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Disk Complexes and Genus Two Heegaard Splittings for NonPrime 3-Manifolds

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Given a genus two Heegaard splitting for a nonprime 3-manifold, we define a special subcomplex of the disk complex for one of the handlebodies of the splitting, and then show that it is contractible. As applications, first we show that the complex of Haken spheres for the splitting is contractible, which refines the results of Lei and Lei-Zhang. Secondly, we classify all the genus two Heegaard splittings for nonprime 3-manifolds, which is a generalization of the result of Montesinos–Safont. Finally, we show that the mapping class group of the splitting, called the Goeritz group, is finitely presented by giving its explicit presentation.

Introduction

Every closed orientable 3-manifold M can be decomposed into two handlebodies V and W by cutting M along a closed orientable surface Σ embedded in it. This is called a *Heegaard splitting* for the manifold M, and denoted by the triple $(V, W; \Sigma)$. The surface Σ is called a *Heegaard surface* and its genus is called the *genus* of the splitting. A

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separating 2-sphere P in M is called a *Haken sphere* for the splitting $(V, W; \Sigma)$ if P intersects the Heegaard surface Σ in a single essential circle. If $(V, W; \Sigma)$ is a genus two Heegaard splitting for M that admits a Haken sphere, then M is one of the 3-sphere, $S^2 \times S^1$, lens spaces or their connected sums. In particular, if the manifold M is nonprime, then M is a connected sum whose summands are lens spaces or $S^2 \times S^1$.

In this paper, we study the genus two Heegaard splittings for nonprime 3-manifolds. Given a genus two Heegaard splitting $(V, W; \Sigma)$ for a closed orientable nonprime 3-manifold M, we define a special subcomplex of the disk complex for each of the handlebodies V and W, which we will call the semi-primitive disk complex, and then show that it is contractible. The semi-primitive disk complex is an analog of the primitive disk complexes studied in the authors' previous works [4–8, 19] to find presentations of certain kinds of mapping class groups, including some Goeritz groups.

Understanding the structure of the semi-primitive disk complexes with their properties, we produce several applications. First, we prove that the complex of Haken spheres is contractible for the genus two Heegaard splitting for any nonprime 3-manifold. The complex of Haken spheres is the simplicial complex whose vertices are isotopy classes of Haken spheres, and it has been an interesting problem to understand the structure of it since Scharlemann [26] showed that the complex for the genus two Heegaard splitting for the 3-sphere is connected. In Lei [20] and Lei-Zhang [21], it was shown that the complexes of Haken spheres are connected for genus two Heegaard splittings for nonprime 3-manifolds. In Theorem 3.1 in this work, we refine their results in an alternative way, showing that those complexes are actually contractible.

Secondly, we classify all the genus two Heegaard splittings for nonprime 3-manifolds. Indeed, any nonprime 3-manifold M admits at most two different genus two Heegaard splittings, and it is known from Montesinos–Safont [23] that, if M is the connected sum of two lens spaces $L(p, q_1)$ and $L(p, q_2)$, then there exists a unique genus two Heegaard surface for M up to homeomorphism if and only if $q_1^2 \equiv 1$ or $q_2^2 \equiv 1 \pmod{p}$. Including this result, we determine all the nonprime 3-manifolds that admit unique Heegaard surfaces up to homeomorphism, which is stated in Theorem 4.2.

The final application is to obtain a presentation of the mapping class group of a genus two Heegaard splitting for a non-Haken 3-manifold, using the semi-primitive disk complex. Such a group is called a (genus two) Goeritz group. Precisely, the *Goeritz group* of a Heegaard splitting $(V, W; \Sigma)$ for a manifold M is the group of isotopy classes of orientation-preserving homeomorphisms of M that preserve V and W setwise. In Theorem 5.1 in this work, we show that the genus two Goeritz groups for any nonprime 3-manifolds are all finitely presented by giving their explicit presentations.

The Goeritz groups have been interesting objects in the study of Heegaard splittings. For example, some interesting questions on Goeritz groups were proposed by Minsky in [11]. A Goeritz group will be "small" when the gluing map of the two handlebodies that defines the Heegaard splitting is sufficiently complicated. Indeed, Namazi [24] showed that the Goeritz group is actually a finite group when the Heegaard splitting has "high" *Hempel distance*. Here, we just simply mention that the Hempel distance is a measure of complexity of the gluing map that defines the splitting. We refer to [14] for its precise definition. Namazi's result is improved by Johnson in [16] showing that the Goeritz group is finite if the Hempel distance of the splitting is at least four. We refer the reader to [17, 18] for related topics. The Goeritz groups of Heegaard splittings of low Hempel distance are not as "small" as in the case of the high Hempel distance.

For example, it is easy to see that the Goeritz group of the genus g Heegaard splitting for $\#_g(S^2 \times S^1)$, which is the double of the genus g handlebody V, is isomorphic to the mapping class group of V. We note that the Hempel distance of this splitting is zero. The mapping class group of a handlebody of genus at least two is, of course, not finite. A finite generating set of this group is obtained by Suzuki [29] and its finite presentation is obtained by Grasse [12] and Wajnryb [30] independently. See also [15, 22].

It is natural to ask if a given Goeritz group is finitely generated or presented, and so finding a generating set or a presentation of it has been an important problem. But beyond the case of $\#_g(S^2 \times S^1)$, the generating sets or the presentations of the groups have been obtained only for few manifolds with their splittings of small genus. In the case of the 3-sphere, it is known that the Goeritz group for the genus two splitting is finitely presented from the works [1, 4, 10, 26]. Further, a finite presentation of the Goeritz group of the genus two Heegaard splitting is obtained for each of the lens spaces $L(p, 1)$ in [5] and $S^2 \times S^1$ in [7]. In addition, finite presentations of the genus two Goeritz groups of some other lens spaces are given in [8]. For the higher genus Goeritz groups of the 3-sphere and lens spaces, it is conjectured that they are all finitely presented but it is still an open problem.

This paper is organized as follows. In Sections 1 and 2, we introduce semi-primitive disks with their various properties, and then show that the semi-primitive disk complexes are contractible, by giving an explicit description of them. In Section 3, the complex of Haken spheres are shown to be contractible (Theorem 3.1), and in Section 4, we give a classification of the genus two Heegaard splittings for nonprime 3-manifolds (Theorem 4.2). In the final section, a finite presentation is given for the Goeritz group of each nonprime 3-manifold with its genus two Heegaard splitting (Theorem 5.1).

By disks, pairs of disks, triples of disks properly embedded in a handlebody, we often mean their isotopy classes throughout the paper. Also, we often speak of Haken spheres of a Heegaard splitting to mean their isotopy classes preserving the Heegaard splitting. When we choose representatives of their isotopy classes, we assume implicitly that they intersect each other minimally and transversely. Moreover, by homeomorphisms we often mean their isotopy classes when it is obvious from context.

We use the standard notation of lens spaces as follows. Let V and W be oriented solid tori. Let (m, l) be the pair of a meridian and a longitude of V. We orient m and l in such a way that the pair (m, l) yields the orientation of ∂V induced by that of V. The homology classes $[m]$ and $[l]$ of m and l induce a basis of $H_1(\partial V)$. In the same manner, we have the pair (m', l') of a meridian and a longitude of W. The lens space $L(p, q)$ is a 3-manifold obtained by identifying the boundaries of V and W using an orientation-reversing homeomorphism $\varphi : \partial V \to \partial W$ that induces an isomorphism $\varphi_* : H_1(\partial V) \to H_1(\partial W)$ represented by $\left(\begin{smallmatrix} q & p \\ s & -r \end{smallmatrix}\right)$, where $qr + ps = 1$. In particular, φ maps m' to a (p, q)-curve with respect to (m, l) on ∂V, that is, $\varphi_*[m'] = p[l] + q[m]$ in $H_1(\partial V)$. We note that the image of m by φ^{-1} is a (p, r)-curve with respect to (m', l') on ∂W. By definition, a lens space is equipped with a canonical orientation induced from those of V and W. This orientation induces a canonical orientation of the connected sum of two lens spaces. Throughout the paper, we will not regard $S^3 = L(1, 0)$ nor $S^2 \times S^1 = L(0, 1)$ as lens spaces.

1 Semi-Primitive Disks

An element of a free group $\mathbb{Z} * \mathbb{Z}$ of rank 2 is said to be *primitive* if it is a member of a generating pair of the group. Primitive elements of $\mathbb{Z} * \mathbb{Z}$ have been well-understood. For example, we refer the reader to [25]. A key property of the primitive elements is that, fixing a generating pair $\{x, y\}$ of $\mathbb{Z} * \mathbb{Z}$, any primitive element has a cyclically reduced form which is a product of terms each of the form $x^\epsilon y^n$ and $x^\epsilon y^{n+1}$, or else a product of terms each of the form $y^\epsilon x^n$ and $y^\epsilon x^{n+1}$, for some $\epsilon \in \{1, -1\}$ and some $n \in \mathbb{Z}$. The following is a direct consequence of this property.

Lemma 1.1. Fix a generating pair $\{x, y\}$ of $\mathbb{Z} * \mathbb{Z}$. Let w be a cyclically reduced word on $\{x, y\}$. If w contains both x and x^{-1}, both y and y^{-1} or both $x^{\pm 2}$ and $y^{\pm 2}$ simultaneously, then the element represented by w is neither trivial nor a power of a primitive element. □

Let V be a genus two handlebody, and let D and E be disjoint disks in V such that $D \cup E$ cuts V into a 3-ball. We fix an orientation on each of ∂D and ∂E, and then assign letters x and y to ∂D and ∂E, respectively. Let l be an oriented simple closed curve on ∂V which intersects $\partial D \cup \partial E$ minimally and transversely. Then l determines a word on $\{x, y\}$ that can be read off by the intersections of l with ∂D and ∂E. We note that this word is well-defined up to cyclic conjugation. The following is a simple criterion for triviality and primitiveness of the elements represented by l, which can be considered as a simpler version of Lemma 2.3 in [6].

Lemma 1.2. In the above setting, if a word w determined by the simple closed curve l contains a subword of the form xy^px^{-1} for some $p \in \mathbb{N}$, or x^2y^2, then any word determined by l is cyclically reduced. Moreover, the element represented by w is neither trivial nor a power of a primitive element. □

The idea of the proof is that, if w contains one of those subwords, then any word determined by l cannot contain $x^{\pm 1}x^{\mp 1}$ and $y^{\pm 1}y^{\mp 1}$, and any cyclically reduced word containing both x and x^{-1} or both x^2 and y^2 cannot represent a power of a primitive element by Lemma 1.1.

Let $(V, W; \Sigma)$ be a genus two Heegaard splitting for a nonprime 3-manifold. Recall that, by [13], the splitting $(V, W; \Sigma)$ admits a Haken sphere. A nonseparating disk D in V is said to be *semi-primitive* if there exists a Haken sphere P of $(V, W; \Sigma)$ disjoint from D. The next lemma follows from the definition.

Lemma 1.3. Let $(V, W; \Sigma)$ be a genus two Heegaard splitting for a nonprime 3-manifold. Let D be a semi-primitive disk in V. Then an element of $\pi_1(W)$ determined by ∂D is either trivial or a power of a primitive element. □

We remark that there is a semi-primitive disk D in V such that ∂D represents the trivial element of $\pi_1(W)$ if and only if the manifold has a $S^2 \times S^1$ summand. In this case, ∂D also bounds a disk in W.

Lemma 1.4. Let $(V, W; \Sigma)$ be a genus two Heegaard splitting for a nonprime 3-manifold. Let D be a nonseparating disk in V. Then D is semi-primitive if and only if there exists a nonseparating disk E' in W disjoint from D. □

Proof. The "only if" part is trivial. Let E' be a non-separating disk in W disjoint from D, and let Σ' be the four-holed sphere obtained by cutting Σ along $\partial D \cup \partial E'$. Let d^+ and d^- (e'^+ and e'^-, respectively) be the two boundary circles of Σ' coming from ∂D

($\partial E'$, respectively). Let α_P be an arbitrary simple arc in Σ' connecting d^+ and d^-. Then, up to isotopy, there exists a unique simple arc α'_P in Σ' connecting e'^+ and e'^- such that $\alpha_P \cap \alpha'_P = \emptyset$. We note that the frontier γ_P of a regular neighborhood of $d^+ \cup \alpha_P \cup d^-$ coincides with the frontier of a regular neighborhood of $e'^+ \cup \alpha'_P \cup e'^-$ in Σ'. It follows that γ_P bounds a disk in each of V and W. This implies that there exists a Haken sphere P of $(V, W; \Sigma)$ such that $P \cap \Sigma = \gamma_P$. ■

In the proof above, every simple closed curve γ_Q in Σ' that separates $d^+ \cup d^-$ and $e'^+ \cup e'^-$ is the frontier of a regular neighborhood of the union of $d^+ \cup d^-$ ($e'^+ \cup e'^-$, respectively) and a simple arc α_Q (α'_Q, respectively) in Σ' connecting d^+ and d^- (e'^+ and e'^-, respectively). Thus every essential, separating, simple closed curve in Σ disjoint from $\partial D \cup \partial E'$ bounds separating disks in both V and W.

1.1 Connected sum of two lens spaces

Throughout this section, we always assume that $(V, W; \Sigma)$ is a genus two Heegaard splitting for the connected sum of two lens spaces.

Lemma 1.5. Let D be a semi-primitive disk in V. Then there is a unique nonseparating disk E' in W disjoint from D. □

Proof. By Lemma 1.4, such a disk E' exists. To see the uniqueness, assume that there exist nonisotopic, nonseparating disks E'_1 and E'_2 in W disjoint from D. We assume that E'_1 and E'_2 intersect each other transversely and minimally. If they have nonempty intersection, a disk obtained from E'_1 by a surgery along an outermost subdisk of E'_2 cut-off by $E'_1 \cap E'_2$ is also a nonseparating disks in W disjoint from D. This disk has fewer intersection with E'_1 than E'_2 had, and so by repeating surgeries if they still have intersection, we obtain a nonseparating disk E' in W disjoint from E'_1 and from D. Since ∂D does not intersects $E'_1 \cup E'$, the circle ∂D bounds a disk D' in W. This implies that $D \cup D'$ is a nonseparating sphere in the connected sum of two lens spaces, whence a contradiction. ■

The next theorem will play an important role in Section 2.

Theorem 1.6. Let D and E be semi-primitive disks in V that intersect each other transversely and minimally. Then at least one of the two disks obtained from E by a surgery along an outermost subdisk of D cut-off by $D \cap E$ is a semi-primitive disk. □

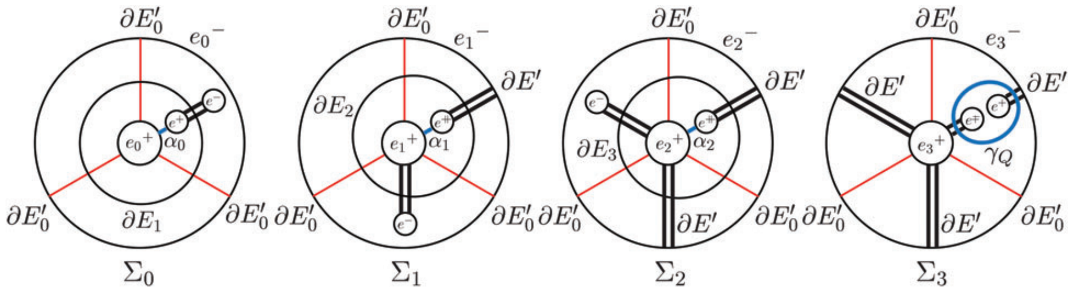

Fig. 1. The case where $(p_1, q_1) = (3, 1)$ and $(p_2, q_2) = (2, 1)$. The circles ∂E_1, ∂E_2 and ∂E_3 determine the words x^2y^3, $x^2yx^2y^2$, $(x^2y)^3$, respectively.

Proof. Let C be an outermost subdisk of D cut-off by $D \cap E$. Each Haken sphere P of $(V, W; \Sigma)$ disjoint from E cuts the handlebody V into two solid tori V_1 and V_2, and W into W_1 and W_2. We assume that E is the meridian disk of V_1, and that $V_1 \cup W_1$ and $V_2 \cup W_2$ are punctured lens spaces. Let E_0, E' and E_0' be the meridian disks of solid tori V_2, W_1 and W_2, respectively, which are disjoint from P. We choose a Haken sphere P among all Haken spheres disjoint from E so that $|C \cap E_0|$ is minimal. Assume that $\partial E'$ ($\partial E_0'$, respectively) is a (p_2, q_2)-curve ((p_1, q_1)-curve, respectively) with respect to the meridian ∂E (∂E_0, respectively) and a fixed longitude on ∂V_1 (∂V_2, respectively). We may assume that $1 \le q_1 < p_1$ and $1 \le q_2 < p_2$. Each element of $\pi_1(W)$ can be represented by a word on $\{x, y\}$, where x and y are determined (up to sign) by the meridian disks E' and E_0', respectively. If E_0 is disjoint from C, then E_0 is one of the disks obtained from E by a surgery along C, and is a semi-primitive disk, so we are done.

Assume that $C \cap E_0 \neq \emptyset$. Let C_0 be an outermost subdisk of C cut-off by $C \cap E_0$ such that $C_0 \cap E = \emptyset$. Let Σ_0 be the four-holed sphere obtained by cutting Σ along $\partial E \cup \partial E_0$. Let e^+ and e^- (${e_0}^+$ and ${e_0}^-$, respectively) be the boundary circles of Σ_0 coming from ∂E (∂E_0, respectively). Then $C_0 \cap \Sigma_0$ is the frontier of a regular neighborhood of the union of one of e^+ and e^-, say e^+, and a simple arc α_0 connecting e^+ and one of ${e_0}^+$ and ${e_0}^-$, say ${e_0}^+$. Up to isotopy, the arc α_0 does not intersect $\partial E_0'$, otherwise a word of ∂D would contain the subword $yx^{p_2}y^{-1}$ (after changing the orientations if necessary), which contradicts Lemmas 1.2 and 1.3. We denote by E_1 the disk obtained from E_0 by a surgery along C_0 that is not E. We remark that $|C \cap E_1| < |C \cap E_0|$ and that ∂E_1 determines a word of the form $x^{p_2}y^{p_1}$ (after changing the orientations if necessary). See Σ_0 in Figure 1.

We define inductively a sequence of disks $E_2, E_3, \ldots, E_{p_1}$ in V as follows. For $i \in \{1, 2, \ldots, p_1 - 1\}$ let Σ_i be the four-holed sphere obtained by cutting Σ along $\partial E \cup \partial E_i$. Let e^+ and e^- (e_i^+ and e_i^-, respectively) be the boundary circles of Σ_i coming from ∂E

(∂E_i, respectively). Then there exists a unique simple arc α_i in Σ_i connecting e^+ and one of e_i^+ or e_i^- such that α_i is disjoint from $\partial E_0'$ and is not parallel to any arc component of $\partial E' \cap \Sigma_i$. We may assume that α_i connects e^+ and e_i^+ by exchanging e_i^+ and e_i^- if necessary. Let E_{i+1} be the disk obtained by the band sum of E and E_i along α_i. The disk E_{i+1} is not isotopic to E_{i-1} since the arc α_i is not parallel to any arc component of $\partial E' \cap \Sigma_i$ (see Figure 1). We note that the circle ∂E_2 determines the word $x^{p_2}y^{q_1}x^{p_2}y^{p_1-q_1}$. The circle ∂E_3 determines the word $x^{p_2}y^{q_1}x^{p_2}y^{q_1}x^{p_2}y^{p_1-2q_1}$ if $1 \leqslant q_1 \leqslant p_1/2$, and $x^{p_2}y^{2p_1-q_1}x^{p_2}y^{p_1-q_1}x^{p_2}y^{p_1-q_1}$ if $p_1/2 < q_1 < p_1$. Also, the circle ∂E_{p_1-1} determines the word $(x^{p_2}y)^{p_1-q_1}y(x^{p_2}y)^{q_1-1}$. Finally, the circle ∂E_{p_1} determines a word of the form $(x^{p_2}y)^{p_1}$, which is apparently a power of a primitive element of $\pi_1(W)$.

We show that E_{p_1} is a semi-primitive disk and in fact there exists a Haken sphere disjoint from E_{p_1} and E. Let Σ_{p_1} be the four-holed sphere obtained by cutting Σ along $\partial E \cup \partial E_{p_1}$. By the construction, the two boundary circles e^+ and e^- of Σ_{p_1} coming from ∂E are contained in the same component of Σ_{p_1} cut-off by $\partial E_0' \cap \Sigma_{p_1}$. Hence, there exists an arc α_Q in Σ_{p_1} connecting e^+ and e^- such that $\alpha_Q \cap \partial E_0' = \emptyset$. We denote by γ_Q the frontier of a regular neighborhood of $\mathrm{e}^+ \cup \alpha_Q \cup \mathrm{e}^-$. Apparently, γ_Q is disjoint from $E \cup E_0'$. See the four-holed sphere Σ_3 in Figure 1. Thus, it follows from the remark right after Lemma 1.4 that there exists a Haken sphere Q in (V, W, Σ) such that $Q \cap \Sigma = \gamma_Q$. In particular, Q is disjoint from E_{p_1}, and hence E_{p_1} is a semi-primitive disk.

Now we claim that, for $i \in \{1, 2, \ldots, p_1 - 1\}$, $C \cap E_i \neq \emptyset$, and E_{i+1} is obtained from E_i by surgery along an outermost subdisk C_i of C cut-off by $C \cap E_i$ such that $C_i \cap E = \emptyset$. The latter claim follows immediately from the former one, since, if C intersects E_i, then $C_i \cap \Sigma_i$ is the frontier of a regular neighborhood of $\mathrm{e}^+ \cup \alpha_i$ in Σ_i, and so the same reason to the case of α_0 implies the latter claim. Suppose that E_i is the first disk disjoint from C for contradiction.

First, assume that $i \in \{1, 2, \ldots, p_1 - 2\}$. Since C does not intersect E_i, the intersection $C \cap \Sigma_i$ is a simple arc with both end points on e^{ϵ_1}, where $\epsilon_1 \in \{+, -\}$. Then $C \cap \Sigma_i$ is the frontier of a regular neighborhood of $e_i^{\epsilon_1} \cup \beta_{\epsilon_1\epsilon_2}$, where $\epsilon_2 \in \{+, -\}$ and $\beta_{\epsilon_1\epsilon_2}$ is a simple arc in Σ_i connecting e^{ϵ_1} and $e_i^{\epsilon_2}$. We see that $\beta_{\epsilon_1\epsilon_2}$ is disjoint from $\partial E_0' \cap \Sigma_i$, otherwise $C \cap \Sigma_i$ would give a word containing $yx^{p_2}y^{-1}$ and hence D is not a semi-primitive disk by Lemmas 1.2 and 1.3, a contradiction. If $\epsilon_1 \neq \epsilon_2$, then we may isotope $C \cap \Sigma_i$ on Σ_i so that $C \cap \Sigma_i$ is disjoint from E_{i-1} (see Figure 2). This contradicts the assumption that C intersects E_{i-1}. Thus, we have $\epsilon_1 = \epsilon_2$. We assumed that $i \leqslant p_1 - 2$, and hence there exists at least one arc component of C cut-off by $\partial E_0'$ that does not intersect $\partial E'$, which means a word determined by $C \cap \Sigma_i$ contains y^2. Therefore, $C \cap \Sigma_i$ gives a word containing $x^{p_2}y^2$, and so containing x^2y^2. Again, this implies that D is not a semi-primitive disk by

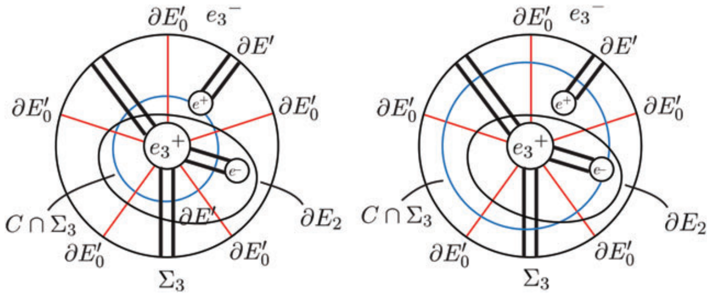

Fig. 2. The case where $(p_1, q_1) = (5, 2)$, $(p_2, q_2) = (2, 1)$ and $i = 3$.

Lemmas 1.2 and 1.3, whence a contradiction. (We note that, when $i = p_1 - 1$, the word determined by $C \cap \Sigma_{p_1-1}$ is of the form $yx^{p_2}yx^{p_2} \cdots yx^{p_2}y$, and so it does not contain y^2.) Next, assume that $i = p_1 - 1$. In this case, C is disjoint from E_{p_1-1} and intersects E_{p_1-2}. Then one of the resulting disks obtained by surgery on E along C is E_{p_1-1}, and the other one is the semi-primitive disk E_{p_1}. In particular, C is disjoint from E_{p_1}. This contradicts the minimality of $|C \cap E_0|$ since we are assuming that $C \cap E_0 \neq \emptyset$. Hence, we get the claim.

However, this is impossible since now we have the inequalities $|C \cap E_{p_1}| < |C \cap E_{p_1-1}| < \cdots < |C \cap E_0|$ and this contradicts, again, the minimality of $|C \cap E_0|$. ■

Lemma 1.7. Let D and E be disjoint, nonisotopic semi-primitive disks in V. Then there exists a unique Haken sphere of $(V, W; \Sigma)$ disjoint from $D \cup E$. □

Proof. The uniqueness follows immediately from Lemma 1.5. To show the existence of a Haken sphere of $(V, W; \Sigma)$ disjoint from $D \cup E$, we choose a Haken sphere P among all Haken spheres disjoint from E so that $|D \cap E_0|$ is minimal as in the proof of Theorem 1.6. Also, we take the disks E' and E_0' in W as in the proof of Theorem 1.6. Each element of $\pi_1(W)$ are represented by a word on $\{x, y\}$, where x and y are determined (up to sign) by the meridian disks E' and E_0'. If $D = E_0$, we are done. Assume that $D \neq E_0$ and $D \cap E_0 = \emptyset$. Then the disk D is the band sum of E and E_0 along an arc, say α_0, which connects ∂E and ∂E_0. Since we assumed that D is semi-primitive, the arc α_0 is disjoint from E_0' by the same reason to the case of the arc α_0 in the proof of Theorem 1.6 (after changing the orientations if necessary). Considering ∂D as a circle lying in the four-holed sphere Σ cut-off by $\partial E \cup \partial E_0$, which is the same case to the circle ∂E_1 in Σ_0 in Theorem 1.6, we observe that a word determined by ∂D must contain a subword of the form x^2y^2. By Lemmas 1.2 and 1.3, the disk D cannot be semi-primitive, a contradiction. Finally, assume that $D \cap E_0 \neq \emptyset$. Then by the same argument as the proof of Theorem 1.6 for the disk D instead of the outermost subdisk C, we can deduce a contradiction. ■

Lemma 1.8. Let D, E and F be pairwise disjoint, pairwise nonisotopic, nonseparating disks in V. If D and E are semi-primitive disks, then F is not a semi-primitive disk. □

Proof. By Lemma 1.7, there exists a (unique) Heken sphere P of $(V, W; \Sigma)$ disjoint from $D \cup E$. Thus, we have the meridian disks D' and E' of the two solid tori W cut-off by $P \cap W$ that are disjoint from P. Then the nonseparating disk F is the band sum of D and E along an arc, say α_0, which connects ∂D and ∂E. This is exactly the case of "$D \neq E_0$ and $D \cap E_0 = \emptyset$" in the proof of Lemma 1.7. Here, D, E, F, D' and E' correspond to E_0, E, D, E_0' and E', respectively, in the proof of Lemma 1.7. Thus, by the same reasoning, we see that F is not semi-primitive. ■

1.2 Connected sum of $S^2 \times S^1$ and a lens space

Throughout this section, we always assume that $(V, W; \Sigma)$ is a genus two Heegaard splitting for the connected sum of $S^2 \times S^1$ and a lens space. A nonseparating disk D in V is called a *reducing disk* if ∂D bounds a disk in W. We remark that a reducing disk is also a semi-primitive one and the boundary circle of a reducing disk represents the trivial element of $\pi_1(W)$.

Lemma 1.9. Let D be a reducing disk in V. Let E be a nonseparating disk in V that is not isotopic to D.

(1) If E is disjoint from D, then there exists a Haken sphere of $(V, W; \Sigma)$ disjoint from $D \cup E$. In particular, E is a semi-primitive disk but is not a reducing disk.

(2) If E intersects D, then E is not a semi-primitive disk. □

Proof. (1) Let Σ' be the four-holed sphere obtained by cutting Σ along $\partial D \cup \partial E$. Let d^+ and d^- be the two boundary circles of Σ' coming from ∂D. Let α_P be an arbitrary simple arc in Σ' connecting d^+ and d^-. Since D is a reducing disk, the frontier γ_P of a regular neighborhood of $d^+ \cup \alpha_P \cup d^-$ bounds a disk in each of V and W. This implies that there exists a Haken sphere P of $(V, W; \Sigma)$ such that $P \cap \Sigma = \gamma_P$, which is disjoint from $D \cup E$.

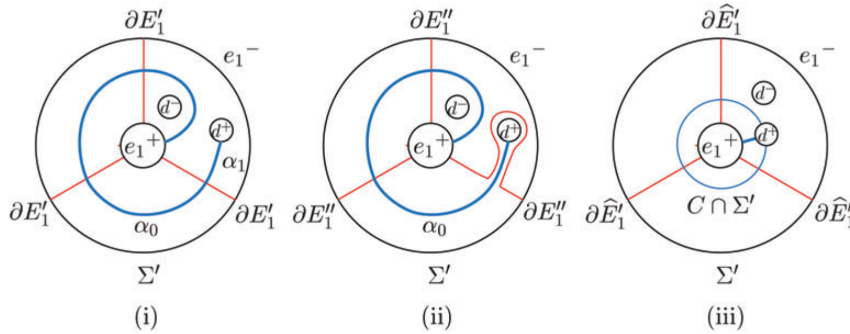

Fig. 3. (i) The arc α_0; (ii) The disk E_1''; (iii) The arc $C \cup \Sigma'$.

(2) Let D' be a disk in W bounded by ∂D. Let C be an outermost subdisk of E cut-off by $D \cap E$. Then a standard cut-and-paste argument allows us to have a non-separating disk E_1 in V that is not isotopic to D and disjoint from $C \cup D$. By (1), E_1 is a semi-primitive disk. Let P be a Haken sphere of $(V, W; \Sigma)$ disjoint from $D \cup E_1$. Let E_1' be the semi-primitive disk in W disjoint from P that is not isotopic to D'. Let Σ' be the four-holed sphere obtained by cutting Σ along $\partial D \cup \partial E_1$. Let d^+ and d^- (${e_1}^+$ and ${e_1}^-$, respectively) be the two boundary circles of Σ' coming from ∂D (∂E_1, respectively). We note that $\partial E_1' \cap \Sigma'$ cuts Σ' into a finite number of rectangles and a single rectangle with two holes d^+ and d^- (see Figure 3(i)). Then $C \cap \Sigma'$ is the frontier of a regular neighborhood of the union of an arc α_0 in Σ' connecting one of d^+ and d^-, say d^+ and one of e_1^+ and e_1^-, say e_1^+, and the boundary circle e_1^+.

Assume that α_0 meets $\partial E_1'$. Let α_1 be a subarc of α_0 connecting d^+ and $\partial E_1'$ such that the interior of α_1 is disjoint from $\partial E_1'$. Let $E_1'' \subset W$ be the band sum of E_1' and D' along α_1. E_1'' is a semi-primitive disk and we have $|\alpha_0 \cap \partial E_1''| < |\alpha_0 \cap \partial E_1'|$ (see Figure 3(ii)). Repeating this process finitely many times, we obtain a semi-primitive disk $\hat{E}_1'$ in W disjoint from both D and α_0.

We give letters x and y to the circles $\partial D'$ and $\partial \hat{E}_1'$, respectively, after fixing an orientation of each of them. Then a word on $\{x, y\}$ determined by ∂E contains a subword of the form xy^px^{-1}, which is determined by the subarc $C \cap \Sigma'$ after changing the orientations if necessary, (see Figure 3 (iii)). By Lemma 1.2, E is neither a reducing disk nor a semi-primitive disk. ■

By Lemma 1.9, $(V, W; \Sigma)$ admits a unique reducing disk. The next lemma follows immediately from the definition of a reducing disk and the proof of Lemma 1.4.

Lemma 1.10. Let D be the reducing disk in V. Then any non-reducing, semi-primitive disk in V is disjoint from D up to isotopy. □

2 The Complex of Semi-Primitive Disks

Let V be a handlebody. The *disk complex* $\mathcal{K}(V)$ of V is the simplicial complex whose vertices are the isotopy classes of essential disks in V such that the collection of distinct $k+1$ vertices spans a k-simplex if they admit a set of pairwise disjoint representatives. The full-subcomplex $\mathcal{D}(V)$ of $\mathcal{K}(D)$ spanned by the vertices corresponding to nonseparating disks is called the *nonseparating disk complex* of V. In [22], it is shown that both $\mathcal{K}(V)$ and $\mathcal{D}(V)$ are contractible. Moreover, we have the following theorem.

Theorem 2.1 ([4, 22])**.** A full subcomplex $\mathcal{L}$ of the disk complex $\mathcal{K}(V)$ is contractible if, given any two representative disks E and D of vertices of $\mathcal{L}$ intersecting each other transversely and minimally, at least one of the disks from surgery on E along an outermost subdisk of D cut-off by $D \cap E$ represents a vertex of $\mathcal{L}$. □

Let M_1 be a lens space or $S^2 \times S^1$, and let M_2 be a lens space. Let $(V, W; \Sigma)$ be a genus two Heegaard splitting for $M_1 \# M_2$. The *semi-primitive disk complex* $\mathcal{SP}(V)$ of V is the full subcomplex of $\mathcal{D}(V)$ spanned by the vertices corresponding to semi-primitive disks of V. We remark that the Goeritz group $\mathcal{G}$ of $(V, W; \Sigma)$ acts on $\mathcal{SP}(V)$ simplicially.

Theorem 2.2. Let M_1 be a lens space or $S^2 \times S^1$, and let M_2 be a lens space. Let $(V, W; \Sigma)$ be a genus two Heegaard splitting for $M_1 \# M_2$.

(1) If M_1 is a lens space, then $\mathcal{SP}(V)$ is a tree.
(2) If $M_1 = S^2 \times S^1$, then $\mathcal{SP}(V)$ is the cone of a tree. □

Proof. (1) That $\mathcal{SP}(V)$ is contractible is a straightforward consequence of Theorems 1.6 and 2.1. That it is a 1-complex follows from Lemma 1.8.

(2) Let D be the unique reducing disk in V. Let $\mathcal{SP}_D(V)$ denote the full subcomplex of $\mathcal{D}(V)$ spanned by the vertices corresponding to nonreducing semi-primitive disks. By Lemmas 1.9 and 1.10, the complex $\mathcal{SP}_D(V)$ is the link of the vertex corresponding to D in $\mathcal{D}(V)$. It is shown in [4, 22] that the link of any vertex of $\mathcal{D}(V)$ is a tree, and hence $\mathcal{SP}_D(V)$ is a tree. ■

3 The Complex of Haken Spheres

Let $(V, W; \Sigma)$ be a genus two Heegaard splitting for a closed orientable 3-manifold M. The *complex $\mathcal{H}$ of Haken spheres* of the splitting $(V, W; \Sigma)$ is defined to be the simplicial complex whose vertices consists of the isotopy classes of Haken spheres such that the collection $P_0, P_1, \ldots, P_k$ of distinct $k+1$ vertices spans a k-simplex if $|P_i \cap \Sigma \cap P_j| = 4$ for all $0 \leqslant i < j \leqslant k$. It is shown that the complex of Haken spheres of the genus two splitting for S^3 is connected by Scharlemann [26], and it turns out that the complex actually deformation retracts to a tree from the works [1, 4]. Lei [20] and Lei-Zhang [21] showed that the complex of Haken spheres of the genus two splitting for a nonprime 3-manifold is connected. In this section, we refine the results of Lei and Lei-Zhang. That is, we show that the complexes of Haken spheres for nonprime 3-manifolds are connected in a new way, and further show that they are actually contractible. We use the results on the semi-primitive disk complexes developed in the previous section.

Theorem 3.1. Let M_1 be a lens space or $S^2 \times S^1$ and let M_2 be a lens space. Let $(V, W; \Sigma)$ be a genus two Heegaard splitting for $M_1 \# M_2$. Then the complex $\mathcal{H}$ of Haken spheres of the splitting $(V, W; \Sigma)$ is contractible. The dimension of $\mathcal{H}$ is 1, that is, $\mathcal{H}$ is a tree, if M_1 is a lens space, and is 3 if M_1 is $S^2 \times S^1$. □

Proof. Let us assume first that M_1 is a lens space. In Theorem 2.2, we have seen that the semi-primitive disk $\mathcal{SP}(V)$ is a tree. Let $\mathcal{SP}'(V)$ be the first barycentric subdivision of the tree $\mathcal{SP}(V)$. The tree $\mathcal{SP}'(V)$ is bipartite, of which we call the vertices of countably infinite valence (the vertices of the original $\mathcal{SP}(V)$) the black vertices, and the vertices of valence 2 the white ones. By Lemma 1.7, the set of the white vertices one-to-one corresponds to the set of Haken spheres.

Let D be a semi-primitive disk in V. We note that D represents a black vertex of the tree $\mathcal{SP}'(V)$. By Lemma 1.5, there exists the unique semi-primitive disk E' in W disjoint from D. The set of white vertices in the link of D in $\mathcal{SP}'(V)$ one-to-one correspond of the set of the Haken spheres disjoint from $D \cup E'$. Let Σ' be the four-holed sphere obtained by cutting Σ off along $\partial D \cup \partial E'$. Let d^+ and d^- (e'^+ and e'^-, respectively) be the two boundary circles of Σ' coming from ∂D ($\partial E'$, respectively). Let $\mathcal{H}_D$ be the full subcomplex of the complex $\mathcal{H}$ spanned by the vertices corresponding to Haken spheres disjoint from D. We assign each vertex of $\mathcal{H}_D$ an element of $\mathbb{Q}_{\mathrm{odd}} \cup \{\infty\}$ in the following way. Fix a Haken sphere P of $(V, W; \Sigma)$ disjoint from $D \cup E'$. Set $\mu = P \cap \Sigma'$ and fix a

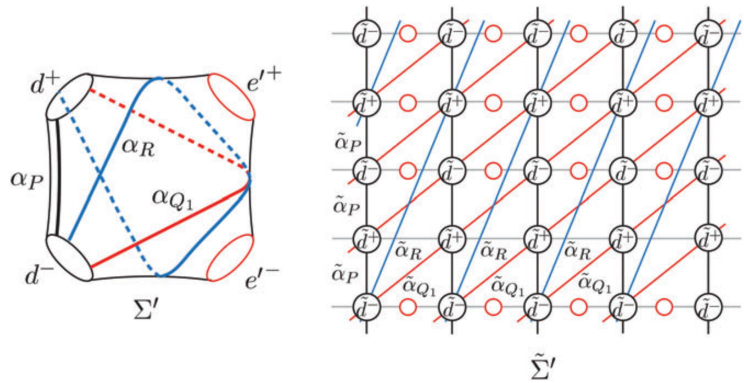

Fig. 4. The surface Σ' and its covering space $\tilde{\Sigma}'$.

separating simple closed curve ν in Σ' such that ν separates $d^+ \cup e'^+$ and $d^- \cup e'^-$, and that $|\mu \cap \nu| = 2$ after minimizing the intersection. Let $\tilde{\Sigma}'$ be the covering space of Σ' such that

(1) the components of the preimage of μ (ν, respectively) are the vertical (horizontal, respectively) lines in the Euclidean plane;
(2) the set of components of the preimage of ∂D correspond to the set of points whose coordinates consist of integers (see Figure 4).

We note that, once we put a lift of d^- at the origin $(0, 0)$, the set of the coordinates corresponding to the lifts of d^+ is $\{(s, t) \mid s \in \mathbb{Z}, t \in \mathbb{Z}_{\text{odd}}\}$, where $\mathbb{Z}_{\text{odd}}$ is the set of odd integers. For each arc connecting d^+ and d^-, we assign the slope $s/t \in \mathbb{Q}_{\text{odd}} \cup \{\infty\}$ of its preimage with respect to the above covering map, where $\mathbb{Q}_{\text{odd}}$ is the set of irreducible rational numbers having odd numerators. Since the set of Haken spheres disjoint from $D \cup E'$ one-to-one corresponds to the set of simple arcs in Σ' connecting d^+ and d^- as in the proof of Lemma 1.4, the above assignment provides an assignment of each vertex of $\mathcal{H}_D$ to an element of $\mathbb{Q}_{\text{odd}} \cup \{1/0\}$.

We now briefly review some well-known facts on the Farey complex. The *Farey complex* $\mathcal{F}$ is the flag complex whose vertex set is $\mathbb{Q} \cup \{1/0\}$. Two vertices s_1/t_1 and s_2/t_2 are connected by an edge if and only if $s_1 t_2 - s_2 t_1 = \pm 1$. See the left-hand side in Figure 5. The assignment of each vertex of $\mathcal{H}_D(V)$ with an element of $\mathbb{Q} \cup \{1/0\}$ described above allows us to get an embedding of $\mathcal{H}_D(V)$ into $\mathcal{F}$. The image of $\mathcal{H}_D(V)$ is the full subcomplex $\mathcal{F}_{\text{odd}}$ of $\mathcal{F}$ spanned by $\mathbb{Q}_{\text{odd}} \cup \{1/0\}$. See the right-hand side in Figure 5. It is easy to check that $\mathcal{F}_{\text{odd}}$ is a tree. It follows that there exists a natural simplicial isomorphism from $\mathcal{H}$ to the simplicial complex obtained from $\mathcal{SP}'(V)$ by replacing the star of each

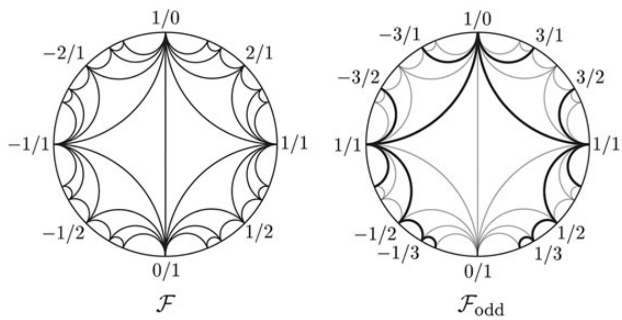

Fig. 5. The Farey complex $\mathcal{F}$ and its subcomplex $\mathcal{F}_{\text{odd}}$.

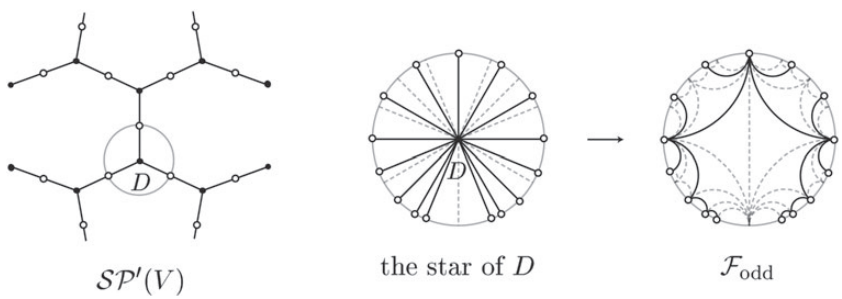

Fig. 6. The shape of the complex $\mathcal{H}$ of Haken spheres when M_1 is a lens space.

black vertex with the tree simplicially isomorphic to $\mathcal{F}_{\text{odd}}$ (see Figure 6). Consequently, $\mathcal{H}$ is a tree.

Next, assume that $M_1 = S^2 \times S^1$. Recall that, by Lemma 1.9, there exists the unique reducing disk D in V. Let Σ_D be the two-holed torus obtained by cutting Σ along ∂D. Let d^+ and d^- be the two components of $\partial \Sigma_D$. Let $\mathcal{A}(\Sigma_D)$ be the simplicial complex whose vertices are isotopy classes of simple arcs in Σ_D connecting d^+ and d^- such that the collection of distinct $k+1$ vertices spans a k-simplex if they admits a set of pairwise disjoint representatives. Each simple arc α_P in Σ_D connecting d^+ and d^- determine a unique Haken sphere P of $(V, W; \Sigma)$. By the uniqueness of D, this correspondence gives a simplicial isomorphism $\mathcal{A}(\Sigma_D) \to \mathcal{H}$. It is shown that $\mathcal{A}(\Sigma_D)$ is a contractible three-dimensional simplicial complex in [9, 27], and so is $\mathcal{H}$. ■

We remark that the argument developed in [22] allows us to show easily that $\mathcal{H}$ is also a tree for the genus two Heegaard splitting $(V, W; \Sigma)$ for $(S^2 \times S^1)\#(S^2 \times S^1)$.

In the remaining of this section, we analyze the action of the Goeritz group on the set of Haken spheres of genus two Heegaard splittings for later works.

Lemma 3.2. Let $(V, W; \Sigma)$ be a genus two Heegaard splitting for the connected sum of two lens spaces $L(p_1, q_1)$ and $L(p_2, q_2)$. For any two Haken spheres P and Q of $(V, W; \Sigma)$ with $|P \cap \Sigma \cap Q| = 4$, there exists an element of the Goeritz group of $(V, W; \Sigma)$ that maps P to Q. □

Proof. The Haken sphere P cuts V into two solid tori V_1 and V_2, and W into W_1 and W_2. We may assume that $V_1 \cup W_1$ and $V_2 \cup W_2$ are punctured lens spaces. Let D and E be the meridian disks of V_1 and V_2, respectively, disjoint from P. Similarly, let D' and E' be the meridian disks of W_1 and W_2, respectively, disjoint from P. ■

Claim. Up to isotopy, Q is disjoint from $D \cup E'$ or $E \cup D'$.

Proof of Claim. Let C_0 be an outermost subdisk of the disk $Q \cap V$ cut-off by $P \cap Q \cap V$, which is contained in either V_1 or V_2. Assume first that C_0 is contained in V_1. Then there exists exactly one more such a subdisk C_1 of $Q \cap V$, and it is also contained in V_1. Since $|P \cap \Sigma \cap Q| = 4$, we have $V_1 \cap Q = C_1 \cup C_2$, and hence Q is disjoint from D. Further, if D_0 is an outermost subdisk of the disk $Q \cap W$ cut-off by $P \cap Q \cap W$, then D_0 must be contained in W_2, otherwise ∂D would bound a meridian disk in W_1, which forms a nonseparating sphere with the disk D in the punctured lens space $V_1 \cup W_1$, a contradiction. Further, by the same reason to the case of C_0 and C_1, there exists exactly one more subdisk D_1 of $Q \cap W$, and it is also contained in W_2. Thus Q is also disjoint from E'. If C_0 is contained in V_2, then, by the same argument, Q is disjoint from $E \cup D'$.

By the claim, we assume that Q is disjoint from $D \cup E'$ without loss of generality. Let Σ' be the four-holed sphere obtained by cutting Σ along $\partial D \cup \partial E'$. Let d^+ and d^- (e'^+ and e'^-, respectively) be the two boundary circles of Σ' coming from ∂D ($\partial E'$, respectively). Then $P \cap \Sigma'$ ($Q \cap \Sigma'$, respectively) is the frontier of a regular neighborhood of the union of $d^+ \cup d^-$ and a simple arc α_P (α_Q, respectively) in Σ' connecting d^+ and d^-. Since $|P \cap \Sigma \cap Q| = 4$, we may assume that $\alpha_P \cap \alpha_Q = \emptyset$. Set $\mu = P \cap \Sigma'$. Let ν be a simple closed curve in Σ such that ν separates $d^+ \cup e'^+$ and $d^- \cup e'^-$, and ν intersects μ transversely in two points (see Figure 7(i)). We note that a half-Dehn twist about μ extends to an orientation-preserving homeomorphism of $L(p_1, q_1) \# L(p_2, q_2)$ that preserves V. Up to a finite number of half-Dehn twists about μ and isotopy, a single Dehn twist τ_ν about ν maps α_P to α_Q (see Figure 7(ii)). However, τ_ν extends to a homeomorphism of neither of V nor W. To see this, recall that each simple closed curve l in Σ determine a (possibly not

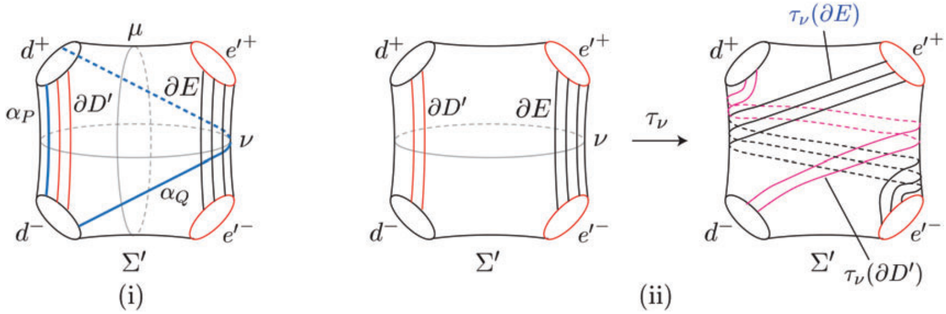

Fig. 7. The arcs α_P and α_Q in Σ'.

reduced) word $w(l)$ on $\{x, y\}$ ($\{z, w\}$, respectively) that can be read off from the intersection of l with $\partial D'$ and $\partial E'$ (∂D and ∂E, respectively) after fixing orientations of the simple closed curves. Note that this word gives the element of $\pi_1(W) = \langle x, y\rangle$ ($\pi_1(V) = \langle z, w\rangle$, respectively) represented by the loop l. On the surface Σ', $\partial D'$ (∂E, respectively) consists of p_1 (p_2, respectively) parallel simple arcs $\delta'_1, \delta'_2, \ldots, \delta'_{p_1}$ ($\epsilon_1, \epsilon_2, \ldots, \epsilon_{p_2}$, respectively). Then the subword $w(\tau_\nu(\delta'_i))$ ($w(\tau_\nu(\epsilon_j))$, respectively) of $w(\tau_\nu(\partial D'))$ ($w(\tau_\nu(\partial E))$, respectively) determined by the subarc $\tau_\nu(\delta'_i)$ ($\tau_\nu(\epsilon_j)$, respectively) of $\tau_\nu(\partial D')$ ($\tau_\nu(\partial E)$, respectively) is x^{p_1} (w^{p_2}, respectively) for each $i \in \{1, 2, \ldots, p_1\}$ ($j \in \{1, 2, \ldots, p_2\}$, respectively). Here, we move $\tau_\nu(\partial D')$ ($\tau_\nu(\partial E)$, respectively) slightly by isotopy so that $\tau_\nu(\partial D')$ ($\tau_\nu(\partial E)$, respectively) and $\partial D'$ (∂E, respectively) intersect each other transversely and minimally at points in the interior of Σ'. See the left-hand side in Figure 8. This implies that $w(\tau_\nu(\partial D')) = x^{{p_1}^2}$ ($w(\tau_\nu(\partial E)) = w^{{p_2}^2}$, respectively). Thus, $\tau_\nu(\partial D')$ ($\tau_\nu(\partial E)$, respectively) cannot bound a disk in W (in V, respectively), and hence τ_ν cannot extend to a homeomorphism of V nor W.

But now we consider the composition $\tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_\nu$. We may choose the Dehn twists $\tau_{\partial D}$ and $\tau_{\partial E'}$ so that the word $w(\tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_\nu(\delta'_i))$ ($w(\tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_\nu(\epsilon_j))$, respectively) is an empty word after cancellation. See the right-hand side in Figure 8. This implies that the word $w(\tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_\nu(\partial D'))$ ($w(\tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_\nu(\partial E))$, respectively) represents the trivial element of $\pi_1(W)$ ($\pi_1(V)$, respectively). Hence by Loop Theorem, $\tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_\nu(\partial D')$ ($\tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_\nu(\partial E)$, respectively) bounds a disk in W (V, respectively). Apparently, $\tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_\nu$ fixes ∂D and $\partial E'$. Consequently both $\tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_\nu(\partial D)$ and $\tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_\nu(\partial E)$ bound disks in V. Therefore by Alexander's trick, this composition extends to a homeomorphism of V. Similarly, $\tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_\nu(\partial D')$ bounds a disk in W and hence this composition extends to a homeomorphism of W. As a consequence, the map $\tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_\nu$ extends to an orientation-preserving homeomorphism of $L(p_1, q_1) \# L(p_2, q_2)$ that preserves V. ∎

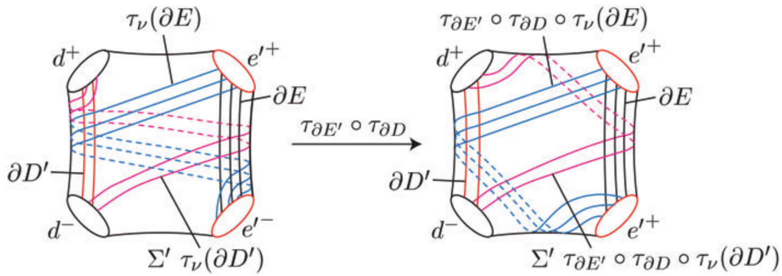

Fig. 8. The maps $\tau_{\partial E'} \circ \tau_{\partial D} \circ \tau_\nu$ extends to homeomorphisms of both V and W.

Lemma 3.3. Let M_1 be a lens space or $S^2 \times S^1$ and let M_2 be a lens space. Let $(V, W; \Sigma)$ be a genus two Heegaard splitting for $M_1 \# M_2$. Then the Goeritz group of $(V, W; \Sigma)$ acts transitively on the set of Haken spheres of $(V, W; \Sigma)$. □

Proof. The case where M_1 is a lens space follows from Theorem 3.1 and Lemma 3.2. Assume $M_1 = S^2 \times S^1$. Let P be a Haken sphere of $(V, W; \Sigma)$. Then P cuts V into two solid tori V_1 and V_2, and W into W_1 and W_2. We may assume that $V_1 \cup W_1$ is a punctured $S^2 \times S^1$. Let D and E be the meridian disks of V_1 and V_2, respectively, disjoint from P. Similarly, let D' and E' be the meridian disks of W_1 and W_2, respectively, disjoint from P. In this case, we may assume that $\partial D = \partial D'$. As we have seen in Section 1.2, D is the unique reducing disk in V. Let Σ_D be a two-holed torus obtained by cutting Σ along ∂D. We denote the boundary circles of Σ_D by d^+ and d^-. Then there exists a simple arc α_P in Σ_D connecting $d+$ and d^- such that $P \cap \Sigma_D$ is the frontier of a regular neighborhood of $d^+ \cup \alpha_P \cup d^-$. Let Q be another Haken sphere of $(V, W; \Sigma)$. By the same argument as above, there exists a simple arc α_Q in Σ_D connecting $d+$ and d^- such that $Q \cap \Sigma_D$ is the frontier of a regular neighborhood of $d^+ \cup \alpha_Q \cup d^-$. Then there exists a hoeomorphism φ of Σ_D defined by pushing d^+ in such a way that φ maps α_P to α_Q (see Figure 9).

The homeomorphism φ extends to a slide of a foot of a handle of each of V and W, and so φ extends to a homeomorphism of $M_1 \# M_2$ that preserves V and W, and takes P to Q up to isotopy. ■

4 Classification of Genus Two Heegaard Splittings

For $i \in \{1, 2\}$, let M_i be a lens space $L(p_i, q_i)$ or $S^2 \times S^1$, and let $(V_i, W_i; \Sigma_i)$ be a genus one Heegaard splitting for M_i. By [3], Σ_i is the unique genus one Heegaard surface for M_i up to isotopy, and there exists an orientation-preserving homeomorphism of M_i that

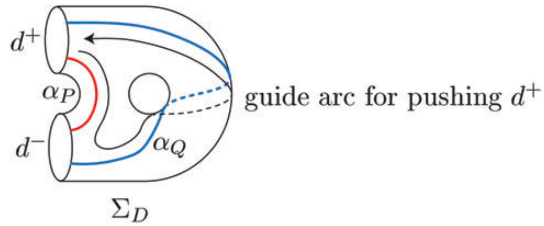

Fig. 9. Pushing d^+ along the guide arc maps α_P to α_Q.

interchanges V_i and W_i if and only if ${q_i}^2 \equiv 1 \pmod{p_i}$ or $M_i = S^2 \times S^1$. Let B_i be a 3-ball embedded in M_i so that $B_i \cap \Sigma_i$ is a single disk properly embedded in B_i. A genus two Heegaard splitting (V, W, Σ) for $M_1 \# M_2$ is created by gluing V_1 and V_2 to obtain V, and W_1 and W_2 to obtain W, by an appropriate orientation-reversing map $\partial B_1 \to \partial B_2$ after removing the interiors of B_1 and B_2 from M_1 and M_2, respectively. Also, another genus two Heegaard splitting $(V', W'; \Sigma')$ for $M_1 \# M_2$ is created by gluing V_1 and W_2 to obtain V', and W_1 and V_2 to obtain W' in the same way. From [13], it is known that each genus two Heegaard surface for $M_1 \# M_2$ is one of the above two Heegaard surfaces Σ and Σ' modulo the homeomorphisms of $M_1 \# M_2$. However, it is shown in [2] that Σ and Σ' do not always coincide modulo homeomorphisms of M. In [23], genus two Heegaard surfaces for $L(p_1, q_1) \# L(p_2, q_2)$ modulo the homeomorphisms of $L(p_1, q_1) \# L(p_2, q_2)$ are classified when $p_1 = p_2$ as follows.

Theorem 4.1 ([23]). Let M be the connected sum of two lens spaces $L(p, q_1)$ and $L(p, q_2)$. Then there exists a unique genus two Heegaard surface for M modulo homeomorphisms of M if and only if ${q_1}^2 \equiv 1$ or ${q_2}^2 \equiv 1 \pmod{p}$, and two Heegard surfaces otherwise. □

The following is a generalization of Theorem 4.1 to the case of all nonprime 3-manifolds which admit genus two Heegaard splittings.

Theorem 4.2. Let M be a connected sum of M_1 and M_2, where M_i is a lens space $L(p_i, q_i)$ or $S^2 \times S^1$ for $i \in \{1, 2\}$. Then there exists a unique genus two Heegaard surface for M modulo homeomorphisms of M if and only if one of M_i is $S^2 \times S^1$ or a lens space $L(p_i, q_i)$ with ${q_i}^2 \equiv 1 \pmod{p_i}$, and two Heegard surfaces otherwise. □

Proof. The "if" part follows trivially from the descriptive comments at the beginning of this section.

Now we prove the "only if" part. Suppose that both of M_i are lens spaces $L(p_i, q_i)$ with $q_i^2 \not\equiv 1 \pmod{p_i}$. Let $(V, W; \Sigma)$ and $(V', W'; \Sigma')$ be the two Heegaard splittings of $M = L(p_1, q_1)\#L(p_2, q_2)$ obtained from the genus one Heegaard splittings $(V_i, W_i; \Sigma_i)$ of $L(p_i, q_i)$, $i = 1, 2$, as described in the beginning of this section. The construction provides the Haken spheres P and P' for the splittings $(V, W; \Sigma)$ and $(V', W'; \Sigma')$, respectively. We give an orientation of P and P' so that the $L(p_1, q_1)$-summand lies in the negative side. Suppose that there exists a homeomorphism f of M that maps Σ' to Σ. Then, by Lemma 3.3, there exists a homeomorphism g of M that preserves Σ and that maps $f(P')$ to P. We may assume that P and $g \circ f(P')$ have the same orientation. Moreover, we may assume that $g \circ f$ induces a homeomorphism of $L(p_1, q_1)$ that preserves V_1 and W_1. Since $q_1^2 \not\equiv 1 \pmod{p_1}$, $g \circ f$ is orientation preserving. Now $g \circ f$ induces an orientation-preserving homeomorphism of $L(p_2, q_2)$ that interchanges V_2 and W_2, contradicting $q_2^2 \equiv 1 \pmod{p_2}$. ■

5 Genus Two Goeritz Groups

Let $(V, W; \Sigma)$ be a genus two Heegaard splitting for the connected sum of two lens spaces. A Haken sphere P of $(V, W; \Sigma)$ is said to be *reversible* if there exists an element g of $\mathcal{G}$ fixing P setwise such that g restricted to P is an orientation-reversing homeomorphism on P. We say that the splitting $(V, W; \Sigma)$ is *symmetric* if it admits a reversible Haken sphere. By Lemma 3.3, if the splitting $(V, W; \Sigma)$ admits a reversible Haken sphere, then every Haken sphere of $(V, W; \Sigma)$ is reversible.

For a genus two Heegaard splitting $(V, W; \Sigma)$ for the connected sum of two lens spaces, we fix the following notations throughout the section (see Figure 10).

(1) Disjoint, nonparallel semi-primitive disks D and E in V,
(2) the disjoint semi-primitive disks D' and E' in W such that $D \cap E' = E \cap D' = \emptyset$ (such D' and E' are determined uniquely by Lemma 1.5),
(3) the Haken sphere P of $(V, W; \Sigma)$ disjoint from $D \cup E$ (the existence and uniqueness of P follows from Lemma 1.7) and
(4) a Haken sphere Q_1 (Q_2, respectively) of $(V, W; \Sigma)$ disjoint from $D \cup E'$ ($E \cup D'$, respectively) such that $|P \cap \Sigma \cap Q_1| = 4$ ($|P \cap \Sigma \cap Q_2| = 4$, respectively) (the existence of Q_1 and Q_2 follows from the proof of Lemma 1.4).

In Figure 10, the four-holed sphere Σ' is obtained by cutting Σ along $\partial D \cup \partial E'$ and the boundary circles d^+ and d^- (e'^+ and e'^-, respectively) come from ∂D ($\partial E'$, respectively). By $\alpha \in \mathcal{G}$, we denote the hyperelliptic involution of both V and W. By $\beta \in \mathcal{G}$, we denote

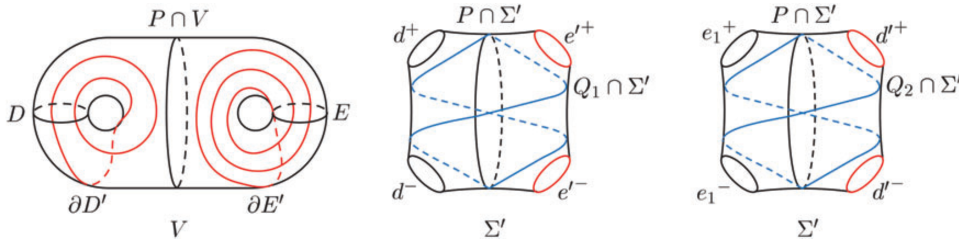

Fig. 10. Fixed notations for the connected sum of two lens spaces.

the extension of a half-Dehn twist about the disk $P \cap V$. By $\gamma_1 \in \mathcal{G}$ ($\gamma_2 \in \mathcal{G}$, respectively), we denote an element of order 2 that preserves $D \cup E'$ ($E \cup D'$, respectively) and that interchanges P and Q_1 (P and Q_2, respectively) (the existence of this element will be proved in Lemma 1.4). When P is reversible, we denote by $\delta \in \mathcal{G}$ an element of order 2 that reverses P.

Also, for a genus two Heegaard splitting $(V, W; \Sigma)$ for the connected sum of $S^2 \times S^1$ and a lens space, we fix the following notations throughout the section (see Figure 11).

(1) The reducing disk D in V and the disk D' in W bounded by ∂D (D is unique by Lemma 1.9),
(2) disjoint, nonisotopic, semi-primitive disks E_1 and E_2 in V,
(3) a semi-primtive disk E' in W such that $\partial E'$ has the same type with respect to E_1 and E_2 (the existence of E' will follow from the proof of Lemma 5.6) and
(4) Haken spheres P and Q of $(V, W; \Sigma)$ disjoint from $D \cup E_1$ such that $|P \cap \Sigma \cap Q| = 4$ (the existence of P and Q follows from the proof of Lemma 1.9(1)).

In Figure 11., the four-holed sphere Σ' is obtained by cutting Σ along $\partial D \cup \partial E_1$ and the boundary circles d^+ and d^- (${e_1}^+$ and ${e_1}^-$, respectively) come from ∂D (∂E_1, respectively). By $\alpha \in \mathcal{G}$, we denote the hyperelliptic involution of both V and W. By $\beta \in \mathcal{G}$ ($\tau \in \mathcal{G}$, respectively), we denote the extension of a half-Dehn twist (Dehn twist, respectively) about the disk $P \cap V$ (D, respectively). By $\gamma \in \mathcal{G}$, we denote an element of order 2 that interchanges P and Q (the existence of this element is proved in Lemma 5.4). By $\sigma \in \mathcal{G}$, we denote an element of order 2 that preserve E' and that interchanges E_1 and E_2 (the existence of this element will be proved in Lemma 5.6).

Now we are ready to state the main theorem, which provides presentations of genus two Goeritz groups of all nonprime 3-manifolds. (Recall that the genus two Goeritz group of $(S^2 \times S^1)\#(S^2 \times S^1)$ is the mapping class group of the genus two handlebody and its presentation is already known.)

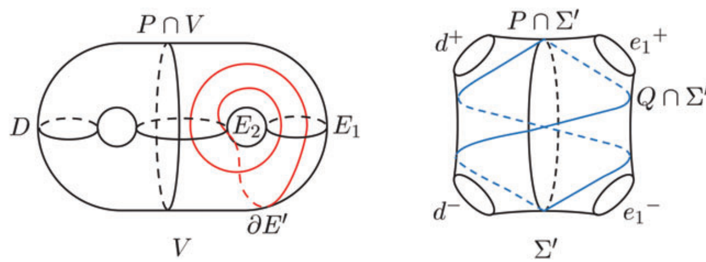

Fig. 11. Fixed notations for the connected sum of $S^2 \times S^1$ and a lens space.

Theorem 5.1. Let M_1 be a lens space or $S^2 \times S^1$, and let M_2 be a lens space. Let $(V, W; \Sigma)$ be a genus two Heegaard splitting for $M_1 \# M_2$. Then the Goeritz group $\mathcal{G}$ of $(V, W; \Sigma)$ has the following presentation:

(1) If M_1 is a lens space,

(a) $\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma_1, \gamma_2 \mid {\gamma_1}^2, {\gamma_2}^2 \rangle$ if $(V, W; \Sigma)$ is not symmetric;

(b) $\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma_1, \delta \mid {\gamma_1}^2, \delta^2, \delta\beta\delta = \alpha\beta \rangle$ if $(V, W; \Sigma)$ is symmetric;

(2) If $M_1 = S^2 \times S^1$, $\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma, \sigma \mid \gamma^2, \sigma^2 \rangle \oplus \langle \tau \rangle$. □

We remark that, from Section 4, once a genus two Heegaard splitting for $M = L(p_1, q_1) \# L(p_2, q_2)$ is given, we may easily determine whether the splitting is symmetric or not. If $L(p_1, q_1) \ncong L(p_2, q_2)$ (as oriented manifolds), no genus two Heegaard splitting of M is symmetric. If $L(p_1, q_1) \cong L(p_2, q_2)$, exactly one genus two Heegaard splitting of M is symmetric and the other, if any, is not.

Throughout the section, for suitable subsets $A_1, A_2, \ldots, A_k$ of $M_1 \# M_2$, we denote by $\mathcal{G}_{\{A_1, A_2, \ldots, A_k\}}$ the subgroup of its Goeritz group $\mathcal{G}$ consisting of elements that preserve each of $A_1, A_2, \ldots, A_k$ setwise.

Lemma 5.2. Let M_1 be a lens space or $S^2 \times S^1$ and let M_2 be a lens space. Let $(V, W; \Sigma)$ be a genus two Heegaard splitting for $M_1 \# M_2$.

(1) If M_1 is a lens space, then $\mathcal{G}_{\{D, P\}} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta \rangle$.

(2) If M_1 is $S^2 \times S^1$, then $\mathcal{G}_{\{D, P\}} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta \rangle \oplus \langle \tau \rangle$. □

Proof. Let g be an element of $\mathcal{G}_{\{D, P\}}$.

(1) Since g preserves D, g is orientation preserving on P. We may assume that g maps each of the disks D, D', E and E' to itself. Moreover if g is orientation preserving on

D (E, respectively), then so is on D' (E', respectively). Hence by taking a composition with α and β, if necessary, we may assume that g fixes $D \cup D' \cup E \cup E'$. Now, Σ cut-off by $D \cup D' \cup E \cup E'$ consists of several disks and a single annulus. By Alexander's trick, boundary-preserving homeomorphisms on a disk is unique up to isotopy. Also, boundary-preserving homeomorphisms on an annulus are determined by Dehn twist about its core circle up to isotopy. This implies that g is a power of β.

(2) Let l be a simple closed curve in Σ disjoint from P that intersects ∂D in a single point. Let g be an element of $\mathcal{G}_{\{D,P\}}$. Since g preserves D, g is orientation preserving on P. We may assume that g maps each of the disks D, D', E and E' to itself. Moreover if g is orientation preserving on D (E, respectively), then so is on D' and l (E' and l, respectively). Hence modulo the action of α and τ, g fixes $D \cup D' \cup l \cup E \cup E'$. The remaining argument is exactly the same as (1). ■

Lemma 5.3. Let M_1 be a lens space or $S^2 \times S^1$, and let M_2 be a lens space. Let $(V, W; \Sigma)$ be a genus two Heegaard splitting for $M_1 \# M_2$.

(1) Suppose that M_1 is a lens space. Let Q_1' be a Haken sphere of $(V, W; \Sigma)$ disjoint from $D \cup E'$ such that $|P \cap \Sigma \cap Q_1'| = 4$. Then a power of β maps Q_1 to Q_1'.

(2) Suppose that M_1 is $S^2 \times S^1$. Let Q' be a Haken sphere of $(V, W; \Sigma)$ disjoint from $D \cup E$ such that $|P \cap \Sigma \cap Q'| = 4$. Then a power of β maps Q to Q'. □

Proof. (1) Let Σ' be the four-holed sphere obtained by cutting Σ along $\partial D \cup \partial E'$. Let d^+ and d^- (e'^+ and e'^-, respectively) be the two boundary circles of Σ' coming from ∂D ($\partial E'$, respectively). Let α_P, α_{Q_1} and $\alpha_{Q_1'}$ be simple arcs in Σ' connecting d^+ and d^- such that the frontiers of regular neighborhoods of $d^+ \cup \alpha_P \cup d^-$, $d^+ \cup \alpha_{Q_1} \cup d^-$ and $d^+ \cup \alpha_{Q_1'} \cup d^-$ are $P \cap \Sigma$, $Q_1 \cap \Sigma$ and $Q_1' \cap \Sigma$, respectively. We may assume that $\alpha_P \cap \alpha_{Q_1} = \alpha_P \cap \alpha_{Q_1'} = \emptyset$ since $|P \cap \Sigma \cap Q_1| = |P \cap \Sigma \cap Q_1'| = 4$. Since α_P cuts Σ' into a pair of pants, a certain power of β carries α_{Q_1} to $\alpha_{Q_1'}$. The proof of (2) is exactly the same as (1). ■

Lemma 5.4. Let M_1 be a lens space or $S^2 \times S^1$ and let M_2 be a lens space. Let $(V, W; \Sigma)$ be a genus two Heegaard splitting for $M_1 \# M_2$.

(1) If M_1 is a lens space, then $\mathcal{G}_{\{D,P,Q\}} = \langle \alpha \mid \alpha^2 \rangle$, and $\mathcal{G}_{\{D,P\cup Q\}} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \gamma_1 \mid {\gamma_1}^2 \rangle$.

(2) If M_1 is $S^2 \times S^1$, then $\mathcal{G}_{\{D,P,Q\}} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \tau \rangle$, and $\mathcal{G}_{\{D,P\cup Q\}} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \gamma \mid \gamma^2 \rangle \oplus \langle \tau \rangle$. □

Proof. (1) We first show the existence of the element $\gamma_1 \in \mathcal{G}$. Let β_1' denote a half-Dehn twist about the sphere Q_1. By Lemma 3.3, there exists an element $g \in \mathcal{G}$ that carries P to Q_1. We may assume without loss of generality that g maps D to D and E' to E'. By Lemma 5.3, a certain power ${\beta_1'}^n$ of β_1' carries $g(Q_1)$ to P. We remark that ${\beta_1'}^n \circ g$ interchanges P and Q_1 and this map carries D to D and E' to E'. Up to isotopy, we may assume that $({\beta_1'}^n \circ g)^2$ fixes $D \cup E' \cup P \cup Q_1$. Then by cutting Σ along $\partial D \cup \partial E'$ and considering simple arcs connecting the two holes coming from ∂D as in the proof of Lemma 5.3, we can easily check that $({\beta_1'}^n \circ g)^2$ restricted to Σ is a power of Dehn twist along $\partial E'$. Hence $({\beta_1'}^n \circ g)^2$ is isotopic to the identity. This implies that ${\beta_1'}^n \circ g$ is the required element γ_1. Since τ is commutative with any element of $\mathcal{G}$ that preserves D, (2) follows from the same argument as (1). ■

Lemma 5.5. Let M_1 be a lens space or $S^2 \times S^1$, and let M_2 be a lens space. Let $(V, W; \Sigma)$ be a genus two Heegaard splitting for $M_1 \# M_2$. Let D be a semi-primitive disk in V.

(1) If M_1 is a lens space, then $\mathcal{G}_{\{D\}} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma \mid \gamma^2 \rangle$.

(2) If M_1 is $S^2 \times S^1$, then $\mathcal{G}_{\{D, E_1\}} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma \mid \gamma^2 \rangle \oplus \langle \tau \rangle$. □

Proof. (1) By Lemma 1.5, E' is the unique non-separating disk in W disjoint from D. This implies that each element of $\mathcal{G}_D$ preserves E'. Let Σ' be the four-holed sphere obtained by cutting Σ along $\partial D \cup \partial E'$. Let d^+ and d^- (e'^+ and e'^-, respectively) be the two boundary circles of Σ' coming from ∂D ($\partial E'$, respectively). As in the proof of Theorem 3.1, let $\mathcal{H}_D$ be the full subcomplex of the complex $\mathcal{H}$ of Haken spheres of $(V, W; \Sigma)$ spanned by the vertices corresponding to Haken spheres disjoint from D. Then $\mathcal{H}_D$ is a tree as we have seen in Lemma 3.1. Let $\mathcal{H}'_D(V)$ be the first barycentric subdivision of $\mathcal{G}_D$. The group $\mathcal{G}_D$ acts on $\mathcal{H}'_D(V)$ simplicially. Moreover, the quotient of $\mathcal{H}'_D(V)$ by the action of $\mathcal{G}_D$ is a single edge. Then by the Bass–Serre theory on groups acting on trees [28], we have $\mathcal{G}_{\{D\}} = \mathcal{G}_{\{D,P\}} *_{\mathcal{G}_{\{D,P,Q_1\}}} \mathcal{G}_{\{D,P\cup Q_1\}}$. Now, (1) follows from Lemmas 5.2 and 5.4.

(2) Cutting Σ along $D \cup E_1$ instead of $D \cup E'$, we get the presentation by almost the same argument as (1). ■

Lemma 5.6. Let $(V, W; \Sigma)$ be the genus two Heegaard splitting for the connected sum of $S^2 \times S^1$ and a lens space. Let E_1 and E_2 be disjoint, nonisotopic, semi-primitive and nonreducing disks in V. Then there exists an element of the Goeritz group $\mathcal{G}$ of the Heegaard splitting $(V, W; \Sigma)$ that interchanges E_1 and E_2. □

Proof. It is easy to see that there exists a nonreducing semi-primitive disk $\hat{E}_2$ in V such that E_1 and $\hat{E}_2$ can be interchanged by an element of $\mathcal{G}$. Thus, it suffices to show that there exists an element of $\mathcal{G}$ that preserves E_1 and that maps E_2 to $\hat{E}_2$. Let Σ_D be a two-holed torus obtained by cutting Σ along ∂D. We denote the boundary circles of Σ_D by d^+ and d^-. Since both E_2 and $\hat{E}_2$ are meridian disks of the solid torus obtained by cutting V along D, there exists a pushing of d^+ in Σ_D that preserve ∂E_1, and that maps ∂E_2 to $\partial \hat{E}_2$. As we have seen in Lemma 3.3, every pushing map of d^+ extends to a slide of a foot of a handle of each of V and W, thus it extends to a homeomorphism of $(S^2 \times S^1)\#L(p,q)$ that preserves V. ■

Finally, the following two lemmas follow from Lemmas 1.7 and 5.5.

Lemma 5.7. Let $(V, W; \Sigma)$ be a genus two Heegaard splitting for the connected sum of two lens spaces.

(1) If $(V, W; \Sigma)$ is not symmetric, then $\mathcal{G}_{\{D,E\}} = \mathcal{G}_{\{D\cup E\}} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta \rangle$.

(2) If $(V, W; \Sigma)$ is symmetric, then $\mathcal{G}_{\{D,E\}} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta \rangle$ and $\mathcal{G}_{\{D\cup E\}} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \delta \mid \delta^2, \delta\beta\delta = \alpha\beta \rangle$. □

Lemma 5.8. Let $(V, W; \Sigma)$ be the genus two Heegaard splitting for the connected sum of $S^2 \times S^1$ and a lens space. Then $\mathcal{G}_{\{D,E_1,E_2\}} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \tau \rangle$ and $\mathcal{G}_{\{D,E_1\cup E_2\}} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \sigma \mid \sigma^2 \rangle \oplus \langle \tau \rangle$. □

Proof of Theorem 5.1. (1a) By Theorem 2.2, $\mathcal{SP}(V)$ is a tree. By Lemmas 3.3, the vertices modulo the action of $\mathcal{G}$ consists of two classes, one contains D and the other contains E. Also, any edge of $\mathcal{SP}(V)$ is equal to the edge $\{D, E\}$ modulo the action of $\mathcal{G}$. Therefore, the quotient of $\mathcal{SP}(V)$ by the action of $\mathcal{G}$ is an edge. Now by the Bass–Serre theory and Lemmas 5.5 and 5.7, we have

$$\begin{aligned}
\mathcal{G} &= \mathcal{G}_{\{D\}} *_{\mathcal{G}_{\{D,E\}}} \mathcal{G}_{\{E\}} \\
&= (\mathcal{G}_{\{D,P\}} *_{\mathcal{G}_{\{D,P,Q_1\}}} \mathcal{G}_{\{D,P\cup Q_1\}}) *_{\mathcal{G}_{\{D,P\}}} (\mathcal{G}_{\{E,P\}} *_{\mathcal{G}_{\{D,E,Q_2\}}} \mathcal{G}_{\{E,P\cup Q_2\}}) \\
&= (\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma_1 \mid {\gamma_1}^2 \rangle) *_{\langle \alpha|\alpha^2\rangle\oplus\langle\beta\rangle} (\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma_2 \mid {\gamma_2}^2 \rangle) \\
&= \langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma_1, \gamma_2 \mid {\gamma_1}^2, {\gamma_2}^2 \rangle.
\end{aligned}$$

(1b) Again by Theorem 2.2, $\mathcal{SP}(V)$ is a tree. Let $\mathcal{SP}'(V)$ be the first barycentric subdivision of $\mathcal{SP}(V)$. We note that the vertices of $\mathcal{SP}'(V)$ consists of the vertices of $\mathcal{SP}(V)$ and the barycenters of the edges of $\mathcal{SP}(V)$, each of which corresponds to an unordered pair of vertices. By Lemmas 3.3, every vertex of $\mathcal{SP}'(V)$ is equal to the vertex D or the barycenter $\{D, E\}$, and any edge of $\mathcal{SP}'(V)$ is equal to the edge $\{D, \{D, E\}\}$ modulo the action of $\mathcal{G}$. Therefore, the quotient of $\mathcal{SP}'(V)$ by the action of $\mathcal{G}$ is an edge. By the Bass–Serre theory and Lemmas 5.5 and 5.7, we have

$$\begin{aligned}\mathcal{G} &= \mathcal{G}_{\{D\}} *_{\mathcal{G}_{\{D,E\}}} \mathcal{G}_{\{D\cup E\}}\\ &= (\mathcal{G}_{\{D,P\}} *_{\mathcal{G}_{\{D,P,Q_1\}}} \mathcal{G}_{\{D,P\cup Q_1\}}) *_{\mathcal{G}_{\{D,P\}}} (\mathcal{G}_{\{D\cup E\}})\\ &= (\langle \alpha \mid \alpha^2\rangle \oplus \langle \beta, \gamma_1 \mid {\gamma_1}^2\rangle) *_{\langle\alpha|\alpha^2\rangle\oplus\langle\beta\rangle} (\langle \alpha \mid \alpha^2\rangle \oplus \langle \beta, \delta \mid \delta^2, \delta\beta\delta = \alpha\beta\rangle)\\ &= \langle \alpha \mid \alpha^2\rangle \oplus \langle \beta, \gamma_1, \delta \mid {\gamma_1}^2, \delta^2, \delta\beta\delta = \alpha\beta\rangle.\end{aligned}$$

(2) We note that $\mathcal{G} = \mathcal{G}_{\{D\}}$. Let $\mathcal{SP}'_D(V)$ be the first barycentric subdivision of $\mathcal{SP}_D(V)$. By Lemma 5.6, the quotient of $\mathcal{SP}'_D(V)$ by the action of $\mathcal{G}$ consists of an edge. By the Bass–Serre theory and Lemmas 5.5 and 5.7, we have

$$\begin{aligned}\mathcal{G}_{\{D\}} &= \mathcal{G}_{\{D,E_1\}} *_{\mathcal{G}_{\{D,E_1,E_2\}}} \mathcal{G}_{\{D,E_1\cup E_2\}}\\ &= (\langle \alpha \mid \alpha^2\rangle \oplus \langle \beta, \gamma \mid \gamma^2\rangle \oplus \langle\tau\rangle) *_{\langle\alpha|\alpha^2\rangle\oplus\langle\tau\rangle} (\langle \alpha \mid \alpha^2\rangle \oplus \langle \sigma \mid \sigma^2\rangle \oplus \langle\tau\rangle)\\ &= \langle \alpha \mid \alpha^2\rangle \oplus \langle \beta, \gamma, \sigma \mid \gamma^2, \sigma^2\rangle \oplus \langle\tau\rangle.\end{aligned}$$

This completes the proof. ■

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References

[1] Akbas, E. "A presentation for the automorphisms of the 3-sphere that preserve a genus two Heegaard splitting." *Pacific Journal of Mathematics* 236, no. 2 (2008): 201–22.

[2] Birman, J. "On the Equivalence of Heegaard splittings of Closed, Orientable 3-Manifolds." *Knots, Groups, and* 3-*Manifolds* (*Papers dedicated to the memory of R. H. Fox*), 137–64. Annals of Mathematics Studies 84, Princeton, NJ, Princeton University Press, 1975.

[3] Bonahon, F. and J.-P. Otal. "Scindements de Heegaard des espaces lenticulaires." *Annales Scientifiques de École Normale Superieure (4)* 16 (1983): 451–66.

[4] Cho, S. "Homeomorphisms of the 3-sphere that preserve a Heegaard splitting of genus two." *Proceedings of the American Mathematical Society* 136, no. 3 (2008): 1113–23.

[5] Cho, S. "Genus two Goeritz groups of lens spaces." *Pacific Journal of Mathematics* 265, no. 1 (2013): 1–16.

[6] Cho, S. and Y. Koda. "Primitive disk complexes for lens spaces." (2012): preprint, arXiv:1206.6243.

[7] Cho, S. and Y. Koda. "The genus two Goeritz group of $S^2 \times S^1$." *Mathematical Research Letters* (2013): preprint, arXiv:1303.7145.

[8] Cho, S. and Y. Koda. "Primitive disk complexes and Goeritz groups of lens spaces." (in preparation).

[9] Cho, S., D. McCullough, and A. Seo. "Arc distance equals level number." *Proceedings of the American Mathematical Society* 137, no. 8 (2009): 2801–7.

[10] Goeritz, L. "Die Abbildungen der Brezelfläche und der Vollbrezel vom Geschlecht 2." *Abhandlungen aus dem Mathematischen Seminar der Universitt Hamburg* 9, no. 1 (1933): 244–59.

[11] Gordon, C. "Problems." *Workshop on Heegaard Splittings*, 401–11. Geometry and Topology Monographs 12. Coventry: Geometry and Topology Publication, 2007.

[12] Grasse, P., "Finite presentation of mapping class groups of certain three-manifolds." *Topology and its Applications* 32, no. 3 (1989): 295–305.

[13] Haken, W. "Some Results on Surfaces in 3-Manifolds." *Studies in Modern Topology*, 39–98. Mathematical Association of America, distributed by Englewood Cliffs, NJ: Prentice-Hall, 1968.

[14] Hempel, J. "3-manifolds as viewed from the curve complex." *Topology* 40, no. 3 (2001): 631–57.

[15] Johannson, K. *Topology and Combinatorics of 3-Manifolds*. Lecture Notes in Mathematics 1599. Berlin: Springer, 1995.

[16] Johnson, J. "Mapping class groups of medium distance Heegaard splittings." *Proceedings of the American Mathematical Society* 138, no. 12 (2010): 4529–35.

[17] Johnson, J. "Heegaard splittings and open books." (2011): preprint, arXiv:1108.5302.

[18] Johnson, J. "Mapping class groups of Heegaard splittings of surface bundles." (2012): preprint, arXiv:1201.2628.

[19] Koda, Y. "Automorphisms of the 3-sphere that preserve spatial graphs and handlebody-knots." (2011): preprint, arXiv:1106.4777.

[20] Lei, F. "Haken spheres in the connected sum of two lens spaces." *Mathematical Proceedings of the Cambridge Philosophical Society* 138, no. 1 (2005): 97–105.

[21] Lei, F. and Y. Zhang. "Haken spheres in genus 2 Heegaard splittings of nonprime 3-manifolds." *Topology and its Applications* 142, no. 1–3 (2004): 101–11.

[22] McCullough, D. "Virtually geometrically finite mapping class groups of 3-manifolds." *Journal of Differential Geometry* 33, no. 1 (1991): 1–65.

[23] Montesinos, J. and C. Safont. "On the Birman invariants of Heegaard splittings." *Pacific Journal of Mathematics* 132, no. 1 (1988): 113–42.

[24] Namazi, H. "Big Heegaard distance implies finite mapping class group." *Topology and its Applications* 154, no. 16 (2007): 2939–49.

[25] Osborne, R. P. and H. Zieschang. "Primitives in the free group on two generators." *Inventiones Mathematicae* 63, no. 1 (1981): 17–24.

[26] Scharlemann, M. "Automorphisms of the 3-sphere that preserve a genus two Heegaard splitting." *Boletin Sociedad Matematica Mexicana* 10 (2004): 503–14.

[27] Seo, A. "Torus leveling of $(1, 1)$-knots." dissertation at the University of Oklahoma, 2008.

[28] Serre, J.-P. *Arbres, amalgames,* SL_2. Astérisque 46. Paris: Société Mathématique de France, 1977.

[29] Suzuki, S. "On homeomorphisms of a 3-dimensional handlebody." *Canadian Journal of Mathematics* 29, no. 1 (1977): 111–24.

[30] Wajnryb, B. "Mapping class group of a handlebody." *Fundamenta Mathematicae* 158, no. 3 (1998): 195–228.