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ADDITIVE ρ–FUNCTIONAL INEQUALITIES AND EQUATIONS

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Abstract. In this paper, we investigate the additive ρ-functional inequalities

$$\left\| f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j) \right\| \leqslant \left\| \rho \left(kf\left(\frac{\sum_{j=1}^{k} x_j}{k}\right) - \sum_{j=1}^{k} f(x_j) \right) \right\| \tag{0.1}$$

and

$$\left\| kf\left(\frac{\sum_{j=1}^{k} x_j}{k}\right) - \sum_{j=1}^{k} f(x_j) \right\| \leqslant \left\| \rho \left(f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j) \right) \right\|, \tag{0.2}$$

where ρ is a fixed complex number with $|\rho| < 1$.

Furthermore, we prove the Hyers-Ulam stability of the additive ρ-functional inequalities (0.1) and (0.2) in complex Banach spaces and prove the Hyers-Ulam stability of additive ρ-functional equations associated with the additive ρ-functional inequalities (0.1) and (0.2) in complex Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [10] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [6] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [8] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [3] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

In [4], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leqslant \|f(xy)\| \tag{1.1}$$

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then f satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)=f(xy)+f(xy^{-1}).$$

See also [9]. Gilányi [5] and Fechner [2] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [7] proved the Hyers-Ulam stability of additive functional inequalities.

In Section 2, we investigate the additive ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.1) in complex Banach spaces. We moreover prove the Hyers-Ulam stability of an additive ρ-functional equation associated with the additive ρ-functional inequality (0.1) in complex Banach spaces.

In Section 3, we investigate the additive ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.2) in complex Banach spaces. We moreover prove the Hyers-Ulam stability of an additive ρ-functional equation associated with the additive ρ-functional inequality (0.2) in complex Banach spaces.

Throughout this paper, let k be a fixed integer with $k \geqslant 2$ and let G be a k-divisible abelian group. Assume that X is a real or complex normed space with norm $\|\cdot\|$ and that Y is a complex Banach space with norm $\|\cdot\|$. Assume that ρ is a fixed complex number with $|\rho|<1$.

2. Additive ρ-functional inequality (0.1)

In this section, we investigate the additive ρ-functional inequality (0.1) in complex Banach spaces.

LEMMA 2.1. *A mapping $f: G \to Y$ satisfies*

$$\left\| f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j)\right\| \leqslant \left\| \rho\left(kf\left(\frac{\sum_{j=1}^{k} x_j}{k}\right) - \sum_{j=1}^{k} f(x_j)\right)\right\| \tag{2.1}$$

for all $x_1, x_2, \cdots, x_k \in G$ if and only if $f: G \to Y$ is additive.

Proof. Assume that $f: G \to Y$ satisfies (2.1).

Letting $x_1 = x_2 = \cdots = x_k = 0$ in (2.1), we get

$$\|(k-1)f(0)\| \leqslant 0.$$

So $f(0)=0$.

Letting $x_1 = x_2 = \cdots = x_k = x$ in (2.1), we get

$$\|f(kx)-kf(x)\| \leqslant 0$$

and so $f(kx)=kf(x)$ for all $x \in G$. Thus

$$f\left(\frac{x}{k}\right) = \frac{1}{k} f(x) \tag{2.2}$$

for all $x \in G$.

It follows from (2.1) and (2.2) that

$$\begin{aligned}\left\| f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j)\right\| &\leqslant \left\|\rho\left(kf\left(\frac{\sum_{j=1}^{k} x_j}{k}\right) - \sum_{j=1}^{k} f(x_j)\right)\right\| \\ &= |\rho| \left\| f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j)\right\|\end{aligned}$$

and so

$$f\left(\sum_{j=1}^{k} x_j\right) = \sum_{j=1}^{k} f(x_j)$$

for all $x_1, x_2, \cdots, x_k \in G$. Hence $f : G \to Y$ is additive.

The converse is obviously true. □

COROLLARY 2.2. *A mapping $f : G \to Y$ satisfies*

$$f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j) = \rho\left(kf\left(\frac{\sum_{j=1}^{k} x_j}{k}\right) - \sum_{j=1}^{k} f(x_j)\right) \tag{2.3}$$

for all $x_1, x_2, \cdots, x_k \in G$ if and only if $f : G \to Y$ is additive.

The equation (2.3) is called an *additive ρ-functional equation.*

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (2.1) in complex Banach spaces.

THEOREM 2.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\left\| f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j)\right\| \leqslant \left\|\rho\left(kf\left(\frac{\sum_{j=1}^{k} x_j}{k}\right) - \sum_{j=1}^{k} f(x_j)\right)\right\| + \theta \sum_{j=1}^{k} \|x_j\|^r \tag{2.4}$$

for all $x_1, x_2, \cdots, x_k \in X$. Then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leqslant \frac{k\theta}{k^r - k} \|x\|^r \tag{2.5}$$

for all $x \in X$.

Proof. Letting $x_1 = x_2 = \cdots = x_k = 0$ in (2.4), we get

$$\|(k-1)f(0)\| \leqslant 0.$$

So $f(0) = 0$.

Letting $x_1 = x_2 = \cdots = x_k = x$ in (2.4), we get

$$\|f(kx) - kf(x)\| \leqslant k\theta\|x\|^r \tag{2.6}$$

for all $x \in X$. So

$$\left\|f(x) - kf\left(\frac{x}{k}\right)\right\| \leqslant \frac{k}{k^r}\theta\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|k^l f\left(\frac{x}{k^l}\right) - k^m f\left(\frac{x}{k^m}\right)\right\| &\leqslant \sum_{j=l}^{m-1}\left\|k^j f\left(\frac{x}{k^j}\right) - k^{j+1} f\left(\frac{x}{k^{j+1}}\right)\right\| \\ &\leqslant \frac{k}{k^r}\sum_{j=l}^{m-1}\frac{k^j}{k^{rj}}\theta\|x\|^r \end{aligned}\tag{2.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{k^n f(\frac{x}{k^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{k^n f(\frac{x}{k^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} k^n f\left(\frac{x}{k^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.7), we get (2.5).

It follows from (2.4) that

$$\begin{aligned}&\left\|h\left(\sum_{j=1}^k x_j\right) - \sum_{j=1}^k h(x_j)\right\| \\ &= \lim_{n\to\infty} k^n \left\|f\left(\frac{\sum_{j=1}^k x_j}{k^n}\right) - \sum_{j=1}^k f\left(\frac{x_j}{k^n}\right)\right\| \\ &\leqslant \lim_{n\to\infty} k^n \left\|\rho\left(kf\left(\frac{\sum_{j=1}^k x_j}{k^{n+1}}\right) - \sum_{j=1}^k f\left(\frac{x_j}{k^n}\right)\right)\right\| + \lim_{n\to\infty}\frac{k^n\theta}{k^{nr}}\sum_{j=1}^k \|x_j\|^r \\ &= \left\|\rho\left(kh\left(\frac{\sum_{j=1}^k x_j}{k}\right) - \sum_{j=1}^k h(x_j)\right)\right\|\end{aligned}$$

for all $x_1, x_2, \cdots, x_k \in X$. So

$$\left\|h\left(\sum_{j=1}^k x_j\right) - \sum_{j=1}^k h(x_j)\right\| \leqslant \left\|\rho\left(kh\left(\frac{\sum_{j=1}^k x_j}{k}\right) - \sum_{j=1}^k h(x_j)\right)\right\|$$

for all $x_1, x_2, \cdots, x_k \in X$. By Lemma 2.1, the mapping $h : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (2.5). Then we have

$$\begin{aligned}\|h(x)-T(x)\| &= k^n \left\|h\left(\frac{x}{k^n}\right)-T\left(\frac{x}{k^n}\right)\right\| \\ &\leqslant k^n\left(\left\|h\left(\frac{x}{k^n}\right)-f\left(\frac{x}{k^n}\right)\right\|+\left\|T\left(\frac{x}{k^n}\right)-f\left(\frac{x}{k^n}\right)\right\|\right) \\ &\leqslant \frac{2k^{n+1}}{(k^r-k)k^{nr}}\theta\|x\|^r,\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (2.5). □

THEOREM 2.4. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.4). *Then there exists a unique additive mapping $h : X \to Y$ such that*

$$\|f(x)-h(x)\| \leqslant \frac{k\theta}{k-k^r}\|x\|^r \tag{2.8}$$

for all $x \in X$.

Proof. Letting $x_1 = x_2 = \cdots = x_k = 0$ in (2.4), we get

$$\|(k-1)f(0)\| \leqslant 0.$$

So $f(0) = 0$.

It follows from (2.6) that

$$\left\|f(x)-\frac{1}{k}f(kx)\right\| \leqslant \theta\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{k^l}f(k^lx)-\frac{1}{k^m}f(k^mx)\right\| &\leqslant \sum_{j=l}^{m-1}\left\|\frac{1}{k^j}f(k^jx)-\frac{1}{k^{j+1}}f(k^{j+1}x)\right\| \\ &\leqslant \sum_{j=l}^{m-1}\frac{k^{rj}}{k^j}\theta\|x\|^r\end{aligned} \tag{2.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.9) that the sequence $\{\frac{1}{k^n}f(k^nx)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{k^n}f(k^nx)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty}\frac{1}{k^n}f(k^nx)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get (2.8).

The rest of the proof is similar to the proof of Theorem 2.3. □

By the triangle inequality, we have

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^{k} x_j\right)-\sum_{j=1}^{k} f(x_j)\right\|-\left\|\rho\left(kf\left(\frac{\sum_{j=1}^{k} x_j}{k}\right)-\sum_{j=1}^{k} f(x_j)\right)\right\| \\ &\leqslant\left\|f\left(\sum_{j=1}^{k} x_j\right)-\sum_{j=1}^{k} f(x_j)-\rho\left(kf\left(\frac{\sum_{j=1}^{k} x_j}{k}\right)-\sum_{j=1}^{k} f(x_j)\right)\right\|.\end{aligned}$$

As corollaries of Theorems 2.3 and 2.4, we obtain the Hyers-Ulam stability results for the additive ρ-functional equation (2.3) in complex Banach spaces.

COROLLARY 2.5. *Let $r>1$ and θ be nonnegative real numbers, and let $f: X \rightarrow Y$ be a mapping such that*

$$\left\|f\left(\sum_{j=1}^{k} x_j\right)-\sum_{j=1}^{k} f(x_j)-\rho\left(kf\left(\frac{\sum_{j=1}^{k} x_j}{k}\right)-\sum_{j=1}^{k} f(x_j)\right)\right\| \leqslant \theta \sum_{j=1}^{k}\|x_j\|^r \tag{2.10}$$

for all $x_1, x_2, \cdots, x_k \in X$. Then there exists a unique additive mapping $h: X \rightarrow Y$ satisfying (2.5).

COROLLARY 2.6. *Let $r<1$ and θ be nonnegative real numbers, and let $f: X \rightarrow Y$ be a mapping satisfying* (2.10). *Then there exists a unique additive mapping $h: X \rightarrow Y$ satisfying* (2.8).

REMARK 2.7. If ρ is a real number such that $-1<\rho<1$ and Y is a real Banach space, then all the assertions in this section remain valid.

3. Additive ρ-functional inequality (0.2)

In this section, we investigate the additive ρ-functional inequality (0.2) in complex Banach spaces.

LEMMA 3.1. *A mapping $f: G \rightarrow Y$ satisfies $f(0)=0$ and*

$$\left\|kf\left(\frac{\sum_{j=1}^{k} x_j}{k}\right)-\sum_{j=1}^{k} f(x_j)\right\| \leqslant\left\|\rho\left(f\left(\sum_{j=1}^{k} x_j\right)-\sum_{j=1}^{k} f(x_j)\right)\right\| \tag{3.1}$$

for all $x_1, x_2, \cdots, x_k \in G$ if and only if $f: G \rightarrow Y$ is additive.

Proof. Assume that $f: G \rightarrow Y$ satisfies (3.1).
Letting $x_1=x$ and $x_2=\cdots=x_k=0$ in (3.1), we get

$$\left\|kf\left(\frac{x}{k}\right)-f(x)\right\| \leqslant 0$$

and so

$$f\left(\frac{x}{k}\right) = \frac{1}{k}f(x) \tag{3.2}$$

for all $x \in G$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}\left\|f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j)\right\| &= \left\|kf\left(\frac{\sum_{j=1}^{k} x_j}{k}\right) - \sum_{j=1}^{k} f(x_j)\right\| \\ &\leqslant |\rho|\left\|f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j)\right\|\end{aligned}$$

and so

$$f\left(\sum_{j=1}^{k} x_j\right) = \sum_{j=1}^{k} f(x_j)$$

for all $x_1, x_2, \cdots, x_k \in G$. Hence $f : G \to Y$ is additive.

The converse is obviously true. □

COROLLARY 3.2. *A mapping $f : G \to Y$ satisfies $f(0) = 0$ and*

$$kf\left(\frac{\sum_{j=1}^{k} x_j}{k}\right) - \sum_{j=1}^{k} f(x_j) = \rho\left(f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j)\right) \tag{3.3}$$

for all $x_1, x_2, \cdots, x_k \in G$ if and only if $f : G \to Y$ is additive.

The equation (3.3) is called an *additive ρ-functional equation.*

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (3.1) in complex Banach spaces.

THEOREM 3.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping with $f(0) = 0$ such that*

$$\left\|kf\left(\frac{\sum_{j=1}^{k} x_j}{k}\right) - \sum_{j=1}^{k} f(x_j)\right\| \leqslant \left\|\rho\left(f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j)\right)\right\| + \theta\sum_{j=1}^{k} \|x_j\|^r \tag{3.4}$$

for all $x_1, x_2, \cdots, x_k \in X$. Then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leqslant \frac{k^r\theta}{k^r - k}\|x\|^r \tag{3.5}$$

for all $x \in X$.

Proof. Letting $x_1 = x$ and $x_2 = \cdots = x_k = 0$ in (3.4), we get

$$\left\| kf\left(\frac{x}{k}\right) - f(x) \right\| \leqslant \theta \|x\|^r \tag{3.6}$$

for all $x \in X$. So

$$\begin{aligned}\left\| k^l f\left(\frac{x}{k^l}\right) - k^m f\left(\frac{x}{k^m}\right) \right\| &\leqslant \sum_{j=l}^{m-1} \left\| k^j f\left(\frac{x}{k^j}\right) - k^{j+1} f\left(\frac{x}{k^{j+1}}\right) \right\| \\ &\leqslant \sum_{j=l}^{m-1} \frac{k^j}{k^{rj}} \theta \|x\|^r \end{aligned} \tag{3.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.7) that the sequence $\{k^n f(\frac{x}{k^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{k^n f(\frac{x}{k^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} k^n f\left(\frac{x}{k^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.7), we get (3.5).

The rest of the proof is similar to the proof of Theorem 2.3. □

THEOREM 3.4. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and* (3.4)*. Then there exists a unique additive mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \leqslant \frac{k^r \theta}{k - k^r} \|x\|^r \tag{3.8}$$

for all $x \in X$.

Proof. It follows from (3.6) that

$$\left\| f(x) - \frac{1}{k} f(kx) \right\| \leqslant \frac{k^r \theta}{k} \|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}\left\| \frac{1}{k^l} f(k^l x) - \frac{1}{k^m} f(k^m x) \right\| &\leqslant \sum_{j=l}^{m-1} \left\| \frac{1}{k^j} f(k^j x) - \frac{1}{k^{j+1}} f(k^{j+1} x) \right\| \\ &\leqslant \frac{k^r}{k} \sum_{j=l}^{m-1} \frac{k^{rj}}{k^j} \theta \|x\|^r \end{aligned} \tag{3.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.9) that the sequence $\{\frac{1}{k^n} f(k^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{k^n} f(k^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} \frac{1}{k^n} f(k^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.9), we get (3.8).

The rest of the proof is similar to the proof of Theorem 2.4. □

By the triangle inequality, we have

$$\left\| kf\left(\frac{\sum_{j=1}^{k} x_j}{k}\right) - \sum_{j=1}^{k} f(x_j)\right\| - \left\| \rho\left(f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j)\right)\right\|$$
$$\leqslant \left\| kf\left(\frac{\sum_{j=1}^{k} x_j}{k}\right) - \sum_{j=1}^{k} f(x_j) - \rho\left(f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j)\right)\right\|.$$

As corollaries of Theorems 3.3 and 3.4, we obtain the Hyers-Ulam stability results for the additive ρ-functional equation (3.3) in complex Banach spaces.

COROLLARY 3.5. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping with $f(0) = 0$ such that*

$$\left\| kf\left(\frac{\sum_{j=1}^{k} x_j}{k}\right) - \sum_{j=1}^{k} f(x_j) - \rho\left(f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j)\right)\right\| \leqslant \theta \sum_{j=1}^{k} \|x_j\|^r \quad (3.10)$$

for all $x_1, x_2, \cdots, x_k \in X$. Then there exists a unique additive mapping $h : X \to Y$ satisfying (3.5).

COROLLARY 3.6. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and* (3.10). *Then there exists a unique additive mapping $h : X \to Y$ satisfying* (3.8).

REMARK 3.7. If ρ is a real number such that $-1 < \rho < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

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