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Crystal $\mathcal{B}(\lambda)$ as a Subset of the Tableau Description of $\mathcal{B}(\infty)$ for the Classical Lie Algebra Types

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Abstract The irreducible highest weight crystal $\mathcal{B}(\lambda)$ is known to appear as a connected component within the crystal graph of $\mathcal{B}(\infty)\otimes\{\mathsf{r}_\lambda\}$. Using the marginally large tableau realization of $\mathcal{B}(\infty)$, we identify the elements belonging to this connected component, for the Lie algebra types C_n, B_n, and D_{n+1}. This gives us a tableau realization of $\mathcal{B}(\lambda)$ that is different from the well known tableau realization by Kashiwara and Nakashima. In particular, our new description no longer involves half-boxes. We further present a description of $\mathcal{B}(\lambda)$ through the Kashiwara embedding.

Keywords Crystal base · Classical simple Lie algebra · Marginally large tableau

1 Introduction

The quantum group $U_q(\mathfrak{g})$ is a q-deformation of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ [5, 10] and the irreducible highest weight $U_q(\mathfrak{g})$-module $V^q(\lambda)$ may also be seen as a deformation of $V(\lambda)$, the irreducible highest weight $\mathfrak{g}$-module. The crystal base $\mathcal{B}(\lambda)$ of $V^q(\lambda)$ reflects the structure of the $U_q(\mathfrak{g})$-module in a combinatorial manner [11, 12] and hence also gives information concerning the structure of the $\mathfrak{g}$-module $V(\lambda)$.

Young tableau realizations of $\mathcal{B}(\lambda)$, for the classical simple Lie algebra types, were given by Kashiwara and Nakashima [15] at an early stage of the crystal base theory development. There, each $\mathcal{B}(\lambda)$ was described as the set $\mathcal{T}(\lambda)$ of all tableaux of shape λ satisfying a certain

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set of conditions. While these conditions were reasonably simple for the A_n type, the same could not be said of the conditions for the other three classical types. The tableau realizations for the C_n, B_n, and D_{n+1} types involved *half-boxes* and the notion of *configurations*.

The current work presents new tableau realizations of the crystals $\mathcal{B}(\lambda)$ for the classical Lie algebra types C_n, B_n, and D_{n+1}. The tableaux in our realization of $\mathcal{B}(\lambda)$ are not necessarily of shape λ and do not involve half-boxes or configurations. Note that similar results for the much simpler A_n and G_2 cases are available through the recent works [18] and [17], respectively.

There are two main ingredients to our new tableau realizations of $\mathcal{B}(\lambda)$. The first is the result by Nakashima [23], which gives the existence of a strict crystal embedding

$$\mathcal{B}(\lambda) \hookrightarrow \mathcal{B}(\infty) \otimes \{\mathsf{r}_\lambda\}, \tag{1}$$

where $\mathcal{B}(\infty)$ is the crystal base of $U_q^-(\mathfrak{g})$, the negative part of the quantum group, and $\{\mathsf{r}_\lambda\}$ is a certain single-element crystal. This implies that the crystal graph of $\mathcal{B}(\infty) \otimes \{\mathsf{r}_\lambda\}$ contains a copy of $\mathcal{B}(\lambda)$ as a connected component.

To identify and present an explicit description of this connected component, one requires a method to express and to compute with the elements of $\mathcal{B}(\infty)$. This functionality is provided by our second ingredient, which is the realization $\mathcal{T}(\infty)$ of $\mathcal{B}(\infty)$ consisting of the marginally large tableaux [7]. Even though the construction of $\mathcal{T}(\infty)$ relied on the existing tableau realization $\mathcal{T}(\lambda)$ of $\mathcal{B}(\lambda)$, the tableaux of $\mathcal{T}(\infty)$ are among the simplest contained in $\mathcal{T}(\lambda)$ and do not involve half-boxes or configurations. Hence, our final realization of $\mathcal{B}(\lambda)$ as an explicitly described subset of $\mathcal{T}(\infty) \otimes \{\mathsf{r}_\lambda\}$ is also free of the complications that were present in the previous realization $\mathcal{T}(\lambda)$.

Note that Nakashima [23] had already expressed the image of the crystal embedding (1) as the set

$$\big\{ b \otimes \mathsf{r}_\lambda \mid \varepsilon_i^*(b) \le \langle h_i, \lambda \rangle \text{ for any } i \in I \big\}, \tag{2}$$

for all symmetrizable Kac-Moody algebras and that this was preceded by an essentially identical result by Kashiwara [14]. However, we had no means of handling the ε_i^* function, which concerns the $*$-crystal structure on $\mathcal{B}(\infty)$, on the realizations $\mathcal{T}(\infty)$, for the C_n, B_n, and D_{n+1} types. This prevented us from attempting to obtain an explicit tableau realization of $\mathcal{B}(\lambda)$ through the description (2) of the image set. On the other hand, since our work provides a description of the set (2), that has been obtained through an independent approach, one could hope for implications in the reverse direction. That is, for a crystal element $T \in \mathcal{B}(\infty)$ that is presented as an element of $\mathcal{T}(\infty)$, the set of conditions associated with our realization of $\mathcal{B}(\lambda)$ must be equivalent to the set of conditions $\varepsilon_i^*(T) \le \langle h_i, \lambda \rangle$ $(i \in I)$, and one could be able to create a formula for computing the $\varepsilon_i^*(T)$ value by studying this equivalence. In fact, we have obtained a candidate formula through this approach and plan to develop this further in a future work.

There are a few other realizations of $\mathcal{B}(\lambda)$ for the finite simple Lie algebra types that are based on the crystal embedding (1). For the A_n type, Kashiwara and Saito[1] gave a matrix form description of $\mathcal{B}(\infty)$ and expressed the image set (2) more concretely in terms of these matrices by computing the ε_i^* values of the matrices explicitly. The embedding was also used by Nakashima [23] to obtain the polyhedral realizations of $\mathcal{B}(\lambda)$ for the A_n and rank-2 types, and analogous results for the remaining finite types were obtained by Hoshino [9].

As a corollary to our main result, we are able to present yet another description of $\mathcal{B}(\lambda)$. There is an embedding of the crystal $\mathcal{B}(\infty)$ into a certain tensor product of crystals, referred

[1] Private communication with Y. Saito (August, 2010).

to as the Kashiwara embedding, and Cliff [4] gave an explicit realization of $\mathcal{B}(\infty)$ by describing the image set of this embedding, for the classical Lie algebra types. Because the realization of $\mathcal{B}(\infty)$ that was used in describing this image set was very similar to the $\mathcal{T}(\infty)$ realization we are using in this work, we were able to identify the subset of the image set that corresponds to $\mathcal{B}(\lambda) \subset \mathcal{B}(\infty) \otimes \{r_\lambda\}$.

The works [2, 3, 16, 22] showed how the Gindikin-Karpelevich formula could be evaluated as a certain summation indexed by the crystal $\mathcal{B}(\infty)$. This sum was further simplified and described combinatorially in [20, 21] by using the marginally large tableau realization $\mathcal{T}(\infty)$ in place of $\mathcal{B}(\infty)$, for the classical Lie algebra types and the G_2 type. The Casselman-Shalika formula is a companion formula to the Gindikin-Karpelevich formula, and one may associate a summation indexed by $\mathcal{B}(\lambda)$ to this formula [1–3]. For the A_n case, the work [19] expressed the sum using semi-standard tableaux. One possible application of our realization of $\mathcal{B}(\lambda)$ could be an analogous treatment of the remaining classical types.

The rest of this paper is organized as follows. In Section 2, we recall the existing results that are used in this paper, and some new notations are introduced in Section 3. The subsequent three sections state and prove the correctness of our realizations of $\mathcal{B}(\lambda)$ for the C_n, B_n, and D_{n+1} types. Section 7 presents the realization of $\mathcal{B}(\lambda)$ associated with the Kashiwara embedding as a corollary to our main realization result. The final section contains a rough explanation of how the conditions for our realizations were obtained.

2 Tableau Description of $\mathcal{B}(\infty)$ and the Embedding of $\mathcal{B}(\lambda)$ in $\mathcal{B}(\infty)$

In this section, we recall the basic notation concerning the crystal theory, the description of $\mathcal{B}(\infty)$ given in terms of the marginally large tableaux, and a crystal embedding of $\mathcal{B}(\lambda)$ into a certain tensor product of two crystals. Even though more general results are available, our review will be restricted to the finite simple Lie algebras types C_n ($n \geq 2$), B_n ($n \geq 2$), and D_{n+1} ($n \geq 3$), since these types are the subject of our study. Unless explicitly stated otherwise, all our future discussions will hold true for each of these types. Notice that the subscript for the D type is different from those of the other two types. This is to simplify our writing and does not imply any restriction on the index range for the D type we are considering.

Standard notation, as may be found in the textbook [6], will be used, and we assume knowledge of the basic notions associated with the crystal base theory, such as the following: index set I, simple root α_i, coroot h_i, fundamental weight Λ_i, set of dominant integral weights P^+, Cartan matrix $\big(\alpha_i(h_j)\big)_{i,j\in I}$, quantum group $U_q(\mathfrak{g})$, abstract crystal with associated Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$, and maps wt, ε_i, φ_i, irreducible highest weight crystal $\mathcal{B}(\lambda)$, tensor product rule, negative part $U_q^-(\mathfrak{g})$ of $U_q(\mathfrak{g})$, and crystal basis $\mathcal{B}(\infty)$ of $U_q^-(\mathfrak{g})$. However, since this work relies heavily on the tensor product rule, we repeat the formula here.

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \tag{3}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \tag{4}$$

Let us now review the marginally large tableau description of $\mathcal{B}(\infty)$ that was presented by [7, 8]. The high-level description of the realization can be given as follows. One starts with the union $\cup_\lambda \mathcal{B}(\lambda)$ and makes certain identifications within this large set, in a manner that is compatible with the Kashiwara operator actions on $\mathcal{B}(\lambda)$. This careful identification allows the set of equivalence classes to be given a crystal structure that is a natural fusion of those structures on the original crystals $\mathcal{B}(\lambda)$, and the resulting crystal $\cup_\lambda \mathcal{B}(\lambda)/\sim$ becomes isomorphic to $\mathcal{B}(\infty)$. Then, each crystal $\mathcal{B}(\lambda)$ is replaced by its Young tableau realization $\mathcal{T}(\lambda)$, given by [15], and the equivalence relation on $\cup_\lambda \mathcal{B}(\lambda)$ is translated to that among the tableaux. Finally, the marginally large tableau description of $\mathcal{B}(\infty)$ is obtained by selecting appropriate representatives from each of the equivalence classes.

The basic crystals that were used in [15] to create the Young tableau realizations $\mathcal{T}(\lambda)$ of $\mathcal{B}(\lambda)$ are as follows, for each Lie algebra type.

$$C_n : \quad \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}} \tag{5}$$

$$B_n : \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{0} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}} \tag{6}$$

$$D_{n+1} : \begin{tikzcd}[row sep=small]
 & & & & \boxed{n+1} \arrow[dr, "n+1"] & & & & \\
\boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \cdots \arrow[r, "n-1"] & \boxed{n} \arrow[ur, "n"] \arrow[dr, "n+1"'] & & \boxed{\bar{n}} \arrow[r, "n-1"] & \cdots \arrow[r, "2"] & \boxed{\bar{2}} \arrow[r, "1"] & \boxed{\bar{1}} \\
 & & & & \boxed{\overline{n+1}} \arrow[ur, "n"'] & & & &
\end{tikzcd} \tag{7}$$

The work [15] utilized a few other crystals, associated with the spin representations, in constructing their tableau realizations, but we will only be using the above crystals in this work.

The set of the above crystal elements, separately collected for each Lie algebra type, will commonly be written as J. Each set J is given an ordering, where one writes $x \prec y$ if and only if y can be reached from x through applications of the $\tilde{f}_i$ operators, possibly of varying colors. For example, the C_n-type basic crystal elements are ordered as

$$J = \{1 \prec 2 \prec \cdots \prec n \prec \bar{n} \prec \cdots \prec \bar{2} \prec \bar{1}\}. \tag{8}$$

In the D_{n+1} case, no order relation exists between the crystal elements $n+1$ and $\overline{n+1}$.

The next definition first appeared in [8] for use with the exceptional Lie algebra types, but can also be used with the C_n, B_n, and D_{n+1} types. Throughout this work, we will refer to the top row of a tableau as its *first* row.

Definition 1 Let $1 \le i \le n$. A *basic i-column* is a single column of i-many boxes, with the box at its x-th row occupied by the basic crystal element $x \in J$, for each $1 \le x \le i$.

The description for the general tableaux appearing in the realization $\mathcal{T}(\lambda)$ for $\lambda \in P^+$ is quite complicated in that it involves *half-boxes* and certain *configuration* conditions, but the marginally large tableaux, recalled from [7] in the next definition, are among the simplest of these tableaux.

Definition 2 A tableau of C_n, B_n, or D_{n+1} type that consists of n rows with its boxes occupied by elements of J is *marginally large*, if it satisfies the following conditions.

1. It is semi-standard, with respect to the ordering $\prec$, in the sense that its box contents weakly increase to the right and strictly increase in the downward direction.
2. It contains exactly one basic i-column, for each $1 \le i \le n$.

3. All entries in its i-th row are less than or equal to $\bar{i}$ with respect to the ordering $\prec$.
4. In the B_n case, the crystal element $0 \in J$ appears at most once in each row.

The set of all marginally large tableaux is denoted by $\mathcal{T}(\infty)$.

Let us clarify a few points that might be slightly confusing in this definition. First, a marginally large tableau consists only of full-sized boxes and contains no half-boxes that are associated with the spin representations. Second, the notion of being semi-standard used here does not involve the configuration conditions that appeared in the original tableau realization [15] of $\mathcal{B}(\lambda)$. Third, in the D_{n+1} case, to satisfy the semi-standard condition, the crystal elements $n+1$ and $\overline{n+1}$ cannot appear in the same row, since there is no order relation defined between them.

It is clear that the first two conditions of Definition 2 imply that the number of i-boxes in the i-th row is greater than the total number of all boxes appearing in the $(i+1)$-th row by exactly one. This was the approach taken by the earlier work [7] in defining marginally large tableaux.

The following result from [7, 8] is true for all finite simple Lie algebra types.

Theorem 3 *The set $\mathcal{T}(\infty)$ of all marginally large tableaux forms a crystal and is isomorphic to the crystal $\mathcal{B}(\infty)$.*

Let us provide an example of marginally large tableau for each of the three Lie algebra types considered in this paper. These figures will be useful to anyone trying to follow through the detailed computations required in our later proofs.

A C_3-type marginally large tableau is of the following form. The shaded parts are optional and can be of varying sizes, but the three non-shaded parts must always be present.

$$\begin{array}{|l|l|l|l|l|l|l|l|l|l|}\hline 1 & \cdots & 1 & \cdots\ 1 & 1 & 2\cdots 2 & 3\cdots 3 & \bar{3}\cdots\bar{3} & \bar{2}\cdots\bar{2} & \bar{1}\cdots\bar{1} \\ \hline 2 & \cdots\ 2 & 2 & 3\cdots 3 & \bar{3}\cdots\bar{3} & \bar{2}\cdots\bar{2} \\ \hline 3 & \bar{3}\cdots\bar{3} \\ \hline \end{array} \tag{9}$$

A B_3-type marginally large tableau is of the following form. The non-shaded parts must exist, whereas the shaded parts are optional. Note that the crystal element 0 may be absent from any row, but may not appear more than once in any row.

$$\begin{array}{|l|l|l|l|l|l|l|l|l|l|l|}\hline 1 & \cdots & 1 & \cdots\ 1 & 1 & 2\cdots 2 & 3\cdots 3 & 0 & \bar{3}\cdots\bar{3} & \bar{2}\cdots\bar{2} & \bar{1}\cdots\bar{1} \\ \hline 2 & \cdots\ 2 & 2 & 3\cdots 3 & 0 & \bar{3}\cdots\bar{3} & \bar{2}\cdots\bar{2} \\ \hline 3 & 0 & \bar{3}\cdots\bar{3} \\ \hline \end{array} \tag{10}$$

A D_4-type marginally large tableau is of the following form. The non-shaded parts must exist, whereas the shaded parts are optional. Either one of 4 or $\bar{4}$ may take the place of each of the letters x, y, and z.

$$\begin{array}{|l|l|l|l|l|l|l|l|l|l|l|}\hline 1 & \cdots & 1 & \cdots\ 1 & 1 & 2\cdots 2 & 3\cdots 3 & x\cdots x & \bar{3}\cdots\bar{3} & \bar{2}\cdots\bar{2} & \bar{1}\cdots\bar{1} \\ \hline 2 & \cdots\ 2 & 2 & 3\cdots 3 & y\cdots y & \bar{3}\cdots\bar{3} & \bar{2}\cdots\bar{2} \\ \hline 3 & z\cdots z & \bar{3}\cdots\bar{3} \\ \hline \end{array} \tag{11}$$

In all three cases discussed above, the marginally large tableau corresponding to the highest weight element $b_\infty \in \mathcal{B}(\infty)$ is

$$T_\infty = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\,. \tag{12}$$

Let us next recall the crystal structure given to the set $\mathcal{T}(\infty)$. We start with the description of the Kashiwara operator actions. The reader is assumed to be familiar with the Kashiwara operator actions on the crystal $\mathcal{T}(\lambda)$, which is to (a) expand a given tableau into its tensor product form through the *far eastern reading*, (b) rely on *tensor product rule* to apply $\tilde{f}_i$ or $\tilde{e}_i$ to one of the boxes, and (c) reorganize the resulting tensor product form into the shape of the original tableau. Note that since the tableaux in $\mathcal{T}(\infty)$ do not involve any half-boxes, we do not have to deal with the somewhat exceptional Kashiwara operator rules that were associated with the spin representations as was required in the original tableau realization of [15].

The reader may be aware that, when performing the above part-(b), one often utilizes the following approach, which is equivalent to the tensor product rule.

1. Under each tensor component, write down its i-signature, where the i-signature of a crystal element b is a sequence of $\varepsilon_i(b)$-many 1's followed by $\varphi_i(b)$-many 0's, reading from left to right.
2. Successively cancel out every occurrence of (0, 1) pairs from the sequence of mixed 0's and 1's, until one arrives at a sequence of 1's followed by 0's, reading from left to right. This is the *i-signature* of the whole tensor product form.
3. To apply $\tilde{f}_i$ to the whole product form, one applies it to the single tensor component corresponding to the leftmost 0 remaining in the shortened i-signature, except that the $\tilde{f}_i$ action is set to zero when no 0's remain.
4. The $\tilde{e}_i$ is similarly applied to the component corresponding to the rightmost 1, or set to zero in the absence of remaining 1's.

Now, the procedure for applying the Kashiwara operator $\tilde{f}_i$ to a marginally large tableau is as follows.

1. Apply $\tilde{f}_i$ to the tableau as usual. That is, write it in tensor product form, use tensor product rule, and reassemble into original tableau form.
2. If the resulting tableau is not marginally large, locate the box $\tilde{f}_i$ has acted upon and insert one basic i-column to its left.

The $\tilde{e}_i$ operation is similar except that the result could be zero and that one may be required to *remove* one basic i-column to retain marginal largeness.

To complete the description of the crystal structure on $\mathcal{T}(\infty)$, it remains to explain the wt, φ_i, and ε_i functions. Let us write the corresponding functions defining the crystal structure on $\mathcal{T}(\lambda) \cong \mathcal{B}(\lambda)$ as $\bar{\mathrm{wt}}$, $\bar{\varphi}_i$, and $\bar{\varepsilon}_i$. Note that any marginally large tableau $T \in \mathcal{T}(\infty)$ belongs to exactly one $\mathcal{T}(\lambda)$ for some $\lambda \in P^+$. Let us write $\mathrm{shape}(T) = \lambda$ when $T \in \mathcal{T}(\lambda)$. The crystal structure on $\mathcal{T}(\infty)$ is such that $\mathrm{wt}(T) = \bar{\mathrm{wt}}(T) - \mathrm{shape}(T)$, $\varepsilon_i(T) = \bar{\varepsilon}_i(T)$, and $\varphi_i(T) = \varepsilon_i(T) + \mathrm{wt}(T)(h_i)$. The definitions imply, in particular, that

$$\varphi_i(T) = \bar{\varphi}_i(T) - \mathrm{shape}(T)(h_i). \tag{13}$$

This work will provide a new expression for the crystal $\mathcal{B}(\lambda)$, for any $\lambda \in P^+$, in terms of the elements of the crystal $\mathcal{T}(\infty)$. The following result, introduced by Nakashima [23], is crucial in achieving this goal.

Theorem 4 *For each $\lambda \in P^+$, there exist a unique strict crystal embedding*

$$\Omega_\lambda : \mathcal{B}(\lambda) \hookrightarrow \mathcal{B}(\infty) \otimes \mathsf{R}_\lambda,$$

that maps b_λ to $b_\infty \otimes \mathsf{r}_\lambda$.

The symbols b_λ and b_∞ appearing in this claim denote the highest weight elements of $\mathcal{B}(\lambda)$ and $\mathcal{B}(\infty)$, respectively, and the crystal R_λ is the single-element set $\{\mathsf{r}_\lambda\}$ with the following crystal structure:

$$\tilde{e}_i(\mathsf{r}_\lambda) = 0, \quad \tilde{f}_i(\mathsf{r}_\lambda) = 0, \quad \mathrm{wt}(\mathsf{r}_\lambda) = \lambda, \quad \varepsilon_i(\mathsf{r}_\lambda) = -\lambda(h_i), \quad \text{and} \quad \varphi_i(\mathsf{r}_\lambda) = 0. \tag{14}$$

This theorem implies that the connected component in the crystal graph of $\mathcal{B}(\infty) \otimes \mathsf{R}_\lambda$ containing the element $b_\infty \otimes \mathsf{r}_\lambda$ is isomorphic to $\mathcal{B}(\lambda)$, and our realization of $\mathcal{B}(\lambda)$ will be based on this fact. After replacing $\mathcal{B}(\infty)$ with its tableau realization $\mathcal{T}(\infty)$ to make the handling of explicit elements possible, we will find the connected component of $\mathcal{T}(\infty)\otimes\mathsf{R}_\lambda$ containing $T_\infty \otimes \mathsf{r}_\lambda$, where T_∞ denotes the highest weight element of $\mathcal{T}(\infty)$.

The embedding of Theorem 4 was also introduced by Kashiwara [14], except that the single-element crystal he used was not the one described by (14). However, the difference is inessential, and the images of both embeddings can be seen as expressing the crystal $\mathcal{B}(\lambda)$.

3 Notation

The Kashiwara operators will be written as $\tilde{f}_*$ and $\tilde{e}_*$ when we wish to leave the operator indices unspecified.

A marginally large tableau of C_n, B_n, or D_{n+1} type consists of n rows of boxes, with each box filled with an element from the basic crystal J associated with the Lie algebra type under consideration. For each row index $1 \le i \le n$ and basic crystal element $x \in J$, we will use symbol $\boxed{x}_{\,i}$ to refer to an x-box situated within the i-th row of a marginally large tableaux, where we are numbering the top row of a tableau as its 1-st row. For each row index $1 \le i \le n$ and basic crystal element $x \succ i$, the number of $\boxed{x}_{\,i}$-boxes in a tableau will be written as $\#\boxed{x}_{\,i}$. We also define $\#\boxed{i}_{\,i} = +\infty$, which is natural in view of the $\tilde{f}_*$ action on marginally large tableaux that inserts basic columns whenever there is a shortage of $\boxed{i}_{\,i}$-boxes.

We next introduce the set of integers $t_{i,x}$, associated with each marginally large tableau. For each row index $1 \le i \le n$ and basic crystal element $x \in J$ such that $x \succ i$, the integer $t_{i,x}$ is defined to be the number of all boxes appearing in the i-th row of the given tableau that contain $k \in J$ such that $k \succeq x$. In particular, $t_{i+1,\bar{i}}$ of a marginally large tableau is always zero for $1 \le i \le n-1$, and we extend this a single step further to $t_{n+1,\bar{n}} = 0$, for notational convenience. We also set $t_{i,i} = +\infty$, for $1 \le i \le n$, as natural consequences of the definition $\#\boxed{i}_{\,i} = +\infty$. It is clear that any full set of $t_{i,x}$ values can correspond to at most one marginally large tableau.

The notations $\#\boxed{x}_{\,i}$ and $t_{i,x}$ do not make the dependences of these values on the tableaux explicit, but our uses of these symbols will always be such that the tableau under consideration is not ambiguous.

For each basic crystal element $x \in J$, in most cases, there is exactly one element that can be reached from x through a single application of some $\tilde{f}_*$ operator, and this unique element will be written as $x+$. More specifically, for the C_n type crystal, we have $j+ = j+1$ and $\bar{j}+ = \overline{j-1}$, in most cases, with the only exceptions being that $n+ = \bar{n}$ and that $\bar{1}+$ is not defined. Likewise, for the B_n type crystal, $n+ = 0$ and $0+ = \bar{n}$ are the exceptional cases. We will not be using the $x+$ notation with the D_{n+1} type crystals.

4 C_n Type

We first present a set of conditions to be applied to marginally large tableaux of C_n type. Our goal will be to show that, for any $\lambda \in P^+$, the set of tableaux satisfying these conditions is a realization of $\mathcal{B}(\lambda)$.

$\boxed{\text{Condition-}C}$

$$\begin{array}{lcl}
\mathsf{A}[i,j]: & t_{i,j} + t_{i+1,\overline{j+}} \le \lambda(h_i) + t_{i+1,j+} + t_{i+1,\bar{j}} & (1 \le i < j \le n) \\
\mathsf{B}[i,j]: & t_{i,j} \le \lambda(h_i) + t_{i+1,j} & (1 \le i < i+1 < j \le n) \\
\mathsf{C}[i,j]: & t_{i,j+} + t_{i,\bar{j}} \le \lambda(h_i) + t_{i,j} + t_{i+1,\overline{j+}} & (1 \le i < j \le n) \\
\mathsf{D}[i,j]: & t_{i,\bar{j}} \le \lambda(h_i) + t_{i+1,\bar{j}} & (1 \le i \le j \le n)
\end{array}$$

It is possible to write Condition-C slightly more compactly. The definitions $\#\boxed{i}_i = +\infty$ and $t_{i,i} = +\infty$ imply that relations that could be labeled as $\mathsf{B}[i, i+1]$ and $\mathsf{C}[i, i]$ are empty conditions. When these are added to the above, the ranges of indices become more uniform, and Condition-C may equivalently be written as follows.

$$\begin{array}{ll}
t_{i,j} \le \lambda(h_i) + \min\left\{ t_{i+1,j+} + \#\boxed{\bar{j}}_{i+1},\, t_{i+1,j} \right\} & (1 \le i < j \le n) \\
t_{i,\bar{j}} \le \lambda(h_i) + \min\left\{ t_{i+1,\overline{j+}} + \#\boxed{j}_{i},\, t_{i+1,\bar{j}} \right\} & (1 \le i \le j \le n)
\end{array}$$

However, for the purpose of presenting the details of our proofs, it will be more convenient to fully expand even the $x+$ notation, separating some of the index ranges into separate conditions, and to work with the following longer listing of Condition-C.

$$\begin{array}{lcl}
\mathsf{A}[i,j]: & t_{i,j} + t_{i+1,\overline{j-1}} \le \lambda(h_i) + t_{i+1,j+1} + t_{i+1,\bar{j}} & (1 \le i < j \le n-1) \\
\mathsf{A}[i,n]: & t_{i,n} + t_{i+1,\overline{n-1}} \le \lambda(h_i) + 2t_{i+1,\bar{n}} & (1 \le i \le n-1) \\
\mathsf{B}[i,j]: & t_{i,j} \le \lambda(h_i) + t_{i+1,j} & (1 \le i < i+1 < j \le n) \\
\mathsf{C}[i,j]: & t_{i,j+1} + t_{i,\bar{j}} \le \lambda(h_i) + t_{i,j} + t_{i+1,\overline{j-1}} & (1 \le i < j \le n-1) \\
\mathsf{C}[i,n]: & 2t_{i,\bar{n}} \le \lambda(h_i) + t_{i,n} + t_{i+1,\overline{n-1}} & (1 \le i \le n-1) \\
\mathsf{D}[i,j]: & t_{i,\bar{j}} \le \lambda(h_i) + t_{i+1,\bar{j}} & (1 \le i \le j \le n-1) \\
\mathsf{D}[n,n]: & t_{n,\bar{n}} \le \lambda(h_n) &
\end{array}$$

Note that we have removed $\mathsf{D}[i,n]$ $(1 \le i \le n-1)$, since these are superfluous conditions that can be obtained by combining $\mathsf{A}[i,n]$ and $\mathsf{C}[i,n]$.

We will prepare two lemmas and our main result for the C_n-type crystals will follow quite easily from the two claims.

Lemma 5 *If a marginally large tableau T of C_n type satisfies* Condition-C, *then $\varphi_i(T) \ge \varepsilon_i(\mathsf{r}_\lambda)$, for all $i \in I$.*

Proof Note that $\varepsilon_n(\mathsf{r}_\lambda) = -\lambda(h_n)$ and that

$$\varphi_n(T) = \bar{\varphi}_n(T) - \text{shape}(T)(h_n) \geq 1 - (1 + t_{n,\bar{n}}) = -t_{n,\bar{n}},$$

so that $\mathsf{D}[n, n]$ implies $\varphi_n(T) \geq \varepsilon_n(\mathsf{r}_\lambda)$.

Similarly, the fact $\varepsilon_{n-1}(\mathsf{r}_\lambda) = -\lambda(h_{n-1})$ and the observation

$$\varphi_{n-1}(T) = \bar{\varphi}_{n-1}(T) - \text{shape}(T)(h_{n-1}) \geq (2t_{n,\bar{n}} + 1) - (1 + t_{n-1,n}) = 2t_{n,\bar{n}} - t_{n-1,n}$$

can be combined with $\mathsf{A}[n-1, n]$, which is $t_{n-1,n} \leq \lambda(h_{n-1}) + 2t_{n,\bar{n}}$, to bring out $\varphi_{n-1}(T) \geq \varepsilon_{n-1}(\mathsf{r}_\lambda)$.

Finally, for $1 \leq i \leq n-2$, one can combine $\varepsilon_i(\mathsf{r}_\lambda) = -\lambda(h_i)$, the observation

$$\begin{aligned}\varphi_i(T) &= \bar{\varphi}_i(T) - \text{shape}(T)(h_i) \geq (1 + t_{i+1,i+2} + t_{i+1,\overline{i+1}}) - (1 + t_{i,i+1}) \\ &= t_{i+1,i+2} + t_{i+1,\overline{i+1}} - t_{i,i+1},\end{aligned}$$

and $\mathsf{A}[i, i+1]$, which is $t_{i,i+1} \leq \lambda(h_i) + t_{i+1,i+2} + t_{i+1,\overline{i+1}}$, to derive $\varphi_i(T) \geq \varepsilon_i(\mathsf{r}_\lambda)$. $\square$

The verification of our next lemma is much more time consuming.

Lemma 6 *Fix a $\lambda \in P^+$ and let T be a marginally large tableau of C_n type that satisfies* Condition-C. *If $\tilde{f}_k(T \otimes \mathsf{r}_\lambda)$ is nonzero, then the marginally large tableau $\tilde{f}_k T$ satisfies* Condition-C. *Similarly, if $\tilde{e}_k(T \otimes \mathsf{r}_\lambda)$ is nonzero, then the marginally large tableau $\tilde{e}_k T$ satisfies* Condition-C.

The proof of this lemma consists of a meticulous verification of whether $\tilde{f}_k T$ and $\tilde{e}_k T$ satisfy every sub-condition of Condition-C, sometimes even further divided into multiple situations. We will first present detailed explanations for two of these situations associated with one sub-condition and then write the arguments for the rest of the cases more concisely.

Let us start with the sub-condition $\mathsf{A}[i, j]$ with indices belonging to the range $1 \leq i < j \leq n-1$. An $\tilde{f}_*$ operation on T has the possibility of incrementing a $t_{*,*}$ value by one, and an $\tilde{e}_*$ operation could decrement a $t_{*,*}$ value by one. Since each Kashiwara operator action has the possibility of changing at most one $t_{*,*}$ value, the only possible manner in which the resulting tableaux $\tilde{f}_* T$ or $\tilde{e}_* T$ may not satisfy $\mathsf{A}[i, j]$ is for one of the following four events to occur through the Kashiwara operator action: (a) $t_{i,j}$ is incremented by an $\tilde{f}_*$ action, (b) $t_{i+1,\overline{j-1}}$ is incremented by an $\tilde{f}_*$ action, (c) $t_{i+1,j+1}$ is decremented by an $\tilde{e}_*$ action, (d) $t_{i+1,\bar{j}}$ is decremented by an $\tilde{e}_*$ action.

We now focus on the situation in which Event-(a) has occurred. Our goal is to show that such a situation can only be possible when the inequality $\mathsf{A}[i, j]$, which is assumed for T, is a strict inequality, so that the new $\mathsf{A}[i, j]$ expression for $\tilde{f}_* T$, which simply has the $t_{i,j}$ of the original expression replaced by $t_{i,j} + 1$, is satisfied. This needs to be done for all choices of indices satisfying $1 \leq i < j \leq n-1$.

Let us first work under the assumption of $j > i+1$. An increase in $t_{i,j}$ is possible only if Kashiwara operator $\tilde{f}_{j-1}$ was used and it converted a $\boxed{j-1}_{\,i}$ into a $\boxed{j}_{\,i}$. Through an application of the tensor product rule, one can argue that the position of this action implies that the numbers of various boxes in T satisfy

$$\#\boxed{j}_{\,i+1} + \#\boxed{\overline{j-1}}_{\,i+1} < \#\boxed{\bar{j}}_{\,i+1} + \#\boxed{j-1}_{\,i}. \tag{15}$$

To actually verify this claim, one must reference the crystal structure (5), and work with the tensor product rule variant that deals with the signatures and the canceling of $(+, -)$-pairs. We clarify that the occurrence of $\tilde{f}_{j-1} : \boxed{j-1}_{\,i} \mapsto \boxed{j}_{\,i}$ implies, not only (15), but also

$\#\boxed{\overline{j-1}}_{i+1} < \#\boxed{j-1}_i$ and many other similar relations, but that the arguments for the current case only calls for (15). Now, relation (15) may be rewritten in terms of the $t_{*,*}$ values as

$$t_{i+1,j} - t_{i+1,j+1} + t_{i+1,\overline{j-1}} - t_{i+1,\overline{j-2}} + 1 \le t_{i+1,\bar{j}} - t_{i+1,\overline{j-1}} + t_{i,j-1} - t_{i,j}. \tag{16}$$

Recalling our temporary assumption of $i < j-1$, we can combine this with $\mathsf{A}[i, j-1]$ for T, which is

$$t_{i,j-1} + t_{i+1,\overline{j-2}} \le \lambda(h_i) + t_{i+1,j} + t_{i+1,\overline{j-1}}, \tag{17}$$

to conclude

$$(t_{i,j} + 1) + t_{i+1,\overline{j-1}} \le \lambda(h_i) + t_{i+1,j+1} + t_{i+1,\bar{j}}. \tag{18}$$

Similarly, in the $j = i+1$ case, an increase of $t_{i,i+1}$ must be associated with the

$$\tilde{f}_i : \boxed{i}_i \mapsto \boxed{i+1}_i \tag{19}$$

action. Recalling (13), we can argue through the tensor product rule that the position of $\tilde{f}_i$ action implies

$$\begin{aligned} \varphi_i(T) &= \bar{\varphi}_i(T) - \mathrm{shape}(T)(h_i) = (1 + t_{i+1,i+2} + t_{i+1,\overline{i+1}}) - (1 + t_{i,i+1}) \\ &= t_{i+1,i+2} + t_{i+1,\overline{i+1}} - t_{i,i+1}. \end{aligned} \tag{20}$$

On the other hand, the assumption of $\tilde{f}_i(T \otimes \mathsf{r}_\lambda)$ being nonzero implies that $\tilde{f}_i$ has acted on the first component, so that the tensor product rule implies

$$\varphi_i(T) > \varepsilon_i(\mathsf{r}_\lambda) = -\lambda(h_i). \tag{21}$$

Combining the two relations, we can claim

$$t_{i,i+1} + 1 \le \lambda(h_i) + t_{i+1,i+2} + t_{i+1,\overline{i+1}}, \tag{22}$$

which is the statement $\mathsf{A}[i, i+1]$ for $\tilde{f}_i T$.

The discussion given so far shows that $\tilde{f}_* T$ satisfies $\mathsf{A}[i, j]$, assuming $\tilde{f}_*(T \otimes \mathsf{r}_\lambda)$ is nonzero, for the case when $T \mapsto \tilde{f}_* T$ increments $t_{i,j}$. The reader that understands the arguments given above should be able to reproduce the above two proof segments from the following condensed expressions.

$\mathsf{A}[i, j]$: $t_{i,j} \uparrow \quad (j > i+1)$

1. $\tilde{f}_{j-1} : \boxed{j-1}_i \mapsto \boxed{j}_i$
2. $\#\boxed{j}_{i+1} + \#\boxed{\overline{j-1}}_{i+1} < \#\boxed{\bar{j}}_{i+1} + \#\boxed{j-1}_i$
3. $\mathsf{A}[i, j-1]$

$\mathsf{A}[i, j]$: $t_{i,j} \uparrow \quad (j = i+1)$

1. $\tilde{f}_i : \boxed{i}_i \mapsto \boxed{i+1}_i$
2. $\varphi_i(T) = t_{i+1,i+2} + t_{i+1,\overline{i+1}} - t_{i,i+1}$
3. $\varphi_i(T) > \varepsilon_i(\mathsf{r}_\lambda)$

The first line describes the targeted inequality and the situation being considered. The above two examples require one to focus on the situation where the $t_{i,j}$ appearing in $\mathsf{A}[i, j]$ is incremented through a Kashiwara operator action. This situation is associated with the more specific operation described by the first item. The position of Kashiwara operator action implies that the second item must be true. After rewriting the second item in terms of $t_{*,*}$, it can be combined with the assumption stated by the third item, and the resulting inequality will be that the targeted inequality still holds true when $t_{i,j}$ is replaced by $t_{i,j}+1$.

The rest of the proof for Lemma 6 is given below in the condensed form, with occasional brief extra explanations.

$\mathsf{A}[i,\, j]$: $t_{i+1,\overline{j-1}} \uparrow$

0. Since $t_{i+1,\bar{i}}$ is always zero, it suffices to treat just the $j > i+1$ case.
1. $\tilde{f}_{j-1} : \boxed{\bar{j}}_{\,i+1} \mapsto \boxed{\overline{j-1}}_{\,i+1}$
2. $\#\boxed{j}_{\,i+1} < \#\boxed{\bar{j}}_{\,i+1}$
3. $\mathsf{B}[i,\, j]$

$\mathsf{A}[i,\, j]$: $t_{i+1,j+1} \downarrow$

1. $\tilde{e}_{j} : \boxed{j+1}_{\,i+1} \mapsto \boxed{j}_{\,i+1}$
2. $\#\boxed{\overline{j+1}}_{\,i+1} + \#\boxed{j}_{\,i} < \#\boxed{j+1}_{\,i+1} + \#\boxed{\bar{j}}_{\,i+1}$
3. $\mathsf{A}[i,\, j+1]$

The $j < n-1$ and $j = n-1$ cases need to be handled separately, when rewriting the above second item into an inequality involving $t_{*,*}$ terms.

$\mathsf{A}[i,\, j]$: $t_{i+1,\bar{j}} \downarrow$

1. $\tilde{e}_{j} : \boxed{\bar{j}}_{\,i+1} \mapsto \boxed{\overline{j+1}}_{\,i+1}$
2. $\#\boxed{j}_{\,i} < \#\boxed{\bar{j}}_{\,i+1}$
3. $\mathsf{B}[i,\, j+1]$

$\mathsf{A}[i,\, n]$: $t_{i,n} \uparrow \quad (i < n-1)$

1. $\tilde{f}_{n-1} : \boxed{n-1}_{\,i} \mapsto \boxed{n}_{\,i}$
2. $\#\boxed{\overline{n-1}}_{\,i+1} + \#\boxed{n}_{\,i+1} < \#\boxed{n-1}_{\,i} + \#\boxed{\bar{n}}_{\,i+1}$
3. $\mathsf{A}[i,\, n-1]$

$\mathsf{A}[i,\, n]$: $t_{i,n} \uparrow \quad (i = n-1)$

1. $\tilde{f}_{n-1} : \boxed{n-1}_{\,n-1} \mapsto \boxed{n}_{\,n-1}$
2. $\varphi_{n-1}(T) = 2t_{n,\bar{n}} - t_{n-1,n}$
3. $\varphi_{n-1}(T) > \varepsilon_{n-1}(\mathsf{r}_\lambda)$

$\mathsf{A}[i,\, n]$: $t_{i+1,\overline{n-1}} \uparrow$

0. Since $t_{n,\overline{n-1}}$ is always zero, it suffices to treat just the $i < n-1$ case.
1. $\tilde{f}_{n-1} : \boxed{\bar{n}}_{\,i+1} \mapsto \boxed{\overline{n-1}}_{\,i+1}$
2. $\#\boxed{n}_{\,i+1} < \#\boxed{\bar{n}}_{\,i+1}$
3. $\mathsf{B}[i,\, n]$

$\mathsf{A}[i,\, n]$: $t_{i+1,\bar{n}} \downarrow$

1. $\tilde{e}_{n} : \boxed{\bar{n}}_{\,i+1} \mapsto \boxed{n}_{\,i+1}$
2. $\#\boxed{n}_{\,i} < \#\boxed{\bar{n}}_{\,i+1}$
3. $\mathsf{C}[i,\, n]$

The calculations for this case are slightly tricky. The second item is easily seen to be equivalent to $t_{i,n} - t_{i,\bar{n}} < t_{i+1,\bar{n}} - t_{i+1,\overline{n-1}}$, but this must be written in the form $2t_{i,n} - 2t_{i,\bar{n}} + 2t_{i+1,\overline{n-1}} \leq 2(t_{i+1,\bar{n}} - 1)$ before being combined with $\mathsf{C}[i,n]$, for one to arrive at $t_{i,n} + t_{i+1,\overline{n-1}} \leq \lambda(h_i) + 2(t_{i+1,\bar{n}} - 1)$.

$\mathsf{B}[i,j]$: $t_{i,j} \uparrow$

1. $\tilde{f}_{j-1} : \boxed{j-1}_i \mapsto \boxed{j}_i$
2. $\# \boxed{\overline{j-1}}_{i+1} < \# \boxed{j-1}_i$
3. $\mathsf{A}[i, j-1]$

$\mathsf{B}[i,j]$: $t_{i+1,j} \downarrow$

1. $\tilde{e}_{j-1} : \boxed{j}_{i+1} \mapsto \boxed{j-1}_{i+1}$
2. $\# \boxed{\bar{j}}_{i+1} < \# \boxed{j}_{i+1}$
3. $\mathsf{A}[i,j]$

Relations involving $t_{*,*}$ need to be written separately for the $j < n$ and $j = n$ cases.

$\mathsf{C}[i,j]$: $t_{i,j+1} \uparrow$

1. $\tilde{f}_j : \boxed{j}_i \mapsto \boxed{j+1}_i$
2. $\# \boxed{\bar{j}}_{i+1} < \# \boxed{j}_i$
3. $\mathsf{D}[i,j]$

$\mathsf{C}[i,j]$: $t_{i,\bar{j}} \uparrow$

1. $\tilde{f}_j : \boxed{\overline{j+1}}_i \mapsto \boxed{\bar{j}}_i$
2. $\# \boxed{j+1}_i + \# \boxed{\bar{j}}_{i+1} < \# \boxed{\overline{j+1}}_i + \# \boxed{j}_i$
3. $\mathsf{C}[i, j+1]$

Relations involving $t_{*,*}$ need to be written separately for the $j < n-1$ and $j = n-1$ cases.

$\mathsf{C}[i,j]$: $t_{i,j} \downarrow$

1. $\tilde{e}_{j-1} : \boxed{j}_i \mapsto \boxed{j-1}_i$
2. $\# \boxed{\bar{j}}_i < \# \boxed{j}_i$
3. $\mathsf{D}[i, j-1]$

$\mathsf{C}[i,j]$: $t_{i+1,\overline{j-1}} \downarrow$

0. Since $t_{i+1,\bar{i}}$ is always zero, it suffices to treat of just the $j > i+1$ case.
1. $\tilde{e}_{j-1} : \boxed{\overline{j-1}}_{i+1} \mapsto \boxed{\bar{j}}_{i+1}$
2. $\# \boxed{j-1}_i + \# \boxed{\bar{j}}_i < \# \boxed{\overline{j-1}}_{i+1} + \# \boxed{j}_i$
3. $\mathsf{C}[i, j-1]$

$\mathsf{C}[i,n]$: $t_{i,\bar{n}} \uparrow$

1. $\tilde{f}_n : \boxed{n}_i \mapsto \boxed{\bar{n}}_i$
2. $\# \boxed{\bar{n}}_{i+1} < \# \boxed{n}_i$
3. $\mathsf{A}[i,n]$

The calculations for this case are slightly tricky. The second item is easily seen to be equivalent to $t_{i+1,\bar{n}} - t_{i+1,\overline{n-1}} < t_{i,n} - t_{i,\bar{n}}$, but this must be written in the form $2(t_{i,\bar{n}} + 1) \le 2t_{i,n} - 2t_{i+1,\bar{n}} + 2t_{i+1,\overline{n-1}}$, before being combined with $\mathsf{A}[i, n]$, for one to arrive at $2(t_{i,\bar{n}} + 1) \le \lambda(h_i) + t_{i,n} + t_{i+1,\overline{n-1}}$.
$\mathsf{C}[i, n]$: $t_{i,n} \downarrow$

1. $\tilde{e}_{n-1} : \boxed{n}_i \mapsto \boxed{n-1}_i$
2. $\#\boxed{\bar{n}}_i < \#\boxed{n}_i$
3. $\mathsf{D}[i, n-1]$

$\mathsf{C}[i, n]$: $t_{i+1,\overline{n-1}} \downarrow$

0. Since $t_{n,\overline{n-1}}$ is always zero, it suffices to treat just the $i < n-1$ case.
1. $\tilde{e}_{n-1} : \boxed{\overline{n-1}}_{i+1} \mapsto \boxed{\bar{n}}_{i+1}$
2. $\#\boxed{n-1}_i + \#\boxed{\bar{n}}_i < \#\boxed{\overline{n-1}}_{i+1} + \#\boxed{n}_i$
3. $\mathsf{C}[i, n-1]$

$\mathsf{D}[i, j]$: $t_{i,\bar{j}} \uparrow$

1. $\tilde{f}_j : \boxed{\overline{j+1}}_i \mapsto \boxed{\bar{j}}_i$
2. $\#\boxed{j+1}_i < \#\boxed{\overline{j+1}}_i$
3. $\mathsf{C}[i, j+1]$

Relations involving $t_{*,*}$ need to be written separately for the $j < n-1$ and $j = n-1$ cases.
$\mathsf{D}[i, j]$: $t_{i+1,\bar{j}} \downarrow$

0. Since $t_{i+1,\bar{i}}$ is always zero, it suffices to treat just the $j > i$ case.
1. $\tilde{e}_j : \boxed{\bar{j}}_{i+1} \mapsto \boxed{\overline{j+1}}_{i+1}$
2. $\#\boxed{j}_i < \#\boxed{\bar{j}}_{i+1}$
3. $\mathsf{C}[i, j]$

$\mathsf{D}[n, n]$: $t_{n,\bar{n}} \uparrow$

1. $\tilde{f}_n : \boxed{n}_n \mapsto \boxed{\bar{n}}_n$
2. $\varphi_n(T) = -t_{n,\bar{n}}$
3. $\varphi_n(T) > \varepsilon_n(\mathsf{r}_\lambda)$

This completes the proof of Lemma 6, and our main result for the C_n type follows directly from the two lemmas we have prepared.

Theorem 7 *Let $\mathcal{T}(\infty)^\lambda$ be the set of all marginally large tableaux of C_n type that satisfy Condition-C, and let $\mathcal{T}(\infty)_\lambda = \{T \otimes \mathsf{r}_\lambda \mid T \in \mathcal{T}(\infty)^\lambda\}$. Then, $\mathcal{T}(\infty)_\lambda$ is the connected component of the crystal $\mathcal{T}(\infty) \otimes \mathsf{R}_\lambda$, containing the maximal element $T_\infty \otimes \mathsf{r}_\lambda$ of weight λ, and is hence isomorphic to the irreducible highest weight crystal $\mathcal{B}(\lambda)$.*

Proof The element $T_\infty \otimes \mathsf{r}_\lambda$ clearly belongs to $\mathcal{T}(\infty)_\lambda$, and the closedness of $\mathcal{T}(\infty)_\lambda$ under Kashiwara operator actions is ensured by Lemma 6. As for the connectedness, note that Lemma 5 implies through the tensor product rule that an $\tilde{e}_k$ action on an element of $\mathcal{T}(\infty)_\lambda$

will always be applied to the first component. Since T_∞ is the unique maximal element of $\mathcal{T}(\infty)$, every element of $\mathcal{T}(\infty)_\lambda$ must be connected to $T_\infty \otimes \mathsf{r}_\lambda$ through $\tilde{e}_k$ actions. $\square$

5 B_n Type

The set of conditions to be applied to marginally large tableaux of B_n type is as follows.

$\boxed{\text{Condition-}B}$

$$
\begin{array}{lll}
\mathsf{A}[i,j]: & t_{i,j} + t_{i+1,\bar{j}+} \le \lambda(h_i) + t_{i+1,j+} + t_{i+1,\bar{j}} & (1 \le i < j \le n) \\
\mathsf{B}[i,j]: & t_{i,j} \le \lambda(h_i) + t_{i+1,j} & (1 \le i < i+1 < j \le n) \\
\mathsf{C}[i,j]: & t_{i,j+} + t_{i,\bar{j}} \le \lambda(h_i) + t_{i,j} + t_{i+1,\bar{j}+} & (1 \le i < j \le n) \\
\mathsf{D}[i,j]: & t_{i,\bar{j}} \le \lambda(h_i) + t_{i+1,\bar{j}} & (1 \le i \le j \le n-1) \\
\mathsf{D}[n,n]: & t_{n,0} + t_{n,\bar{n}} \le \lambda(h_n) &
\end{array}
$$

Note that, as with Condition-C, conditions $\mathsf{A}[i,j]$ and $\mathsf{B}[i,j]$ for the B_n type may also be merged into

$$t_{i,j} \le \lambda(h_i) + \min\left\{ t_{i+1,j+} + \#\boxed{\bar{j}}_{\,i+1}, t_{i+1,j} \right\} \qquad (1 \le i < j \le n). \tag{23}$$

However, the combination of $\mathsf{A}[i,n]$ and $\mathsf{C}[i,n]$, results in

$$t_{i,n+} + t_{i,\bar{n}} \le 2\lambda(h_i) + t_{i+1,n+} + t_{i+1,\bar{n}}, \quad (1 \le i \le n-1), \tag{24}$$

which is not compatible with $\mathsf{D}[n,n]$, so that $\mathsf{C}[i,j]$ and $\mathsf{D}[i,j]$ cannot be combined into a single expression as naturally as in the C_n case.

Understanding the detailed proofs to be given below will be easier, when the following slightly longer presentation of Condition-B is referenced.

$$
\begin{array}{lll}
\mathsf{A}[i,j]: & t_{i,j} + t_{i+1,\overline{j-1}} \le \lambda(h_i) + t_{i+1,j+1} + t_{i+1,\bar{j}} & (1 \le i < j \le n-1) \\
\mathsf{A}[i,n]: & t_{i,n} + t_{i+1,\overline{n-1}} \le \lambda(h_i) + t_{i+1,0} + t_{i+1,\bar{n}} & (1 \le i \le n-1) \\
\mathsf{B}[i,j]: & t_{i,j} \le \lambda(h_i) + t_{i+1,j} & (1 \le i < i+1 < j \le n) \\
\mathsf{C}[i,j]: & t_{i,j+1} + t_{i,\bar{j}} \le \lambda(h_i) + t_{i,j} + t_{i+1,\overline{j-1}} & (1 \le i < j \le n-1) \\
\mathsf{C}[i,n]: & t_{i,0} + t_{i,\bar{n}} \le \lambda(h_i) + t_{i,n} + t_{i+1,\overline{n-1}} & (1 \le i \le n-1) \\
\mathsf{D}[i,j]: & t_{i,\bar{j}} \le \lambda(h_i) + t_{i+1,\bar{j}} & (1 \le i \le j \le n-1) \\
\mathsf{D}[n,n]: & t_{n,0} + t_{n,\bar{n}} \le \lambda(h_n) &
\end{array}
$$

As in the previous section, we provide two lemmas from which our main result for the B_n type follows.

Lemma 8 *If a marginally large tableau T of B_n type satisfies* Condition-B, *then $\varphi_i(T) \ge \varepsilon_i(\mathsf{r}_\lambda)$, for all $i \in I$.*

Proof Note that $\varepsilon_n(\mathsf{r}_\lambda) = -\lambda(h_n)$ and that

$$\varphi_n(T) = \bar{\varphi}_n(T) - \mathrm{shape}(T)(h_n) \ge (2 + t_{n,0} - t_{n,\bar{n}}) - 2(1 + t_{n,0}) = -(t_{n,0} + t_{n,\bar{n}}),$$

so that $\mathsf{D}[n,n]$ implies $\varphi_n(T) \ge \varepsilon_n(\mathsf{r}_\lambda)$. Similarly, the fact $\varepsilon_{n-1}(\mathsf{r}_\lambda) = -\lambda(h_{n-1})$ and the observation

$$\varphi_{n-1}(T) = \bar{\varphi}_{n-1}(T) - \mathrm{shape}(T)(h_{n-1}) \ge (1 + t_{n,0} + t_{n,\bar{n}}) - (1 + t_{n-1,n}) = t_{n,0} + t_{n,\bar{n}} - t_{n-1,n}$$

can be combined with $\mathsf{A}[n-1, n]$, which is $t_{n-1,n} \leq \lambda(h_{n-1}) + t_{n,0} + t_{n,\bar{n}}$, to bring out $\varphi_{n-1}(T) \geq \varepsilon_{n-1}(\mathsf{r}_\lambda)$. For the remaining $1 \leq i \leq n-2$ cases, the final sentence of the proof of Lemma 5 holds true, word for word. In all cases, we have $\varphi_i(T) \geq \varepsilon_i(\mathsf{r}_\lambda)$. □

The next lemma provides the closedness of Condition-B under the Kashiwara operator actions.

Lemma 9 *Fix a $\lambda \in P^+$ and let T be a marginally large tableau of B_n type that satisfies* Condition-B. *If $\tilde{f}_k(T \otimes \mathsf{r}_\lambda)$ is nonzero, then the marginally large tableau $\tilde{f}_k T$ satisfies* Condition-B. *Similarly, if $\tilde{e}_k(T \otimes \mathsf{r}_\lambda)$ is nonzero, then the marginally large tableau $\tilde{e}_k T$ satisfies* Condition-B.

As in the C_n case, the proof of this lemma consists of a case by case verification of whether $\tilde{f}_k T$ and $\tilde{e}_k T$ satisfy every sub-condition of Condition-B. However, the proof details for many of the cases are identical to the corresponding cases previously discussed for the C_n type.

$\mathsf{A}[i, j]$: $t_{i,j} \uparrow \quad (j > i + 1)$

Proof is identical to the corresponding C_n case.

$\mathsf{A}[i, j]$: $t_{i,j} \uparrow \quad (j = i + 1)$

Proof is identical to the corresponding C_n case.

$\mathsf{A}[i, j]$: $t_{i+1,\overline{j-1}} \uparrow$

Proof is identical to the corresponding C_n case.

$\mathsf{A}[i, j]$: $t_{i+1,j+1} \downarrow$

Proof is identical to the corresponding C_n case.

$\mathsf{A}[i, j]$: $t_{i+1,\bar{j}} \downarrow$

Proof is identical to the corresponding C_n case.

$\mathsf{A}[i, n]$: $t_{i,n} \uparrow \quad (i < n - 1)$

Proof is identical to the corresponding C_n case.

$\mathsf{A}[i, n]$: $t_{i,n} \uparrow \quad (i = n - 1)$

1. $\tilde{f}_{n-1} : \boxed{n-1}_{\,n-1} \mapsto \boxed{n}_{\,n-1}$
2. $\varphi_{n-1}(T) = t_{n,0} + t_{n,\bar{n}} - t_{n-1,n}$
3. $\varphi_{n-1}(T) > \varepsilon_{n-1}(\mathsf{r}_\lambda)$

$\mathsf{A}[i, n]$: $t_{i+1,\overline{n-1}} \uparrow$

Proof is identical to the corresponding C_n case.

$\mathsf{A}[i, n]$: $t_{i+1,0} \downarrow$

1. $\tilde{e}_n : \boxed{0}_{\,i+1} \mapsto \boxed{n}_{\,i+1}$
2. $2\#\boxed{n}_{\,i} + \#\boxed{0}_{\,i} < \#\boxed{0}_{\,i+1} + 2\#\boxed{\bar{n}}_{\,i+1}$ and $\#\boxed{0}_{\,i+1} = 1$
3. $\mathsf{C}[i, n]$

$\mathsf{A}[i, n]$: $t_{i+1,\bar{n}} \downarrow$

1. $\tilde{e}_n : \boxed{\bar{n}}_{\,i+1} \mapsto \boxed{0}_{\,i+1}$
2. $2\#\boxed{n}_{\,i} + \#\boxed{0}_{\,i} < \#\boxed{0}_{\,i+1} + 2\#\boxed{\bar{n}}_{\,i+1}$ and $\#\boxed{0}_{\,i+1} = 0$
3. $\mathsf{C}[i, n]$

$\mathsf{B}[i, j]$: $t_{i,j} \uparrow$

Proof is identical to the corresponding C_n case.

$\mathsf{B}[i, j]$: $t_{i+1,j} \downarrow$
Proof is identical to the corresponding C_n case.
$\mathsf{C}[i, j]$: $t_{i,j+1} \uparrow$
Proof is identical to the corresponding C_n case.
$\mathsf{C}[i, j]$: $t_{i,\bar{j}} \uparrow$
Proof is identical to the corresponding C_n case.
$\mathsf{C}[i, j]$: $t_{i,j} \downarrow$
Proof is identical to the corresponding C_n case.
$\mathsf{C}[i, j]$: $t_{i+1,\overline{j-1}} \downarrow$
Proof is identical to the corresponding C_n case.
$\mathsf{C}[i, n]$: $t_{i,0} \uparrow$

1. $\tilde{f}_n : \boxed{n}_i \mapsto \boxed{0}_i$
2. $\#\boxed{0}_{i+1} + 2\#\boxed{\bar{n}}_{i+1} < 2\#\boxed{n}_i + \#\boxed{0}_i \quad \text{and} \quad \#\boxed{0}_i = 0$
3. $\mathsf{A}[i, n]$

$\mathsf{C}[i, n]$: $t_{i,\bar{n}} \uparrow$

1. $\tilde{f}_n : \boxed{0}_i \mapsto \boxed{\bar{n}}_i$
2. $\#\boxed{0}_{i+1} + 2\#\boxed{\bar{n}}_{i+1} < 2\#\boxed{n}_i + \#\boxed{0}_i \quad \text{and} \quad \#\boxed{0}_i = 1$
3. $\mathsf{A}[i, n]$

$\mathsf{C}[i, n]$: $t_{i,n} \downarrow$
Proof is identical to the corresponding C_n case.
$\mathsf{C}[i, n]$: $t_{i+1,\overline{n-1}} \downarrow$
Proof is identical to the corresponding C_n case.
$\mathsf{D}[i, j]$: $t_{i,\bar{j}} \uparrow$
Proof is identical to the corresponding C_n case.
$\mathsf{D}[i, j]$: $t_{i+1,\bar{j}} \downarrow$
Proof is identical to the corresponding C_n case.
$\mathsf{D}[n, n]$: $t_{n,0} \uparrow$

1. $\tilde{f}_n : \boxed{n}_n \mapsto \boxed{0}_n$
2. $\varphi_n(T) = -(t_{n,0} + t_{n,\bar{n}}) \quad \text{and} \quad \#\boxed{0}_n = 0$
3. $\varphi_n(T) > \varepsilon_n(\mathsf{r}_\lambda)$

$\mathsf{D}[n, n]$: $t_{n,\bar{n}} \uparrow$

1. $\tilde{f}_n : \boxed{0}_n \mapsto \boxed{\bar{n}}_n$
2. $\varphi_n(T) = -(t_{n,0} + t_{n,\bar{n}}) \quad \text{and} \quad \#\boxed{0}_n = 1$
3. $\varphi_n(T) > \varepsilon_n(\mathsf{r}_\lambda)$

This completes the proof of Lemma 9. As before, our main result for the B_n type follows directly from the previous two lemmas, with a proof identical to that of Theorem 7.

Theorem 10 *Let $\mathcal{T}(\infty)^\lambda$ be the set of all marginally large tableaux of B_n type that satisfy Condition-B, and let $\mathcal{T}(\infty)_\lambda = \{T \otimes \mathsf{r}_\lambda \,|\, T \in \mathcal{T}(\infty)^\lambda\}$. Then, $\mathcal{T}(\infty)_\lambda$ is the connected component of the crystal $\mathcal{T}(\infty) \otimes \mathsf{R}_\lambda$, containing the maximal element $T_\infty \otimes \mathsf{r}_\lambda$ of weight λ, and is hence isomorphic to the irreducible highest weight crystal $\mathcal{B}(\lambda)$.*

6 D_{n+1} Type

The set of conditions to be applied to marginally large tableaux of D_{n+1} type is as follows.
$\boxed{\text{Condition-}D}$

$$
\begin{array}{rll}
\mathsf{A}[i,j]: & t_{i,j}+t_{i+1,\overline{j-1}} \le \lambda(h_i)+t_{i+1,j+1}+t_{i+1,\bar{j}} & (1\le i<j\le n-1)\\
\mathsf{A}[i,n]: & t_{i,n}+t_{i+1,\overline{n-1}} \le \lambda(h_i)+t_{i+1,n+1}+t_{i+1,\overline{n+1}} & (1\le i\le n-1)\\
\mathsf{B}[i,j]: & t_{i,j}\le \lambda(h_i)+t_{i+1,j} & (1\le i<i+1<j\le n)\\
\mathsf{C}[i,j]: & t_{i,j+1}+t_{i,\bar{j}}\le \lambda(h_i)+t_{i,j}+t_{i+1,\overline{j-1}} & (1\le i<j\le n-1)\\
\mathsf{C}[i,n]: & t_{i,n+1}+t_{i,\overline{n+1}}\le \lambda(h_i)+t_{i,n}+t_{i+1,\overline{n-1}} & (1\le i\le n-1)\\
\mathsf{D}[i,j]: & t_{i,\bar{j}}\le \lambda(h_i)+t_{i+1,\bar{j}} & (1\le i\le j\le n-1)\\
\mathsf{E}[i,n+1]: & t_{i,n+1}\le \lambda(h_i)+t_{i+1,\overline{n+1}} & (1\le i\le n-1)\\
\mathsf{E}[n,n+1]: & t_{n,n+1}\le \lambda(h_n) & \\
\mathsf{F}[i,n+1]: & t_{i,\overline{n+1}}\le \lambda(h_i)+t_{i+1,n+1} & (1\le i\le n-1)\\
\mathsf{F}[n,n+1]: & t_{n,\overline{n+1}}\le \lambda(h_{n+1}) &
\end{array}
$$

Let us provide the two lemmas from which our main result for the D_{n+1} type will follow.

Lemma 11 *If a marginally large tableau T of D_{n+1} type satisfies* Condition-D*, then $\varphi_i(T)\ge\varepsilon_i(\mathsf{r}_\lambda)$, for all $i\in I$.*

Proof The observations

$$\varphi_{n+1}(T)\ge(1+\#\boxed{n+1}_{\,n})-(1+\#\boxed{n+1}_{\,n}+\#\boxed{\overline{n+1}}_{\,n}+t_{n,\bar{n}})=-t_{n,\overline{n+1}}$$
$$\varphi_{n}(T)\ge(1+\#\boxed{\overline{n+1}}_{\,n})-(1+\#\boxed{n+1}_{\,n}+\#\boxed{\overline{n+1}}_{\,n}+t_{n,\bar{n}})=-t_{n,n+1}$$

may be combined with $\mathsf{F}[n,n+1]$ and $\mathsf{E}[n,n+1]$ to show $\varphi_{n+1}(T)\ge\varepsilon_{n+1}(\mathsf{r}_\lambda)$ and $\varphi_n(T)\ge\varepsilon_n(\mathsf{r}_\lambda)$, respectively. The observation

$$\varphi_{n-1}(T)\ge(1+t_{n,n+1}+t_{n,\overline{n+1}})-(1+t_{n-1,n})=t_{n,n+1}+t_{n,\overline{n+1}}-t_{n-1,n}$$

may be combined with $\mathsf{A}[n-1,n]$ to show $\varphi_{n-1}(T)\ge\varepsilon_{n-1}(\mathsf{r}_\lambda)$. For the remaining $1\le i\le n-2$ cases, the final sentence of the proof of Lemma 5 holds true, word for word. In all cases, we have $\varphi_i(T)\ge\varepsilon_i(\mathsf{r}_\lambda)$. □

The closedness of Condition-D under the Kashiwara operator actions is given next.

Lemma 12 *Fix a $\lambda\in P^+$ and let T be a marginally large tableau of D_{n+1} type that satisfies* Condition-D*. If $\tilde{f}_k(T\otimes\mathsf{r}_\lambda)$ is nonzero, then the marginally large tableau $\tilde{f}_kT$ satisfies* Condition-D*. Similarly, if $\tilde{e}_k(T\otimes\mathsf{r}_\lambda)$ is nonzero, then the marginally large tableau $\tilde{e}_kT$ satisfies* Condition-D*.*

Proofs for some of the cases are identical to the corresponding proofs for the C_n type.

$\mathsf{A}[i,j]$: $t_{i,j}\uparrow$ $\quad(j>i+1)$

Proof is identical to the corresponding C_n case.

$\mathsf{A}[i,j]$: $t_{i,j}\uparrow$ $\quad(j=i+1)$

Proof is identical to the corresponding C_n case.

$\mathsf{A}[i, j]$: $t_{i+1,\overline{j-1}} \uparrow$
Proof is identical to the corresponding C_n case.
$\mathsf{A}[i, j]$: $t_{i+1,j+1} \downarrow$
Proof is identical to the corresponding C_n case.
$\mathsf{A}[i, j]$: $t_{i+1,\bar{j}} \downarrow$
Proof is identical to the corresponding C_n case.
$\mathsf{A}[i, n]$: $t_{i,n} \uparrow \quad (i < n-1)$
Proof is identical to the corresponding C_n case.
$\mathsf{A}[i, n]$: $t_{i,n} \uparrow \quad (i = n-1)$

1. $\tilde{f}_{n-1} : \boxed{n-1}_{n-1} \mapsto \boxed{n}_{n-1}$
2. $\varphi_{n-1}(T) = t_{n,n+1} + t_{n,\overline{n+1}} - t_{n-1,n}$
3. $\varphi_{n-1}(T) > \varepsilon_{n-1}(\mathsf{r}_\lambda)$

$\mathsf{A}[i, n]$: $t_{i+1,\overline{n-1}} \uparrow$
Proof is identical to the corresponding C_n case.
$\mathsf{A}[i, n]$: $t_{i+1,n+1} \downarrow \quad (\#\boxed{n+1}_{i+1} > 0)$

1. $\tilde{e}_n : \boxed{n+1}_{i+1} \mapsto \boxed{n}_{i+1}$
2. $\#\boxed{n}_i + \#\boxed{\overline{n+1}}_i < \#\boxed{n+1}_{i+1} + \#\boxed{\bar{n}}_{i+1}$
3. $\mathsf{E}[i, n+1]$

$\mathsf{A}[i, n]$: $t_{i+1,n+1} = t_{i+1,\bar{n}} \downarrow \quad (\#\boxed{n+1}_{i+1} = 0)$

1. $\tilde{e}_n : \boxed{\bar{n}}_{i+1} \mapsto \boxed{\overline{n+1}}_{i+1}$
2. $\#\boxed{n}_i + \#\boxed{\overline{n+1}}_i < \#\boxed{\bar{n}}_{i+1}$
3. $\mathsf{E}[i, n+1]$

$\mathsf{A}[i, n]$: $t_{i+1,\overline{n+1}} \downarrow \quad (\#\boxed{\overline{n+1}}_{i+1} > 0)$

1. $\tilde{e}_{n+1} : \boxed{\overline{n+1}}_{i+1} \mapsto \boxed{n}_{i+1}$
2. $\#\boxed{n}_i + \#\boxed{n+1}_i < \#\boxed{\overline{n+1}}_{i+1} + \#\boxed{\bar{n}}_{i+1}$
3. $\mathsf{F}[i, n+1]$

$\mathsf{A}[i, n]$: $t_{i+1,\overline{n+1}} = t_{i+1,\bar{n}} \downarrow \quad (\#\boxed{\overline{n+1}}_{i+1} = 0)$

1. $\tilde{e}_{n+1} : \boxed{\bar{n}}_{i+1} \mapsto \boxed{n+1}_{i+1}$
2. $\#\boxed{n}_i + \#\boxed{n+1}_i < \#\boxed{\bar{n}}_{i+1}$
3. $\mathsf{F}[i, n+1]$

$\mathsf{B}[i, j]$: $t_{i,j} \uparrow$
Proof is identical to the corresponding C_n case.
$\mathsf{B}[i, j]$: $t_{i+1,j} \downarrow$
Proof is identical to the corresponding C_n case.
$\mathsf{C}[i, j]$: $t_{i,j+1} \uparrow$
Proof is identical to the corresponding C_n case.
$\mathsf{C}[i, j]$: $t_{i,\bar{j}} \uparrow$
Proof is identical to the corresponding C_n case.

C$[i, j]$: $t_{i,j} \downarrow$
Proof is identical to the corresponding C_n case.

C$[i, j]$: $t_{i+1,\overline{j-1}} \downarrow$
Proof is identical to the corresponding C_n case.

C$[i, n]$: $t_{i,n+1} \uparrow \quad (\#\boxed{\overline{n+1}}_i = 0)$

1. $\tilde{f}_n : \boxed{n}_i \mapsto \boxed{n+1}_i$
2. $\#\boxed{n+1}_{i+1} + \#\boxed{\bar{n}}_{i+1} < \#\boxed{n}_i = t_{i,n} - t_{i,n+1}$
3. F$[i, n+1]$

C$[i, n]$: $t_{i,n+1} = t_{i,\bar{n}} \uparrow \quad (\#\boxed{\overline{n+1}}_i > 0)$

1. $\tilde{f}_n : \boxed{\overline{n+1}}_i \mapsto \boxed{\bar{n}}_i$
2. $\#\boxed{n+1}_{i+1} + \#\boxed{\bar{n}}_{i+1} < \#\boxed{n}_i + \#\boxed{\overline{n+1}}_i = t_{i,n} - t_{i,\bar{n}}$
3. F$[i, n+1]$

C$[i, n]$: $t_{i,\overline{n+1}} \uparrow \quad (\#\boxed{n+1}_i = 0)$

1. $\tilde{f}_{n+1} : \boxed{n}_i \mapsto \boxed{\overline{n+1}}_i$
2. $\#\boxed{\overline{n+1}}_{i+1} + \#\boxed{\bar{n}}_{i+1} < \#\boxed{n}_i = t_{i,n} - t_{i,\overline{n+1}}$
3. E$[i, n+1]$

C$[i, n]$: $t_{i,\overline{n+1}} = t_{i,\bar{n}} \uparrow \quad (\#\boxed{n+1}_i > 0)$

1. $\tilde{f}_{n+1} : \boxed{n+1}_i \mapsto \boxed{\bar{n}}_i$
2. $\#\boxed{\overline{n+1}}_{i+1} + \#\boxed{\bar{n}}_{i+1} < \#\boxed{n}_i + \#\boxed{n+1}_i = t_{i,n} - t_{i,\bar{n}}$
3. E$[i, n+1]$

C$[i, n]$: $t_{i,n} \downarrow$
Proof is identical to the corresponding C_n case.

C$[i, n]$: $t_{i+1,\overline{n-1}} \downarrow$
Proof is identical to the corresponding C_n case.

D$[i, j]$: $t_{i,\bar{j}} \uparrow$
Proof is identical to the corresponding C_n case.

D$[i, j]$: $t_{i+1,\bar{j}} \downarrow$
Proof is identical to the corresponding C_n case.

E$[i, n+1]$: $t_{i,n+1} \uparrow \quad (\#\boxed{\overline{n+1}}_i = 0)$

1. $\tilde{f}_n : \boxed{n}_i \mapsto \boxed{n+1}_i$
2. $\#\boxed{n+1}_{i+1} + \#\boxed{\bar{n}}_{i+1} < \#\boxed{n}_i = t_{i,n} - t_{i,n+1}$
3. A$[i, n]$

E$[i, n+1]$: $t_{i,n+1} = t_{i,\bar{n}} \uparrow \quad (\#\boxed{\overline{n+1}}_i > 0)$

1. $\tilde{f}_n : \boxed{\overline{n+1}}_i \mapsto \boxed{\bar{n}}_i$
2. $\#\boxed{n+1}_{i+1} + \#\boxed{\bar{n}}_{i+1} < \#\boxed{n}_i + \#\boxed{\overline{n+1}}_i = t_{i,n} - t_{i,\bar{n}}$

3. $\mathsf{A}[i,n]$

$\mathsf{E}[i,n+1]$: $t_{i+1,\overline{n+1}} \downarrow \quad (\#\boxed{\overline{n+1}}_{\,i+1} > 0)$

1. $\tilde{e}_{n+1} : \boxed{\overline{n+1}}_{\,i+1} \mapsto \boxed{n}_{\,i+1}$
2. $\#\boxed{n}_{\,i} + \#\boxed{n+1}_{\,i} < \#\boxed{\overline{n+1}}_{\,i+1} + \#\boxed{\bar{n}}_{\,i+1}$
3. $\mathsf{C}[i,n]$

$\mathsf{E}[i,n+1]$: $t_{i+1,\overline{n+1}} = t_{i+1,\bar{n}} \downarrow \quad (\#\boxed{\overline{n+1}}_{\,i+1} = 0)$

1. $\tilde{e}_{n+1} : \boxed{\bar{n}}_{\,i+1} \mapsto \boxed{n+1}_{\,i+1}$
2. $\#\boxed{n}_{\,i} + \#\boxed{n+1}_{\,i} < \#\boxed{\bar{n}}_{\,i+1}$
3. $\mathsf{C}[i,n]$

$\mathsf{E}[n,n+1]$: $t_{n,n+1} \uparrow \quad (\#\boxed{\overline{n+1}}_{\,n} = 0)$

1. $\tilde{f}_n : \boxed{n}_{\,n} \mapsto \boxed{n+1}_{\,n}$
2. $\varphi_n(T) = -t_{n,n+1}$
3. $\varphi_n(\bar{T}) > \varepsilon_n(\mathsf{r}_\lambda)$

$\mathsf{E}[n,n+1]$: $t_{n,n+1} = t_{n,\bar{n}} \uparrow \quad (\#\boxed{\overline{n+1}}_{\,n} > 0)$

1. $\tilde{f}_n : \boxed{\overline{n+1}}_{\,n} \mapsto \boxed{\bar{n}}_{\,n}$
2. $\varphi_n(T) = -t_{n,n+1}$
3. $\varphi_n(\bar{T}) > \varepsilon_n(\mathsf{r}_\lambda)$

$\mathsf{F}[i,n+1]$: $t_{i,\overline{n+1}} \uparrow \quad (\#\boxed{n+1}_{\,i} = 0)$

1. $\tilde{f}_{n+1} : \boxed{n}_{\,i} \mapsto \boxed{\overline{n+1}}_{\,i}$
2. $\#\boxed{\overline{n+1}}_{\,i+1} + \#\boxed{\bar{n}}_{\,i+1} < \#\boxed{n}_{\,i} = t_{i,n} - t_{i,\overline{n+1}}$
3. $\mathsf{A}[i,n]$

$\mathsf{F}[i,n+1]$: $t_{i,\overline{n+1}} = t_{i,\bar{n}} \uparrow \quad (\#\boxed{n+1}_{\,i} > 0)$

1. $\tilde{f}_{n+1} : \boxed{n+1}_{\,i} \mapsto \boxed{\bar{n}}_{\,i}$
2. $\#\boxed{\overline{n+1}}_{\,i+1} + \#\boxed{\bar{n}}_{\,i+1} < \#\boxed{n}_{\,i} + \#\boxed{n+1}_{\,i} = t_{i,n} - t_{i,\bar{n}}$
3. $\mathsf{A}[i,n]$

$\mathsf{F}[i,n+1]$: $t_{i+1,n+1} \downarrow \quad (\#\boxed{n+1}_{\,i+1} > 0)$

1. $\tilde{e}_n : \boxed{n+1}_{\,i+1} \mapsto \boxed{n}_{\,i+1}$
2. $\#\boxed{n}_{\,i} + \#\boxed{\overline{n+1}}_{\,i} < \#\boxed{n+1}_{\,i+1} + \#\boxed{\bar{n}}_{\,i+1}$
3. $\mathsf{C}[i,n]$

$\mathsf{F}[i,n+1]$: $t_{i+1,n+1} = t_{i+1,\bar{n}} \downarrow \quad (\#\boxed{n+1}_{\,i+1} = 0)$

1. $\tilde{e}_n : \boxed{\bar{n}}_{\,i+1} \mapsto \boxed{\overline{n+1}}_{\,i+1}$

2. $\#\boxed{n}_i + \#\boxed{\overline{n+1}}_i < \#\boxed{\bar{n}}_{i+1}$
3. $\mathsf{C}[i, n]$

$\mathsf{F}[n, n+1]$: $t_{n,\overline{n+1}} \uparrow \quad (\#\boxed{n+1}_n = 0)$

1. $\tilde{f}_{n+1} : \boxed{n}_n \mapsto \boxed{\overline{n+1}}_n$
2. $\varphi_{n+1}(\bar{T}) = -t_{n,\overline{n+1}}$
3. $\varphi_{n+1}(\bar{T}) > \varepsilon_{n+1}(\mathsf{r}_\lambda)$

$\mathsf{F}[n, n+1]$: $t_{n,\overline{n+1}} = t_{n,\bar{n}} \uparrow \quad (\#\boxed{n+1}_n > 0)$

1. $\tilde{f}_{n+1} : \boxed{n+1}_n \mapsto \boxed{\bar{n}}_n$
2. $\varphi_{n+1}(\bar{T}) = -t_{n,\overline{n+1}}$
3. $\varphi_{n+1}(\bar{T}) > \varepsilon_{n+1}(\mathsf{r}_\lambda)$

This completes the proof of Lemma 12, and we can state our main result for the D_{n+1} type.

Theorem 13 *Let $\mathcal{T}(\infty)^\lambda$ be the set of all marginally large tableaux of D_{n+1} type that satisfy Condition-D, and let $\mathcal{T}(\infty)_\lambda = \{T \otimes \mathsf{r}_\lambda \mid T \in \mathcal{T}(\infty)^\lambda\}$. Then, $\mathcal{T}(\infty)_\lambda$ is the connected component of the crystal $\mathcal{T}(\infty) \otimes \mathsf{R}_\lambda$, containing the maximal element $T_\infty \otimes \mathsf{r}_\lambda$ of weight λ, and is hence isomorphic to the irreducible highest weight crystal $\mathcal{B}(\lambda)$.*

7 Description of $\mathcal{B}(\lambda)$ Through the Kashiwara Embedding

In the previous sections, we realized $\mathcal{B}(\lambda)$ by describing the connected component within the crystal graphs of $\mathcal{T}(\infty) \otimes \mathsf{R}_\lambda$ that contains the element $T_\infty \otimes \mathsf{r}_\lambda$. In this section, we give another realization of $\mathcal{B}(\lambda)$ by expressing it as a subset of the image of a Kashiwara embedding for $\mathcal{B}(\infty)$. There will be a trivial correspondence between this subset and our previous connected component.

Kashiwara [13] showed the existence of a unique strict crystal embedding

$$\Psi_\iota : \mathcal{B}(\infty) \hookrightarrow \mathcal{B}(\infty) \otimes \mathcal{B}_{i_k} \otimes \mathcal{B}_{i_{k-1}} \otimes \cdots \otimes \mathcal{B}_{i_1}, \tag{25}$$

for certain sequences $\iota = (i_k, i_{k-1}, \ldots, i_1)$ of indices from I, and we will refer to this as the Kashiwara embedding. Here, the crystals $\mathcal{B}_i = \{b_i(m) \mid m \in \mathbf{Z}\}$, appearing on the righthand side, are defined for each $i \in I$. The Kashiwara operators $\tilde{f}_i$ and $\tilde{e}_i$ act on elements of $\mathcal{B}_i$ by decrementing or incrementing the inner index m, and operators $\tilde{f}_j$ and $\tilde{e}_j$ of non-matching indices $j \neq i$ map every element of $\mathcal{B}_i$ to zero. The exact crystal structure will not be required for our discussion.

Cliff [4] fixed an explicit choice of sequence ι for each of the classical Lie algebra types and described the full image of the Kashiwara embedding Ψ_ι to give a realization of $\mathcal{B}(\infty)$. The notion of large tableaux for the classical types, which the work introduced, took an important part in this description. To express the image $\Psi_\iota(b)$ for an element $b \in \mathcal{B}(\infty)$, the crystal element b was first represented as a large tableau and a set of integers that give the numbers of certain boxes appearing in this tableau were defined. Then, the element $\Psi_\iota(b)$ could be written in the form

$$b_\infty \otimes b_{i_k}(m_k) \otimes \cdots \otimes b_{i_1}(m_1), \tag{26}$$

where the m_i values were determined by the set of box-count integers. Finally, since the first tensor product term b_∞ of $\Psi_\iota(b)$ was always fixed, the image $\Psi_\iota\big(\mathcal{B}(\infty)\big)$ could be described through a set of relations concerning the m_i indices.

The tableau representation of $b \in \mathcal{B}(\infty)$ used by [4] in the process described above is essentially identical to that used by [7] for the tableau realization $\mathcal{T}(\infty)$ of $\mathcal{B}(\infty)$. In fact, when the differences contained in the details are ignored, the relationship between the two works can be described as follows. The former work [4] was satisfied with using any choice of large tableau for each $b \in \mathcal{B}(\infty)$ and used the tableau representation only as a tool for expressing $\Psi_\iota(b)$. On the other hand, the later work [7] focused on the tableau representations themselves, specified for the *marginally* large tableau to be used for each b, gathered them into one set $\mathcal{T}(\infty)$, and gave the set a crystal structure.

Once this connection between [4] and [7] is understood, the result of [4] that realizes $\mathcal{B}(\infty)$ as the image of the Kashiwara embedding Ψ_ι can easily be rewritten in the language of this paper. Below, we give concise presentations of these for the three classical types considered in this paper.

- C_n type
  1. $\iota = (\iota_1, \dots, \iota_{n-1}, \iota_n)$ with $\iota_i = (i, \dots, n-1, n, n-1, \dots, i)$
  2. $\beta_i = b_i(-t_{i,\bar{i}}) \otimes \cdots \otimes b_{n-1}(-t_{i,\overline{n-1}}) \otimes b_n(-t_{i,\bar{n}}) \otimes b_{n-1}(-t_{i,n}) \otimes b_{n-2}(-t_{i,n-1}) \otimes \cdots \otimes b_i(-t_{i,i+1})$
  3. $0 \le t_{i,\bar{i}} \le t_{i,\overline{i+1}} \le \cdots \le t_{i,\bar{n}} \le t_{i,n} \le t_{i,n-1} \le \cdots \le t_{i,i+1}$
- B_n type
  1. $\iota = (\iota_1, \dots, \iota_{n-1}, \iota_n)$ with $\iota_i = (i, \dots, n-1, n, n-1, \dots, i)$
  2. $\beta_i = b_i(-t_{i,\bar{i}}) \otimes \cdots \otimes b_{n-1}(-t_{i,\overline{n-1}}) \otimes b_n(-(t_{i,0} + t_{i,\bar{n}})) \otimes b_{n-1}(-t_{i,n}) \otimes b_{n-2}(-t_{i,n-1}) \otimes \cdots \otimes b_i(-t_{i,i+1})$
  3. $0 \le t_{i,\bar{i}} \le t_{i,\overline{i+1}} \le \cdots \le t_{i,\overline{n-1}} \le \frac{t_{i,0}+t_{i,\bar{n}}}{2} \le t_{i,n} \le t_{i,n-1} \le \cdots \le t_{i,i+1}$
- D_{n+1} type
  1. $\iota = (\iota_1, \dots, \iota_{n-1}, \iota_n)$ with $\iota_i = (i, \dots, n-1, n+1, n, n-1, \dots, i)$ and $\iota_n = (n+1, n)$
  2. $\beta_i = b_i(-t_{i,\bar{i}}) \otimes \cdots \otimes b_{n-2}(-t_{i,\overline{n-2}}) \otimes b_{n-1}(-t_{i,\overline{n-1}}) \otimes b_{n+1}(-t_{i,\overline{n+1}}) \otimes b_n(-t_{i,n+1}) \otimes b_{n-1}(-t_{i,n}) \otimes \cdots \otimes b_i(-t_{i,i+1})$ and $\beta_n = b_{n+1}(-t_{n,\overline{n+1}}) \otimes b_n(-t_{n,n+1})$
  3. $0 \le t_{i,\bar{i}} \le \cdots \le t_{i,\overline{n-1}} \le t_{i,\bar{n}} = \min\big(t_{i,n+1}, t_{i,\overline{n+1}}\big) \le \max\big(t_{i,n+1}, t_{i,\overline{n+1}}\big) \le t_{i,n} \le \cdots \le t_{i,i+1}$ and $t_{n,\bar{n}} = \min\big(t_{n,n+1}, t_{n,\overline{n+1}}\big) \le \max\big(t_{n,n+1}, t_{n,\overline{n+1}}\big)$

Some explanations need to be given. The first item for each Lie algebra type gives the sequence ι, broken into sub-sequences ι_i, that were used by [4]. For any sequence η of indices from I, there is a corresponding product of crystals $\mathcal{B}_k$ that could naturally be denoted as $\mathcal{B}_\eta$. Each β_i given by the second items should be interpreted as an element of the crystal product $\mathcal{B}_{\iota_i}$. Finally, the image $\Psi_\iota\big(\mathcal{B}(\infty)\big)$ for each Lie algebra type is being stated as the set of elements $b_\infty \otimes \beta_1 \otimes \beta_2 \otimes \cdots \otimes \beta_n$ satisfying the conditions listed by the corresponding third item.

The following claim, written with the notation explained in this section, is now a direct corollary to Theorem 7, Theorem 10, and Theorem 13.

Corollary 14 *The image of the connected component in $\mathcal{B}(\infty) \otimes \mathsf{R}_\lambda$ containing $b_\infty \otimes \mathsf{r}_\lambda$, under the strict crystal embedding*

$$\Psi_\iota \otimes \mathrm{id} : \mathcal{B}(\infty) \otimes \mathsf{R}_\lambda \hookrightarrow \mathcal{B}(\infty) \otimes \mathcal{B}_{\iota_1} \otimes \mathcal{B}_{\iota_2} \otimes \cdots \otimes \mathcal{B}_{\iota_n} \otimes \mathsf{R}_\lambda,$$

is the set of elements $b_\infty \otimes \beta_1 \otimes \beta_2 \otimes \cdots \otimes \beta_n \otimes \mathsf{r}_\lambda$, where each β_i satisfies one of the three third items previously listed and one of Condition-*C*, Condition-*B*, *and* Condition-*D*, *as is appropriate for the Lie algebra type under consideration.*

Since the connected component mentioned in this claim is isomorphic to $\mathcal{B}(\lambda)$, we have obtained another description of crystal $\mathcal{B}(\lambda)$.

8 Discussion

Let us briefly explain how the various conditions that specify our realizations of $\mathcal{B}(\lambda)$ were obtained.

The most direct approach to obtaining our tableau description of $\mathcal{B}(\lambda)$ would be to work out examples of the appropriate connected components within the crystal graph of $\mathcal{T}(\infty) \otimes \mathsf{R}_\lambda$ for specific choices of $\lambda \in P^+$ and to generalize whatever patterns that could be recognized. We had tried this approach, but were not successful in obtaining meaningful results. Examples that were small enough for us to hand-compute were too small to contain all characteristics of the general situation, and examples that were large enough, generated through computer programming, were too complex for us to understand and extract patterns from.

Our set of conditions were instead *devised* to make it possible to prove our lemmas. Our experience with the A_n case allowed us to have some confidence in the presence of a few of the conditions. These were the conditions that were, in some sense, related to our three lemmas concerning the satisfaction of $\varphi_i(T) \geq \varepsilon_i(\mathsf{r}_\lambda)$ properties, and accounted for a very small part of our full set of conditions for any one type. The other conditions were formulated so that the proof of closedness under Kashiwara operator actions could be written down for the full set of conditions.

This was not a straightforward process, since the presence of one condition did not imply the presence of another condition. The only arguments that could be used were in the opposite direction, where a guessed condition could imply that a condition we had more confidence in would be preserved under the Kashiwara operator actions. The full set of conditions was obtained by incrementally adding one guessed condition at a time that would enable us to prove that the existing conditions are preserved under the Kashiwara operator actions. In many of these steps, we were confronted with multiple conditions that were equally reasonable in view of the conditions we had collected up to that point, and the whole process was a tedious repetition of guessed trials and back stepping. Computer-generated examples were occasionally helpful in ruling out incorrectly guessed conditions and in giving us confidence in any candidate for a full set of conditions, but they were of little use in constructing the conditions themselves.

Our eventual success in discovering the correct full set of conditions is largely due to our previous experience with the A_n case and the fact that the A_n-type basic crystal shares some similarities with the basic crystals for our three targeted types. Attempts are currently being made to extend the results of this paper to the exceptional finite simple Lie algebra types that remain untreated. However, the basic crystals for these cases are extremely complicated in comparison to those cases that have been treated, and it is not clear at this point as to how much can be achieved in this direction.

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