Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 8 (2015), 218–223
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Approximate ternary quadratic derivation on ternary Banach algebras and C^*-ternary rings: revisited

Choonkill Park[a], Jung Rye Lee[b,*]

[a]Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Korea
[b]Department of Mathematics, Daejin University, Kyeonggi 487-711, Korea

Communicated by P. Găvruta

Abstract

Recently, Shagholi et al. [S. Shagholi, M. Eshaghi Gordji, M. B. Savadkouhi, J. Comput. Anal. Appl., 13 (2011), 1097–1105] defined ternary quadratic derivations on ternary Banach algebras and proved the Hyers-Ulam stability of ternary quadratic derivations on ternary Banach algebras. But the definition was not well-defined.

Using the fixed point method, Bodaghi and Alias [A. Bodaghi, I. A. Alias, Adv. Difference Equ., 2012 (2012), 9 pages] proved the Hyers-Ulam stability and the superstability of ternary quadratic derivations on ternary Banach algebras and C^*-ternary rings. There are approximate $\mathbb{C}$-quadraticity conditions in the statements of the theorems and the corollaries, but the proofs for the $\mathbb{C}$-quadraticity were not completed. In this paper, we correct the definition of ternary quadratic derivation and complete the proofs of the theorems and the corollaries.

Keywords: Hyers-Ulam stability; algebra-C^*-ternary ring, fixed point, quadratic functional equation, algebra-ternary Banach algebra, ternary quadratic derivation.
2010 MSC: 39B52, 13N15, 47H10, 47B47.

1. Introduction

The study of stability problems for functional equations is related to a question of Ulam [9] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [4]. Subsequently, the result of Hyers was generalized by Aoki [1] for additive mappings and by Th. M. Rassias [6] for linear mappings by considering an unbounded Cauchy difference.

*Corresponding author
Email addresses: baak@hanyang.ac.kr (Choonkill Park), jrlee@daejin.ac.kr (Jung Rye Lee)

Received 2013-7-11

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic mapping. The Hyers-Ulam stability problem for the quadratic functional equation (1.1) was proved by Skof for mappings $f : A \to B$, where A is a normed space and B is a Banach space (see [8]).

In [7], Shagholi et al. defined a ternary quadratic derivation D from a ternary Banach algebra A into a ternary Banach algebra B such that

$$D[x,y,z]=[D(x),y^2,z^2]+[x^2,D(y),z^2]+[x^2,y^2,D(z)]$$

for all $x, y, z \in A$. But x^2, y^2, z^2 are not defined and the brackets of the right side are not defined, since A is not an algebra and $D(x) \in B$ and $y^2, z^2 \in A$. So we correct them as follows.

Definition 1.1. Let A be a complex algebra-ternary Banach algebra with norm $\|\cdot\|$ or a complex algebra-C^*-ternary ring with norm $\|\cdot\|$. A $\mathbb{C}$-linear mapping $D : A \to A$ is called a ternary quadratic derivation if

(1) D is a quadratic mapping,

(2) $D[x,y,z]=[D(x),y^2,z^2]+[x^2,D(y),z^2]+[x^2,y^2,D(z)]$ for all $x, y, z \in A$.

There are approximate $\mathbb{C}$-quadraticity conditions in the statements of the theorems and the corollaries in [2], but the proofs for the $\mathbb{C}$-quadraticity were not completed.

In this paper, we complete the proofs of the theorems and the corollaries given in [2].

Throughout this paper, let A be a complex algebra-ternary Banach algebra with norm $\|\cdot\|$ or a complex algebra-C^*-ternary ring with norm $\|\cdot\|$.

2. Stability of ternary quadratic derivations

We need the following lemma to obtain the main results.

Lemma 2.1. *Let $f : A \to A$ be a quadratic mapping such that $f(\mu x) = \mu^2 f(x)$ for all $x \in A$ and $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $f : A \to A$ satisfies $f(\mu x) = \mu^2 f(x)$ for all $x \in A$ and all $\mu \in \mathbb{C}$.*

The proof is similar to the proof of the corresponding lemma given in [5].

Proof. Let r be a rational number. it is easy to show that $f(rx) = r^2 f(x)$ for all $x \in A$.

By the same reasoning as in the proof of main theorem of [6], one can show that $f(rx) = r^2 f(x)$ for all $x \in A$ and all $r \in \mathbb{R}$. So

$$f(\mu x)=f\left(|\mu|\frac{\mu}{|\mu|}x\right)=|\mu|^2 f\left(\frac{\mu}{|\mu|}x\right)=|\mu|^2\cdot\frac{\mu^2}{|\mu|^2}f\left(x\right)=\mu^2 f(x)$$

for all $\mu \in \mathbb{C}\setminus\{0\}$ and all $x \in A$. Since $f(0) = 0$, $f(\mu x) = \mu^2 f(x)$ for all $x \in A$ and all $\mu \in \mathbb{C}$. □

We recall a fundamental result in fixed point theory.

Theorem 2.2. ([3]) *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x)=\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

Theorem 2.3. *Let A be a complex algebra-C^*-ternary ring. Let $f : A \to A$ be a mapping with $f(0) = 0$ and let $\varphi : A^5 \to [0, \infty)$ be a function such that*

$$\left\| 2f\left(\mu\frac{a+b}{2}\right) + 2f\left(\mu\frac{a-b}{2}\right) - \mu^2(f(a)+f(b)) \right\| \leq \varphi(a, b, 0, 0, 0), \tag{2.1}$$

$$\|f([x, y, z]) - [f(x), y^2, z^2] - [x^2, f(y), z^2] - [x^2, y^2, f(z)]\| \leq \varphi(0, 0, x, y, z) \tag{2.2}$$

for all $\mu \in \mathbb{T}^1$ and all $a, b, x, y, z \in A$. Assume that there exists a constant $M \in (0, 1)$ such that

$$\varphi(2a, 2b, 2x, 2y, 2z) \leq 4M\varphi(a, b, x, y, z) \tag{2.3}$$

for all $a, b, x, y, z \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique ternary quadratic derivation $D : A \to A$ such that

$$\|f(a) - D(a)\| \leq \frac{M}{1-M}\varphi(a, 0, 0, 0, 0) \tag{2.4}$$

for all $a \in A$.

Proof. It follows from (2.3) that

$$\lim_{j\to\infty} \frac{\varphi(2^j a, 2^j b, 2^j x, 2^j y, 2^j z)}{4^j} = 0$$

for all $a, b, x, y, z \in A$.

Putting $b = 0$ and $\mu = 1$ and replacing a by $2a$ in (2.1), we get

$$\|4f(a) - f(2a)\| \leq \varphi(2a, 0, 0, 0, 0) \leq 4M\varphi(a, 0, 0, 0, 0)$$

and so

$$\left\| f(a) - \frac{1}{4}f(2a) \right\| \leq M\varphi(a, 0, 0, 0, 0) \tag{2.5}$$

for all $a \in A$.

We consider the set $\Omega := \{h : A \to A \mid h(0) = 0\}$ and introduce the generalized metric d on Ω as follows:

$$d(h_1, h_2) := \inf\{K \in [0, \infty) : \|h_1(a) - h_2(a)\| \leq K\varphi(a, 0, 0, 0, 0), \forall a \in A\}$$

if there exists such constant K, and $d(h_1, h_2) = \infty$, otherwise. One can easily show that (Ω, d) is complete. We define the linear mapping $J : \Omega \to \Omega$ by

$$J(h)(a) = \frac{1}{4}h(2a) \tag{2.6}$$

for all $a \in A$.

Given $h_1, h_2 \in \Omega$, let $K \in \mathbb{R}_+$ be an arbitrary constant with $d(h_1, h_2) \leq K$, that is

$$\|h_1(a) - h_2(a)\| \leq K\varphi(a, 0, 0, 0, 0) \tag{2.7}$$

for all $a \in A$. Replacing a by $2a$ in (2.7))and using (2.3) and (2.6), we have

$$\|(Jh_1)(a) - (Jh_2)(a)\| = \frac{1}{4}\|h_1(2a) - h_2(2a)\| \leq \frac{1}{4}K\varphi(2a, 0, 0, 0, 0) \leq KM\varphi(a, 0, 0, 0, 0)$$

for all $a \in A$ and so $d(Jh_1, Jh_2) \leq KM$. Thus we conclude that $d(Jh_1, Jh_2) \leq Md(h_1, h_2)$ for all $h_1, h_2 \in \Omega$. It follows from (2.5) that

$$d(Jf, f) \leq M. \tag{2.8}$$

By Theorem 2.2, the sequence $\{J^n f\}$ converges to a unique fixed point $D : A \to A$ in the set $\Omega_1 := \{h \in \Omega, d(f, h) < \infty\}$, i.e.,

$$\lim_{n\to\infty} \frac{f(2^n a)}{4^n} = D(a)$$

for all $a \in A$. By Theorem 2.2 and (2.8), we have

$$d(f, D) \le \frac{d(Jf, f)}{1 - M} \le \frac{M}{1 - M}.$$

The last inequality shows that (2.4) holds for all $a \in A$. Replacing a, b by $2^n a, 2^n b$ in (2.1), respectively, and dividing both sides of the resulting inequality by 4^n, and letting n tend to infinity, we obtain

$$2D\left(\mu\frac{a+b}{2}\right) + 2D\left(\mu\frac{a-b}{2}\right) = \mu^2 D(a) + \mu^2 D(b) \tag{2.9}$$

for all $a, b \in A$ and all $\mu \in \mathbb{T}^1$. Putting $\mu = 1$ in (2.9), we have

$$2D\left(\frac{a+b}{2}\right) + 2D\left(\frac{a-b}{2}\right) = D(a) + D(b)$$

for all $a, b \in A$. Hence D is a quadratic mapping. It follows from (2.9) that $D(\mu a) = \mu^2 D(a)$ for all $a \in A$ and $\mu \in \mathbb{T}^1$. By Lemma 2.1 and the same reasoning as in the proof of main theorem of [6], one can show that $D(\mu a) = \mu^2 D(a)$ for all $a \in A$ and $\mu \in \mathbb{C}$.

Replacing x, y, z by $2^n x, 2^n y, 2^n z$ in (2.2), respectively, and dividing by 4^{3n}, we obtain

$$\begin{aligned}&\left\|f([2^n x, 2^n y, 2^n z]) - [f(2^n x), 4^n y^2, 4^n z^2] - [4^n x^2, f(2^n y), 4^n z^2] - [4^n x^2, 4^n y^2, f(2^n z)])\right\| \\ &\quad\le \frac{\varphi(0, 0, 2^n x, 2^n y, 2^n z)}{4^{3n}} \le \frac{\varphi(0, 0, 2^n x, 2^n y, 2^n z)}{4^n},\end{aligned}$$

which tends to zero as $n \to \infty$. So

$$D([x, y, z]) = [D(x), y^2, z^2] + [x^2, D(y), z^2] + [x^2, y^2, D(z)]$$

for all $x, y, z \in A$. So D is a ternary quadratic derivation. □

Corollary 2.4. *Let p, θ be nonnegative real numbers with $p < 2$ and let A be a complex algebra-C^*-ternary ring. Let $f : A \to A$ be a mapping such that*

$$\left\|2f\left(\mu\frac{a+b}{2}\right) + 2f\left(\mu\frac{a-b}{2}\right) - \mu^2(f(a) + f(b))\right\| \le \theta(\|a\|^p + \|b\|^p), \tag{2.10}$$

$$\|f([x, y, z]) - [f(x), y^2, z^2] - [x^2, f(y), z^2] - [x^2, y^2, f(z)]\| \le \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.11}$$

for all $\mu \in \mathbb{T}^1$ and all $a, b, x, y, z \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique ternary quadratic derivation $D : A \to A$ such that

$$\|f(a) - D(a)\| \le \frac{2^p\theta}{4 - 2^p}\|a\|^p$$

for all $a \in A$.

Proof. The result follows from Theorem 2.3 by putting $\varphi(a, b, x, y, z) = \theta(\|a\|^p + \|b\|^p + \|x\|^p + \|y\|^p + \|z\|^p)$. □

Now we prove the superstability of ternary quadratic derivations on complex algebra-C^*-ternary rings.

Corollary 2.5. *Let p, θ be nonnegative real numbers with $p < \frac{2}{3}$ and let A be a complex algebra-C^*-ternary ring. Let $f : A \to A$ be a mapping such that*

$$\left\| 2f\left(\mu\frac{a+b}{2}\right) + 2f\left(\mu\frac{a-b}{2}\right) - \mu^2(f(a)+f(b)) \right\| \le \theta(\|a\|^p \cdot \|b\|^p), \tag{2.12}$$

$$\|f([x,y,z]) - [f(x),y^2,z^2] - [x^2,f(y),z^2] - [x^2,y^2,f(z)]\| \le \theta(\|x\|^p \cdot \|y\|^p \cdot \|z\|^p) \tag{2.13}$$

for all $\mu \in \mathbb{T}^1$ and all $a,b,x,y,z \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then $f : A \to A$ is a ternary quadratic derivation.

Proof. Putting $a = b = 0$ in (2.12), we get $f(0) = 0$. Letting $b = 0$, $\mu = 1$ and replacing a by $2a$ in (2.13), we get $f(2a) = 4f(a)$ for all $a \in A$. It is easy to show that $f(2^n a) = 4^n f(a)$ and so $f(a) = \frac{f(2^n a)}{4^n}$ for all $a \in A$. It follows from Theorem 2.3 that $f : A \to A$ is a quadratic mapping. The result follows from Theorem 2.3 by putting $\varphi(a,b,x,y,z) = \theta(\|a\|^p \cdot \|b\|^p + \|x\|^p \cdot \|y\|^p \cdot \|z\|^p)$. □

Theorem 2.6. *Let A be a complex algebra-ternary Banach algebra. Let $f : A \to A$ be a mapping with $f(0) = 0$ and let $\varphi : A^5 \to [0,\infty)$ be a function satisfying (2.2) and*

$$\left\| f(\mu(a+b)) + f(\mu(a-b)) - 2\mu^2(f(a)+f(b)) \right\| \le \varphi(a,b,0,0,0) \tag{2.14}$$

for all $\mu \in \mathbb{T}^1$ and all $a,b,x,y,z \in A$. Assume that there exists a constant $M \in (0,1)$ such that

$$\varphi(2a,2b,2x,2y,2z) \le 4M\varphi(a,b,x,y,z) \tag{2.15}$$

for all $a,b,x,y,z \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique ternary quadratic derivation $D : A \to A$ such that

$$\|f(a) - D(a)\| \le \frac{1}{4(1-M)}\varphi(a,a,0,0,0)$$

for all $a \in A$.

Proof. It follows from (2.15) that

$$\lim_{j\to\infty} \frac{\varphi(2^j a, 2^j b, 2^j x, 2^j y, 2^j z)}{4^j} = 0$$

for all $a,b,x,y,z \in A$.

Putting $b = a$ and $\mu = 1$ in (2.14), we get

$$\|4f(a) - f(2a)\| \le \varphi(a,a,0,0,0)$$

and so

$$\left\| f(a) - \frac{1}{4}f(2a) \right\| \le \frac{1}{4}\varphi(a,a,0,0,0)$$

for all $a \in A$.

We consider the set $\Omega := \{h : A \to A \mid h(0) = 0\}$ and introduce the generalized metric d on Ω as follows:

$$d(h_1,h_2) := \inf\{K \in [0,\infty) : \|h_1(a) - h_2(a)\| \le K\varphi(a,a,0,0,0), \forall a \in A\}$$

if there exists such constant K, and $d(h_1,h_2) = \infty$, otherwise. One can easily show that (Ω, d) is complete. We define the linear mapping $J : \Omega \to \Omega$ by

$$J(h)(a) = \frac{1}{4}h(2a)$$

for all $a \in A$.

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.7. *Let p, θ be nonnegative real numbers with $p < 2$ and let A be a complex algebra-ternary Banach algebra. Let $f : A \to A$ be a mapping satisfying* (2.11) *and*

$$\left\| f\left(\mu(a+b)\right) + f\left(\mu(a-b)\right) - 2\mu^2(f(a)+f(b))\right\| \leq \theta(\|a\|^p + \|b\|^p) \tag{2.16}$$

for all $\mu \in \mathbb{T}^1$ and all $a, b, x, y, z \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique ternary quadratic derivation $D : A \to A$ such that

$$\|f(a) - D(a)\| \leq \frac{2\theta}{4 - 2^p}\|a\|^p$$

for all $a \in A$.

Proof. The result follows from Theorem 2.6 by putting $\varphi(a, b, x, y, z) = \theta(\|a\|^p + \|b\|^p + \|x\|^p + \|y\|^p + \|z\|^p)$. □

Remark 2.8. Bodaghi and Alias [2] provided the conditions (2.1), (2.10), (2.12), (2.14) and (2.16), which are approximate $\mathbb{C}$-quadraticity conditions. But they only proved the quadraticity of the resulting mappings. In this paper, the $\mathbb{C}$-quadraticity has been proved for each case.

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2** (1950), 64–66. 1

[2] A. Bodaghi, I. A. Alias, *Approximate ternary quadratic derivation on ternary Banach algebras and C^*-ternary rings*, Adv. Difference Equ., **2012** (2012), 9 pages. 1, 2.8

[3] J. Diaz, B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc., **74** (1968), 305–309. 2.2

[4] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci., **27** (1941), 222–224. 1

[5] C. Park, *Homomorphisms between Poisson JC^*-algebras*, Bull. Braz. Math. Soc., **36** (2005), 79–97. 2

[6] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces,* Proc. Amer. Math. Soc., **72** (1978), 297–300. 1, 2, 2

[7] S. Shagholi, M. Eshaghi Gordji, M. B. Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras*, J. Comput. Anal. Appl., **13** (2011), 1097–1105. 1

[8] F. Skof, *Propriet locali e approssimazione di operatori,* Rend. Sem. Mat. Fis. Milano, **53** (1983), 113–129. 1

[9] S. M. Ulam, *Problems in Modern Mathematics, Chapter VI*. Science ed., Wiley, New York, (1940). 1