Exact cosmological solutions of 4D string gravity in the string frame

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Abstract

We have discussed a particular class of exact cosmological solutions of the 4-dimensional low energy string gravity in the string frame. In the vacuum without matter and the 2-form fields, the exact cosmological solutions always give monotonically shrinking universes if the dilaton field is not a constant. However, in the presence of the 2-form fields and/or the radiation-like fluid in the string frame, the exact cosmological solutions show a minimum size of the universe in the evolution, but with an initial cosmological curvature singularity in the string frame.

1. Introduction

The low energy limit of string theory provides a definite form of dilaton gravity action [1–4] with a specific dilaton coupling and additional 2-form fields [5]. Thus, it may be worth examining the exact cosmological solutions of the 4-dimensional low energy string gravity in the framework of dilaton gravity, instead of General Relativity.

There have been many studies on the cosmological evolution of the low energy string gravity with 2-form fields [6–10]. In the early cosmological evolution, it is necessary to consider the effect of matter as well as the 2-form fields in the low energy string gravity. Without matter, it would be convenient to choose the Einstein frame redefining the metric as usual. However, in

the presence of matter, the frame changes of metric make the dilaton field couple with matter fields differently depending on their spins. Thus, the matter energy–momentum tensors would be non-trivial functions of the dilaton field in a different frame.

Therefore, we have considered a fluid-like matter distribution in the fundamental string frame [11], in which the gravitational constant determined by the dilaton field evolves in time. Because the relativistic matter should be dominant over the non-relativistic matter in the early evolution of the universe, we have examined the exact cosmological solutions with the 2-form fields and/or the radiation-like fluid in the string frame.

2. Low energy string gravity in the string frame

The vanishing beta functions originated from the world sheet conformal symmetry in the closed string sector of string theory give the well-known low energy string gravity action in the critical D-dimension [12]. Assuming a Ricci-flat compactification of the internal $(D-4)$-dimensional space decoupled from our 4-dimensional spacetime [13,14], and adding the matter Lagrangian $\mathcal{L}_m$ which is decoupled from the dilaton field in the string frame, we have the following form of action in 4-dimensional spacetime from the low energy limit of string theory [5,12,15–17],

$$S = -\frac{1}{2} \int d^4 x \sqrt{-g} \left[\Phi^2 R + 4(\partial \Phi)^2 - \frac{1}{12} \Phi^2 H_{\alpha\mu\nu} H^{\alpha\mu\nu} \right] + \int d^4 x \sqrt{-g} \mathcal{L}_m, \tag{1}$$

where we have redefined the dilaton field ϕ with Φ as

$$\Phi \equiv \frac{1}{\kappa_0} e^{-\phi} \tag{2}$$

in the string frame, and $H_{\alpha\mu\nu}$ are the field strengths of the 2-form antisymmetric fields $B_{\mu\nu}$ in NS–NS sector as $\mathbf{H} = d\mathbf{B}$.

The equations of motion for the 2-form fields $B_{\mu\nu}$, the dilaton field Φ, and the metric $g_{\mu\nu}$ are obtained as

$$\nabla_\alpha (\Phi^2 H^{\alpha\mu\nu}) = \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} \Phi^2 H^{\alpha\mu\nu}) = 0, \tag{3}$$

$$4\nabla^2 \Phi = \Phi R - \frac{1}{12} \Phi H_{\alpha\mu\nu} H^{\alpha\mu\nu}, \tag{4}$$

$$\Phi^2 G_{\mu\nu} - 2\Phi \nabla_\mu \nabla_\nu \Phi + 2g_{\mu\nu} \Phi \nabla^2 \Phi + 2\nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{4} \Phi^2 H_{\mu\alpha\beta} H_\nu^{\alpha\beta}$$
$$+ \frac{1}{24} g_{\mu\nu} \Phi^2 H_{\alpha\beta\lambda} H^{\alpha\beta\lambda} = T_{\mu\nu}^{(m)}, \tag{5}$$

where $T_{\mu\nu}^{(m)}$ is the matter energy–momentum tensor. Taking the trace part of Eq. (5), we have an expression for $\Phi^2 R$. Plugging it into the dilaton equation (4), we can rewrite the dilaton equation of motion equivalently as

$$2(\Phi \nabla^2 \Phi + (\nabla \Phi)^2) = \frac{1}{6} \Phi^2 H_{\alpha\mu\nu} H^{\alpha\mu\nu} + T^{(m)}, \tag{6}$$

where $T^{(m)}$ is the trace of the matter energy–momentum tensor.

As we can see from Eq. (6), the driving force of the dilaton field is given by the antisymmetric 2-form field strengths and the trace of matter energy–momentum tensor.

$H_{\alpha\mu\nu}$ solution of Eq. (3) is found as

$$H^{\alpha\mu\nu} = \Phi^{-2}\epsilon^{\alpha\mu\nu\lambda}\partial_\lambda\psi, \tag{7}$$

where ψ is a new smooth scalar field, $\epsilon^{\alpha\mu\nu\lambda}$ is the covariant constant antisymmetric tensor such that $\epsilon^{\alpha\mu\nu\lambda} = \varepsilon^{\alpha\mu\nu\lambda}/\sqrt{-g}$, with $\varepsilon^{0123} = +1$. The Bianchi identity $d\mathbf{H} = 0$ for the antisymmetric field strengths $H_{\alpha\mu\nu}$ gives the equation of motion for the scalar field ψ as [8–10,17–19]

$$\nabla^\alpha\left(\Phi^{-2}\partial_\alpha\psi\right) = 0. \tag{8}$$

Defining the dual strength vector h_α as $H^{\mu\nu\beta} \equiv \epsilon^{\mu\nu\beta\alpha}h_\alpha$, we find that

$$\frac{1}{12}H_{\alpha\mu\nu}H^{\alpha\mu\nu} = -\frac{1}{2}\Phi^{-4}(\nabla\psi)^2 = -\frac{1}{2}h^\alpha h_\alpha. \tag{9}$$

We examine the cosmological evolution of the Eqs. (3), (4), (5) with the spatially flat Robertson–Walker metric

$$ds^2 = dt^2 - S^2(t)\,d\vec{\mathbf{r}}^2, \tag{10}$$

and the cosmological Ansatz: $\Phi = \Phi(t)$, $\psi = \psi(t)$. Then, Eq. (8) gives

$$\frac{d\psi}{dt} = \Phi^2\frac{A}{S^3}, \quad \text{i.e. } H^{ijk} = \epsilon^{ijk0}\frac{A}{S^3}, \text{ and } H^{0ij} = 0, \tag{11}$$

where A is a constant which measures the strength of the 2-form fields. The solution (11) gives a uniform non-vanishing dual strength vector such as

$$h_i = 0, \qquad h_0 = -\frac{A}{S^3}. \tag{12}$$

Using the metric (10) and the explicit solution of $H_{\mu\nu\lambda}$ from Eqs. (9), (12), Eqs. (5), (6) are reduced to a set of cosmological equations. Denoting the time derivative as an over-dot, the cosmological equations with uniform matters in the string frame are given as

$$3\Phi^2\mathcal{H}^2 + 6\mathcal{H}\Phi\dot\Phi + 2\dot\Phi^2 = \rho_m + \frac{1}{4}\Phi^2\frac{A^2}{S^6}, \tag{13}$$

$$2\mathbf{a}\Phi^2 + \mathcal{H}^2\Phi^2 + 4\mathcal{H}\Phi\dot\Phi + 2\Phi\ddot\Phi = -p_m - \frac{1}{4}\Phi^2\frac{A^2}{S^6}, \tag{14}$$

$$2\left(\Phi\ddot\Phi + 3\mathcal{H}\Phi\dot\Phi + \dot\Phi^2\right) = \rho_m - 3p_m - \Phi^2\frac{A^2}{S^6}, \tag{15}$$

where the Hubble parameter is $\mathcal{H} \equiv \dot{S}/S$, and the cosmic acceleration parameter is $\mathbf{a} \equiv \ddot{S}/S$. With the uniform matter density ρ_m and the uniform matter pressure p_m, the energy–momentum tensors are $T_0^{(m)0} = \rho_m$, and $T_1^{(m)1} = T_2^{(m)2} = T_3^{(m)3} = -p_m$.

In Eqs. (13), (14), it seems that the 2-form fields play a role of matter with the energy density $\Phi^2\frac{A^2}{4S^6}$ and the pressure $\Phi^2\frac{A^2}{4S^6}$ having the equation of state $w = 1$. However, the $\Phi^2 H_{\alpha\mu\nu}H^{\alpha\mu\nu}$ term in the action (1) gives an additional contribution, the conformal density $-\Phi^2\frac{A^2}{2S^6}$ defined as $\Phi\partial\mathcal{L}/\partial\Phi$, to the right-hand side of Eq. (15).

By taking a time derivative Eq. (13) and using Eqs. (14), (15), it is found that the matter energy conservation holds as

$$\frac{d}{dt}\rho_m + 3\mathcal{H}(\rho_m + p_m) = 0, \tag{16}$$

which is independent of the 2-from fields $B_{\mu\nu}$ and the dilaton field Φ in the string frame. With the equation of state for matter w, Eq. (16) gives the matter density evolving as

$$\rho_m(t) = \frac{C}{S^{3(w+1)}}, \tag{17}$$

where C is a positive constant which measures the strength of the matter density.

The 2-form fields having $\frac{A^2}{S^6}$ behavior in Eqs. (13), (14), (15) dominate the universe with a small scale factor $S(t)$. As the universe expands, the rapidly decreasing 2-form fields lay down their role in the late time evolution of the universe, and matter with the density (17) or a cosmological constant will take over the dominance.

3. The exact vacuum solutions

First, let us begin with the vacuum solution of Eqs. (13), (14), (15), without the 2-form fields and matter. Eq. (15) can be written as

$$S^3 \frac{d}{dt}\left(S^3 \frac{d}{dt}\Phi^2\right) = 0. \tag{18}$$

With a new time function $\tau(t)$ such that $d\tau \equiv dt/S^3(t)$, this equation becomes a simple equation having a general linear solution in τ,

$$\frac{d^2}{d\tau^2}\Phi^2 = 0, \qquad \Phi^2 = \alpha\tau + \beta, \tag{19}$$

where α and β are two positive integration constants which should be determined by an initial condition. If $\alpha = 0$, then we have the trivial vacuum solution such that $\Phi^2 = \beta$ is a constant, and the scale factor S is also a constant with $\mathcal{H} = 0 = \mathbf{a}$. If $\alpha \neq 0$, then we have

$$\frac{\dot{\Phi}}{\Phi} = \frac{\alpha}{2S^3(\alpha\tau + \beta)}. \tag{20}$$

The new time function τ is a monotonically increasing function of time t. As the time t elapses, τ increases rapidly for a small scale factor, but very slowly for a large scale factor S. Thus, the gravitational constant, which is the inverse of Φ^2, decreases as $1/(\alpha\tau + \beta)$ in vacuum.

From Eq. (13), we obtain $dS/d\tau$ and its integration as

$$\frac{dS}{d\tau} = \eta\frac{\alpha S}{\alpha\tau + \beta} < 0, \quad S = \sigma_0(\alpha\tau + \beta)^\eta, \quad \eta \equiv -\frac{\sqrt{3} \mp 1}{2\sqrt{3}}, \tag{21}$$

where σ_0 is a positive integration constant.

Thus, using $d\tau/dt = 1/S^3(t)$, we obtain the cosmological time t as a function of τ as

$$t - t_0 = \frac{\sigma_0^3}{\alpha(3\eta + 1)}(\alpha\tau + \beta)^{3\eta+1}, \tag{22}$$

where t_0 is a constant time. Thus, we find the exact vacuum solutions without matter and the 2-form fields as

$$S = \sigma_0\left(\frac{\alpha}{\sigma_0^3}(3\eta + 1)(t - t_0)\right)^{\eta/(3\eta+1)}, \qquad \Phi^2 = \left(\frac{\alpha}{\sigma_0^3}(3\eta + 1)(t - t_0)\right)^{1/(3\eta+1)}, \tag{23}$$

where $3\eta + 1 = -(1 \mp \sqrt{3})/2$ and $\eta/(3\eta + 1) = \mp 1/\sqrt{3}$.

Therefore, the both vacuum solutions with $\alpha \neq 0$ give an ever shrinking universe to $S = 0$ in the future. Especially, the solution with the lower signs meets a curvature singularity at a finite time $t = t_0$.

4. The exact solutions with the 2-form fields only

Let us consider the solution of Eqs. (13), (14), (15) with the 2-form fields only. Eq. (15) can be written as

$$S^3 \frac{d}{dt}\left(S^3 \frac{d}{dt}\Phi^2\right) = -A^2 \Phi^2. \tag{24}$$

With a new time function $\tau(t)$ such that $d\tau \equiv dt/S^3(t)$, this equation becomes the simple harmonic equation in τ,

$$\frac{d^2}{d\tau^2}\Phi^2 = -A^2 \Phi^2, \qquad \Phi^2 = \Phi_0^2 \sin(A\tau), \qquad \frac{\dot{\Phi}}{\Phi} = \frac{A}{2S^3}\cot(A\tau), \tag{25}$$

where Φ_0 is an integration constant, and $A \neq 0$. We find that τ should be in a compact domain, $0 \leqslant \tau \leqslant \pi/A$.

From Eq. (13), we have the solution of $\mathcal{H}$ as follows:

$$\mathcal{H} = -\frac{\dot{\Phi}}{\Phi} \pm \frac{1}{\sqrt{3}}\sqrt{\left(\frac{\dot{\Phi}}{\Phi}\right)^2 + \frac{A^2}{4S^6}}. \tag{26}$$

Using the solution (25) of Φ, from Eq. (26), we find the velocity of the cosmic scale factor as follows

$$\frac{dS}{d\tau} = -\frac{AS}{2}\left(\cot(A\tau) \mp \frac{1}{\sqrt{3}}\csc(A\tau)\right). \tag{27}$$

We can integrate Eq. (27) to have the scale factor S as a function of τ

$$S_\pm(\tau) = S_0 \frac{\tan^{\pm\frac{1}{2\sqrt{3}}}(A\tau/2)}{\sqrt{\sin(A\tau)}}, \tag{28}$$

for the upper and the lower signs of Eq. (26) respectively.

The cosmological time $0 \leqslant t < \infty$ for each solution is given by the integration over $0 \leqslant \tau < \pi/A$,

$$t(\tau) = \int_0^\tau S_\pm^3(\tau)\,d\tau. \tag{29}$$

The infinite future of the cosmological time t corresponds to $\tau = \pi/A$.

The cosmological time t_E and the scale factor S_E in the Einstein frame are obtained as

$$t_E(\tau) = \int_0^\tau \Phi(\tau)S_\pm^3(\tau)\,d\tau, \qquad S_E(\tau) = \Phi(\tau)S_\pm(\tau) = |\Phi_0|S_0 \tan^{\pm\frac{1}{2\sqrt{3}}}(A\tau/2). \tag{30}$$

Because the time derivative of S_E about t_E is given as

$$\frac{dS_E}{dt_E} = \pm \frac{A}{2\sqrt{3}} \frac{\csc(A\tau)}{S_\pm^2(\tau)}, \tag{31}$$

it is found that the upper/lower sign solution corresponds to an ever expanding/shrinking solution, respectively, in the Einstein frame. The second derivative of S_E about t_E is the same for the both solutions, and always negative as

$$\frac{d^2 S_E}{dt_E^2} = -\frac{A^2}{6|\Phi_0|} \frac{\csc^{5/2}(A\tau)}{S_\pm^5(\tau)}. \tag{32}$$

Thus, both solutions decelerate always in the Einstein frame.

The exact solutions in 4-spacetime dimension seem to be consistent with the phase-plane analysis [9], and with the analytic Fourier mode solutions [10].

5. The exact solutions with the radiation-like fluid only in the string frame

Next, let us examine the cosmological solution of Eqs. (13), (14), (15) with the radiation-like fluid in the string frame. Because the uniform relativistic matter has the equation of state $w = 1/3$ with $p_m = w\rho_m$, the trace of the matter energy–momentum tensors vanishes. Thus, without the 2-form fields $A = 0$, the dilaton equation (15) gives the same solution (19) as in the vacuum. If $\alpha = 0$, then we have the well-known decelerating expansion solution with a constant dilaton field $\Phi^2 = \beta$ as in General Relativity.

Thus, let us consider the $\alpha \neq 0$ case, $\Phi^2 = \alpha\tau + \beta$. Using a relativistic matter density ρ_m and a pressure density p_m such that $\rho_m = 3p_m = C/S^4$ from Eq. (17), and with the dilaton solution (19), (20), we obtain the following equation for the scale factor S from Eq. (13),

$$\frac{dS}{d\tau} = -\frac{\alpha S}{2\sqrt{3}(\alpha\tau + \beta)} \left(\sqrt{3} \mp \sqrt{1+\gamma} \right), \quad \gamma(\tau) \equiv \frac{4C S^2}{\alpha^2}(\alpha\tau + \beta). \tag{33}$$

Introducing another time function T which is also an increasing function of the time t, we can rewrite this equation as follows:

$$\frac{d\gamma}{dT} = \pm\sqrt{1+\gamma}, \qquad dT \equiv \frac{4C}{\sqrt{3}\alpha} S^2 d\tau = \frac{4C}{\sqrt{3}\alpha S} dt. \tag{34}$$

Integrating this, we have the solution for the time parameters T as

$$T = T_0 \pm 2\sqrt{1+\gamma}, \tag{35}$$

where T_0 is an integration constant. Thus, the range for the time T is determine as

$$T \geqslant T_0 + 2 \quad \text{for the upper sign solution,}$$
$$T \leqslant T_0 - 2 \quad \text{for the lower sign solution.} \tag{36}$$

From Eq. (35), we have a relation between the two time parameters T and τ as

$$\frac{dT}{d\tau} = \frac{\alpha}{4\sqrt{3}} \frac{(T-T_0)^2 - 4}{\alpha\tau + \beta}. \tag{37}$$

Integrating this equation, the solution for the upper signs is obtained as

$$\Phi^2 = \epsilon \left(\frac{T - T_0 - 2}{T - T_0 + 2} \right)^{\sqrt{3}}, \qquad S(T) = \frac{\alpha}{4\sqrt{\epsilon C}} \frac{(T - T_0 + 2)^{(\sqrt{3}+1)/2}}{(T - T_0 - 2)^{(\sqrt{3}-1)/2}}, \tag{38}$$
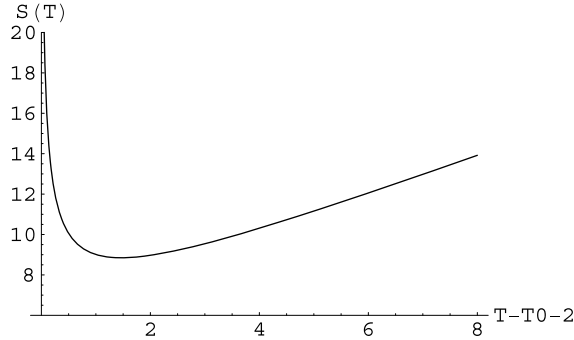
Fig. 1. S versus $(T - T_0 - 2)$ with $\frac{\alpha}{4\sqrt{\epsilon C}} = 1$ in Eq. (38).
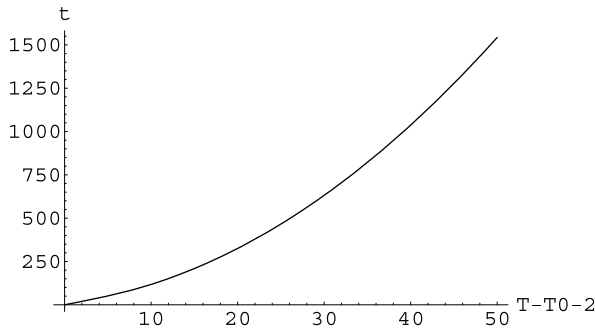

Fig. 2. t versus $(T - T_0 - 2)$ from Eq. (39).

where ϵ is a positive integration constant, and the integration constant T_0 may be chosen as -2 such that $T > T_0 + 2 = 0$. The scale factor $S(T)$ for the upper signs is a downward concave function in time, having a minimum as shown in Fig. 1.

The cosmological time t is related to the time parameter T as follows

$$t = \int_{T_0+2}^{T} \frac{\sqrt{3}\alpha^2}{16\sqrt{\epsilon C^3}} \frac{(T - T_0 + 2)^{(\sqrt{3}+1)/2}}{(T - T_0 - 2)^{(\sqrt{3}-1)/2}} \, dT. \tag{39}$$

Integrating this equation, it is found that the cosmological time t is a monotonically increasing function of the time parameter T with $t = 0$ at $T = T_0 + 2$ as shown in Fig. 2.

For the upper sign bounce solution (38), as the time T approaches to $T_0 + 2$, the dilaton field vanishes and the scale factor grows infinitely. Therefore, the gravitational constant grows infinitely as the cosmological time approaches to the initial cosmological time $t = 0$.

We can examine the cosmological evolution with the help of Eq. (14) also. Using the solution (19) of Φ and Eq. (33), we find the acceleration of the scale factor from Eq. (14) as

$$\frac{d^2 S}{dt^2} = -\frac{\alpha^2}{6S^5(\alpha\tau + \beta)^2} \left(1 + \frac{\gamma}{2} \mp \sqrt{3 + 3\gamma} \right). \tag{40}$$

For the lower signs in Eqs. (33), (40), the velocity and the acceleration of the scale factor are both negative always, giving an ever shrinking universe to meet the singularity with the scale factor $S = 0$ in the future.

However taking the upper signs gives the interesting bounce solution as we have obtained in Eq. (38). From Eqs. (33), (40), the velocity and the acceleration of the scale factor in the time t show the following behavior,

$$\dot{S}=0 \quad \text{at } \gamma(\tau)=2, \qquad \dot{S}<0 \quad \text{for } \gamma(\tau)<2, \qquad \dot{S}>0 \quad \text{for } \gamma(\tau)>2,$$
$$\ddot{S}=0 \quad \text{at } \gamma(\tau)=4+2\sqrt{6}, \qquad \ddot{S}>0 \quad \text{for } \gamma(\tau)<4+2\sqrt{6},$$
$$\ddot{S}<0 \quad \text{for } \gamma(\tau)>4+2\sqrt{6}. \tag{41}$$

Thus, the scale factor should have a downward concave form in time as seen in Fig. 1, having a minimum size at $\gamma(\tau)=2$. If a universe started its evolution at a finite size with $\gamma(\tau)<2$, it would shrink with a positive acceleration to the minimum size at $\gamma=2$, which is the turning point. Then, the universe begins an accelerating expansion again until $\gamma<4+2\sqrt{6}$, and continues a decelerating expansion afterward. If we choose S_m, τ_m as the value satisfying the equation $2CS_m^2(\alpha\tau_m+\beta)=\alpha^2$. Then, the dilaton field Φ^2 increases from the value Φ_m^2 at the minimum size S_m, and approaches to a constant value $\sqrt{\epsilon}$ finally in the expanding universe. At the minimum size S_m, the matter density has the maximum value ρ_{max} as

$$\rho_{max}=\frac{16\epsilon^2 C^3}{\alpha^4}\frac{(\sqrt{3}-1)^{2(\sqrt{3}-1)}}{(\sqrt{3}+1)^{2(\sqrt{3}+1)}}, \qquad S_m=\frac{\alpha}{2\sqrt{\epsilon C}}\frac{(\sqrt{3}+1)^{(\sqrt{3}+1)/2}}{(\sqrt{3}-1)^{(\sqrt{3}-1)/2}}. \tag{42}$$

Unlike other examples of bouncing scenarios in string cosmology [20], the bouncing solution found with the radiation-like fluid decoupled from the dilaton field in the string frame does not satisfy the property of duality symmetry which is typical of string cosmology [21].

6. The exact solutions both with the 2-form fields and the radiation-like fluid in the string frame

Finally, we consider the solutions of Eqs. (13), (14), (15) with both the radiation-like fluid and the 2-form fields.

Eq. (15) with the 2-form fields and the radiation-like fluid gives the same solution (25) of the case with the 2-form fields only. From Eq. (13), we have the solution of $\mathcal{H}$ as follows:

$$\mathcal{H}=-\frac{\dot{\Phi}}{\Phi}\pm\frac{1}{\sqrt{3}}\sqrt{\left(\frac{\dot{\Phi}}{\Phi}\right)^2+\frac{A^2}{4S^6}+\frac{C}{S^4\Phi^2}}. \tag{43}$$

Using the solution (25) of Φ, from Eq. (43) we find the velocity of the cosmic scale factor as follows

$$\frac{dS}{d\tau}=-\frac{AS}{2\sqrt{3}}\csc(A\tau)\left(\sqrt{3}\cos(A\tau)\mp\sqrt{1+\gamma_A}\right), \quad \gamma_A(\tau)\equiv\frac{4CS^2}{A^2\Phi_0^2}\sin(A\tau). \tag{44}$$

Similarly as in Eq. (34), introducing a new time function T_A, we can rewrite this equation as

$$\frac{d\gamma_A}{dT_A}=\pm\sqrt{1+\gamma_A}, \qquad dT_A\equiv\frac{4C}{\sqrt{3}\Phi_0^2 A}S^2 d\tau=\frac{4C}{\sqrt{3}\Phi_0^2 AS}dt. \tag{45}$$

Integrating this, we have the solution for the upper signs as

$$\tan\left(\frac{A\tau}{2}\right) = \epsilon\left(\frac{T_A - T_0 - 2}{T_A - T_0 + 2}\right)^{\sqrt{3}},$$

$$S(T_A) = \frac{\Phi_0 A}{4\sqrt{2C}}\left(\epsilon\frac{(T_A - T_0 + 2)^{\sqrt{3}+1}}{(T_A - T_0 - 2)^{\sqrt{3}-1}} + \frac{1}{\epsilon}\frac{(T_A - T_0 - 2)^{\sqrt{3}+1}}{(T_A - T_0 + 2)^{\sqrt{3}-1}}\right)^{1/2}, \qquad (46)$$

where ϵ is a positive integration constant, and the integration constant T_0 may be also chosen as -2 such that $T_A > T_0 + 2 = 0$. This scale factor $S(T_A)$ for the upper signs is a downward concave function in time, having a minimum size at $\cos(A\tau) = \sqrt{(1 + \gamma_A)/3}$, like the previous upper sign solution for the relativistic matter only case. Thus, it is found that adding the 2-form fields does not change the qualitative behaviour of the exact solutions (38) for the radiation-like fluid only.

7. Conclusions and discussion

We have discussed a particular class of exact cosmological solutions of the 4-dimensional low energy string gravity with the 2-form fields and/or the uniform relativistic matter in the string frame (see [7] for the general exact solutions with arbitrary equation of state of the fluid sources). In the vacuum without matter and the 2-form fields, the exact cosmological solutions always give monotonically shrinking universes if the dilaton field is not a constant. However, in the presence of a radiation-like fluid and/or the 2-form fields, the exact solutions exhibit a minimum size of the universe in the string frame. Throughout the evolution, the exact solutions do not show any singularity, except the initial cosmological curvature singularity in the string frame.

As further studies, in the early stage of cosmological evolution, it may be necessary to consider the effects of the cosmological constant or a dilaton potential with the radiation-like fluid. Furthermore, in the late stage of the cosmological evolution, the effects of the dominant non-relativistic fluid should be considered also. Especially, after the transition from the radiation-dominated to the matter-dominated regime, the qualitative behaviour of the considered solutions is expected to change in the latest stage of the cosmological evolution, and the dilaton field might be relevant for phenomenological applications, such as a possible interpretation of the dilaton as the cosmic dark-energy field [22].

Acknowledgements

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