Lo'lo' et al. *Journal of Inequalities and Applications* 2014, 2014:517
http://www.journalofinequalitiesandapplications.com/content/2014/1/517

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Existence of a common solution of an integral equations system by (ψ,α,β)-weakly contractions

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Abstract

In this paper, we consider a system of integral equations and apply the coincidence and common fixed point theorems for four mappings satisfying a (ψ,α,β)-weakly contractive condition in ordered metric spaces to prove the existence of a common solution to integral equations. Also we furnish suitable examples to demonstrate the validity of the hypotheses of our results.

MSC: 54H25; 47H10

Keywords: coincidence point; common fixed point; partially weakly increasing mappings; compatible pair of mappings; weakly compatible pair of mappings; semi-compatible pair of mapping; (ψ,α,β)-weakly contraction; reciprocally continuous mappings; f-weak reciprocally continuous mappings

1 Introduction and preliminary

Fixed point theory has wide and endless applications in many fields of engineering and science. Its core, the Banach contraction principle (see [1]), has attracted many researchers who tried to generalize it in different aspects. In particular, Alber and Guerre-Delabriere [2] introduced the concept of weak contractions in Hilbert spaces. Rhoades [3] showed that the result which Alber *et al.* had proved in Hilbert spaces was also valid in complete metric spaces. Eshaghi Gordji *et al.* [4] proved a new coupled fixed point theorem related to the Pata contraction for mappings having the mixed monotone property in partially ordered metric spaces. Singh *et al.* [5] obtained coincidence and common fixed point theorems for a class of Suzuki hybrid contractions involving two pairs of single-valued and multi-valued maps in a metric space.

Definition 1.1 ([6]) The function $\psi:[0,+\infty)\to[0,+\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is continuous and non-decreasing,

(ii) $\psi(t)=0$ if and only if $t=0$.

Definition 1.2 ([3]) Let (X,d) be a metric space. A mapping $f:X\to X$ is said to be weakly contractive if

$$d(fx,fy)\le d(x,y)-\varphi\bigl(d(x,y)\bigr)\quad\text{for each } x,y\in X,$$

where $\varphi:[0,+\infty)\to[0,+\infty)$ is an altering distance function.

In [3], Rhoades proved that if X is complete, then every weak contraction has a unique fixed point.

The weak contraction principle, its generalizations and extensions and other fixed point results for mappings satisfying weak contractive type inequalities have been considered in a number of recent works.

In 2008, Dutta and Choudhury [7] proved the following theorem.

Theorem 1.3 ([7]) *Let (X,d) be a complete metric space and $f : X \to X$ be such that*

$$\psi\big(d(fx,fy)\big) \le \psi\big(d(x,y)\big) - \varphi\big(d(x,y)\big) \quad \text{for each } x,y \in X,$$

where $\psi,\varphi : [0,+\infty) \to: [0,+\infty)$ are altering distance functions. Then f has a fixed point in X.

In [8], Eslamian and Abkar introduced the concept of (ψ,α,β)-weak contraction. They stated the following theorem as a generalization of Theorem 1.3.

Theorem 1.4 ([8]) *Let (X,d) be a complete metric space and $f : X \to X$ be a mapping satisfying*

$$\psi\big(d(fx,fy)\big) \le \alpha\big(d(x,y)\big) - \beta\big(d(x,y)\big)$$

for all $x,y \in X$, where $\psi,\alpha,\beta : [0,+\infty) \to [0,+\infty)$ are such that ψ is an altering distance function, α is continuous, β is lower semi-continuous, and

$$\psi(t) - \alpha(t) + \beta(t) > 0 \quad \text{for each } t > 0,$$

and $\alpha(0) = \beta(0) = 0$. Then f has a unique fixed point.

Aydi *et al.* [9] proved that Theorem 1.4 is a consequence of Theorem 1.3. (Define $\varphi : [0,+\infty) \to [0,+\infty)$ by $\varphi(t) = \psi(t) - \alpha(t) + \beta(t)$ for all $t \ge 0$.)

It is also known that common fixed point theorems are generalizations of fixed point theorems. Recently, many researchers have been interested in generalizing fixed point theorems to coincidence point theorems and common fixed point theorems.

Definition 1.5 ([10]) Let X be a non-empty set, N be a natural number such that $N \ge 2$ and $f_1,f_2,\ldots,f_{N-1},f_N : X \to X$ be given self-mappings of X. If $w = f_1x = f_2x = \cdots = f_{N-1}x = f_Nx$ for some $x \in X$, then x is called a coincidence point of $f_1,f_2,\ldots,f_{N-1}$ and f_N, and w is called a point of coincidence of $f_1,f_2,\ldots,f_{N-1}$ and f_N. If $w = x$, then x is called a common fixed point of $f_1,f_2,\ldots,f_{N-1}$ and f_N.

On the other hand, compatibility of two mappings introduced by Jungck [11, 12] is an important concept in the context of common fixed point problems in metric spaces.

Definition 1.6 ([11]) Let (X,d) be a metric space and $f,g : X \to X$ be given self-mappings on X. The pair (f,g) is said to be compatible if $\lim_{n\to\infty} d(fgx_n,gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.

Definition 1.7 ([12]) Two mappings $f, g : X \to X$, where (X, d) is a metric space, are weakly compatible if they commute at their coincidence points, that is, if $ft = gt$ for some $t \in X$ implies that $fgt = gft$.

It is clear that if the pair (f, g) is compatible, then (f, g) is weakly compatible.

Recently, fixed point theory has developed rapidly in partially ordered metric spaces (for example, see [13–23] and the references therein). Harjani and Sadarangani in [19, 20] extended Theorem 1.3 in the framework of partially ordered metric spaces in the following way. In 2012, Choudhury and Kundu [24] established the (ψ, α, β)-weak contraction principle to coincidence point and common fixed point results in partially ordered metric spaces and proved the following fixed point theorem as a generalization of Theorem 1.4.

Theorem 1.8 ([24]) *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let $f, g : X \to X$ be such that $fX \subseteq gX$, f is g-non-decreasing, gX is closed and*

$$\psi\big(d(fx, fy)\big) \le \alpha\big(d(gx, gy)\big) - \beta\big(d(gx, gy)\big) \quad \text{for all } x, y \in X \text{ such that } gx \preceq gy,$$

where $\psi, \alpha, \beta : [0, +\infty) \to [0, +\infty)$ are such that ψ is continuous and monotone non-decreasing, α is continuous, β is lower semi-continuous,

$$\psi(t) - \alpha(t) + \beta(t) > 0 \quad \text{for all } t > 0,$$

and $\psi(t) = 0$ if and only if $t = 0$ and $\alpha(0) = \beta(0) = 0$. Also, if any non-decreasing sequence $\{x_n\}$ in X converges to z, then we assume $x_n \preceq z$ for all $n \in \mathbb{N} \cup \{0\}$. If there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point.

Altun and Simsek [15] introduced the concept of weakly increasing mappings as follows.

Definition 1.9 Let f, g be two self-maps on a partially ordered set $(X, \preceq)$. A pair (f, g) is said to be

(i) weakly increasing if $fx \preceq g(fx)$ and $gx \preceq f(gx)$ for all $x \in X$ [15],
(ii) partially weakly increasing if $fx \preceq g(fx)$ for all $x \in X$ [13].

Note that a pair (f, g) is weakly increasing if and only if the ordered pairs (f, g) and (g, f) are partially weakly increasing.

Nashine and Samet [25] introduced weakly increasing mappings with respect to another map as follows.

Definition 1.10 ([25]) Let $(X, \preceq)$ be a partially ordered set and $f, g, h : X \to X$ be given mappings such that $fX \subseteq hX$ and $gX \subseteq hX$. We say that f and g are weakly increasing with respect to h if and only if for all $x \in X$, we have

$$fx \preceq gy, \quad \forall y \in h^{-1}(fx)$$

and

$$gx \preceq fy, \quad \forall y \in h^{-1}(gx),$$

where $h^{-1}(x) := \{u \in X \mid hu = x\}$ for $x \in X$.

If $f = g$, we say that f is weakly increasing with respect to h.

If $h : X \to X$ is the identity mapping ($hx = x$ for all $x \in X$), then f and g being weakly increasing with respect to h implies that f and g are weakly increasing mappings.

Nashine *et al.* [26] proved some new coincidence point and common fixed point theorems for a pair of weakly increasing mappings with respect to another map.

In [17], Esmaily *et al.* gave the following definition.

Definition 1.11 ([17]) Let $(X, \preceq)$ be a partially ordered set and $f, g, h : X \to X$ be given mappings such that $fX \subseteq hX$. We say that (f, g) is partially weakly increasing with respect to h if and only if for all $x \in X$, we have

$$fx \preceq gy, \quad \forall y \in h^{-1}(fx).$$

Theorem 1.12 ([17]) *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let $f, g, S, T : X \to X$ be given mappings satisfying the following*:

(i) $fX \subseteq TX$, $gX \subseteq SX$,
(ii) f, g, S *and* T *are continuous,*
(iii) *the pairs* (f, S) *and* (g, T) *are compatible,*
(iv) (f, g) *is partially weakly increasing with respect to* T *and* (g, f) *is partially weakly increasing with respect to* S.

Suppose that for every $x, y \in X$ *such that* Sx *and* Ty *are comparable, we have*

$$\psi\big(d(fx, gy)\big) \le \psi\big(M(x, y)\big) - \phi\big(N(x, y)\big), \tag{1}$$

where

$$M(x, y) = \max\left\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), \frac{1}{2}\big[d(Sx, gy) + d(fx, Ty)\big]\right\},$$

$$N(x, y) = \max\big\{d(Sx, Ty), d(Sx, gy), d(Ty, fx)\big\},$$

and $\psi : [0, +\infty) \to [0, +\infty)$ *is an altering distance function, and* $\phi : [0, +\infty) \to [0, +\infty)$ *is a continuous function with* $\phi(t) = 0$ *if only if* $t = 0$. *Then the pairs* (f, S) *and* (g, T) *have a coincidence point* $u \in X$; *that is,* $fu = Su$ *and* $gu = Tu$. *Moreover, if* Su *and* Tu *are comparable, then* $u \in X$ *is a coincidence point* f, g, S *and* T.

Definition 1.13 ([25]) Let $(X, d, \preceq)$ be an ordered metric space. We say that X is regular if the following hypothesis holds: if $\{x_n\}$ is a non-decreasing sequence in X with respect to $\preceq$ such that $x_n \to x \in X$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Theorem 1.14 ([17]) *Let $(X, d, \preceq)$ be a partially ordered complete metric space such that X is regular. Let $f, g, S, T : X \to X$ be given mappings satisfying the following*:

(i) $fX \subseteq TX$, $gX \subseteq SX$,
(ii) SX *and* TX *are closed subsets of* (X, d),
(iii) *pairs* (f, S) *and* (g, T) *are weakly compatible,*
(iv) (f, g) *is partially weakly increasing with respect to* T *and* (g, f) *is partially weakly increasing with respect to* S.

Suppose that for every $x, y \in X$ such that Sx and Ty are comparable, (1) *holds. Then the pairs (f, S) and (g, T) have a coincidence point $u \in X$.*

In this paper, an attempt is made to derive some coincidence and common fixed point theorems for four mappings on complete ordered metric spaces, satisfying a (ψ, α, β)-weak contractive condition, which generalizes the existing results. Our results are supported by some examples.

2 Coincidence and common fixed point results

We begin our study with the following result.

Theorem 2.1 *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let $f, g, S, T : X \to X$ be given mappings satisfying*:
- (i) $fX \subseteq TX$, $gX \subseteq SX$,
- (ii) *f, g, S and T are continuous,*
- (iii) *the pairs (f, S) and (g, T) are compatible,*
- (iv) *(f, g) is partially weakly increasing with respect to T and (g, f) is partially weakly increasing with respect to S.*

Suppose that for every $x, y \in X$ such that Sx and Ty are comparable, we have

$$\psi\big(d(fx, gy)\big) \le \alpha\big(M(x, y)\big) - \beta\big(N(x, y)\big), \tag{2}$$

where

$$M(x, y) = \max\left\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), \frac{1}{2}\big[d(Sx, gy) + d(fx, Ty)\big]\right\},$$
$$N(x, y) = \max\big\{d(Sx, Ty), d(Sx, fx), d(Ty, gy)\big\},$$

and $\psi, \alpha, \beta : [0, +\infty) \to [0, +\infty)$ are such that ψ is a continuous and monotone non-decreasing function, α is an upper semi-continuous function, β is a lower semi-continuous function and for all $t > 0$,

$$\psi(t) - \alpha(t) + \beta(t) > 0. \tag{3}$$

Then the pairs (f, S) and (g, T) have a coincidence point $u \in X$; that is, $fu = Su$ and $gu = Tu$. Moreover, if Su and Tu are comparable, then $u \in X$ is a coincidence point of f, g, S and T.

Proof Let x_0 be an arbitrary point in X. Since $fX \subseteq TX$, there exists $x_1 \in X$ such that $Tx_1 = fx_0$. Since $gX \subseteq SX$, there exists $x_2 \in X$ such that $Sx_2 = gx_1$. Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X defined by

$$y_{2n} = fx_{2n} = Tx_{2n+1}, \qquad y_{2n+1} = gx_{2n+1} = Sx_{2n+2}, \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{4}$$

By construction we have $x_{2n+1} \in T^{-1}(fx_{2n})$. Then, using the fact that (f, g) is partially weakly increasing with respect to T, we obtain

$$Tx_{2n+1} = fx_{2n} \preceq gx_{2n+1} = Sx_{2n+2}, \quad \forall n \in \mathbb{N} \cup \{0\}.$$

On the other hand, we have $x_{2n+2} \in S^{-1}(gx_{2n+1})$. Then, using the fact that (g,f) is partially weakly increasing with respect to S, we obtain

$$Sx_{2n+2} = gx_{2n+1} \preceq fx_{2n+2} = Tx_{2n+3}, \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Therefore, we can then write

$$Tx_1 \preceq Sx_2 \preceq Tx_3 \preceq \cdots \preceq Tx_{2n+1} \preceq Sx_{2n+2} \preceq Tx_{2n+3} \preceq \cdots$$

or

$$y_0 \preceq y_1 \preceq y_2 \preceq \cdots \preceq y_{2n} \preceq y_{2n+1} \preceq y_{2n+2} \preceq \cdots. \tag{5}$$

We will prove our result in four steps.

Step 1.

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{6}$$

Since Sx_{2n} and Tx_{2n+1} are comparable, by applying inequality (2), we have

$$\begin{aligned}\psi\big(d(y_{2n}, y_{2n+1})\big) &= \psi\big(d(fx_{2n}, gx_{2n+1})\big) \\ &\leq \alpha\big(M(x_{2n}, x_{2n+1})\big) - \beta\big(N(x_{2n}, x_{2n+1})\big),\end{aligned} \tag{7}$$

where

$$\begin{aligned}M(x_{2n}, x_{2n+1}) &= \max\Big\{d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, fx_{2n}), d(Tx_{2n+1}, gx_{2n+1}), \\ &\qquad \frac{1}{2}\big[d(Sx_{2n}, gx_{2n+1}) + d(fx_{2n}, Tx_{2n+1})\big]\Big\} \\ &= \max\Big\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\ &\qquad \frac{1}{2}\big[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})\big]\Big\} \\ &= \max\Big\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \frac{1}{2}d(y_{2n-1}, y_{2n+1})\Big\}.\end{aligned}$$

Since $\frac{1}{2}d(y_{2n-1}, y_{2n+1}) \leq \frac{1}{2}[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]$, it follows that

$$M(x_{2n}, x_{2n+1}) = \max\big\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\big\} \tag{8}$$

and

$$\begin{aligned}N(x_{2n}, x_{2n+1}) &= \max\big\{d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, fx_{2n}), d(Tx_{2n+1}, gx_{2n+1})\big\} \\ &= \max\big\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\big\} \\ &= \max\big\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\big\}.\end{aligned} \tag{9}$$

If $d(y_{2n-1}, y_{2n}) < d(y_{2n}, y_{2n+1})$, then it follows from (8) and (9) that

$$M(x_{2n}, x_{2n+1}) = N(x_{2n}, x_{2n+1}) = d(y_{2n}, y_{2n+1}).$$

Therefore, (7) implies that

$$\psi\bigl(d(y_{2n}, y_{2n+1})\bigr) \le \alpha\bigl(d(y_{2n}, y_{2n+1})\bigr) - \beta\bigl(d(y_{2n}, y_{2n+1})\bigr). \tag{10}$$

By (3), we have $d(y_{2n}, y_{2n+1}) = 0$; that is, $y_{2n} = y_{2n+1}$, and consequently we obtain

$$M(x_{2n+2}, x_{2n+1}) = N(x_{2n+2}, x_{2n+1}) = d(y_{2n+1}, y_{2n+2}).$$

Now, by applying inequality (2), we have

$$\begin{aligned}\psi\bigl(d(y_{2n+2}, y_{2n+1})\bigr) &= \psi\bigl(d(fx_{2n+2}, gx_{2n+1})\bigr) \le \alpha\bigl(M(x_{2n+2}, x_{2n+1})\bigr) - \beta\bigl(N(x_{2n+2}, x_{2n+1})\bigr) \\ &= \alpha\bigl(d(y_{2n+1}, y_{2n+2})\bigr) - \beta\bigl(d(y_{2n+1}, y_{2n+2})\bigr),\end{aligned}$$

and (3) implies that $d(y_{2n+1}, y_{2n+2}) = 0$; that is, $y_{2n+1} = y_{2n+2}$. Repeating the above process inductively, one obtains $y_k = y_{2n}$ for all $k \ge 2n$, which implies that (6) holds. On the other hand, if

$$d(y_{2n}, y_{2n+1}) \le d(y_{2n-1}, y_{2n}), \tag{11}$$

by a similar calculation we obtain

$$d(y_{2n+1}, y_{2n+2}) \le d(y_{2n}, y_{2n+1}). \tag{12}$$

Thus by (11) and (12) we obtain

$$d(y_{n+1}, y_{n+2}) \le d(y_n, y_{n+1}),$$

which implies that the sequence $\{d(y_n, y_{n+1})\}$ is monotonically non-increasing. Hence, there exists $r \ge 0$ such that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = r.$$

Taking the upper limit on both sides of (7) and using (8), (9), the upper semi-continuity of α, the lower semi-continuity of β and the continuity of ψ, we obtain $\psi(r) \le \alpha(r) - \beta(r)$, which by (3) implies that $r = 0$. So equation (6) holds and the proof of Step 1 is completed.

Step 2. We claim that $\{y_n\}$ is a Cauchy sequence in X. By (6), it suffices to show that the subsequence $\{y_{2n}\}$ of $\{y_n\}$ is a Cauchy sequence in X. If not, then there exists $\epsilon > 0$ for which we can find two subsequences $\{y_{2m(k)}\}$ and $\{y_{2n(k)}\}$ of $\{y_{2n}\}$ such that $n(k)$ is the smallest integer and, for all $k > 0$,

$$n(k) > m(k) > k, \quad d(y_{2m(k)}, y_{2n(k)}) \ge \epsilon. \tag{13}$$

This means that

$$d(y_{2m(k)}, y_{2n(k)-2}) < \epsilon. \tag{14}$$

Therefore we use (13), (14) and the triangular inequality to get

$$\begin{aligned}\epsilon \le d(y_{2m(k)}, y_{2n(k)}) &\le d(y_{2m(k)}, y_{2n(k)-2}) + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}) \\ &< \epsilon + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}).\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (6), we obtain

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) = \epsilon. \tag{15}$$

Again, using the triangular inequality, we have

$$\left|d(y_{2m(k)-1}, y_{2n(k)}) - d(y_{2m(k)}, y_{2n(k)})\right| \le d(y_{2m(k)-1}, y_{2m(k)}).$$

Letting again $k \to \infty$ in the above inequality and using (6), (15), we get

$$\lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)}) = \epsilon. \tag{16}$$

On the other hand we have

$$d(y_{2m(k)}, y_{2n(k)}) \le d(y_{2m(k)}, y_{2n(k)+1}) + d(y_{2n(k)+1}, y_{2n(k)}).$$

Thanks to (6), (15), letting $k \to \infty$, we have from the above inequality that

$$\epsilon \le \lim_{n\to\infty} d(y_{2m(k)}, y_{2n(k)+1}). \tag{17}$$

Also, by the triangular inequality, we have

$$d(y_{2m(k)}, y_{2n(k)}) \le d(y_{2m(k)}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2n(k)+1}) + d(y_{2n(k)+1}, y_{2n(k)}).$$

Letting again $k \to \infty$ in the above inequality and using (6) and (15), we obtain

$$\epsilon \le \lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)+1}).$$

Similarly, we can show that $\lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)+1}) \le \epsilon$, so

$$\lim_{n\to\infty} d(y_{2m(k)-1}, y_{2n(k)+1}) = \epsilon. \tag{18}$$

From (2) we have

$$\begin{aligned}\psi\left(d(y_{2m(k)}, y_{2n(k)+1})\right) &= \psi\left(d(fx_{2m(k)}, gx_{2n(k)+1})\right) \\ &\le \alpha\left(M(x_{2m(k)}, x_{2n(k)+1})\right) - \beta\left(N(x_{2m(k)}, x_{2n(k)+1})\right),\end{aligned} \tag{19}$$

where

$$\begin{aligned}M(x_{2m(k)},x_{2n(k)+1}) &= \max\Big\{d(Sx_{2m(k)},Tx_{2n(k)+1}),d(Sx_{2m(k)},fx_{2m(k)}),d(Tx_{2n(k)+1},gx_{2n(k)+1}),\\&\qquad\frac{1}{2}\big[d(Sx_{2m(k)},gx_{2n(k)+1})+d(fx_{2m(k)},Tx_{2n(k)+1})\big]\Big\}\\&= \max\Big\{d(y_{2m(k)-1},y_{2n(k)}),d(y_{2m(k)-1},y_{2m(k)}),d(y_{2n(k)},y_{2n(k)+1}),\\&\qquad\frac{1}{2}\big[d(y_{2m(k)-1},y_{2n(k)+1})+d(y_{2m(k)},y_{2n(k)})\big]\Big\}\end{aligned}$$

and

$$\begin{aligned}N(x_{2m(k)},x_{2n(k)+1}) &= \max\big\{d(Sx_{2m(k)},Tx_{2n(k)+1}),d(Sx_{2m(k)},fx_{2m(k)}),d(Tx_{2n(k)+1},gx_{2n(k)+1})\big\}\\&= \max\big\{d(y_{2m(k)-1},y_{2n(k)}),d(y_{2m(k)-1},y_{2m(k)}),d(y_{2n(k)},y_{2n(k)+1})\big\}.\end{aligned}$$

Since ψ is a non-decreasing function, (17) implies that

$$\psi(\epsilon)\le\psi\Big(\lim_{k\to\infty}d(y_{2m(k)},y_{2n(k)+1})\Big). \tag{20}$$

Taking the upper limit on both sides of (19) and using (6), (15), (16), (18), (20) and the upper semi-continuity of α, the lower semi-continuity of β and the continuity of ψ, we obtain

$$\psi(\epsilon)\le\alpha\left(\max\left\{\epsilon,0,0,\frac{1}{2}(\epsilon+\epsilon)\right\}\right)-\beta\big(\max\{\epsilon,0,0\}\big).$$

By (3), we have $\epsilon=0$, which is a contradiction. Thus $\{y_{2n}\}$ is a Cauchy sequence in X, and hence $\{y_n\}$ is a Cauchy sequence.

Step 3. Existence of a coincidence point for (f,S) and (g,T).

From the completeness of (X,d), there is $u\in X$ such that

$$\lim_{n\to\infty}y_n=u. \tag{21}$$

From (4) and (21), we obtain

$$\begin{aligned}&d(Sx_{2n},u)\to 0,\qquad d(fx_{2n},u)\to 0,\\&d(Sx_{2n+2},u)\to 0,\qquad d(gx_{2n+1},u)\to 0,\qquad d(Tx_{2n+1},u)\to 0.\end{aligned} \tag{22}$$

Since the pairs (f,S) and (g,T) are compatible,

$$d\big(S(fx_{2n}),f(Sx_{2n})\big)\to 0,\qquad d\big(T(gx_{2n+1}),g(Tx_{2n+1})\big)\to 0. \tag{23}$$

Using the continuity of f, g, S, T and (22), we have

$$\begin{aligned}&d\big(f(Sx_{2n}),fu\big)\to 0,\qquad d\big(g(Tx_{2n+1}),gu\big)\to 0,\\&d\big(S(Tx_{2n+1}),Su\big)\to 0,\qquad d\big(T(Sx_{2n+2}),Tu\big)\to 0.\end{aligned} \tag{24}$$

The triangular inequality and (4) yield

$$d(Su,fu) \le d\big(Su, S(Tx_{2n+1})\big) + d\big(S(fx_{2n}), f(Sx_{2n})\big) + d\big(f(Sx_{2n}), fu\big),$$
$$d(Tu,gu) \le d\big(Tu, T(Sx_{2n+2})\big) + d\big(T(gx_{2n+1}), g(Tx_{2n+1})\big) + d\big(g(Tx_{2n+1}), gu\big).$$

Taking $n \to \infty$ and using (23) and (24), we obtain

$$d(Su,fu) \le 0, \qquad d(Tu,gu) \le 0,$$

which means that $Su = fu$ and $Tu = gu$.

Step 4. The existence of a coincidence point for f, g, S and T.

Since Su and Tu are comparable, we can apply inequality (2)

$$\psi\big(d(fu,gu)\big) \le \alpha\big(M(u,u)\big) - \beta\big(N(u,u)\big),$$

where

$$\begin{aligned} M(u,u) &= \max\left\{d(Su,Tu), d(Su,fu), d(Tu,gu), \frac{1}{2}\big[d(Su,gu) + d(fu,Tu)\big]\right\} \\ &= \max\left\{d(Su,Tu), 0, 0, \frac{1}{2}\big[d(Su,Tu) + d(Su,Tu)\big]\right\} = d(Su,Tu) \end{aligned}$$

and

$$\begin{aligned} N(u,u) &= \max\big\{d(Su,Tu), d(Su,fu), d(Tu,gu)\big\} \\ &= \max\big\{d(Su,Tu), 0, 0\big\} = d(Su,Tu). \end{aligned}$$

Therefore we have

$$\psi\big(d(Su,Tu)\big) \le \alpha\big(d(Su,Tu)\big) - \beta\big(d(Su,Tu)\big).$$

By (3), we have $d(Su,Tu) = 0$; that is, $Su = Tu$. Therefore u is a coincidence point of f, g, S and T. □

Now, we relax the conditions of Theorem 2.1, the continuity of f, g, S and T and the compatibility of the pairs (f,S) and (g,T), and we replace them by other conditions in order to find the same result. This will be the purpose of the next theorems.

Theorem 2.2 *Let $(X,d,\preceq)$ be a partially ordered complete metric space such that X is regular. Let $f,g,S,T : X \to X$ be given mappings satisfying*:

(i) *$fX \subseteq TX$, $gX \subseteq SX$,*
(ii) *SX and TX are closed subsets of (X,d),*
(iii) *pairs (f,S) and (g,T) are weakly compatible,*
(iv) *(f,g) is partially weakly increasing with respect to T and (g,f) is partially weakly increasing with respect to S.*

Suppose that for every $x,y \in X$ such that Sx and Ty are comparable, (2) *holds.*

Then the pairs (f,S) and (g,T) have a coincidence point $u \in X$; that is, $fu = Su$ and $gu = Tu$. Moreover, if Su and Tu are comparable, then $u \in X$ is a coincidence point of f, g, S and T.

Proof We take the same sequences $\{x_n\}$ and $\{y_n\}$ as in the proof of Theorem 2.1. In particular, $\{y_n\}$ is a Cauchy sequence in (X,d). Hence, there exists $v \in X$ such that

$$\lim_{n\to\infty} y_n = v. \tag{25}$$

Since SX and TX are closed subsets of (X,d), there exist $u_1, u_2 \in X$ such that

$$y_{2n} = Tx_{2n+1} \to Tu_1, \qquad y_{2n+1} = Sx_{2n+2} \to Su_2.$$

Therefore $v = Tu_1 = Su_2$.

Since $\{y_n\}$ is a non-decreasing sequence and X is regular, it follows from (25) that $y_n \preceq v$ for all $n \in \mathbb{N} \cup \{0\}$. Hence,

$$Tx_{2n+1} = y_{2n} \preceq v = Su_2.$$

Applying inequality (2), we have

$$\begin{aligned}\psi\bigl(d(fu_2,y_{2n+1})\bigr) &= \psi\bigl(d(fu_2,gx_{2n+1})\bigr)\\ &\le \alpha\bigl(M(u_2,x_{2n+1})\bigr) - \beta\bigl(N(u_2,x_{2n+1})\bigr),\end{aligned} \tag{26}$$

where

$$\begin{aligned}M(u_2,x_{2n+1}) = \max\biggl\{&d(Su_2,Tx_{2n+1}), d(Su_2,fu_2), d(Tx_{2n+1},gx_{2n+1}),\\ &\frac{1}{2}\bigl[d(Su_2,gx_{2n+1}) + d(fu_2,Tx_{2n+1})\bigr]\biggr\}\\ = \max\biggl\{&d(v,y_{2n}), d(v,fu_2), d(y_{2n},y_{2n+1}), \frac{1}{2}\bigl[d(v,y_{2n+1}) + d(fu_2,y_{2n})\bigr]\biggr\}\end{aligned}$$

and

$$\begin{aligned}N(u_2,x_{2n+1}) &= \max\bigl\{d(Su_2,Tx_{2n+1}), d(Su_2,fu_2), d(Tx_{2n+1},gx_{2n+1})\bigr\}\\ &= \max\bigl\{d(v,y_{2n}), d(v,fu_2), d(y_{2n},y_{2n+1})\bigr\}.\end{aligned}$$

Letting $n \to \infty$ in (26) and using (25), we obtain

$$\psi\bigl(d(fu_2,v)\bigr) \le \alpha\biggl(\max\biggl\{0, d(v,fu_2), 0, \frac{1}{2}\bigl[0 + d(fu_2,v)\bigr]\biggr\}\biggr) - \beta\bigl(\max\bigl\{0, d(v,fu_2), 0\bigr\}\bigr)$$

or

$$\psi\bigl(d(v,fu_2)\bigr) \le \alpha\bigl(d(v,fu_2)\bigr) - \beta\bigl(d(v,fu_2)\bigr).$$

By (3), we have $d(v, fu_2) = 0$, and hence $v = fu_2$. Similarly, we have

$$Sx_{2n} = y_{2n-1} \preceq v = Tu_1.$$

Therefore we can apply inequality (2) to obtain

$$\begin{aligned}\psi\big(d(y_{2n}, gu_1)\big) &= \psi\big(d(fx_{2n}, gu_1)\big)\\ &\leq \alpha\big(M(x_{2n}, u_1)\big) - \beta\big(N(x_{2n}, u_1)\big),\end{aligned} \tag{27}$$

where

$$\begin{aligned}M(x_{2n}, u_1) &= \max\bigg\{d(Sx_{2n}, Tu_1), d(Sx_{2n}, fx_{2n}), d(Tu_1, gu_1),\\ &\quad \frac{1}{2}\big[d(Sx_{2n}, gu_1) + d(fx_{2n}, Tu_1)\big]\bigg\}\\ &= \max\bigg\{d(y_{2n-1}, v), d(y_{2n-1}, y_{2n}), d(v, gu_1), \frac{1}{2}\big[d(y_{2n-1}, gu_1) + d(y_{2n}, v)\big]\bigg\}\end{aligned}$$

and

$$\begin{aligned}N(x_{2n}, u_1) &= \max\big\{d(Sx_{2n}, Tu_1), d(Sx_{2n}, fx_{2n}), d(Tu_1, gu_1)\big\}\\ &= \max\big\{d(y_{2n-1}, v), d(y_{2n-1}, y_{2n}), d(v, gu_1)\big\}.\end{aligned}$$

Letting $n \to \infty$ in (27) and using (25), we obtain

$$\psi\big(d(v, gu_1)\big) \leq \alpha\bigg(\max\bigg\{0, 0, d(v, gu_1), \frac{1}{2}\big[d(v, gu_1) + 0\big]\bigg\}\bigg) - \beta\big(\max\big\{0, 0, d(v, gu_1)\big\}\big)$$

or

$$\psi\big(d(v, gu_1)\big) \leq \alpha\big(d(v, gu_1)\big) - \beta\big(d(v, gu_1)\big).$$

By (3), we have $d(v, gu_1) = 0$ and hence $v = gu_1$.

Therefore we have obtained

$$v = Su_2 = fu_2, \qquad v = Tu_1 = gu_1.$$

Now, if (f, S) and (g, T) are weakly compatible, then $fv = fSu_2 = Sfu_2 = Sv$ and $gv = gTu_1 = Tgu_1 = Tv$, and v is a coincidence point of (f, S) and (g, T).

The rest of the conclusion follows as in the proof of Theorem 2.1. □

Definition 2.3 ([27]) Let (X, d) be a metric space and $f, g : X \to X$ be given self-mappings on X. The pair (f, g) is said to be semi-compatible if the two conditions hold:

(i) $ft = gt$ implies $fgt = gft$,

(ii) $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$, implies $\lim_{n\to\infty} fgx_n = gt$.

Singh and Jain [28] observe that (ii) implies (i). Hence, they defined the semi-compatibility by condition (ii) only. It is clear that if the pair (f, g) is semi-compatible, then (f, g) is weakly compatible.

Definition 2.4 ([29]) Let (X,d) be a metric space and $f,g: X \to X$ be given self-mappings on X. The pair (f,g) is said to be reciprocally continuous if $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Definition 2.5 ([30]) Let (X,d) be a metric space and $f,g: X \to X$ be given self-mappings on X. The pair (f,g) is said to be f-weak reciprocally continuous if $\lim_{n\to\infty} fgx_n = ft$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

In the next theorem, the concepts of semi-compatibility and f-weakly reciprocal continuity are used.

Theorem 2.6 *Let $(X,d,\preceq)$ be a partially ordered complete metric space. Let $f,g,S,T: X \to X$ be given mappings satisfying*:

(i) $fX \subseteq TX$, $gX \subseteq SX$,
(ii) *the pair (f,S) is f-weak reciprocally continuous and semi-compatible*,
(iii) *the pair (g,T) is g-weak reciprocally continuous and semi-compatible*,
(iv) *(f,g) is partially weakly increasing with respect to T and (g,f) is partially weakly increasing with respect to S.*

Suppose that for every $x,y \in X$ such that Sx and Ty are comparable, (2) *holds*.

Then the pairs (f,S) and (g,T) have a coincidence point $u \in X$; that is, $fu = Su$ and $gu = Tu$. Moreover, if Su and Tu are comparable, then $u \in X$ is a coincidence point f, g, S and T.

Proof We take the same sequences $\{x_n\}$ and $\{y_n\}$ as in the proof of Theorem 2.1. In particular, $\{y_n\}$ is a Cauchy sequence in (X,d). Hence, there exists $u \in X$ such that

$$\lim_{n\to\infty} y_n = u. \tag{28}$$

From (4) and (28), we obtain

$$d(Sx_{2n},u) \to 0 \quad \text{and} \quad d(fx_{2n},u) \to 0.$$

Hence by (ii) we deduce that

$$\lim_{n\to\infty} fSx_{2n} = fu \quad \text{and} \quad \lim_{n\to\infty} fSx_{2n} = Su,$$

which implies that $fu = Su$. Similarly, we can apply (4) and (28) to obtain

$$d(gx_{2n+1},u) \to 0 \quad \text{and} \quad d(Tx_{2n+1},u) \to 0.$$

Hence by (iii) we deduce that

$$\lim_{n\to\infty} gTx_{2n+1} = gu \quad \text{and} \quad \lim_{n\to\infty} gTx_{2n+1} = Tu,$$

which implies that $gu = Tu$. Therefore, we have proved that u is a coincidence point of (f,S) and (g,T).

The rest of the conclusion follows as in the proof of Theorem 2.1. □

Now, we shall prove the existence and uniqueness theorem of a common fixed point.

Theorem 2.7 *If, in addition to the hypotheses of Theorems* 2.1, 2.2 *and* 2.6, *we suppose that Tu with T^2u and Su with S^2u are comparable, where u is a coincidence point of f, g, S and T, then f, g, S and T have a common fixed point in X. Moreover, if a set of fixed points of one of the mappings f, g, S and T is totally ordered, then f, g, S and T have a unique common fixed point.*

Proof We set

$$w := Su = fu = Tu = gu. \tag{29}$$

Since the pair (g, T) is compatible in Theorem 2.1, the pair (g, T) is weakly compatible in Theorem 2.2 and the pair (g, T) is semi-compatible in Theorem 2.6, we have

$$gw = gTu = Tgu = Tw. \tag{30}$$

Since Tu and TTu are comparable, it follows that Su and Tw are comparable. Applying inequality (2) and using (29) and (30), we obtain

$$\psi\big(d(w,gw)\big) = \psi\big(d(fu,gw)\big) \le \alpha\big(M(u,w)\big) - \beta\big(N(u,w)\big), \tag{31}$$

where

$$\begin{aligned} M(u,w) &= \max\left\{d(Su,Tw), d(Su,fu), d(Tw,gw), \frac{1}{2}\big[d(Su,gw) + d(fu,Tw)\big]\right\} \\ &= \max\left\{d(w,gw), d(w,w), d(gw,gw), \frac{1}{2}\big[d(w,gw) + d(w,gw)\big]\right\} \\ &= \max\big\{d(w,gw), 0, 0, d(w,gw)\big\} = d(w,gw) \end{aligned}$$

and

$$\begin{aligned} N(u,w) &= \max\big\{d(Su,Tw), d(Su,fu), d(Tw,gw)\big\} \\ &= \max\big\{d(w,gw), d(w,w), d(gw,gw)\big\} = d(w,gw). \end{aligned}$$

Therefore, (31) implies that

$$\psi\big(d(w,gw)\big) \le \alpha\big(d(w,gw)\big) - \beta\big(d(w,gw)\big).$$

By (3), we have $d(w,gw) = 0$, that is, $w = gw$. Then, by (30), we have

$$w = gw = Tw. \tag{32}$$

Similarly, we can show that

$$w = fw = Sw. \tag{33}$$

Hence, by (32) and (33), we deduce that $w = fw = gw = Sw = Tw$. Therefore w is a common fixed point of f, g, S and T.

Now, suppose that the set of fixed points of f is totally ordered. Assume on the contrary that $fp = gp = Sp = Tp = p$ and $fq = gq = Sq = Tq$ but $p \neq q$. Since p and q contain a set of fixed points of f, we obtain $p = Sp$ and $q = Tq$ are comparable, by inequality (2), we have

$$\psi\bigl(d(p,q)\bigr) = \psi\bigl(d(fp,gq)\bigr) \le \alpha\bigl(M(p,q)\bigr) - \beta\bigl(N(p,q)\bigr), \tag{34}$$

where

$$\begin{aligned} M(p,q) &= \max\left\{d(Sp,Tq), d(Sp,fp), d(Tq,gq), \frac{1}{2}\bigl[d(Sp,gq) + d(fp,Tq)\bigr]\right\} \\ &= \max\left\{d(p,q), d(p,p), d(q,q), \frac{1}{2}\bigl[d(p,q) + d(p,q)\bigr]\right\} = d(p,q) \end{aligned}$$

and

$$\begin{aligned} N(p,q) &= \max\bigl\{d(Sp,Tq), d(Sp,fp), d(Tq,gq)\bigr\} \\ &= \max\bigl\{d(p,q), d(p,p), d(q,q)\bigr\} = d(p,q). \end{aligned}$$

Therefore, (34) implies that

$$\psi\bigl(d(p,q)\bigr) \le \alpha\bigl(d(p,q)\bigr) - \beta\bigl(d(p,q)\bigr),$$

by (3), $d(p,q) = 0$, a contradiction. Therefore f, g, S and T have a unique common fixed point. Similarly, the result follows when the set of fixed points of g, S or T is totally ordered. This completes the proof of Theorem 2.7. □

3 Some examples

In this section we present some examples which illustrate our results.

Now, we present an example to illustrate the obtained result given by the previous theorems.

Example 3.1 Let $X = [0,+\infty)$. We define an order $\preceq$ on X as $x \preceq y$ if and only if $x \ge y$ for all $x, y \in X$. We take the usual metric $d(x,y) = |x - y|$ for $x, y \in X$. It is easy to see that $(X, d, \preceq)$ is a partially ordered complete metric space. Let $f, g, S, T : X \to X$ be defined by

$$fx = \ln(1+x), \qquad gx = \ln\left(1 + \frac{x}{3}\right), \qquad Sx = e^{3x} - 1, \qquad Tx = e^{x} - 1.$$

Define $\psi, \alpha, \beta : [0,+\infty) \to [0,+\infty)$ by $\psi(t) = t$,

$$\alpha(t) = \begin{cases} \frac{1}{3}t & \text{if } 0 \le t < 1, \\ \frac{1}{3}t + \frac{1}{2} & \text{if } t \ge 1 \end{cases} \quad \text{and} \quad \beta(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1, \\ \frac{1}{2} & \text{if } t > 1. \end{cases}$$

Then f, g, S, T, ψ, α and β satisfy all the hypotheses of Theorems 2.1 and 2.7.

Proof The proof of (i) and (ii) is clear. To prove (iii), let $\{x_n\}$ be any sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$. Since $fx_n = \ln(1 + x_n)$ and $Sx_n = e^{3x_n} - 1$, we have $x_n \to e^t - 1$ and $x_n \to \frac{1}{3}\ln(1 + t)$. By the uniqueness of limit, we get that $e^t - 1 = \frac{1}{3}\ln(1 + t)$ and hence $t = 0$. Thus, $x_n \to 0$ as $n \to \infty$. Since f and S are continuous, we have $fx_n \to f0 = 0$ and $Sx_n \to S0 = 0$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} d\bigl(S(fx_n), f(Sx_n)\bigr) = d(S0, f0) = d(0,0) = 0.$$

Thus, the pair (f,S) is compatible. Similarly, one can show that the pair (g,T) is compatible.

To prove that (f,g) is partially weakly increasing with respect to T, let $x, y \in X$ be such that $y \in T^{-1}(fx)$. Then $Ty = fx$. By the definition of f and T, we have $\ln(1 + x) = e^y - 1$. So, we have $y = \ln(1 + \ln(1 + x))$. Now, since $e^{3x} - 1 \geq 3x \geq x \geq \ln(1 + x)$, we have

$$1 + x \geq 1 + \frac{1}{3}\ln\bigl(1 + \ln(1 + x)\bigr)$$

or

$$fx = \ln(1 + x) \geq \ln\left(1 + \frac{1}{3}\ln\bigl(1 + \ln(1 + x)\bigr)\right) = \ln\left(1 + \frac{y}{3}\right) = gy.$$

Therefore, $fx \preceq gy$. Thus, we have proved that (f,g) is partially weakly increasing with respect to T. Similarly, one can show that (g,f) is partially weakly increasing with respect to S.

Now, we prove that ψ, α and β do satisfy the inequality of (3). If $t > 1$, then $\psi(t) - \alpha(t) + \beta(t) = t - \frac{1}{3}t - \frac{1}{2} + \frac{1}{2} = \frac{2}{3}t > 0$; if $t = 1$, then $\psi(1) - \alpha(1) + \beta(1) = 1 - \frac{1}{3} - \frac{1}{2} = \frac{1}{6} > 0$. And if $0 \leq t < 1$, then $\psi(t) - \alpha(t) + \beta(t) = t - \frac{1}{3}t = \frac{2}{3}t > 0$.

In order to show that f, g, S, T, ψ, α and β do satisfy the contractive condition (2) in Theorem 2.1, using a mean value theorem, we have, for $x, y \in X$,

$$|fx - gy| = \left|\ln(1 + x) - \ln\left(1 + \frac{y}{3}\right)\right| \leq \frac{1}{3}|3x - y| \leq \frac{1}{3}\left|e^{3x} - e^y\right| = \frac{1}{3}|Sx - Ty| \leq \frac{1}{3}M(x,y).$$

Then the following cases are possible.

Case I. $M(x,y) \geq 1$.

In this case, we have $N(x,y) > 1$ or $N(x,y) \leq 1$. If $N(x,y) > 1$, then $\alpha(M(x,y)) = \frac{1}{3}M(x,y) + \frac{1}{2}$ and $\beta(N(x,y)) = \frac{1}{2}$. Therefore, we have

$$\psi\bigl(d(fx,gy)\bigr) = |fx - gy| \leq \frac{1}{3}M(x,y) = \frac{1}{3}M(x,y) + \frac{1}{2} - \frac{1}{2} = \alpha\bigl(M(x,y)\bigr) - \beta\bigl(N(x,y)\bigr).$$

If $N(x,y) \leq 1$, then $\alpha(M(x,y)) = \frac{1}{3}M(x,y) + \frac{1}{2}$ and $\beta(N(x,y)) = 0$. Therefore, we have

$$\psi\bigl(d(fx,gy)\bigr) = |fx - gy| \leq \frac{1}{3}M(x,y) < \frac{1}{3}M(x,y) + \frac{1}{2} - 0 = \alpha\bigl(M(x,y)\bigr) - \beta\bigl(N(x,y)\bigr).$$

Therefore in this case (2) is satisfied.

Case II. $M(x,y) < 1$.

In this case, since $N(x,y) \leq M(x,y)$, we obtain $N(x,y) < 1$. Therefore, we have $\alpha(M(x,y)) = \frac{1}{3}M(x,y)$ and $\beta(N(x,y)) = 0$. So, we obtain

$$\psi\bigl(d(fx,gy)\bigr) = |fx - gy| \leq \frac{1}{3}M(x,y) - 0 = \alpha\bigl(M(x,y)\bigr) - \beta\bigl(N(x,y)\bigr).$$

Therefore in this case (2) is satisfied.

Thus, f, g, S, T, ψ and φ satisfy all the hypotheses of Theorems 2.1. Therefore, f, g, S and T have a coincidence point. Moreover, since f, g, S and T satisfy all the hypotheses of Theorem 2.7, we obtain that f, g, S and T have a unique common fixed point. In fact, 0 is the unique common fixed point of f, g, S and T. □

Clearly, the above example satisfies all the hypotheses of Theorem 2.6.

Example 3.2 Let $X = \{1,2,3,4\}$. Let $d : X \times X \to \mathbb{R}$ be given as

$$d(x,y) = \begin{cases} 0, & x = y, \\ x + y, & x \neq y \end{cases}$$

and $\preceq := \{(1,1),(2,2),(3,3),(4,4),(1,4),(2,4),(3,4)\}$ on X. Clearly, $(X,d,\preceq)$ is a partially ordered complete metric space.

Let $\{x_n\}$ be a non-decreasing sequence in X with respect to $\preceq$ such that $x_n \to x$. By the definition of metric d, there exists $k \in \mathbb{N}$ such that $x_n = x$ for all $n \geq k$. So $(X,d,\preceq)$ is regular.

Let $\psi,\alpha,\beta : [0,\infty) \to [0,\infty)$ be defined by $\psi(t) = t$,

$$\alpha(t) = \begin{cases} \frac{3}{4}t + \frac{1}{t}, & t \geq 5, \\ 1, & t < 5 \end{cases} \quad \text{and} \quad \beta(t) = \begin{cases} \frac{1}{t}, & t > 5, \\ 1 + t^2, & t \leq 5 \end{cases}$$

and self-maps f, g, S and T on X be given by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 3 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 3 \end{pmatrix},$$
$$S = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}.$$

It is easy to see that f, g, S and T satisfy all the conditions given in Theorem 2.2. Thus 1, 2 and 3 are coincidence points of the pairs (f,S) and (g,T). Since $S1 = 2$ and $T1 = 2$ are comparable, 1 is a coincidence point f, g, S and T. Moreover, since $S1 = 2$ and $SS1 = 1$ are not comparable, so Theorem 2.7 is not applicable for this example. It is observed that 1 is not a common fixed point f, g, S and T.

4 Application: existence of a common solution to integral equations

Consider the integral equations:

$$\begin{aligned} x(t) &= \int_a^b K_1\bigl(t,s,x(s)\bigr)\,ds, \\ x(t) &= \int_a^b K_2\bigl(t,s,x(s)\bigr)\,ds, \end{aligned} \tag{35}$$

where $b > a \geq 0$. The purpose of this section is to give an existence theorem for a solution of (35) using Theorem 2.1 or 2.2.

Theorem 4.1 *Consider the integral equations* (35).

(i) $K_1, K_2 : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ *are continuous*;

(ii) *for all* $t, s \in [a,b]$,

$$K_1\bigl(t,s,x(s)\bigr) \leq K_2\left(t,s,\int_a^b K_1\bigl(s,\tau,x(\tau)\bigr)\,d\tau\right),$$

$$K_2\bigl(t,s,x(s)\bigr) \leq K_1\left(t,s,\int_a^b K_2\bigl(s,\tau,x(\tau)\bigr)\,d\tau\right),$$

(iii) *for all* $s, t \in [a,b]$ *and comparable* $u, v \in \mathbb{R}$,

$$\bigl|K_1(t,s,u) - K_2(t,s,v)\bigr|^2 \leq p(t,s)\log\bigl(1 + |u-v|^2\bigr),$$

where $p : [a,b] \times [a,b] \to [0,+\infty)$ *is a continuous function satisfying*

$$\sup_{a\leq t\leq b}\int_a^b p(s,t)\,ds < \frac{1}{b-a}.$$

Then integral equations (35) *have a solution* $x \in C[a,b]$.

Proof Let $X := C[a,b]$ (the set of continuous functions defined on $C[a,b]$ and taking value in $\mathbb{R}$) with the usual supremum norm, that is, $\|x\| = \sup_{a\leq t\leq b}|x(t)|$, for $x \in C[a,b]$. Consider on X the partial order defined by

$$x, y \in X, \quad x \preceq y \quad \Longleftrightarrow \quad x(t) \leq y(t), \quad \forall t \in [a,b].$$

Then $(X, \preceq)$ is a partially ordered set and regular. Also $(X, \|\cdot\|)$ is a complete metric space. Define $f, g : X \to X$ by

$$fx(t) = \int_a^b K_1\bigl(t,s,x(s)\bigr)\,ds, \quad \forall t \in [a,b]$$

and

$$gx(t) = \int_a^b K_2\bigl(t,s,x(s)\bigr)\,ds, \quad \forall t \in [a,b].$$

Now, let $x, y \in X$ such that $x \preceq y$. From condition (iii), for all $t \in [a,b]$, we can write

$$\begin{aligned}
\bigl|fx(t) - gy(t)\bigr|^2 &\leq \left(\int_a^b \bigl|K_1\bigl(t,s,x(s)\bigr) - K_2\bigl(t,s,y(s)\bigr)\bigr|\,ds\right)^2 \\
&\leq \int_a^b 1^2\,ds \int_a^b \bigl|K_1\bigl(t,s,x(s)\bigr) - K_2\bigl(t,s,y(s)\bigr)\bigr|^2\,ds \\
&\leq (b-a)\int_a^b p(t,s)\log\bigl(1 + \bigl|x(s) - y(s)\bigr|^2\bigr)\,ds
\end{aligned}$$

$$\begin{aligned}
&\leq (b-a)\int_a^b p(t,s)\log\bigl(1+d(x,y)^2\bigr)\,ds \\
&= (b-a)\left(\int_a^b p(t,s)\,ds\right)\log\bigl(1+d(x,y)^2\bigr) \\
&< \log\bigl(1+d(x,y)^2\bigr) \leq \log\bigl(1+M(x,y)^2\bigr) \\
&= \frac{4}{3}M(x,y)^2 - \bigl(M(x,y)^2 - \log\bigl(1+M(x,y)^2\bigr)\bigr).
\end{aligned}$$

Since $M(x,y) \geq N(x,y)$ and $\phi(t) = t^2 - \log(1+t^2)$ is a non-decreasing function in $[0,\infty)$, we have

$$\Bigl(\sup_{a\leq t\leq b}\bigl|fx(t) - gy(t)\bigr|\Bigr)^2 \leq \frac{4}{3}M(x,y)^2 - \bigl(N(x,y)^2 - \log\bigl(1+N(x,y)^2\bigr)\bigr).$$

Put $\psi(t) = t^2$, $\alpha(t) = \frac{4}{3}t^2$ and $\beta(t) = t^2 - \log(1+t^2)$, we get

$$\psi\bigl(d(fx,gy)\bigr) \leq \alpha\bigl(M(x,y)\bigr) - \beta\bigl(N(x,y)\bigr),$$

and $\psi(t) - \alpha(t) + \beta(t) > 0$ for each $t > 0$. By taking $S = T = I_X$ (the identity mapping on X), all the required hypotheses of Theorem 2.1 (or Theorem 2.2) are satisfied. Then there exists $x \in X$, a common fixed point of f and g, that is, x is a solution to (35). $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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Received: 12 August 2014 Accepted: 9 December 2014 Published: 23 Dec 2014

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10.1186/1029-242X-2014-517
Cite this article as: Lo'lo' et al.: Existence of a common solution of an integral equations system by (ψ, α, β)-weakly contractions. *Journal of Inequalities and Applications* 2014, 2014:517