Note that the third identity follows from 2.5 above. Since the duality maps are isomorphisms, our conclusion follows by 3.2 above. □

Remark. We note that an argument is missing in our discussions of the paper which shows that our tangential Thom class $U \in H^n(\Gamma \times \Gamma; \mathbb{Z}\Gamma\Delta^{w_\iota})$ has a property by which U may be regarded as the Thom class of a spherical fibration inverse to the Spivak fibration. There is only circumstantial evidence, consisting of 2.1, 2.2, 2.3, 2.4 and 3.1 above, which shows that U resembles the Thom class of the normal fibration of the Poincaré embedding structure on the diagonal $\Delta : X \to X \times X$ where $X = K(\Gamma, 1)$. So far we are successful in this regards only when $K(\Gamma, 1)$ is the type of a finite complex whose universal cover is forward tame and simply connected at infinity (see 4.1, 5.1, 6.2 and 6.3 of [3]).

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